Theorem   2000. 1/26. pm 3:00 at Essen . Chin-Lung Wang   P. 1

let X and X' be (K-equiv) smooth v.

and assume that the pluricanonical system is free.

then $\chi(X, \underline{\underline{K_X^{\otimes r}}} \otimes \Omega_X^k) = \chi(X', \underline{\underline{K_{X'}^{r}}} \otimes \Omega_{X'}^k)$

$\forall\ r, k \in \mathbb{Z}$.

Rmk: In all places can replace X by any $\mathbb{h}^p$. the only problem is that now for ∞ set of r does not immed implies vali. for all $r \in \mathbb{Q}$ st. $K^{\otimes r}$ is actually Cartie

pf: the statement is true for curves and surfaces. since K-equiv $\Rightarrow$ isom. in codim 1.

By induction, suppose true for all dim $\leq n-1$.

let $\dim X = \dim X' = n$:

By Serre duality, may assume that $k \geq 0$.

since they are poly in r, so if prove for an ∞ set of number r, then done. (for any $r \in \mathbb{Q}$!)

In fact can take $D \longleftrightarrow D'$ be any pair of K-equiv div. i.e. $D =_K D'$ (then so is $K_X|_D \leftrightarrow K_{X'}'|_D$ as line bundles so usually may think first prove this case then ext. to line bundles).

Suppose sK_X, $sK_{X'}$ are free.

let $D \in |sK_X|$, $D' \in |sK_{X'}|$ be corr. divisor. (smooth)

then $K_D = (K_X + D)|_D = (s+1) K_X|_D$

$K_{D'} = (K_{X'} + D')|_{D'} = (s+1) K_{X'}|_{D'}$

\begin{tikzcd}
 & Y \arrow[dl, "\varphi"'] \arrow[dr, "\varphi'"] & \\
X & & X'
\end{tikzcd}

$$\varphi^* K_D = (s+1)\, \varphi^* K_X|_{\tilde{D}}$$

$$= (s+1)\, \varphi'^* K_{X'}|_{\tilde{D}} = \varphi'^* K_{D'}$$

so D, D' are still K-equiv.

notice. $\nexists$ ample K-equiv divisors.

claim: $\chi(X, \Omega_X^k(-\ell D)) = \chi(X', \Omega_{X'}^k(-\ell D'))$ $\forall\ \ell \geq 0$.

pf: for $\ell = 0$, this follows from the equivalence of Hodge numbers. (proved via Motivic integration).

for $\ell = 1$. consider

$$0 \to \Omega_X^k(-D) \to \Omega_X^k \to \Omega_X^k|_D \to 0.$$

then $\chi(\Omega_X^k) = \chi(\Omega_X^k(-D)) + \chi(\Omega_X^k|_D)$

by induction (on ℓ). $\chi(\Omega_X^k) = \chi(\Omega_{X'}^k)$

so only have to prove that $\chi(D, \Omega_X^k|_D) = \chi(D', \Omega_{X'}^k|_{D'})$

from $0 \to T_D \to T_X|_D \to N_D \to 0$

$$N_D = \mathcal{O}_D(D)$$

$\Rightarrow$ $0 \to \mathcal{O}_D(-D) \to \Omega_X^1|_D \to \Omega_D^1 \to 0$

$\Rightarrow$ $\Lambda^k(\Omega_X^1|_D) \cong \underbrace{\Omega_D^k} \oplus \underbrace{\Omega_D^{k-1} \otimes \mathcal{O}_D(-D)}$

i.e. $\chi(\Omega_X^k|_D) = \chi(\Omega_D^k) + \chi(\Omega_D^{k-1} \otimes \underline{\mathcal{O}_D(-D)})$

$$\underline{\mathcal{O}_D(-D)} = K_X|_D \otimes K_D^{-1}$$

by induction, $\chi(\Omega_D^k) = \chi(\Omega_{D'}^k)$

Also $D \sim sK_X$. $K_D = (s+1)K_X|_D \sim (1+\frac{1}{s}) D|_D$

so $\mathcal{O}_D(-D) \cong -\frac{s}{s+1} K_D$

i.e. $\chi(\Omega_D^{k-1} \otimes \mathcal{O}_D(-D)) = \chi(\Omega_D^{k-1} \otimes K_D^{(-\frac{s}{s+1})})$

by induction (on $\underline{\dim = X}$), get

$\chi(\Omega_D^{k-1} \otimes \mathcal{O}_D(-D)) = \chi(\Omega_{D'}^{k-1} \otimes \mathcal{O}_{D'}(-D'))$

combine $\Rightarrow$ $\chi(\Omega_X^k|_D) = \chi(\Omega_{X'}^k|_{D'})$. done.

for general ℓ is similar.

Suppose have been proved for $0, 1, \cdots, \ell$. then

$0 \to \Omega_X^k(-(\ell+1)D) \to \Omega_X^k(-\ell D) \to \Omega_X^k(-\ell D)|_D \to 0$

$\Rightarrow$ $\chi(\Omega_X^k(-\ell D)) = \chi(\Omega_X^k(-(\ell+1)D)) + \chi(\Omega_X^k(-\ell D)|_D)$

already know — want to show — need only to show this.

but $\Omega_X^k|_D = \Omega_D^k \oplus \Omega_D^{k-1} \otimes \mathcal{O}_D(-D)$

$\Rightarrow$ $\Omega_X^k(-\ell D)|_D = (\Omega_D^k \otimes \mathcal{O}_D(-\ell D)) \oplus (\Omega_D^{k-1} \otimes \mathcal{O}_D(-(\ell+1)D))$

ie, only need to show the RHS 2 terms are the same for D & D'.

Again this is by the result for dim $= n-1$

bec. any $\Omega_D^k \otimes \mathcal{O}_D(\lambda D) = \Omega_D^k \otimes K_D\left(-\frac{\lambda s}{s+1}\right)$ $\in \mathbb{Q}$.

It is OK. $\square$

this show that all smooth K-equiv. v. with free pluri-can. system have the same "twisted χ_y" genus!

ie. all chern numbers $c_1^{\,i} \cdot \chi_k^{\,n-i}$ are the same.

$$\underline{\chi_k := ch(\Omega^k).td(X)}$$

this generalizes totaro's result, in the case of good smooth min. models.

Notice that k negative means $\wedge^k T$.

in particular, $\chi(X, T)$ <u>allowed</u>.

- How to remove the minimal model assumption?
- Can one prove $\chi(X, \Omega_X^k) = \chi(X', \Omega_{X'}^k)$ directly without using equiv. of Hodge numbers?
- Instead of $\wedge^k(\Omega)$, how about $S^k(\Omega)$?

Q: Is

for X, X' general type.

$$\begin{tikzcd} & Y \arrow[dl, "\varphi"'] \arrow[dr, "\varphi'"] & \\ X \arrow[dr, "|sK_X|"', "f"] & & X' \arrow[dl, "|sK_{X'}|"] \\ & \bar{X} & \end{tikzcd}$$

if E has coeff $(k+1)$.

coh. small!

then k = codim of $\varphi(E)$ in X. so fiber dim $= n-1-k$

(wisniewski $\Rightarrow$ $n-1+ \cdots$ no!) if dim fiber $f = 2a$ (~~real dim~~)

$\to$ only for extr. rays? $?\ 2n-2(n-k-a) > 4a$

ie. $k+a > 2a$, ie $\underline{k > a}$?