ON THE MINIMALITY OF EXTRA CRITICAL POINTS OF GREEN FUNCTIONS ON FLAT TORI

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ABSTRACT. This is a sequel to [8, 2] to study the geometry of flat tori. In [8], we showed that the solvability of the mean field equation (MFE)

\[ \Delta u + e^u = \rho \delta_0 \]

on a flat torus \( E_\tau \) with \( \rho = 8\pi \) is equivalent to the existence of extra pair of critical points \( \pm p \) of the Green function \( G \). And such a pair, if exists, is unique. It was also announced there that \( G \) actually attains its minimum at \( \pm p \). Here our first main result is to confirm this statement by way of the variational form of the MFE. It implies that the solution \( u \) is a minimizer of the corresponding non-linear functional \( J_{8\pi}(u) \) (c.f. (1.1)), hence settles the existence problem of minimizers posed in [12].

We also prove the uniqueness of solution to the MFE when \( 0 < \rho < 8\pi \) and get the exact counting result of the number of solutions in terms of the number of critical points of \( G \) when \( \rho \) is close to \( 8\pi \). This allows us to analyze the bifurcation structure of the MFE when \( \rho \) crosses \( 8\pi \).

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0. INTRODUCTION

Consider the flat torus \( E = E_\tau = \mathbb{C}/\Lambda_\tau, \tau = a + bi, b > 0 \) and \( \Lambda = \Lambda_\tau = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \) with \( \omega_1 = 1 \) and \( \omega_2 = \tau \). Let \( G \) be the Green function on \( E \):

\[
\begin{cases}
-\Delta G = \delta_0 - \frac{1}{|E|} & \text{on } E, \\
\int_E G = 0,
\end{cases}
\]

where \( \delta_0 \) is the Dirac measure at the lattice point \( 0 \in E \). We continue our study, initiated in [8, 2, 6], on the critical points of \( G \):

\[ \nabla G(z) = 0. \]
Since $G$ is an even function on $E$, all the half-periods $\omega_k^2$ are critical points of $G$. A critical point $p$ is called a \textit{non-trivial critical point} of $G$ if $p$ is not one of the three half-periods. Clearly, non-trivial critical points appear in pair $\pm p$. It is natural to ask: how many pairs of non-trivial critical points might $G$ have? This has been answered completely in our previous paper [8]:

\textbf{Theorem A.} For any $\tau \in \mathcal{H}$, the Green function $G(z; \tau)$ on the flat torus $E_{\tau}$ has at most one pair of non-trivial critical points.

Thus $G$ has either 3 or 5 critical points. Following [8] we denote by $\Omega_3$ (resp. $\Omega_5$) the subset of the moduli $\mathcal{M}_1 = \mathbb{H}/\text{SL}(2, \mathbb{Z})$ where $G(z; \tau)$ on the flat torus $E_{\tau}$ has exactly 3 (resp. 5) critical points. See [8, 7] for the actual shape of the (simply connected) domain $\Omega_5$.

What is the nature of those extra critical points? We answer it in the following theorem, which had been announced in [8, §1 Theorem A]:

\textbf{Theorem 0.1.} Suppose that the pair of non-trivial critical points $\{\pm p\}$ of $G$ exists, then $\pm p$ are the minimal points of $G$.

We will present a proof of it in §1 based on the mean field equation

\begin{equation} \triangle u + e^u = 8\pi \delta_0 \quad \text{on $E$.} \end{equation}

In fact our proof shows that any solution to (0.3) must be a minimizer of the non-linear functional

$$J_{8\pi}(u) = \frac{1}{2} \int_E |\nabla u|^2 - 8\pi \log \int_E e^{-8\pi G + u}$$

on $u \in H^1(E) \cap \{ u \mid \int_E u = 0 \}$. This completely solves the existence problem on minimizers raised in [12] when the two vortex points collapse into one.

One important application of Theorem 0.1 is the following result:

\textbf{Corollary 0.2.} Suppose that the Green function $G$ has non-trivial critical points, then all the three half periods are saddle points of $G$. That is, the Hessian of $G$ is non-positive: $\det D^2 G(\omega_k^2) \leq 0$ for $k = 1, 2, 3$.

\textbf{Remark 0.3.} Based on Corollary 0.2, a stronger result is proved in [7]. Namely $\det D^2 G(\omega_k^2) < 0$ for $k = 1, 2, 3$ if $G$ satisfies the hypothesis of Corollary 0.2.

From the Weierstrass elliptic curve model $y^2 = 4x^3 - g_2x - g_3$ of $E_{\tau}$, we know that the half periods $E_{\tau}[2]$ are precisely the branch points of the map $x = \wp(z) : E_{\tau} \to \mathbb{P}^1$. A quantity $D(q)$ defined at any branch point is strongly related to the geometry of $E_{\tau}$ at $q$. In [1, 6] it was proved that if $u_k$ is a bubbling sequence of solutions to (0.3) with $\rho = \rho_k \to 8\pi$ (as $k \to \infty$), $\rho_k \neq 8\pi$ for large $k$, and with $q$ the blow-up point, then $q$ must be a half period point. In fact, asymptotically

\begin{equation} \rho_k - 8\pi = (D(q) + o(1)) e^{-\lambda_k} \end{equation}
where \( \lambda_k = \max_{E_r} u_k \) and

\[
D(q) := \int_{E_r} \frac{h(z)e^{8\pi(\tilde{G}(z,q) - \phi(q))} - h(q)}{|z - q|^4} - \int_{E_r} \frac{h(q)}{|z - q|^4}.
\]

Here \( h(z) = e^{-8\pi G(z)} \), \( \tilde{G}(z,q) \) is the regular part of the Green function, and \( \phi(q) = \tilde{G}(q,q) \). See §2 for more details. \( D(q) \) plays an important role in the construction of bubbling solutions to (0.3), as well as in other non-linear PDEs, with \( \rho_k \rightarrow 8\pi \). The sign of \( D(q) \) determines the direction where the bubbling may take place, namely \( \rho_k < 8\pi \) or \( \rho_k > 8\pi \). If \( q \) is a not half-period critical point then \( D(q) \) is still defined. But then \( D(q) = 0 \) since \( \rho_k = 8\pi \) for all \( k \) (large).

In general it is difficult to compute \( D(q) \) for a given torus. Nevertheless we will prove the following result in §2:

**Theorem 0.4.** For any half period \( q \in E_r \), \( \tau = a + bi \), we have

\[
(0.5) \quad D(q) = -4\pi^2 b e^{-8\pi G(q)} \det D^2 G(q).
\]

By Remark 0.3, we have \( D(q) > 0 \) if \( q \) is a saddle point. In particular if \( \tau \in \Omega_5 \) then \( D(q) > 0 \) for all half-periods. For any \( \tau \in \mathbb{H} \), \( D(q) \leq 0 \) if and only if \( q \) is the minimal point.

Combining with a recent technique in analyzing uniqueness of blow-up solutions [11], we will be able to classify all solutions to (0.3) for \( \rho \) in the range \((0,8\pi + \varepsilon_0)\) for some \( \varepsilon_0 > 0 \):

**Theorem 0.5.** For any torus \( E_r \), there is a small number \( \varepsilon_0 > 0 \) such that

(i) If \( \tau \in \Omega_3 \) then (0.3) has only one solution for \( \rho < 8\pi \), no solution for \( \rho = 8\pi \), and two solutions for \( 8\pi < \rho < 8\pi + \varepsilon_0 \).

(ii) If \( \tau \in \Omega_5 \) then (0.3) has only one solution for \( \rho < 8\pi \), infinitely many solutions for \( \rho = 8\pi \), and four solutions for \( 8\pi < \rho < 8\pi + \varepsilon_0 \).

In particular, the topological Leray–Schauder degree \( d_\rho \), which is 2 for \( \rho \in (8\pi,16\pi) \) [3, 4, 5, 6], does not reflect the actual number of solutions. The proof is presented in §3, which relies also on the theory of Lamé equations in [2] accompanied with (0.3) as well as the blow-up analysis in [4, 11].

**Remark 0.6.** In [9] (see also [2]), we proved that (0.3) with \( \rho = 12\pi \) has exactly two solutions on \( E_r \) for \( \tau \neq e^{\pi i/3} \). By Theorem 0.5, we see that when \( \tau \in \Omega_5 \) the bifurcation diagram of (0.3) is complicate for \( \rho \) ranging from \( 8\pi \) to \( 12\pi \). It is a natural question to ask if (0.3) has exactly two solutions for \( \rho \in (8\pi,16\pi) \) when \( \tau \in \Omega_3 \). Theorem 0.5 also reflects the difficulty in the study the corresponding Lamé equations for the case \( n \notin \frac{1}{2} \mathbb{N} \).

1. ON THE MINIMALITY OF EXTRA CRITICAL POINTS

**Theorem 1.1.** Let \( p \) be a critical point of \( G \) which is not a half period point, then \( p \) is a minimal point of \( G \).
Proof. Consider the even, normalized, \( L^1 \) Sobolev space

\[
H^1_{ev}(E) = \{ u \in H^1(E) \mid u(-z) = u(z), \int_E u = 0 \}
\]

and the non-linear functional

\begin{equation}
J_\rho(u) = \frac{1}{2} \int_E |\nabla u|^2 - \rho \log \int_E e^{-\rho G + u}, \quad u \in H^1_{ev}(E).
\end{equation}

It is well known that, as a consequence of the Moser–Trudinger inequality, \( J_\rho \) attains its minimum for \( \rho < 8\pi \). Let \( v_\rho \) be a minimizer of \( J_\rho \). Then \( v_\rho \) is an even solution of

\[
\triangle v + \rho \left( \frac{e^{-\rho G + v}}{\int_E e^{-\rho G + v}} - \frac{1}{|E|} \right) = 0 \quad \text{in } E.
\]

By the result of [8], when \( \rho \to 8\pi \), \( v_\rho \) converges to a smooth function \( v \) which satisfies

\begin{equation}
\triangle v + 8\pi \left( \frac{e^{-8\pi G + v}}{\int_E e^{-8\pi G + v}} - \frac{1}{|E|} \right) = 0 \quad \text{in } E.
\end{equation}

It is then obvious that

\[
u(z) = -8\pi G(z) + v(z) - \log \int_E e^{-8\pi G + v}
\]

is an even solution to the Liouville equation

\[
\triangle u + 8\pi e^u = 8\pi \delta_0 \quad \text{in } E.
\]

Since

\[
J_\rho(v_\rho) = \inf_{\varphi \in H^1_{ev}} J_\rho(\varphi),
\]

we have

\[
J_{8\pi}(v) = \inf_{\varphi \in H^1_{ev}} J_{8\pi}(\varphi).
\]

Let \( f \) be the developing map of \( u \), that is,

\[
u(z) = \log \frac{8\pi |f'(z)|^2}{(1 + |f(z)|^2)^2} \quad \text{for } z \in E.
\]

As before, for \( \lambda \in \mathbb{R} \) we define \( u^\lambda \) and \( v^\lambda \) by

\begin{equation}
u^\lambda(z) := \log \frac{8\pi e^{2\lambda}|f'(z)|^2}{(1 + e^{2\lambda}|f(z)|^2)^2} =: 8\pi G(z) + v^\lambda(z) + c^\lambda,
\end{equation}

where the constant \( c^\lambda \) is chosen so that \( \int_E v^\lambda = 0 \). Thus \( v^\lambda \) is also a solution to (1.2) and \( v^\lambda(z) \) blows up at \( z = p \) as \( \lambda \to +\infty \) (i.e. \( p \) is a zero of \( f \)).
Next we would like to compute \( J_{8\pi}(v^\lambda) \). By differentiation with respect to \( \lambda \), we have by (1.2)

\[
\frac{d}{d\lambda} J_{8\pi}(v^\lambda) = \int_E \nabla v^\lambda \cdot \nabla \left( \frac{\partial v^\lambda}{\partial \lambda} \right) - 8\pi \int_E e^{-8\pi \tilde{G} + v^\lambda} \frac{\partial v^\lambda}{\partial \lambda}
\]

\[
= -\int_E (\Delta v^\lambda) \frac{\partial v^\lambda}{\partial \lambda} - 8\pi \int_E e^{-8\pi \tilde{G} + v^\lambda} \frac{\partial v^\lambda}{\partial \lambda}
\]

\[
= -8\pi \left| E \right| \int_E \frac{\partial v^\lambda}{\partial \lambda} = 0.
\]

That is, \( J_{8\pi}(v^\lambda) \) is independent of \( \lambda \). In particular,

\[
(1.4) \quad \lim_{\lambda \to +\infty} J_{8\pi}(v^\lambda) = \inf_{\varphi \in H^1_{ev}} J_{8\pi}(\varphi).
\]

Using (1.4), we shall obtain an upper bound of \( \lim J_{8\pi}(v^\lambda) \) by a choice of suitable test function \( \varphi_\epsilon \).

We fix a half period point \( q \in E \) and small \( \delta > 0 \). For any \( \epsilon > 0 \) we define

\[
\varphi_\epsilon(z) = \begin{cases} 
2\log \frac{\epsilon^2/\delta^2 + 1}{\epsilon^2 + |z - q|^2} + 8\pi \tilde{G}(z, q), & \text{if } z \in B_\delta(q), \\
8\pi G(z, q), & \text{if } z \in E \setminus B_\delta(q), 
\end{cases}
\]

where

\[
\tilde{G}(z, q) = G(z - q) + \frac{1}{2\pi} \log |z - q|
\]

is the regular part of \( G(z, q) \) which is defined on \( z \in T(0) \), the fundamental domain of \( E \) centered at \( q \). Notice that the above two expressions for \( \varphi_\epsilon(z) \) coincide when \( |z - q| = \delta \). Since \( \tilde{G}(z, q) \) depends only on \( w = z - q \), we also denote \( \tilde{G}(z, q) = \tilde{G}(z - q) = \tilde{G}(w) \), which is defined on the fundamental domain \( T(0) \) centered at 0.

Obviously \( \varphi_\epsilon \) is an even function. Since \( \int_E G = 0 \), direct integration gives

\[
(1.5) \quad c_\epsilon := \frac{1}{|E|} \int_E \varphi_\epsilon = \frac{2}{|E|} \int_{B_\delta(q)} \log \frac{(\epsilon^2/\delta^2 + 1) |z - q|^2}{\epsilon^2 + |z - q|^2} = O(\epsilon^2 \log \epsilon),
\]

where the notation \( O \) is with respect to the limit \( \epsilon \to 0 \). Thus \( \varphi_\epsilon - c_\epsilon \in H^1_{ev}(E) \) and

\[
J_{8\pi}(\varphi_\epsilon - c_\epsilon) = \frac{1}{2} \int_E |\nabla (\varphi_\epsilon - c_\epsilon)|^2 - 8\pi \log \int_E e^{-8\pi \tilde{G} + \varphi_\epsilon} + O(\epsilon^2 \log \epsilon).
\]

We will estimate the energy term and the non-linear term separately.
By Green’s theorem, we have for $w = z - q$,

$$
\int_E |\nabla \varphi_e|^2 = \int_{B_{\delta}(q)} |\nabla \varphi_e|^2 + (8\pi)^2 \int_{E \setminus B_{\delta}(q)} |\nabla G(z - q)|^2
- 32\pi \int_{B_{\delta}(0)} \log \frac{1}{e^2 + |w|^2} \Delta \tilde{G}(w) + 32\pi \int_{\partial B_{\delta}(0)} \log \frac{1}{e^2 + |w|^2} \frac{\partial \tilde{G}(w)}{\partial \nu}
- (8\pi)^2 \int_{B_{\delta}(0)} \tilde{G} \Delta \tilde{G} + (8\pi)^2 \int_{\partial B_{\delta}(0)} \tilde{G} \frac{\partial \tilde{G}}{\partial \nu}
- (8\pi)^2 \int_{E \setminus B_{\delta}(0)} G \Delta G - (8\pi)^2 \int_{\partial B_{\delta}(0)} G \frac{\partial G}{\partial \nu}.
$$

To estimate these terms, we first notice that (for $\delta > 0$ fixed)

$$
\int_{B_{\delta}(0)} \frac{16|w|^2}{(e^2 + |w|^2)^2} = 16\pi \log(1 + \delta^2/e^2) - 16\pi \delta^2/(e^2 + \delta^2)
= 16\pi (\log(1 + \delta^2/e^2) - 1) + O(\delta),
\int_{B_{\delta}(0)} \log \frac{1}{e^2 + |w|^2} = O(\delta).
$$

Since $\Delta G = \delta_0 - 1/|E|$, $\Delta \tilde{G} = -1/|E|$, and $\int_E G = 0$, it is easy to see that each of three integrals involving $G$ or $\tilde{G}$ is $O(\delta)$ and all boundary terms are $O(\delta)$ except

$$
\frac{32\pi}{\delta} \int_{\partial B_{\delta}(0)} G = 32\pi ( - \log \delta + 2\pi \gamma) + O(\delta),
$$

where $\gamma = \tilde{G}(0) = \tilde{G}(q, q)$ is a constant independent of $q$.

Next we compute the non-linear term.

Since both $\nabla G(q) = 0$ and $\nabla \tilde{G}(z, q)|_{z=q} = \nabla \tilde{G}(0) = 0$, we have

$$
\tilde{G}(z, q) - G(z) = \gamma - G(q) + O(|z - q|^2)
$$

and

$$
\int_{B_{\delta}(q)} e^{-8\pi G(z) + \varphi_e(z)} = e^{8\pi(\gamma - G(q))} \int_{B_{\delta}(0)} \frac{(e^2/\delta^2 + 1)^2}{(e^2 + |w|^2)^2} + O(e^2 \log e)
= e^{8\pi(\gamma - G(q))} \left( \frac{\pi}{e^2} - \frac{\pi}{\delta^2 + e^2} \right) + O(e^2 \log e).
$$
On $E \setminus B_{\delta}(q)$, by (1.8) and direct estimate we have
\[
\int_{E \setminus B_{\delta}(q)} e^{-8\pi G + \varphi} = \int_{E \setminus B_{\delta}(q)} e^{8\pi(G(z,q) - G(z))} = \int_{T(q) \setminus B_{\delta}(q)} \frac{\pi e^{8\pi(\gamma - G(q))}}{|z - q|^4} + O(1),
\]
where $O(1)$ denotes a bounded number which is independent of $\delta$ and $\epsilon$.

By taking into account of (1.6)–(1.10), we get for $0 < \epsilon < \delta$
\[
J_{8\pi}(\varphi - c_\epsilon) = 8\pi(\log(1 + \delta^2 / \epsilon^2) - 1) + 16\pi(-\log \delta + 2\pi \gamma) - 8\pi \log(\pi) - 2\pi \gamma + 8\pi(1 + \log \pi) + O(\delta) + O(\epsilon^2 \log \epsilon).
\]

Let $\epsilon \to 0$ and then let $\delta \to 0$. From (1.4) we conclude that
\[
J_{8\pi}(v^\lambda) \leq 64\pi^2 G(q) - 32\pi^2 \gamma - 8\pi(1 + \log \pi).
\]

From (1.3), $u^\lambda$ blows up at $p$ as $\lambda \to +\infty$. By using the explicit expression (1.3), a similar calculation as the above shows that
\[
\lim_{\lambda \to +\infty} J_{8\pi}(v^\lambda) = 64\pi^2 G(p) - 32\pi^2 \gamma - 8\pi(1 + \log \pi).
\]

Therefore (1.11) implies
\[
G(p) \leq G(q),
\]
which finishes the proof. \hfill \Box

**Corollary 1.2.** Suppose that $G$ has five critical points. Then any half-period is a saddle point of $G$.

**Proof.** Since the extra critical point $p$ (reps. $-p$) is a discrete minimal point, the index of $\nabla G$ at $p$ (reps. $-p$) is 1. By the Hopf–Poincaré index theorem,
\[
-1 = \chi(E \setminus \{0\}) = 2 + \sum_{i=1}^{3} \text{ind}_{\frac{1}{2} \omega_i} \nabla G.
\]

Since $\frac{1}{2} \omega_i$ is non-degenerate, $\nabla G$ has index $\pm 1$ at it. Hence the index must be $-1$ for all $i = 1, 2, 3$. This implies that $\frac{1}{2} \omega_i$ is a saddle point for all $i$. \hfill \Box

2. **Computations for $D(\frac{1}{2} \omega_j)$**

Let $u_k$ be a sequence of blowup solutions to
\[
\triangle u_k + e^{u_k} = \rho_k \delta_0
\]
in $E_r$ and $\rho_k \to 8\pi$. Suppose that $\rho_k \neq 8\pi$. In \cite[Theorem 0.7.5]{2}, it was proved that $u_k$ blows up at a half period $q$. Let

$$\lambda_k := \max_{E_r} u_k.$$ 

We recall a result in \cite{6}:

**Theorem 2.1.** Let $\tilde{G}(z, q)$ be the regular part of $G(z, q)$, namely $\tilde{G}(z, q) = G(z - q) + \frac{1}{2\pi} \log |z - q|$. Let $\phi(q) := \tilde{G}(q, q)$ and $h(z) = e^{-8\pi \tilde{G}(z)}$. Then

$$\rho_k - 8\pi = (D(q) + o(1))e^{-\lambda_k},$$

where

$$D(q) := \int_{E_r} \frac{h(z)e^{8\pi(\tilde{G}(z, q) - \phi(q))} - h(q)}{|z - q|^4} - \int_{E_r} \frac{h(q)}{|z - q|^4}.$$ 

The quantity $D(q)$ is well defined for any critical point of $G(z, q)$. However, if $q$ is not a half period then $D(q) = 0$ since such a blow-up can only occur for $\rho_k = 8\pi$. When $q$ is a half period, $D(q)$ has a geometric interpretation. Indeed,

$$D(q) = \lim_{r \to 0} \left( \int_{E_r \setminus B_r(q)} \frac{e^{-8\pi G(z)}e^{8\pi(\tilde{G}(z, q) - \phi(q))}}{|z - q|^4} - \int_{\mathbb{R}^2 \setminus B_r(q)} \frac{e^{-8\pi G(q)}}{|z - q|^4} \right)$$

$$= \lim_{r \to 0} \left( \int_{E_r \setminus B_r(q)} e^{-8\pi \phi(q)}e^{8\pi(\tilde{G}(z - q) - G(z))} - \int_{\mathbb{R}^2 \setminus B_r(q)} e^{-8\pi G(q)} \right).$$

Note that $8\pi(G(z - q) - G(z))$ is a doubly periodic harmonic function in $\mathbb{R}^2$ with singularities $-4 \log |z - q|$ at $z = q$ and $4 \log |z|$ at $z = 0$. Thus

$$8\pi(G(z - q) - G(z)) = 2 \log |\varphi(z - q) - \varphi(q)| + C$$

for the constant $C = 8\pi(\phi(q) - G(q))$. (The identity does not hold if $q$ is not a half period.) Therefore,

$$e^{8\pi(\tilde{G}(z - q) - G(z))} = e^{8\pi(\phi(q) - G(q))}|\varphi(z - q) - \varphi(q)|^2,$$

and

$$D(q) = e^{-8\pi G(q)} \lim_{r \to 0} \left( \int_{E_r \setminus B_r(0)} |\varphi(z) - \varphi(q)|^2 - \int_{\mathbb{R}^2 \setminus B_r(0)} \frac{1}{|z|^4} \right).$$

Let $T$ be a fundamental domain of $E_r$ with $0 \notin \partial T$. Let $\gamma$ be the image of $\Gamma := \partial T$ under the map

$$\Sigma(z) := -\zeta(z) - \varphi(q)z.$$ 

Denote by $\Lambda_+(q)$ the union of components bounded by $\gamma$ and covered by $T$ under $\Sigma$, and by $\Lambda_-(q)$ the union of components bounded by $\gamma$ but not covered by $T$ under $\Sigma$. Then obviously

$$|\Lambda_+(q)| - |\Lambda_-(q)| = \lim_{r \to 0} \left( \int_{E_r \setminus B_r(0)} |\varphi(z) - \varphi(q)|^2 - \int_{\mathbb{R}^2 \setminus B_r(0)} \frac{1}{|z|^4} \right).$$

\begin{equation}
|\Lambda_+(q)| - |\Lambda_-(q)| = \lim_{r \to 0} \left( \int_{E_r \setminus B_r(0)} |\varphi(z) - \varphi(q)|^2 - \int_{\mathbb{R}^2 \setminus B_r(0)} \frac{1}{|z|^4} \right),
\end{equation}

\end{equation}
and so

\begin{equation}
D(q) = e^{-8\pi G(q)}(|\Lambda_+(q)| - |\Lambda_-(q)|).
\end{equation}

We will give another characterization of $D(q)$ in terms of the Hessian of $G$ at $q$, hence establish a correspondence between the geometric interpretation and the degeneracy structure of the Green function. Recall [8, (7.7)]:

\begin{equation}
\det D^2 G = \frac{-1}{4\pi^2} \left( |(\log \vartheta)_{zz}|^2 + \frac{2\pi}{b} \text{Re} (\log \vartheta)_{zz} \right).
\end{equation}

To write it in the Weierstrass theory we use $(\log \vartheta)_z(z) = \zeta(z) - \eta_1 z$ and

\begin{equation}
(\log \vartheta)_{zz} \left( \frac{1}{2} \omega_i \right) = - \varphi \left( \frac{1}{2} \omega_i \right) - \eta_1 = -(\epsilon_i + \eta_1).
\end{equation}

**Theorem 2.2.** For any half period $q$,

$$|\Lambda_+(q)| - |\Lambda_-(q)| = -4\pi^2 b \det D^2 G(q).$$

**Proof.** Without loss of generality, we assume that $q = \frac{1}{2} \omega_1 = \frac{1}{2}$ and denote $\Lambda_+(q)$ and $\Lambda_-(q)$ by $\Lambda_+$ and $\Lambda_-$ respectively. By (2.2), we have

$$|\Lambda_+| - |\Lambda_-| = \lim_{r \to 0} \left( \int_{E_+\setminus B_0} |\varphi(z)|^2 - \int_{\mathbb{R}^2 \setminus B_0} \frac{1}{|z|^4} \right)$$

$$- \lim_{r \to 0} \int_{E_+\setminus B_0} (\varphi(z)z_1 + \bar{\varphi}(z)e_1) + b|e_1|^2,$$

where $\tau = a + bi$.

To compute the first term, write the Weierstrass zeta function as $\zeta = u + iv$ and then $\varphi = -\zeta' = -u_x - iv_x = -u_x + iu_y$. Hence

$$|\varphi|^2 = u_x^2 + u_y^2 = \partial_x (uu_x) + \partial_y (uu_y).$$

Using integration by parts, and noticing that the singularity at $z = 0$ is cancelled out by the second integral, the first limit term then becomes

$$\int_{\Gamma} uu_x \, dy - uu_y \, dx = \int_{\Gamma} u(v_x \, dx + v_y \, dy) = \int_{\Gamma} u \, dv.$$

This can be calculated easily as

$$-\frac{1}{2} \text{Im} \int_{\Gamma} \bar{\zeta} d\bar{\zeta} = \frac{1}{2} \text{Im} (\eta_1 \eta_2 - \eta_1 \bar{\eta}_2).$$

Applying the Legendre relation $\eta_2 = \eta_1 \tau - 2\pi i$, we get

$$\bar{\eta}_1 \eta_2 - \eta_1 \bar{\eta}_2 = \bar{\eta}_1 (\eta_1 \tau - 2\pi i) / (\eta_1 \bar{\tau} + 2\pi i) = 2ib|\eta_1|^2 - 2\pi i(\eta_1 + \bar{\eta}_1).$$

Consequently,

\begin{equation}
\lim_{r \to 0} \left( \int_{E_+\setminus B_0} |\varphi(z)|^2 - \int_{\mathbb{R}^2 \setminus B_0} \frac{1}{|z|^4} \right) = b|\eta_1|^2 - \pi(\eta_1 + \bar{\eta}_1).
\end{equation}
For the second limit term, we first compute
\[
\int_{E \setminus B_r(0)} \varphi(z) = \frac{i}{2} \int_{T \setminus B_r(0)} \varphi \, dz \wedge d\bar{z} = -\frac{i}{2} \int_{T \setminus B_r(0)} d(\zeta \, d\bar{z})
\]
\[
= -\frac{i}{2} \left( \left. \int_{T \setminus B_r(0)} \zeta \, d\bar{z} \right|_{\partial B_r(0)} \right).
\]
This first integral gives \(\eta_1 \tau - \eta_2 = \eta_1 \tau - \eta_1 \tau + 2\pi i = -2b_1 + 2\pi i\). For the second integral, in the limit \(r \to 0\) it tends to \(\int_0^{2\pi} e^{-i\theta} e^{-i\theta} (-i) \, d\theta = 0\).
Hence
\[
\lim_{r \to 0} \int_{E \setminus B_r(0)} \varphi(z) = -\eta_1 b + \pi.
\]
Putting everything together we get (c.f. (2.4) and (2.5))
\[
|\Lambda_+| - |\Lambda_-|
= b|\eta_1|^2 - \pi(\eta_1 + \bar{\eta}_1) + (\eta_1 b - \pi)\bar{e}_1 + (\bar{\eta}_1 b - \pi)e_1 + b|e_1|^2
= b|e_1 + \eta_1|^2 - \pi((e_1 + \eta_1) + (\bar{e}_1 + \eta_1))
= -4\pi^2 b \det D^2 G(\frac{1}{2}; \tau).
\]
The proof is completed. \(\square\)

**Corollary 2.3.** Let \(u_k\) be a sequence of blow-up solutions to (2.1) with \(\rho_k \to 8\pi\) and \(q\) the blow-up point.

(1) \(q\) is a half period and a saddle point of \(G(z; \tau)\) if and only if \(\rho_k > 8\pi\).

(2) \(q\) is a half period and a minimal point of \(G(z; \tau)\) if and only if \(\rho_k < 8\pi\).

### 3. Uniqueness of solutions

In this section we classify all solutions to

(3.1) \[\triangle u + e^u = \rho \delta_0 \quad \text{on } E\]

for \(0 < \rho \leq 8\pi + \epsilon_0\) where \(\epsilon_0\) is a small positive number.

Recall in [8] we showed that equation (3.1) has a unique solution for \(\rho = 4\pi\), and a unique even solution for \(4\pi \leq \rho \leq 8\pi\). Here we prove the uniqueness result without the evenness assumption.

**Lemma 3.1.** Equation (3.1) has a unique solution for \(0 < \rho \leq 4\pi\).

**Proof.** We first show that for any solution \(u\) to (3.1) with \(\rho \leq 4\pi\), the linearized equation

(3.2) \[\triangle \varphi + e^u \varphi = 0 \quad \text{on } E\]

has only trivial solution \(\varphi = 0\).

Suppose that \(\varphi\) is a solution to (3.2). Then a straightforward computation shows that \((\varphi_{zz} - u_z \varphi_z)z = 0\). Since
\[
u(z) \sim \frac{\rho}{2\pi} \log |z|,
\]

\( \phi_{zz} - u_z \phi_z \) is an elliptic function on \( E \) whose only singularity is a pole of order one at 0. This forces that \( \phi_z(0) = 0 \) and

\[
\phi_{zz} - u_z \phi_z = c_1 \quad \text{on} \quad E
\]

for some constant \( c_1 \), or equivalently

\[
(e^{-u} \phi_z)_z = c_1 e^{-u}.
\]

Notice that

\[
|e^{-u} \phi_z(z)| \leq c_2 |z|^{1-\rho/2\pi}
\]

for some constant \( c_2 > 0 \). Thus if \( \rho < 4\pi \),

\[
\lim_{r \to 0} \int_{E \setminus B_r(0)} (e^{-u} \phi_z)_z = \frac{1}{2} \lim_{r \to 0} \int_{\partial B_r(0)} e^{-u} \phi_z \frac{z}{|z|} ds = 0,
\]

and if \( \rho = 4\pi \) the above limit is finite. If \( c_1 \neq 0 \), this implies that

\[
\int_{E} e^{-u} = \begin{cases} 0 & \text{if} \ \rho < 4\pi, \\ -\infty & \text{if} \ \rho = 4\pi, \end{cases}
\]

which leads to a contradiction. So we have \( c_1 = 0 \) and \( e^{-u} \phi_z \) is an elliptic function. By (3.3) this again implies that \( e^{-u} \phi_z = c_3 \) is a constant.

If \( \phi \neq 0 \) then \( \phi \) has a maximum point \( p \) and a minimum point \( q \) with \( p \neq q \). One of \( p, q \) is not a lattice point where \( \phi_z = 0 \). This implies that \( c_3 = 0 \) and hence \( \phi_z = 0 \). This leads to \( \phi \equiv 0 \) which is a contradiction to \( \phi \neq 0 \). Hence we must have \( \phi \equiv 0 \).

Now the uniqueness follows from the fact that (3.1) has only one solution at \( \rho = 4\pi \). \( \square \)

Remark 3.2. In [8] we showed that the unique even solution to (3.1) with \( \rho \in [4\pi, 8\pi] \) is non-degenerate in the class of \( H^1_{E_0} = \{ u \in H^1 \mid u(-z) = u(z) \} \).

Now the proof of Lemma 3.1 allows us to remove the evenness assumption: \( u \) is non-degenerate in the whole space \( H^1 \), provided that \( 0 < \rho < 8\pi \).

To see this, we may assume that the solution \( \phi \) is odd. Therefore \( \phi_{zz} - u_z \phi_z \) is odd and by exactly the same calculation we have

\[
\phi_{zz} - u_z \phi_z = c_1 = 0.
\]

This implies that \( e^{-u} \phi_z \) is an elliptic function on \( E \), with 0 being its only pole. However, since \( \phi \) is odd, \( \phi_z \) is even and the estimate (3.3) can be improved to

\[
|e^{-u} \phi_z(z)| \leq c_2 |z|^{2-\rho/2\pi}.
\]

If \( \rho < 8\pi \), we find \( 2 - \rho/2\pi > -2 \). This implies that \( e^{-u} \phi_z \) is a constant. If \( \phi \neq 0 \), by evaluating it at a maximum or minimum point, with one of it not a lattice point, we conclude that \( e^{-u} \phi_z \equiv 0 \), and then \( \phi \equiv 0 \) follows. (Notice that if \( \rho = 8\pi \) then \( e^{-u} \phi_z = c \phi(z) \) for some constant \( c \neq 0 \).)

Now we may conclude that the unique even solution \( u \) is always a minimum point of the non-linear functional \( J_\rho \) in (1.1) for \( 0 < \rho \leq 8\pi \). In fact we can prove a stronger result, namely Theorem 0.5.
Lemma 3.3. Let $u$ be a solution to (3.1) with $\rho \not\in 8\pi N$. Then $u$ is even.

Proof. This was proved in [2] for $\rho = 4\pi l$ with $l$ being a positive odd integer, so we assume that $\rho \not\in 4\pi N$.

Let $f(z)$ be a multi-valued developing map of $u$. The readers are referred to [2, §8] for the details to treat these multi-valued functions as global analytic functions $f(\zeta)$, which are defined on the universal cover $\overline{\zeta} \in \mathbb{H} \rightarrow E^\times$. In particular the cusp $\zeta = 0$ is mapped to the cusp $z = 0$ in $E^\times$.

As in [2], we have $S(f) = 2(\eta(\eta + 1)\varphi + B)$ for $\eta = \rho/8\pi$ for some $B \in \mathbb{C}$. Thus $f = w_1/w_2$ and $\tilde{f} = w_1/w_2$ for two linearly independent solutions $w_1$ and $w_2$ to the Lamé equation

$$(3.4) \quad w'' = (\eta(\eta + 1)\varphi + B)w.$$ 

Since $\tilde{w}_i(z) := w_i(-z)$ are also two linearly independent solutions to (3.4), $\tilde{f} := \tilde{w}_1/\tilde{w}_2$ also defines a global analytic function $\tilde{f}$ and we have

$$\tilde{f} = Sf = \frac{af + b}{cf + d}, \text{ for some } S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C}).$$

Consider the covering transformations on $\mathbb{H}$: $g_1, g_2 \in \text{SL}(2, \mathbb{R})$ determined by the two free generators of $\pi_1(E^\times) \cong \mathbb{Z} \ast \mathbb{Z}$. Let $\Gamma \subset \text{SL}(2, \mathbb{R})$ be the rank two free subgroup generated by $g_1$ and $g_2$, and $r : \Gamma \rightarrow \text{PSU}(2)$ be the unitary representation associated to the solution $u$. The mapping $(-1) : z \mapsto -z$ on $E^\times$ lifts to a map $i$ on $\mathbb{H}$ which is not a covering map for $\mathbb{H} \rightarrow E^\times$. Nevertheless the composition $i \circ i$, namely we apply $(-1)$ twice, does give a covering map for $\mathbb{H} \rightarrow E^\times$. That is, the matrix $S^2$ can be represented as an element generated by $S_1 := r(g_1)$ and $S_2 := r(g_2)$.

By considering the action of $(-1)$ in a simply connected neighborhood $U$ of $0 \in E$, we see that $S^2 f = f(e^{2\pi i}z) = \beta f(z)$ for some $\beta \in \text{PSU}(2, \mathbb{C})$. Indeed, $\beta = r(g_2^{-1}S_1^{-1}g_2g_1) = S_2^{-1}S_1^{-1}S_2S_1$. Under some normalization on $f$, the matrix $\beta$ is calculated in [2, Lemma 8.3.4, p.262] (suppress all index $k$ in the formula in the bottom of p.262) as

$$\beta = \begin{pmatrix} p|2\alpha + |q|^2\bar{\alpha} & -\bar{q}(\alpha - \bar{\alpha}) \\ \bar{p}q(\alpha - \bar{\alpha}) & |p|^2\alpha + |q|^2\bar{\alpha} \end{pmatrix},$$

where $f(0) := \lim_{z \rightarrow 0} f(\zeta) = q/p$ with $|p|^2 + |q|^2 = 1$, $p, q \neq 0$, and $\alpha = e^{2\pi i/\eta}$. Clearly $\alpha \neq \bar{\alpha}$ since $\eta \notin \frac{1}{2}\mathbb{Z}$. In particular $\beta \neq \pm I_2$.

We claim that $S \in \text{PSU}(2, \mathbb{C})$. To prove it, we choose a new unitary basis to diagonalize $\beta$ to $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ for some $e^{i\theta} \neq \pm 1$. Since

$$S^2 = \beta = \begin{pmatrix} a^2 + bc & b(a + d) \\ c(a + d) & d^2 + bc \end{pmatrix},$$

$S^2 = \beta$ implies that either $a + d = 0$ or both $b = 0$ and $c = 0$. If $a + d = 0$ then $a^2 + bc = -ad + bc = -1$ and $d^2 + bc = -1$, which leads to $e^{i\theta} = -1$, a contradiction. Hence $b = c = 0$ and $ad = 1$, $a^2 = e^{i\theta}, d^2 = e^{-i\theta}$. Therefore,
$S \in \text{PSU}(2, \mathbb{C})$ and $\tilde{f}(z) = f(-z)$ gives rise to the same solution $u$. Hence $u(-z) = u(z)$ and the lemma follows.

**Proof of Theorem 0.5.** By Lemma 3.1 and 3.3, the uniqueness of solution holds for $0 < \rho < 8\pi$. The statements for $\rho = 8\pi$ was proved in [8].

For $\tau \in \Omega_3$, the unique solutions $u_\rho$ blows up as $\rho \nearrow 8\pi$ (since equation (3.1) has no solutions at $\rho = 8\pi$). The blow-up point of $u_\rho$ must be the minimum point which is one of the half periods. The other two half periods $q_1$ and $q_2$ are saddle critical point of $G$. By Theorem 0.4 and Remark 0.3, we have $\det D^2G(q_i) < 0$ and then $D(q_i) > 0$. Under these conditions, by the method in [4] we can construct a bubbling sequence of solutions $u_{\rho,i}$ to (3.1), for each $i = 1, 2$, with $\rho > 8\pi$ which blows up at $q_i$.

**Remark 3.4.** In [4] the non-degenerate condition $D(q_i) \neq 0$ was replaced by some other non-degenerate condition. Nevertheless the similar process as there still works in our current case (see e.g. the remark in [6] concerning with the degree counting formula).

Indeed, for the Chern–Simons–Higgs equation, the same non-degenerate conditions $D(q) < 0$ and $\det D^2G(q) \neq 0$ were recently used to construct such kind of bubbling solutions [10].

Now we need the following uniqueness theorem:

**Theorem 3.5.** Suppose that $u_k$ and $\tilde{u}_k$ are two sequences of solutions to (3.1) with $\rho_k \to 8\pi$, and both sequences have the same blow-up point $q$.

If $D(q) \neq 0$, i.e. $q$ is a non-degenerate critical point of $G$ by Theorem 0.4, then $u_k = \tilde{u}_k$ for large $k$.

This is recently proved in [11] for the Chern–Simons–Higgs equation

$$\triangle u + \frac{1}{\epsilon} e^u (1 - e^u) = 8\pi \delta_0,$$

but the proof given there also works for (3.1).

By Theorem 3.5, $u_{\rho,i}$ are exactly all the solutions to equation (3.1) for $8\pi < \rho < 8\pi + \epsilon_0$. This proves (i).

For $\tau \in \Omega_5$, all the three half periods are saddle points of $G$. By Theorem 3.5 again, we must have three bubbling solutions. On the other hand, (3.1) has a unique even solution $u$ for $\rho = 8\pi$ whose linearized equation in the class of even functions is non-degenerate. Therefore for $8\pi < \rho < 8\pi + \epsilon_0$ there is a unique even solution $u_\rho$ which converges to $u$ as $\rho \searrow 8\pi$.

By Lemma 3.3, (3.1) has only even solutions for $8\pi < \rho < 8\pi + \epsilon_0$, we conclude that (3.1) has the only one even solution $u_\rho$ which converges to $u$ as $\rho \searrow 8\pi$. Hence there are four solutions in total. This proves (ii) and thus completes the proof Theorem 0.5. □

**References**


