A REPORT ON HIRZEBRUCH SIGNATURE FORMULA AND MILNOR'S EXOTIC SEVEN SPHERES

CHIN-LUNG WANG

December 1993

Introduction: In 1956 J. Milnor constructed a non-standard smooth structures on S^7 which startled the whole mathematical society because it is the first known example that a manifold can admit more than one smooth structures! This article is an attemption to understand this construction.

Start from the sphere bundle $S(E)$ of an oriented four plane bundle E over S^4, can show the total space M of $S(E)$ is a topological S^7 if the Euler number $e(E) = 1$. In this case, if M is diffeomorphic to the standard S^7, we can attach an 8-disk to the disk bundle $D(E)$ along the boundary M via this diffeomorphism to get a smooth closed 8-manifold W. By applying the Hirzebruch signature formula to W, we will obtain some divisibility condition on its Pontryagin numbers. By a detailed computation of the characteristic classes, this can not be true for some E, and we get the "exotic spheres".

The main construction is in §4, the computation of characteristic classes is done in §1 (1.2) (1.3). Other part of this paper is devoted to a proof of Hirzebruch signature formula. Since I adopt the topological approach, Thom's cobordism theorem is discussed in §2.

§1 Topological preliminaries.

(1.0) Cohomology group of grassmanian manifolds. Let $\mathbf{G}_n(\mathbf{C}^m)$ be the grassemannian of all n dimensional complex linear subspaces of $\mathbf{C}^m$. It is known as the classfying space of $\mathbf{C}^m$ bundles in the sense that any $\mathbf{C}^m$ bundle $E \to M$ arises from $f^*\gamma^n$ for some $f\colon M \to \mathbf{G}_n(\mathbf{C}^m)$ with m large enough, where γ^n is the "universal bundle" with fiber $\gamma^n_{[X]}$ the vector space X. In order to attach a "natural characteristic class" $c(E)$ to E, it must satisfies $c(f^*\gamma) = f^*c(\gamma)$. So it is necessary to study cohomology of $\mathbf{G}_n(\mathbf{C}^m)$. To begin with, we construct a cell decomposition as follows. Let

$$\mathbf{C}^0 \subset \mathbf{C}^1 \subset \mathbf{C}^2 \subset \cdots \subset \mathbf{C}^m$$

be a fixed filtration. For any $X \in \mathbf{G}_n(\mathbf{C}^m)$, the sequence of m numbers

$$0 \le \dim_{\mathbf{C}}(X \cap \mathbf{C}^1) \le \cdots \le \dim_{\mathbf{C}}(X \cap \mathbf{C}^m) = n$$

has n jumps, and denote the sequence of jumps by $j(X)$. We call a sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$ with $1 \le \sigma_1 < \sigma_2 < \cdots < \sigma \le m$ a Schubert symbol. For each Schubert symbol σ, we associate a subset $e(\sigma)$ in $\mathbf{G}_n(\mathbf{C}^m)$ as the collection of all X with $j(X) = \sigma$. $e(\sigma)$ is topologically a cell of complex dimension

$$(\sigma_1 - 1) + (\sigma_2 - 2) + \cdots + (\sigma_n - n)$$

This dimension formula is easy to see by deform X a little bit, but it needs some work to show it is really an open cell (cf. [MS] p.76), anyway it is elementary. There are totally $\binom{m}{n}$ such cells, and the whole collection of these cells form a CW complex structure of $\mathbf{G}_n(\mathbf{C}^m)$.

In particular, there is no cell of (real) odd dimension, and for cells of even dimension $2r$, they corresponds to those σ such that (let $\tau_i = \sigma_i - 1$):

$$0 \le \tau_1 \le \tau_2 \le \cdots \le \tau_n \le m - n$$
$$\tau_1 + \tau_2 + \cdots + \tau_n = r.$$

When m is large, say $m - n \ge r$, and $n \ge r$, the number of all such σ is exactly the partition number $p(r)$ of r. Since no cells are of odd dimension, the cohomology groups are then clear:

$$\mathbf{H}^{2r+1}(\mathbf{G}_n(\mathbf{C}^m); \mathbf{Z}) = 0$$
$$\mathbf{H}^{2r}(\mathbf{G}_n(\mathbf{C}^m); \mathbf{Z}) \simeq \mathbf{Z}^{p(i)}.$$

In the case m, n both large, one will suspect that this cohomology ring is a polynomial ring with each even dimension $2i$ a generator c_i (the universal chern class) for the following reason: If this is true, than for $c_1^{s_1} \cdots c_r^{s_r}$ $(= c_{i_1} \cdots c_{i_\ell}$ with $i_1 \le \cdots \le i_\ell)$ to be of degree $2r$, there are exactly $p(r)$ terms of such monomials! (corresponds to the partions $I = \{i_1, \dots, i_\ell\}$.) This is maybe the origine how chern classes invented. Since the construction of characteristic are not the goal of this article, I will just list the formal axioms of chern classes and define the Pontryagin classes from it. And then compute them in some special cases that will be used later.

(1.1) Characteristic classes. Start from the following (without proof)

Theorem. For any complex vector bundle E over a manifold M, we can uniquely associate an element $c(E) = \sum_{i\geq 0} c_i(E) \in \mathbf{H}^{2*}(M;\mathbf{Z})$, the (total) chern class of E, such that

(1) $c_0(E) = 1, c_i(E) \in \mathbf{H}^{2i}(M;\mathbf{Z})$ and $c_i(E) = 0$ for $i > \operatorname{rank}(E)$.
(2) Naturality: $c(f^*E) = f^*c(E)$ for any $f\colon M' \to M$.
(3) Whitney sum formula: $c(E \oplus F) = c(E) \cdot c(F)$.
(4) Normalization: $c(\gamma^1) = 1 - g$, where γ^1 is the universal line bundle over $\mathbf{P}^n(\mathbf{C})$ and g is the Poincare dual of the hyperplane class.

Remark. In fact $c_n(E) = e(E_{\mathbf{R}})$, the Euler class of the underlying real oriented vector bundle. This is the first step to define the chern class.

For a complex manofold M, we denote $c(TM)$ by $c(M)$, for example let's compute $c(\mathbf{P}^n(\mathbf{C}))$. Let ϵ be the trivial line bundle, it can be show that $T\mathbf{P}^n(\mathbf{C}) \oplus \epsilon \simeq \bigoplus^{n+1} \bar{\gamma}^1$ and we have $c_i(\bar{E}) = (-1)^i c_i(E)$, so apply the sum formula, we get $c(\mathbf{P}^n(\mathbf{C})) = c(T\mathbf{P}^n(\mathbf{C}) \oplus \epsilon) = c(\bar{\gamma}^1)^{n+1} = (1+g)^{n+1}$.

Now we define the Pontryagin classes $p_i(E)$ of a real vector bundle E by

$$p_i(E) = (-1)^i c_{2i}(E \otimes \mathbf{C}) \in \mathbf{H}^{4i}(M;\mathbf{Z}).$$

and $p(E) = \sum p_i(E)$ the total Pontryagin class. It has similar properties as (1) to (4), with (3′) $p(E \oplus F) = p(E) \cdot p(F)$ (mod 2-torsion). and (4′) $p(\gamma^1_{\mathbf{R}}) = 1 + g^2$. These are all easy consequences of $E \otimes \mathbf{C} \simeq_{\mathbf{C}} \overline{E \otimes \mathbf{C}}$ if E is real, and $E_{\mathbf{R}} \otimes \mathbf{C} \simeq E \oplus \bar{E}$ if E is complex. For example, $c(\gamma^1_{\mathbf{R}} \otimes \mathbf{C}) = (1-g)(1+g) = 1 - g^2$, so $p(\gamma^1_{\mathbf{R}}) = 1 + g^2$. Also for the tangent bundle, $p(\mathbf{P}^n(\mathbf{C})) = (1+g^2)^{n+1}$.

Consider the carnonical quaternion line bundle γ over $\mathbf{P}^m(\mathbf{H}) = (\mathbf{H}^{m+1} - 0)/\mathbf{H}^\times$, by the right action of $\mathbf{H}^\times$. It has the sphere bundle $S(\gamma) \simeq S^{4m+3} \subset \mathbf{H}^{m+1}$. (But the total space $E(\gamma)$ is not $\mathbf{H}^{m+1} - 0$!). Notice that $S(\gamma)$ has the fiber $S^3 = \mathbf{Sp_1} =$ the unit group of $\mathbf{H}$, hence $S(\gamma)$ is in fact a S^3 principal bundle. This is the Hopf fibration in the quaternion case. Since γ has a natural underlying real and complex bundle structure, we will compute $c(\gamma_{\mathbf{C}})$ and $p(\gamma_{\mathbf{R}})$.

The principal bundle $S(\gamma)$:

$$\begin{array}{ccc} S^3 & \xrightarrow{i} & S^{4m+3} \\ & & \Big\downarrow{\scriptstyle p_0} \\ & & \mathbf{P}^m(\mathbf{H}) \end{array}$$

has a homotopy long exact sequence end at $\pi_0(S^3)$ term,

$$\pi_k(S^3)\xrightarrow{i_*}\pi_k(S^{4m+3})\xrightarrow{p_{0*}}\pi_k(\mathbf{P}^m(\mathbf{H}))\xrightarrow{\partial}\pi_{k-1}(S^3)$$

When k =1,2, and 3 we get $\pi_k(\mathbf{P}^m(\mathbf{H}))$ = trivial. Now from the cohmology Gysin sequence:

$$\mathbf{H}^{i-1+4}(S^{4m+3}) \longrightarrow \mathbf{H}^i(\mathbf{P}^m(\mathbf{H}))\xrightarrow{\cup e}\mathbf{H}^{i+4}(\mathbf{P}^m(\mathbf{H}))\xrightarrow{p_0^*}\mathbf{H}^{i+4}(S^{4m+3})$$

where $e = e(\gamma_{\mathbf{R}}) = c_2(\gamma_{\mathbf{C}}) \in \mathbf{H}^4(\mathbf{P}^m(\mathbf{H}))$. We have for $i+3, i+4 \neq 4m+3$, ie. $i \neq 4m, 4m-1$, $\cup e$ gives an isomorphism $\mathbf{H}^i(\mathbf{P}^m(\mathbf{H})) \simeq \mathbf{H}^{i+4}(\mathbf{P}^m(\mathbf{H}))$. In particular, $\mathbf{H}^{4i}(\mathbf{P}^m(\mathbf{H})) = e^i \cdot \mathbf{Z}$. Combine with the above result on π_1, π_2, and π_3, using Hurwicz theorem, we conclude $\mathbf{H}^i(\mathbf{P}^m(\mathbf{H})) = 0$ for i = 1,2, and 3. This then implies $\mathbf{H}^i(\mathbf{P}^m(\mathbf{H})) = 0$ for $i \not| 4$. That is, $\mathbf{H}^*(\mathbf{P}^m(\mathbf{H})) = \mathbf{Z}[e]/e^{m+1}$ as a truncated polynomial ring generated by e. (Note. If one feel a cell decomposition of $\mathbf{P}^m(\mathbf{H})$ is visible, then this result is in hand.) So $c(\gamma_{\mathbf{C}}) = 1 + e$, use the eailier technique, have

$$\begin{aligned}(1 - p_1 + p_2 - \cdots) &= (1 - c_1 + c_2 - \cdots)(1 + c_1 + c_2 + \cdots)\\ &= (1 + c_2)(1 + c_2)\\ &= 1 + 2e + e^2.\end{aligned}$$

so $p(\gamma_{\mathbf{R}}) = 1 - 2e + e^2$.

Now we specialize to m=1, then $\mathbf{P}^1(\mathbf{H}) \equiv S^4$. Denote e by u (for we will compute the Euler class of other bundles later). We summarize what we have done: The Hopf bundle γ has $e(\gamma) = u, p_1 = -2u$. Its sphere bundle $S(\gamma)$ is a S^3 bundle over S^4 with total space S^7.

There are still many other S^3 bundles over S^4. We will classify them and compute all their characteristic classes.

(1.2) $\mathbf{SO}(4)$ bundles over S^4. To begin with, those vector bundles are classified by the homotopy group $\pi_3(\mathbf{SO}(4))$ by glueing two trivial bundles along the equator S^3. So let's compute it and find its generators.

Recall a well known result ([S] p.115), $\mathbf{SO}(3) \simeq \mathbf{P}^3(\mathbf{R})$, this is proved by consider the map $\rho: S^3 \to \mathbf{SO}(3)$ defined by using quaternion multiplication

$$(*) \qquad \rho(u)v = uvu^{-1}$$

here $v \in S^2$ is considered as the unit sphere of the space spanned by i, j, and k in the quaternion $\mathbf{H}$ (this map ρ in fact says $S^3 = \mathbf{Sp}_1 = \mathbf{Spin}(3)$). So $\pi_1(\mathbf{SO}(3)) = \mathbf{Z}/2\mathbf{Z}$ and $\pi_i(\mathbf{SO}(3)) = \pi_i(S^3)$ for $i \geq 2$. Now consider the principal bundle structure:

$$\begin{tikzcd} \mathbf{SO}(3) \arrow[r, "j"] & \mathbf{SO}(4) \arrow[d, "p"] \\ & S^3 \end{tikzcd}$$

where p is defined by $p(g) = g \cdot 1$, and define a map $\sigma\colon S^3 \to \mathbf{SO}(4)$ by

$$(**) \qquad \sigma(u)v = uv$$

also use the quaternion multiplication. Since $p(\sigma(u)) = \sigma(u) \cdot 1 = u \cdot 1 = u$, σ is a section, and by standard result on principal bundle we conclude $\mathbf{SO}(4) \simeq S^3 \times \mathbf{SO}(3)$. So we have

$$\pi_3(\mathbf{SO}(4)) \simeq \pi_3(S^3) \times \pi_3(\mathbf{SO}(3)) = \mathbf{Z} \oplus \mathbf{Z}$$

and its two generators are $[\sigma]$ and $[j \circ \rho]$, it is reasonable to still denote the later one by ρ because the behavior of j is compatible with the quaternion structure under consideration. In view of (*) and (**) we can now describe the general form of all such bundles, they all come from the form $f_{hj}\colon S^3 \to \mathbf{SO}(4)$, where

$$(***) \qquad f_{hj}(u)v = u^h v u^j$$

This is just a corollary of the following

Lemma. Let G be a topological group, then for $k \geq 2$ the (pointwise) multiplication of two homotopy classes in $\pi_k(G)$ corresponds to the composition law of homotopy classes.

Proof. Let $\phi_1, \phi_2 : (I^k, \partial I^k) \to (G, e)$ and let ϕ_0 be the constant map e, then clearly have homotopies

$$\phi_1 + \phi_0 \sim \phi_1, \quad \phi_0 + \phi_2 \sim \phi_2$$

Multiply these two homotopies, we get

$$(\phi_1 + \phi_0) \cdot (\phi_0 + \phi_1) \sim \phi_1 \cdot \phi_2$$

By the definition of composition law, the left hand side is exactly $\phi_1 + \phi_2$. This proves the lemma. **Qed.**

In fact, it is very clear that σ corresponds to the Hopf bundle discussed before because we use right action there, and as a $S^3 = \mathbf{Sp}_1$ bundle its coordinate transform will be the left transformation. So we have an even simpler set of generators, namely the right Hopf bundle γ and the left Hopf bundle $\bar{\gamma}$ defined by left action. It correspons to the homotopy class $\bar{\sigma} := \sigma\rho^{-1}$, that is

$$(**)' \qquad \bar{\sigma}(u)v = vu.$$

This complete the description of the $\mathbf{SO}(4)$ bundles over S^4.

Since γ, $\bar{\gamma}$ are isomorphic as real bundles, they have the same Euler class $e = u$, but since the "quaternion orientation" is changed, we will have $p_1 = 2u$. Now we will use a general argument to compute the characteristic classes of all these bundles.

(1.3) Characteristic classes of $\mathbf{SO}(4)$ bundles over S^4. In general, $\mathbf{SO}(n)$ bundles over a space X are classified by the set of homotopy classes $[X, \tilde{\mathbf{G}}_n(\mathbf{R}^\infty)]$. Where $\tilde{\mathbf{G}}_n(\mathbf{R}^m)$ is the grassmanian of all oriented n planes in $\mathbf{R}^m$, or more explicitly, it is $\mathbf{SO}(m)/\mathbf{SO}(n) \times \mathbf{SO}(m-n)$. When X happens to be a sphere S^m, it is just as what we have done in (1.2) that this set $\simeq \pi_m(\tilde{\mathbf{G}}_n(\mathbf{R}^\infty)) \simeq \pi_{m-1}(\mathbf{SO}(n))$, which is also easily deduced from the following fibration structure:

$$\begin{array}{ccc} \mathbf{SO}(n) & \xrightarrow{i} & \mathbf{SO}(N+n)/\mathbf{SO}(N) \\ & & \Big\downarrow p \\ & & \tilde{\mathbf{G}}_n(\mathbf{R}^{N+n}) \end{array}$$

Now we will show both maps e and $p_1\colon \pi_4(\tilde{\mathbf{G}}_4) \to \mathbf{H}^4(S^4)$ are group homomorphisms. For example, let $[f] \in \pi_4(\tilde{\mathbf{G}}_4)$, p_1 is the map

$$[f] \mapsto p_1(f^*\tilde{\gamma}^4)([S^4])$$

so $p_1(f^*\tilde{\gamma}_4)([S^4]) = f^*(p_1\tilde{\gamma}^4)([S^4]) = p_1\tilde{\gamma}^4(f_*[S^4])$ by the definition of f^* and f_*, and the last map $[f] \mapsto f_*([S^4])$ is exactly the Hurwicz homomorphism

$$\pi_4(\tilde{\mathbf{G}}_4) \to \mathbf{H}_4(\tilde{\mathbf{G}}_4).$$

Combine with the isomorphism $\pi_4(\tilde{\mathbf{G}}_4) \simeq \pi_3(\mathbf{SO}(4))$, we furnish the computations:

Proposition: The $\mathbf{SO}(4)$ bundle E_{hj} defined by f_{hj} has $e(E_{hj}) = (h+j)u$ and $p_1(E_{hj}) = 2(h-j)u$. In another word, for $k \equiv l \pmod 2$, there is an unique $\mathbf{SO}(4)$ bundle E such that $p_1(E) = 2ku$ and $e(E) = lu$.

Proof. Obviously by looking at the characteristic classes of the right and left hopf bundles γ and $\bar{\gamma}$. Another way to see this is to see the tangent bundle TS^4, which has $e(TS^4) = 2u$ (the Euler number) and $p_1(TS^4) = 0$ since $TS^4 \oplus \epsilon = TS^4 \oplus NS^4 = \bigoplus^5 \epsilon$ and by the sum formula. (TS^4 corresponds to f_{11}, the "sum" of γ and $\bar{\gamma}$.) **Qed.**

(1.4) Characteristic numbers. For a complex manifold M of dimension n, it is easy to see (described below) there are exactly $p(n)$ terms of products of c_i's to be of the top degree $2n$ (the real dimension of M). Also for a $4n$ dimensional real manifold M, there are exactly $p(n)$ terms of products of p_i's to of the top degree $4n$. Here all characteristic class are understood to be of the tangent bundle TM.

Let $I = \{i_1, \ldots, i_r\}$ be any partition of n with $i_1 \geq \cdots \geq i_r$. Define $c_I = c_{i_1} \cdots c_{i_r}$ and $p_I = p_{i_1} \cdots p_{i_r}$ they are all of top degree classes. By evaluating these top degree classes on $[M]$, we get some intergers and called them the "characteristic numbers". (chern numbers and Pontryagin numbers) We will see in the next section that the Pontryagin numbers have strong relation to the cobordism problem. Here we compute some examples that will be used later.

Again let $I = \{i_1, \ldots, i_r\}$ be any partition of n with $i_1 \geq \cdots \geq i_r$. Define

$$M^I_{\mathbf{C}} = \mathbf{P}^{i_1}(\mathbf{C}) \times \cdots \times \mathbf{P}^{i_r}(\mathbf{C})$$
$$M^I_{\mathbf{R}} = \mathbf{P}^{2i_1}(\mathbf{C}) \times \cdots \times \mathbf{P}^{2i_r}(\mathbf{C})$$

so $\dim(M^I_{\mathbf{C}}) = 2n$, $\dim(M^I_{\mathbf{R}}) = 4n$ as real manifolds. We want to show, the characteristic numbers are good enough invariants to distinguish these manifolds, that is

Lemma. The $p(n) \times p(n)$ matrix $[c_I(M^J_{\mathbf{C}})]_{IJ}$ of characteristic numbers is nonsingular. And the same statement holds for $[p^I(M^J_{\mathbf{R}})]_{IJ}$.

Proof. It is convinient (although not strictly necessary) to introduce another set of characteristic classes, the chern character $\mathrm{ch} = \sum_{i\geq 0} \mathrm{ch}_i$, which is defined to be

$$\sum_{i=1}^n e^{x_i} = \sum_{i=1}^n (1 + x_i + \frac{1}{2}x_i^2 + \cdots) = n + c_1 + \frac{c_1^2 - 2c_2}{2} + \cdots$$

that is, $\mathrm{ch}_1 = c_1$, $\mathrm{ch}_2 = (c_1^2 - 2c_2)/2, \ldots$. We don't need to know the exact formula between c and ch, we only have to know they are equivelent $\mathbf{Q}$-bases, which is quite obvious. The advantage to use ch is the following fact, $\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$, which is clear by definition. In our case it reads

$$(*) \qquad \mathrm{ch}(M_1 \times M_2) = \mathrm{ch}(M_1) + \mathrm{ch}(M_2),$$

this linearity allows us to handle product space much easier. We only have to prove the matrix formed by $a(I, J) = ch_I(M^J_{\mathbf{C}})$ is nonsingular. Then the first part of the lemma follows. (ch_I is defined by the same way as c_I)

Since $\mathrm{ch}_i(M) = 0$ if $i > \dim_{\mathbf{C}}(M)$, by the property (*), we get $a(I, J) = 0$ if $|I| < |J|$. (At least one i_s is larger than all j_t.) Furthermore, even in the case $|I| = |J|$, $a(I, J) = 0$ unless $I = J$. Introduce a total order on all partitions which extends the partial order $|I|$, then the matrix $[a(I, J)]$ is in a triangular form. So to prove it to be nonsingular, only have to prove the diagonal elements (ie. $I = J$) are nonzero. Let $I = J = i_1, \ldots, i_r$, then $a(I, I)$ is

$$\prod_{\ell=1}^r \mathrm{ch}_{i_\ell}(\mathbf{P}^{i_1}(\mathbf{C}) \times \cdots \times \mathbf{P}^{i_r}(\mathbf{C}))[M^I_{\mathbf{C}}]$$
$$= \prod_{\ell=1}^r (\mathrm{ch}_{i_\ell}(\mathbf{P}^{i_1}(\mathbf{C})) + \cdots + \mathrm{ch}_{i_\ell}(\mathbf{P}^{i_\ell}(\mathbf{C})))[M^I_{\mathbf{C}}]$$

where the obvious zero terms are omitted. By evaluating on the $[M^I_{\mathbf{C}}]$ from $\ell = 1, \cdots, r$ gradually, again use $\mathrm{ch}_i(M) = 0$ if $i > \dim_{\mathbf{C}}(M)$, we find

$$a(I, I) = \prod_{\ell=1}^r \mathrm{ch}_{i_\ell}(\mathbf{P}^{i_\ell}(\mathbf{C}))[\mathbf{P}^{i_\ell}(\mathbf{C})].$$

so we only have to compute $\mathrm{ch}_n(\mathbf{P}^n(\mathbf{C}))$. By the formula $T\mathbf{P}^n(\mathbf{C})\oplus\epsilon \simeq \oplus^{n+1}\bar{\gamma}$ which is already used in (1.1), get $\mathrm{ch}_n(\mathbf{P}^n(\mathbf{C})) = (n+1)\mathrm{ch}_n(\bar{\gamma})$. Since $\bar{\gamma}$ is a line bundle, by def $\mathrm{ch}_n(\bar{\gamma}) = c_1(\bar{\gamma})^n/n!$ and by the definition of chern class the later term is g^n, which is the generator of $\mathbf{H}^{2n}(\mathbf{P}^n(\mathbf{C}))$ and $g^n([\mathbf{P}^n(\mathbf{C})])$ is not zero. This proves the (first part) of the lemma.

For the Pontryagin case, we define for a real bundle E, $\mathrm{ph}_i(E) = \mathrm{ch}_{2i}(E\otimes\mathbf{C})$, then for M a complex manifold, $\mathrm{ph}_i(TM) = \mathrm{ch}_{2i}((TM)_{\mathbf{R}}\otimes\mathbf{C}) = \mathrm{ch}_{2i}(TM\oplus\overline{TM})$ $= \mathrm{ch}_{2i}(TM) + \mathrm{ch}_{2i}(\overline{TM}) = 2\mathrm{ch}_{2i}(TM)$. Then by the same manner we also get the matrix with entries $\mathrm{ph}_I(M^J_{\mathbf{R}})$ is nonsingular. Since these classes ph_i are an equivalent basis of p_i over $\mathbf{Q}$, the result follows. **Qed**.

(1.5) Cohomology ring of $\tilde{\mathbf{G}}_n(\mathbf{R}^\infty)$.

For later use, we state the following theorem and refer its proof to the literature ([MS] p.179).

Theorem. Let R be an integral domain with 2 to be invertible, and denote p_i, e the characteristic classes of the universal bundle $\tilde{\xi}$, then

$$\begin{array}{l}\mathbf{H}^*(\tilde{\mathbf{G}}_{2n+1}(\mathbf{R}^\infty);R) = R[p_1,\ldots,p_n]\\ \mathbf{H}^*(\tilde{\mathbf{G}}_{2n}(\mathbf{R}^\infty);R) = R[p_1,\ldots,p_{n-1},e]\end{array}.$$

§2 Thom's Cobordism Theorem

(2.0) Introduction. We will consider the oriented case only. Two closed n-manifolds M_1, M_2 are said to be cobordant if there is an $(n+1)$-dimensional compact oriented manifold-with-boundary W such that $M_1 - M_2 = \partial W$. This equality means an orientation preserving diffeomorphism. This relation clearly defines an equivalence relation on all closed n-manifolds. Using the disjoint union as addition law, the set of cobordism classes form an abelian group, denoted by $\mathbf{\Omega}_n$. Toghther with the cartisian product as multiplication law, the set $\mathbf{\Omega} = \bigoplus_{n \geq 0} \mathbf{\Omega}_n$ then becomes a commutative graded ring, with the class of all manifolds that bound as the zero element and $[pt]$ the multiplication unit. We have to check that the product is well-defined: If $M_1 - M_2 = \partial W_1$ and $M_1' - M_2' = \partial W_2'$, then we have

$$M_1 \times M_1' - M_2 \times M_2' = \partial(W_1 \times M_1' - M_2 \times W_2')$$

so the map: $\mathbf{\Omega}_m \times \mathbf{\Omega}_n \to \mathbf{\Omega}_{m+n}$ is actually well-defined. The commutativity and associativity are clear.

The goal of this section is to study the structure of the ring $\mathbf{\Omega}$, or actually $\mathbf{\Omega} \otimes \mathbf{Q}$. We will start from the Thom transversality theorem which will lead us to a representation of $\mathbf{\Omega}_n$ as a certain homotopy group of the Thom space of a universal bundle, we then need some results of rational homotopy groups or Serre's theory of C-ismorphism to transform these homotopy groups into homology groups of classifying spaces. In this step we have to consider $\mathbf{\Omega} \otimes \mathbf{Q}$ or $\mathbf{\Omega}(\mathrm{mod}\mathcal{C})$ to get a satisfictory result.

An important observation is that by using the deRham cohomology and Stokes' theorem, corbodant manifolds have the same Pontryagin numbers, (say all the pontryagin numbers are zero if a manifold bounds). The result in (1.4) about the Pontryagin numbers of (products of) complex projective spaces then shows, the various products of $\mathbf{P}^{2n_j}(\mathbf{C})$'s with $j_1 + \cdots + j_r = m$ are all in distinct cobordism classes in $\mathbf{\Omega}_{4m}$, and the total number of such products is $p(m)$.

Since the cohomology rings of classifying spaces are known to be generated by characteristic classes (Pontryagin classes in our case), and by counting the dimension, we can finally conclude that the ring $\mathbf{\Omega} \otimes \mathbf{Q}$ is freely generated by $\{[\mathbf{P}^{2n}(\mathbf{C})] | n \geq 0\}$. This is exactly the context of Thom's Cobordism Theorem in the oriented case. And we will use this result to prove the Hirzebruch's Signature Theorem in the next section.

The $\mathbf{P}^{2k+1}(\mathbf{C})$'s do not appear because they bound some manifolds in the following manner: Consider the indentification of $\mathbf{C}^{2k+1}$ and $\mathbf{H}^{k+1}$, then taking projectilization (the compatibility of both action by $\mathbf{C}^\times$ and $\mathbf{H}^\times$ is obvious), we get a fiber bundle $f\colon \mathbf{P}^{2k+1}(\mathbf{C}) \to \mathbf{P}^k(\mathbf{H})$ with fibre $\mathbf{P}^1(\mathbf{C}) \simeq S^2$, this bundle has structure group $\mathbf{SO}(3)$, so this bundle is in fact the sphere bundle of some vector bundle and then it bounds the disk bundle.

Another remark is the determination of cobordant relation by Pontryagin numbers. In this fashion, Thom's cobordism theorem can also be stated as "Two manifolds are coborbant if and only if they have the same pointryagin numbers". The result of this section can implies this statement to be true when we ignore the torsion part.

(2.1) Transversality. Let $f: X \to Y$ be a smooth map between two smooth manifolds, A a subset of X, and Z a submanifold of Y. f is said to be transversal to Z on A (denoted by $f \pitchfork_A Z$ or write $f \pitchfork Z$ on A), if $\forall x \in f^{-1}(Z) \cap A$, have the following surjectivity condition

$$Df(T_x X) + T_{f(x)} Z = T_{f(x)} Y.$$

We dorp A if $A = X$. In the case Z reduced to be a point, and $A = X$, this is just the usual definition of a regular value. The reason to study the transversality is due to following observation: If $f \pitchfork Z$ and Z is of codimension k in Y, then by the implicit function theorem we have $f^{-1}(Z)$ a smooth submanifold of X of codimension k (empty set is allowed). And the normal bundle of $f^{-1}(Z)$ in X is isomorphic to the pull back bundle $f^*(N)$ of the normal bundle N of Z in Y.

Now we begin to prove the transversality theorem which says that any smooth map can be approximated by transversal ones.

Theorem (Thom). Let $f: X \to Y$ be smooth, and $f \pitchfork_A Z$, where A is a closed set of X and Z is a submanifold of Y, let d be any metric compatible with the underlying topology of Y and $\epsilon > 0$, then there is a smooth map $g: X \to Y$ such that $g \pitchfork Z, \quad d(f(x), g(x)) < \epsilon$ and $f|_A = g|_A$.

Proof. There are several steps:

(1) Since transversality is an open condition, there is an open set $U \subset A$ such that $f \pitchfork_U Z$. Now we will choose some appropriate coordinate coverings to reduce the problem to the case of Euclidean sapces. First of all, let $Y_0 = Y - Z$ and Y_i be charts cover Z with $Z \cap Y_i$ coordinate planes. Secondly, choose V_i charts such that $\{V_i\}$ is a refinement of both $\{X - A, U\}$ and $\{f^{-1}(Y_i)\}$. Since we can not modify f on A, we need an even more refine covering of those V_i that do not interesect A. We choose (by paracompactness) $\{W_i\}$ a family of locally finite relatively compact charts with $\{\bar{W}_i\}$ finer than $\{V_j\}$. Since the index will no longer be preserved, we re-indexing V_i (so some V_i may equals V_j). Finally we disgard those W_i which are contained in U. Still denote the final family by $\{W_i\}$. It is claerly a covering of $X - U$.

(2) We will construct f_i inductively such that

a) $d(f_i(x), f_{i-1}(x)) \le \epsilon/2^i \quad \forall x \in X$.
b) $f_i \equiv f_{i-1}$ outside a compact neighborhood of $\bar{W}_i$ in V_i.
c) $f_i \pitchfork Z$ on $f_i^{-1}(Z) \cap (\bar{W}_1 \cup \cdots \cup \bar{W}_i)$.

Once this is done, we get $\lim f_i = g: X \to Y$ the desired smooth map. Notice there is no limit process since the covering is chosen to be locally finite. Actually when X is compact, $\{W_i\}$ is a finite family. We remark also that in the whole process, we never change the value of f near A.

(3) For each $i \geq 1, f_{i-1}(V_i) \subset Y_{j(i)}$ for some $j(i)$. Since V_i, Y_j are all coordinate charts, by the induction steps, we only have to treat everything in the Euclidean spases. Namely, $K \equiv \bar{W} \subset V \subset \mathbf{R}^n$, $Z = Y \cap \mathbf{R}^q \subset Y \subset \mathbf{R}^p$, and $f\colon V \to Y$ smooth with $f \pitchfork Z$ on a relatively closed set S (S is to be thought as $\bar{W}_1 \cup \cdots \cup \bar{W}^{i-1}$).

Consider the projection $p\colon \mathbf{R}^p \to \mathbf{R}^{p-q}$, then $Z = p|_Y^{-1}(0)$, so $p \circ f\colon V \to \mathbf{R}^{p-q}$ has 0 as a regular value if and omly if f is transversal to Z, thus it suffices to consider the case $Z = 0$, that is, a point.

(4) Since 0 is now a regular value of f on S, what we have to do is to modify it to be regular on all $S \cup K$. Using a smooth partition of unity, construct a smooth map $\lambda\colon V \to [0,1]$ which equals 1 on a neighborhood of K and equals 0 outside a compact neighborhood K' of K in V. By sard's theorem, there are always points arbitrarily near 0 and are still regular values, pick y with $|y| < \epsilon$ and consider

$$g(x) = f(x) - \lambda(x)y.$$

then by the definition of λ, clearly have

1. g has 0 as a regular value (ie. $g \pitchfork 0$) on K.
2. $g \equiv f$ outside K'.
3. $|g(x) - f(x)| < \epsilon$.

Since y can be chosen arbitrarily near 0 and $|\frac{\partial}{\partial x_i}\lambda|$ is globally bounded, we can make g, Dg near f, Df uniformly. By 2. $g \pitchfork 0$ on $S - K'$, so we only have to care about the set $(S \cap K') \cap g^{-1}(0)$. Notice $S \cap K'$ is a compact set, so when $|y|$ small we have Df_x onto $\Rightarrow Dg_x$ onto. So $g \pitchfork 0$ on $S \cup K$ as required. **Qed.**

(2.2) Thom homorphism: $\tau\colon \pi_{k+n}(\mathbf{T}(\xi)) \to \mathbf{\Omega}_n$.

Let ξ be an oriented k plane bundle over the base manifold B equipped with a bundle metric, we define the Thom space $\mathbf{T}(\xi)$ of ξ to be $D(\xi)/S(\xi)$, that is, identify all vectors with length ≥ 1 to a single point, we always denote this point by t_0 and refer to it the base point. We notice that different metrics give the same Thom space. When B is compact, $\mathbf{T}(\xi)$ is just the one point compatification of the total space $E(\xi)$ of ξ. Actually, this point of view is used more often, and for simplicity, we assume that B is compact.

Given a map $f\colon S^{k+n} \to \mathbf{T}(\xi)$ with $\infty \mapsto t_0$, have $f^{-1}(B) \subset S^{k+n} - f^{-1}(t_0)$, so $f^{-1}(B)$ can be considered to be in some open set U with $\bar{U} \subset S^{k+n} - f^{-1}(t_0) \simeq \mathbf{R}^{k+n}$. We can operate everything inside U, that is we will not change f outside U. Notice that by definition f is surely transverse to B outside U. By transversality theorem, we may modify f in its homotopy class and keep it unchanged outside U, so we may assume $f \pitchfork B$, then $f^{-1}(B)$ is an n dimensional closed oriented submanifold of U. That is we get an element of $\mathbf{\Omega}_n$.

To prove that τ is well defined, suppose $F: S^{k+n} \times [0,1] \to \mathbf{T}(\xi)$ be a homotopy such that $F(x,0) = f_0(x)$, $F(x,1) = f_1(x)$. We may choose F such that $F(\cdot, [0, \frac{1}{3}]) = f_0$, $F(\cdot, [\frac{2}{3}, 1]) = f_1$. Apply the transversality theorem to $X = F^{-1}(E(\xi)) \cap (S^{k+n} \times (0,1))$, get a map $\tilde{F}$ coinside with f_0 near 0, and coinside with f_1 near 1, and $\tilde{F}^{-1}(B)$ is a manifold with boundary $f_1^{-1}(B) - f_0^{-1}(B)$. So τ is well defined as a set map.

To prove τ is a group homomorphism we go back to the definition of the addition rule in homotopy groups, clearly it corresponds to the disjoint union of manifolds in our construction of τ, say, one in the north half sphere and one in the south half sphere. This complete the proof.

In the following consider in particular $B = \tilde{G}_k(\mathbf{R}^{k+p})$ and $\xi = \tilde{\xi}_p^k$ the oriented universal k plane bundle. Denote the bundle projection by π. Then we have

(2.3) Theorem (Thom). τ is an ismorphism for $k \geq n+2$ and $p \geq n$.

Proof. Surjectivity: we only need $k \geq n$, $p \geq n$ in this part.

Let $[M] \in \mathbf{\Omega}_n$. By Whitney embedding theorem, M can be embeded in $\mathbf{R}^{k+n}$. (Whitney's theorem is much easier to prove if $k \geq n+1$.) Let ν be the normal bundle of M in $\mathbf{R}^{k+n}$. By the existence of tubular neighborhood of M, denoted by U, we can construct the generalized Gauss map:

$$g: U \simeq E(\nu) \longrightarrow E(\tilde{\xi}_n^k) \hookrightarrow E(\xi)$$

Complete g to be a map $g: S^{n+k} \to \mathbf{T}(\xi)$ by sending all points outside U to t_0. Then do the same as before, can assume g to be transversal to B. It is then clear that $\tau([g]) = [M]$.

Injectivity: This is much more involved than the surjective part. In fact we don't need this part in the remaining sections. Even more, when we finally prove the Thom's cobordism theorem for $\mathbf{\Omega} \otimes \mathbf{Q}$, we obtain the injectivity for those homotopy classes that are not torsion elements as a corollary. Anyway, for completeness we give the proof.

So let $g: S^{k+n} \to \mathbf{T}(\xi)$ be transversal to B and $g^{-1}(B) = M = \partial N$, have to show $g \sim$ constant map. M is an closed n manifold in $\mathbf{R}^{k+n} \subset S^{k+n}$.

Claim 1: Can embed N in $\mathbf{R}^{k+n} \times [0, \frac{1}{2}]$ such that $N \cap (\mathbf{R}^{k+n} \times [0, \frac{1}{4}]) = M \times [0, \frac{1}{4}]$ (it's true that near ∂N, N has a product structure, the collar theorem, but how one get such embedding globally?)

Assume this, then let V be a tubular neighborhood of N in $\mathbf{R}^{k+n} \times [0,1]$ with $d(V, N) < \epsilon$ and then $U = V \cap (\mathbf{R}^{k+n} \times \{0\})$ is the tubular neighborhood of M in $\mathbf{R}^{k+n}$.

Claim 2: We can deform the map g such that $g|_U: U \to E(\xi)$ is a bundle map.

Once this is done, g can be extended to be a bundle map $\tilde{g}: V \to E(\xi)$ by the classification theory of vector bundles (cf. [Hirsh] p.100). Define $\tilde{g}$ on the whole $\mathbf{R}^{k+n} \times [0,1]$ by sending the complement of V to t_0. Then $\tilde{g}$ gives the desired homotopy from g to the constant map t_0.

To prove claim 1, let $h: M \times [0,1)$ diffeomorphic onto a neighborhood of ∂N (the collar). Define $\beta: \mathbf{R} \to [0,1]$ to be a smooth increasing cut-off function such that $\beta(x) = 0$ for $x < \frac{1}{2} + \epsilon$ and $\beta(x) = 1$ for $x > 1 - \epsilon$. Then define $h_1: N \to \mathbf{R}^{k+n} \times [0, \frac{1}{2}]$ by

$$h_1(y) = \begin{cases} (x, \frac{1}{2}s) & \text{for } y \in h(M \times [0, \frac{1}{2}]) \\ p & \text{for } y \notin h(M \times [0,1)) \\ (1-\beta(s))(x, \frac{1}{4}) + \beta(s)p & \text{for } \frac{1}{2} \leq s < 1 \end{cases}$$

where $p = (x_0, \frac{1}{2})$ with $x_0 \in M$ an arbitrary point. Although h_1 is a smooth map, the resulting image $h_1(N)$ is never a manifold. It looks like the roof of an old fashioned house. But it still contains the collar $M \times [0, \frac{1}{4}]$.

Since $k \geq n+2, k+n+1 \geq 2(n+1)+1 = 2\dim(N)+1$. We can apply the smooth approximation theorem which says in such a dimension, embeddings are dense in the space of smooth maps. (cf. []) So we find a map

$$h_2: N \hookrightarrow \mathbf{R}^{k+n+1}$$

and $h_2 \equiv h_1$ in $h(M \times [0, \frac{1}{4}])$. Claim 1 is proved.

To prove claim 2, we first deform the map g such that it sends all points outside U to t_0 and keep g unchanged on a smaller neighborhood of $M = g^{-1}(B)$. Recall the universal bundle $\pi: E(\xi) \to B$ and denote the bundle projection of $U \to M$ by p. Let $x \in U$, by using the linear structures on $U_{p(x)}$, which is induced by the normal bundle ν (it is just a rescaling of each fiber), and the linear structure on $\xi_{\pi(g(x))}$, can define for $s \in [0,1]$,

$$H(x,s) = \frac{g(s\vec{x})}{s},$$

with $H(x,0) = Dg_{\pi(x)}(\vec{x})$, which is surely a linear map on the normal space over $p(x)$ (since Dg is linear on the whole tangent space), and this linear map is an isomorphism by the transversality of g Thus $g_0 \equiv H(\cdot, 0)$ defines a bundle map $U \to E(\xi)$ and $H(\cdot, s), s \in [0,1]$ defines the homotopy from $g \equiv g_1$ to g_0. The construction behavior well on points near ∂U, so claim 2 is proved. This complete the proof of this "Thom isomorphism". **Qed.**

(2.4) Rational Homotopy Groups.

Although we have established the ismorphism of Thom homomorphism, in general it is still difficult to compute the higher homotopy groups. The Hurwicz theorem, which established a good relation between the homology group and homotopy group, is such a type of theorem that we need now, but its original form needs strong assumptions which are surely not satisfied in our situation. Anyway, since we only concern with the non-torsion part $\mathbf{\Omega} \otimes \mathbf{Q}$, so we only have to compute the so-called rational homotopy groups. (But it is still not possible to include the theory here.) In this case, we have (cf. [DFN] p.129)

Theorem. Let X be a finite CW complex which is r-connected with $r \geq 1$, then after $\otimes \mathbf{Q}$, the Hurwicz homomorphism

$$\pi_i(X) \otimes \mathbf{Q} \longrightarrow \mathbf{H}_i(X; \mathbf{Q})$$

is an isomorphism for $i \leq 2r$.

In order to apply this to our case, we must show $\mathbf{T}(\xi)$ have some higher connectivity, but since $B = \tilde{G}_k(\mathbf{R}^{k+p})$ whose cell decomposition is well known, say e_l is an open j cell of it, then the inverse image $\pi^{-1}(e_l)$ is clearly an open $j+k$ cell of $\mathbf{T}(\xi)$, Together with the zero cell t_0, we obtain a CW complex structure of $\mathbf{T}(\xi)$ without cells of dimension between 0 and k, that is, $\mathbf{T}(\xi)$ is $(k-1)$-connected. So we get, for $n \leq k-2$:

$$\pi_{k+n}(\mathbf{T}(\xi)) \otimes \mathbf{Q} \simeq \mathbf{H}_{k+n}(\mathbf{T}(\xi); \mathbf{Q}).$$

Now we have the Thom isomorphism

$$\mathbf{H}_{k+n}(\mathbf{T}(\xi), t_0; \mathbf{Z}) \simeq \mathbf{H}_{k+n}(D(\xi), S(\xi); \mathbf{Z}) \simeq \mathbf{H}_n(B; \mathbf{Z}).$$

So by connecting the three ismorphisms, we finally obtain for $k \geq n+2$,

$$\mathbf{\Omega}_n \otimes \mathbf{Q} \simeq \mathbf{H}_n(B; \mathbf{Q}).$$

As noted (1.5), by letting k large enough, $\mathbf{H}^n(B; \mathbf{Q})$ is freely generated by Pontryagin classes of the universal bundle ξ, so it is zero when $4 \not| n$, and is of dimension $p(m)$ if $n = 4m$. Toghther with the calculation on products of $\mathbf{P}^{2n}(\mathbf{C})$'s. which shows the various products of $\mathbf{P}^{2n_j}(\mathbf{C})$'s with $j_1 + \cdots + j_r = m$ are all in distinct cobordism classes, since the total number of such products is exactly $p(m)$, we finally get the

Thom's Cobordism Theorem:

$$\mathbf{\Omega} \otimes \mathbf{Q} = \mathbf{Q}[\mathbf{P}^2(\mathbf{C}), \mathbf{P}^4(\mathbf{C}), \mathbf{P}^6(\mathbf{C}), \ldots]$$

Qed.

Remark. In the proof we do not need each step to be isomorphism. We only need

$$\begin{aligned} p(r) &= \dim \mathbf{H}_n(B; \mathbf{Q}) = \dim \mathbf{H}_{k+n}(\mathbf{T}(\xi), t_0; \mathbf{Q}) \\ &\geq \dim(\pi_{k+n}(\mathbf{T}(\xi)) \otimes \mathbf{Q}) \geq \dim(\mathbf{\Omega}_n \otimes \mathbf{Q}) \\ &\geq p(r) \end{aligned}$$

So we need only the surjective part of the Thom homomorphism τ (which is the easier part), and the injective part of the rational Hurwicz homomorphism.

§3 Hirzebruch's signature theorem

(3.0) Genera. Let R be a commutative ring over $\mathbf{Q}$. An R-genus is defined to be a ring homomorohism $\phi : \Omega \otimes \mathbf{Q} \to R$. Since as a ring, $\Omega \otimes \mathbf{Q}$ is generated by the projective spaces $\mathbf{P}^{2i}(\mathbf{C}), i \geq 0$, we only have to know the value of ϕ on these spaces. This section is in fact a realization of this simple observation.

(3.1) Signature. Let M be a $2n$ dimensional oriented closed manifold. By Poincare duality, the intersection form q_M on $\mathbf{H}_n(M, \mathbf{Z})$ is a nondegenerate pairing, it is alternating when n is odd, symmetric when n is even, in the later case, it is unimodular. Since we have some classification theory of integarl quadratic forms and alternating forms, it is very hopeful that the study of the intersection form will provide some important information about the topology of M. Now we define the signature $\sigma(M)$ of M to be zero when $4 \not| \dim(M)$ and to be the signature of q_M (that is, the number of positive eigenvalues σ^+ minus the number of negative eigenvalues σ^-). We notice that we can also define the signature by using cohomology, the intersection pairing is then the cup product and evaluated on the fundamental class $[M]$. Using deRham cohomology, the intersection pairing then can be viewed as the integration of wedge of closed differential forms over M. This view-point will be very useful in some cases.

As an example, we will now show that the signature of manifolds is a genus.

Lemma. The signature is a $\mathbf{Z}$-genus, that is,

(1) $\sigma(V + W) = \sigma(V) + \sigma(W)$, $\sigma(-V) = -\sigma(V)$.
(2) $\sigma(V \times W) = \sigma(V) \times \sigma(W)$.
(3) $\sigma(M) = 0$ if M bounds some compact oriented manifolds.

Proof. (1) is clear since the intersection form of disjoint union of manifolds splits as the direct sum of the individual ones.

(2) We use the cohomology with coefficient in $\mathbf{R}$. Let $M^{4k} = V^n \times W^m$, by the Kunneth formula,

$$\mathbf{H}^{2k}(M) = \bigoplus_{s+t=2k} \mathbf{H}^s(V) \otimes \mathbf{H}^t(W).$$

Let $\{v_i^s\}, \{w_j^t\}$ be the basis of $\mathbf{H}^s(V), \mathbf{H}^t(W)$, such that $v_i^s v_j^{n-s} = \delta_{ij}, w_i^t w_j^{m-t} = \delta_{ij}$ for $s \neq \frac{n}{2}, t \neq \frac{m}{2}$. (We can not do this in the middle dimension in general) Let $A = \mathbf{H}^{\frac{n}{2}}(V) \otimes \mathbf{H}^{\frac{m}{2}}(W)$ when n, m are both even, and let $A = 0$ in other cases. Let $B = A^{\perp}$ in $\mathbf{H}^{2k}(M)$, that is, the space spanned by elements not in A (elements in A can not be orthogonal to A because the intersection product on A is the tensor product $q_V \otimes q_W$ which is also nondegenerate). The set

$$\{v_i^s \otimes w_j^t | s + t = 2k, s \neq \frac{n}{2} (t \neq \frac{m}{2})\}$$

is an orthogonal basis of B. We will show in fact $\sigma(B) = 0$, and $\sigma(A) = 0$ when $4 \not| n (4 \not| m)$. Then $\sigma(M) = \sigma(A) + \sigma(B) = \sigma(A) = \sigma(V) \times \sigma(W)$ as required.

To prove $\sigma(B)=0$, observe $(v_i^s \otimes w_j^t)\cdot(v_{i'}^{s'} \otimes v_{j'}^{t'}) \neq 0$ only when $i=i', j=j'$ and $s+s'=n, (t+t'=m)$, and it equals ± 1 in this case. The intersection matrix thus has $\pm\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ as its building blocks. It is then clear that $\sigma(B)=0$.

If $4 \not| n$ (so $4 \not| m$), we have to show $\sigma(A)=0$. This amounts to say that the symmetric bilinear form obtained from the tensor product of two alternating forms must has zero signature. By the structure theorem of nondegenerate alternating form over $\mathbf{R}$, we know it has a matrix representation with $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ as the building block. And the tensor product of two such matrix gives

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

so the intersection matrix on A is a direct sum of the above matrix, it clearly has zero signature.

(3) Let $i : M^{4k} \to W^{4k+1}$ be the boundary inclusion. We have the following commutative diagram:

$$\begin{array}{ccccccc}
\cdots \to & \mathbf{H}^{2k}(W) & \xrightarrow{i^*} & \mathbf{H}^{2k}(M) & \xrightarrow{\partial^*} & \mathbf{H}^{2k+1}(W,M) & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \mathbf{H}_{2k+1}(W,M) & \xrightarrow{\partial_*} & \mathbf{H}_{2k}(M) & \xrightarrow{i_*} & \mathbf{H}_{2k}(W) & \to \cdots
\end{array}$$

The verticle maps are all isomorphisms by Poincare-Lefschetz duality. Let $A = \mathrm{Im}\ i^*, B = \mathrm{Ker}\ i_*$. Since i^*, i_* are dual vector space maps, A is dual to $\mathbf{H}_{2k}(M)/B$ under the duality between $\mathbf{H}^{2k}(M)$ and $\mathbf{H}_{2k}(M)$. So $\dim(A) = \dim \mathbf{H}_{2k}(M) - \dim(B)$. By the exactness of the above diagram, $\dim(A) = \dim(B)$, so we get $\dim(A) = \dim(B) = \frac{1}{2}\dim \mathbf{H}_{2k}(M)$. Now by Stokes' theorem, for any $\omega \in A$,

$$\int_M (i^*\omega)^2 = \int_W d(i^*\omega \wedge i^*\omega) = 0$$

so the zero cone of the intersection form contains A. Since A has half the dimension, this can happen only when $\sigma^+ = \sigma^-$, that is, $\sigma(M)=0$. This complete the proof of this lemma. **Qed.**

(3.2) Multiplicative sequences. Let ϕ be a genus, it is natural to consider the generating power series

$$P_\phi(x) = \sum_{n=0}^{\infty} \phi(\mathbf{P}^{2n}(\mathbf{C}))x^{2n}.$$

In the case of signature, It is easy to see that $\sigma(\mathbf{P}^{2n}(\mathbf{C})) = 1$, so the generating power series is $P_\sigma(x) = 1 + x^2 + x^4 + \cdots = 1/(1-x^2)$. What we want to do now is to find a polynomial $K(p) = K(p_1, p_2, \ldots)$ with (formal) Pnotryagin classes as its variables, such that $K(p(M))[M] = \phi(M)$, the left hand side means we substitute the pontryagin classes of M into the formal variables p, then evaluate at the fundamental class $[M]$ by using the term of degree n when $\dim(M) = 4n$, it is a linear combination of Pontryagin numbers. Unfortunately this polynomial can not be obtained very directly from P_ϕ. The method to do this was invented by Hirzebruch (also the definition of genera is due to him).

Let's start from an arbitrary even power series $Q(x) = 1 + q_1x^2 + q_2x^4 + \cdots$ with coefficients in R (we use even power series in order to make some expression clear, see later), and form

$$\begin{aligned}\prod\nolimits_{i=1}^{n} Q(x_i) &= 1 + q_2 \sum\nolimits_{i=1}^{n} x_i^2 + \cdots \\ &= 1 + K_1(p_1) + k_2(p_1, p_2) + \cdots \\ &\qquad + K_n(p_1, \ldots, p_n) + K_{n+1}(p_1, \ldots, p_n, 0) + \cdots\end{aligned}$$

here we regard the variables x_i's as have weight 2, and the weight $4r$ parts of the first product is a symmetric polynomial of x_i^2. Let p_i be the i-th elementary symmetric polynomial of x_i^2, then we can transformed the weight $4r$ part into the an unigue polynomial of these p_i's, this is the definition of K_r. By the theory of symmetric polynomials, we know that K_r is independent of the number of variables n if $n \geq r$. In the following, we will always assume n is large enough, in fact we can take $n = \infty$, the resulting series is denated by $K(p)$, here the variable p denates $1 + p_1 + p_2 + \cdots$, the formal total pontryagin classes. In the special case $p = 1 + p_1 = 1 + x^2$, we have $K(1 + x^2) = Q(x)$.

Lemma. The function $\phi_Q(M) := K(p(M))[M]$ is an R genus.

Proof. (1) We first check that if $M^{4n} = \partial W^{4n+1}$ then $\phi_Q(M) = 0$. In this case we have $TW|_M = TM \oplus \epsilon$, where ϵ is the trivial normal bundle of M in W. Then $p(M) = p(W)|_M$, that is, all pontryagin classes of M are restriction of those of W, since any Pontryagin number p_I of M is the integration of the corresponding closed (wedge of) Pointryagin form ω_I on $M = \partial W$, by Stokes' theorem,

$$p_I = \int_M \omega_I = \int_W d\tilde{\omega}_I = 0$$

where $\tilde{\omega}_I$ is the Pontryagin form on W such that $\tilde{\omega}_I|_M = \omega_I$. This proves (1).

(2) We have to show the adivitity and multiplicativity. The aditivity is obvious, the multiplicativity is more subtle. We have to prove: Let $p' = 1 + p_1' + p_2' + \cdots$, $p'' = 1 + p_1'' + p_2'' + \cdots$, ($p_k', p_k''$ of weight $4k$). If $p = p'p'' = 1 + p_1 + p_2 + \cdots$ by

collecting corresponding terms, then

$$K(p) \equiv K(p'p'') = K(p')K(p'').$$

(This is why people call K_r the multiplicative sequences.) Let $Q(x) = \sum_{i=0}^{\infty} q_i x^{2i}$ as before. For any partition $I = \{i_1, \ldots, i_r\}$ of n, let $q_I = q_{i_1} \cdots q_{i_r}$, and let

$$s_I(p_1, \ldots, p_n) = \sum x_1^{2j_1} \cdots x_r^{2j_r}$$

where the sum is over all distinct permutations $(j_1, \ldots, j_r)$ of I. Since it is symmetric in x_i^2's, it is uniquely represented by a polynomial in $p_i's$, this is the definition of s_I. Then we have

$$s_I(p'p'') = \sum\nolimits_{\{H,J\}=I} s_H(p')s_J(p''),$$

which is just a partition of standard monomials into two parts of fewer variables. Again by comparing corrseponding terms, we have

$$K_r(p_1, \ldots, p_n) = \sum\nolimits_I q_I s_I(p_1, \ldots, p_n).$$

Take summation over all n, we get

$$\begin{aligned} K(p'p'') &= \sum\nolimits_I q_I s_I(p'p'') \\ &= \sum\nolimits_I \sum\nolimits_{H,J=I} q_{H,J} s_H(p')s_J(p'') \\ &= \sum\nolimits_{H,J} q_H q_J s_H(p')s_J(p'') \\ &= \sum\nolimits_H q_H s_H(p') \sum\nolimits_J q_J s_J(p'') \\ &= K(p')K(p''). \end{aligned}$$

This is what we want. **Qed.**

Remark. Actually we have already proved: Given any even power series $Q(x)$ begins with 1, there exists an unique multiplicative sequence K such that $K(1+x^2) = Q(x)$. We have proved the existense, for the uniqueness, use a formal decomposition $p = \prod_i(1 + x_i^2)$, then $K(p) = \prod_i Q(x_i)$. This is exactly our construction.

(3.3) We have the following very important lemma. Let $f(x) = x/Q(x) = x + \cdots \in \mathbf{R}[\![x]\!]$, it is clearly invertible. Let $g(y) = f^{-1}(y)$, then:

Lemma. $g'(y) = \sum_{n=0}^{\infty} \phi_Q(\mathbf{P}^n(\mathbf{C}))y^n$.

Proof. Strat from $p(\mathbf{P}^n(\mathbf{C})) = (1 + p_1)^{n+1}$ (cf. (1.1)), have $K(p) = K(1 + p_1)^{n+1} =$

$Q(p_1)^{n+1}$. Thus

$$\begin{aligned}
\phi_Q(\mathbf{P}^n(\mathbf{C})) &= Q(p_1)^{n+1}[\mathbf{P}^n(\mathbf{C})] \\
&= \text{ coefficient of } x^n \text{ in } Q(x)^{n+1} = \left(\frac{x}{f(x)}\right)^{n+1} \\
&= \text{ residue at 0 of } \frac{1}{f(x)^{n+1}}dx \\
&= \frac{1}{2\pi i}\int_C \frac{1}{f(x)^{n+1}}dx = \frac{1}{2\pi i}\int_{f(C)} \frac{g'(y)}{y^{n+1}}dy \\
&= \text{coefficient of } y^n \text{ in } g'(y)
\end{aligned}$$

Since $f(x) = x + \cdots$, when the circle C around 0 is small enough, $f(C)$ is also a curve around 0 with winding number 1. In fact we even don not need f to be convergent since the above argument is essentially a sequence of substitution of formal power seris. This complete the proof. **Qed.**

This lemma says $g'(y)$ is exactly the generating power seris of ϕ_Q! So if we start from a genus ϕ, with generating power series P_ϕ, then we set $g = \int P_\phi$ with constant term zero, and set $f = g^{-1}$, then $Q = x/f(x)$ is the fundamental power series whose associated multiplicative sequence K defines the genus $\phi_Q = \phi$.

Apply this to the signature σ, since $P_\sigma(y) = 1 + y^2 + y^4 + \cdots = 1/(1-y^2)$, so $g(y) = \int \frac{1}{1-y^2} = \tanh^{-1}(y)$, and then $f(x) = \tanh(x)$, finally we get the fundamental power series $Q(x) = x/\tanh(x)$. Hirzebruch gave the corresponding $K = \sum K_r$ a name, the "L polynomials": $L = \sum L_r$, and called signature the "L genus".

To actually compute the L polynomials L_r's, we need the Taylor expansion of $\frac{x}{\tanh(x)}$. It reads

$$\frac{x}{\tanh(x)} = 1 + \frac{1}{3}x^2 - \frac{1}{45}x^4 + \cdots + (-1)^{k+1}\frac{2^{2k}B_k}{(2k)!}x^{2k} + \cdots$$

where B_k's are the Bernouli numbers ([MS] App.B), the first few terms are 1/6, 1/30, 1/42, 1/30, 5/66, 691/2730, $\cdots$. Then a direct computation gives the L polynomials:

$$\begin{aligned}
L_1(p_1) =& \frac{1}{3}P_1 \\
L_2(p_1, p_2) =& \frac{1}{45}(7p_1 - p_1^2) \\
& \vdots
\end{aligned}$$

Since we will need only L_2 in the following sections, we don't list the higher L polynomials here.

As a conclusion, we have already established the

Hirzebrich Signature Theorem: $\sigma(M) = L(M)$.

(3.4) Remarks (about the theorem.)

(1) Using other $Q(x)$ we can get other interesting genus (cf. [ ]), for example take $Q(x) = \frac{x/2}{\sinh(x/2)}$, we get the so-called "$A$-roof genus", $\hat{A}(M)$, which appears in the Atiyah-Singer Index Theorem.

(2) The Signature Theorem (as well as the Gauss-Bonnet-Chern Theorem and the Hirzebruch's Riemann-Roch Theorem) is in fact a special case of a more general theorem, namely Atiyah-Singer Index Theorem, but historically the Index Theorem was first proved by a "twisted version" of Hirzebruch Signature Theorem. Now there are several new methods to prove the Index Theorem without involve the Signature Theorem, so it is actually a corollary of the Index Theorem.

§4 An Exotic Seven Sphere.

(4.0) As showed in §1, the Hopf fibrations are typical examples of spheres which can be realized as a "sphere fibered by sphere". Milnor in his 1956' paper ([M1]) described a lot of S^3 bundles over S^4, with total space the seven sphere, but with different smooth structures from standard S^7. In this section we will describe these "exotic spheres". We will see the power of the Signature Theorem. Although we have already discussed some topological properties of such sphere fibration, we will start with the most naive way in (4.1) and put the results of §1 into consideration after (4.6).

(4.1) Let D^4_+, D^4_- denote the upper and lower hemi-sphere of S^4, then any (oriented) vector bundle on S^4 can be described as identifying two trivial vector bundles over D^4_+, D^4_- (contractible space!) along their common boundary, the equator S^3, this identification is given by a map $f : S^3 \to \mathbf{SO}(4)$, and this map is unique up to homotopy.

Now consider $f_{hj} : S^3 \to \mathbf{SO}(4)$ by the rule: (identify $\mathbf{R}^4 = \mathbf{H}$ and using quaternion multiplication)

$$f_{hj}(u) \cdot v = u^h v u^j.$$

This defines an $\mathbf{SO}(4)$ bundle E_{hj} on S^4 with fiber $\mathbf{R}^4$, Let ξ_{hj} be the sphere bundle of E_{hj}, that is, $\partial D(E_{hj})$, we will show when $h+j=1$, (so $h-j=k$ is odd), the total spase of ξ_{hj}, denoted by M^7_k, is a topological seven sphere. (In (1.3) we have already proved these two numbers $h+j$, $2(h-j)$ correspond to e, p_1.) Since $h+j=1$, k determines the pair (h,j) uniquely, so in the following we write the lower indices as k instead of hj.

(4.2) The idea is to construct a "Morse function" f on M^7_k with exactly two critical points. (Recall that a Morse function is a real valued smooth function with discrete critical points and the hessian of each critical point is a nondegenerate quadratic form.) Once this is done, since our manifold is compact, this implies the two critical points, say y_0, y_1, are exactly the minimal and maximal points of f. We may assume that $f(y_0)=0, f(y_1)=1$. By considering the gradient flow:

$$\frac{dx}{dt} = \nabla f(x).$$

we know that $f^{-1}([0,a])$ are all diffeomorphic for $0 < a < 1$. When a is small, by Morse lemma, there exists a coordinate system $(x_1, \cdots, x_7)$ about y_0, such that $f(x_1, \ldots, x_7) = x_1^2 + \cdots + x_7^2$, so $f^{-1}([0,a])$ is clearly diffeomorphic to the seven disk D^7. By the flow, we conclude that $M^7_k - y_1 = f^{-1}([0,1))$ is diffeomorphic to D^7, so M^7_k is a smooth "topological sphere". (Remark: the above argument surely works for all dimensions.)

(4.3) Now we will show M_k^7 can be realized as an identification of two $\mathbf{R}^4 \times S^3$ along $(\mathbf{R}^4 - 0) \times S^3$ via the diffeorphism g of $(\mathbf{R}^4 - 0) \times S^3$:

$$g\colon (u,v) \mapsto (u',v') = \left(\frac{u}{|u|^2}, \frac{u^h v u^j}{|u|}\right)$$

this makes sense because (we should verify that $v' \in S^3$)

$$\left|\frac{u^h v u^j}{|u|}\right| = \frac{|u|^h |v| |u|^j}{|u|} = \frac{|u|^{h+j}}{|h|} = 1.$$

Here $h + j = 1$ is essentially used. The formula $u' = u/|u|^2$ is nothing but the coordinate change of the two stereographic projections: $S^4 \to \mathbf{R}^4$, one from the south and one from the north. It suffices to check this in the 1 dimensional case, and it is trivially done by the similar triangle rule.

To check the glueing really gives M_k^7, we consider the equator S^3, that is, $|u| = |u'| = 1$, in fact $u = u'$. The map g restrict on this equator then defines a map $\tilde{g} : S^3 \to \mathbf{SO}(4)$ by $\tilde{g}(u)v = u^h v u^j$. which is exactly the map f_{hj}, since any bundle over S^4 is classified by this map as mentioned before, this space is exactly M_k^7.

(4.4) To construct the desired function f on M_k^7, consider the following two coordinate charts, (u,v) and (u'',v'), where $u'' = u'(v')^{-1}$. Define $f\colon M_k^7 \to \mathbf{R}$ by

$$f(x) = \frac{\mathrm{Re}(v)}{\sqrt{1+|u|^2}} = \frac{\mathrm{Re}(u'')}{\sqrt{1+|u''|^2}}$$

Check equality: later term =

$$\frac{\mathrm{Re}(u'(v')^{-1})}{\sqrt{1+|u'|^2}} = \frac{\mathrm{Re}(|u|u'(v')^{-1})}{\sqrt{1+|u|^2}}$$

$(\cdots) = u(|u|v')^{-1} = u(u^h v u^j)^{-1} = u \cdot u^{1-j} v^{-1} u^{-h} = u^h v^{-1} u^{-h}$. When we represent $\mathbf{H}$ as 4×4 matrix over $\mathbf{R}$, we have $\mathrm{Re}(x) = \frac{1}{4}\mathrm{trace}(x)$, so $\mathrm{Re}(u^h v^{-1} u^{-h}) = \frac{1}{4}\mathrm{trace}(u^h v^{-1} (u^h)^{-1}) = \frac{1}{4}\mathrm{trace}(v^{-1}) = \mathrm{Re}(v^{-1})$. Since $|v| = 1, v^{-1} = \bar{v}$, we have $\mathrm{Re}(v^{-1}) = \mathrm{Re}(\bar{v}) = \mathrm{Re}(v)$. So left = right.

Now we consider the critical points. From the right expression of f we easily see that no critical points exists in the chart (u'',v'): the function $x_1/\sqrt{1+|x|^2} \nearrow$ in the direction x_1, so $\partial_1 f(x) > 0$. Hence all critical points lie in the (u,v) chart, and in fact lie in the set $(0,v)$. But in this set $f(x)$ reduces to be $\mathrm{Re}(v)$ (the height function) on S^3, the unit sphere of $\mathbf{H}$. So the critical points are clearly the two points $v = \pm 1$, that is, $(0, \pm 1)$. By (5.2), M_k^7 is a topological sphere.

(4.5) Now we will show some M_k^7 are not diffeomorphic to the standard S^7. Suppose M_k^7 is diffeomorphic to S^7, then we can attach an standard 8 dimensional disk D^8

to the boundary of the total space of the disk bundle $D(E_k)$ along $M_k^7 \simeq S^7$ via the assummed diffeomorphism. Denote the resulting closed 8 dimensional manifold by W_k^8. We compute the signature $\sigma(W_k^8)$ as follows:

We notice W_k^8 is nothing but the Thom space $\mathbf{T}(E_k)$, by the Thom isomorphism theorem, we get (by excision and $\cup e(E_k)$):

$$\mathbf{H}^i(S^4) \simeq \mathbf{H}^{4+i}(D(E_k), S(E_k)) \simeq \mathbf{H}^{4+i}(\mathbf{T}(E_k), t_0).$$

The integral cohomology groups of W_k^8 therefore equal $\mathbf{Z}$ in dimension 0, 4, and 8, and zero in other dimensions. Actually, it is $\mathbf{Z} \oplus \mathbf{Z}e(E_k) \oplus \mathbf{Z}e(E_k)^2$. This implies $\sigma(W_k^8) = \pm 1$. Choosing an orientation, may assume $\sigma(W_k^8) = 1$. Now apply the Hirzebruch signature theorem, we have

$$1 = \sigma = \frac{7p_2 - p_1^2}{45}.$$

Thus all we have to do now is to compute the pontryagin classes of W_k^8.

(4.6) Recall the result of (1.3), which says

$$e(E_{hj}) = (h+j)u, \quad p_1(E_{hj}) = 2(h-j)u.$$

In the present case, $e(E_k) = u$ and $p_1(E_k) = 2ku$.

To pass this result to W_k^8, denote by $\pi\colon E_k \to S^4$ the bundle projection, we always have $TE_k \simeq \pi^*(TS^4) \oplus \pi^*(E_k)$, so apply the Whitney sum formula and naturality as in (1.1), and $p(TS^4) = 1$, we get $p(TE_k) = \pi^* p(E_k)$ and $p_1(TE_k) = \pi^* p_1(E_k) = \pi^*(2ku) = 2ku = 2ke(E_K)$. So $p_1^2(TW_k^8) = p_1^2(TE_k) = 4k^2$. This is true because of naturality, the pontryagin classes of E_k are the restriction of Pontryagin classes of W_k^8 which is a smooth closed manifold and only have one more point than E_k, and so have the same value when evaluted on the fundamental class.

Put it into the signature formula, we get $4k^2 + 45 = 7p_2 \equiv 0 \pmod 7$, (Pontryagin numbers are integers!) this implies $4(k^2 - 1) \equiv 0 \pmod 7$ and so $k \equiv \pm 1 \pmod 7$. But k can be any odd integers! We get a contradiction for those E_k with $k \not\equiv \pm 1 \pmod 7$, that is, the hypothesis in (4.5) is wrong: M_k^7 is not diffeomorphic to S^7!

(4.7) Final remarks.

(1) There is a quick way to prove: $E(\xi_{hj})$ is a topological sphere if and only if $h + j = 1$. First, calculate its cohomology by using Gysin sequence as in (1.1), but then we need Smale's throrem on the generalized Poincare conjecture to conclude the result.

(2) There is still something not so good: although there are a lot of exotic seven spheres, they may be diffeomorphic! For example, Are M_3^7 and M_5^7 diffeomorphic? In Milnor's original approach, he put everything in the category of manifolds with boundary, and from this he constructed a diffeomorphism invariant

which is exactly $k-1 \pmod 7$, in this way he can distinguish some of these exotic spheres.

(3) But there is still another even more sophesticated question: How many smooth structures can a topological sphere have? The following section is a summary of Kervaire and Milnor's result to this question. I do not include the proofs here. Instead, I will describe Brieskorn and Hirzebruch's construction of these exotic spheres (including higher dimensional exotic spheres) in later sections.

§5 Summary of Kervaire/Milnor's results ([ ])

All manifolds are smooth oriented with dimension ≥ 5 and all bundles are smooth oriented in this section. Two manifolds M_1, M_2 are said to be **h-cobordant** if $M_1 - M_2 = \partial W$ and M_1, M_2 are both deformation retracts of W. It defines an equivalence relation on manifolds.

The connected sum $M_1 \# M_2$ is well defined up to orientation preserving diffeomorphism. It is commutative, associative and compatible with the relation of h-cobordism. All closed n-manifolds form a commutative monoid under $\#$, with identity the standard sphere S^n. We are interested in those closed manifolds which have the same homotopy type as a sphere, called the homotopy spheres. we have

(5.1) The set of h-cobordant classes of homotopy n-spheres form an abelian group under connected sum, denoted by Θ_n.

(5.2) Θ_n is finite.

By Smale's h-cobordism theorem, which says "h-cobordant $\Rightarrow$ diffeomorphic", and the truth of generalized Poincare conjecture of dimension ≥ 5, we have

(5.3) Θ_n is the group of all smooth structures on S^n.

We will not actually use (5.2), (5.3) in the sequal, what we really concern is a smaller subgroup $bP_{n+1} \subset \Theta_n$, to define it, we need the concept of parallelizable manifolds. M is called parallelizable if TM is trivial, and called s-parallelizable if $TM \oplus \epsilon$ is trivial, where ϵ is the trivial line bundle over M.

We need the following basic facts:

(5.4) Lemma. *Let ξ be a k plane bundle over M^n, $k \geq n$. If $\xi \oplus \epsilon^r$ is trivial, then ξ is trivial.*

Proof. Only have to consider the case $r = 1$. The isomorphism $\xi \oplus \epsilon \cong \epsilon^{k+1}$ gives rise to a bundle map

$$\begin{array}{ccc} \xi & \longrightarrow & \gamma^k \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & S^k \end{array}$$

where γ^k is the universal oriented k plane bundle over the oriented grassmannian $\tilde{G}(k, k+1) = S^k$. Since $k \geq n$, f is null homotopic, so ξ is trivial. ■

(5.5) Corollary. *Let M^n be a submanifold of S^{n+k}, $k \geq n$, then M is s-parallelizable iff the normal bundle is trivial.*

Proof. The bundle $T \oplus N \oplus \epsilon$ is always trivial, where ϵ is the (trivial) normal bundle of S^{n+k} in $\mathbf{R}^{n+k+1}$. If the normal bundle N is trivial, apply (5.4) to $(T \oplus \epsilon) \oplus N$, we get $T \oplus \epsilon$ is trivial. Conversely, if M is s-parallelizable, apply (5.4) to $N \oplus (T \oplus \epsilon)$, we get that N is trivial. ∎

(5.6) Corollary. *A connected manifold with nonempty boundary is s-parallelizable iff it is parallelizable.*

Proof. We need Morse Theory to conclude that a smooth manifold admits a CW complex structure, and if the boundary is not empty, the dimension of this CW complex can be choosen to be $< n = \dim(M)$. In the proof of Lemma (5.4), we need only $k \geq$ the CW complex dimension, so the result follows. ∎

(5.7) Corollary. *Any oriented submanifold M of $\mathbf{R}^n$ with $\partial M \neq \emptyset$ is parallelizable.*

Proof. Such manifold has trivial normal bundle. If we take n large, then M becomes s-parallelizable by (5.5). So it is parallelizable by (5.6). ∎

Now we define the set $bP_{n+1} \subset \Theta_n$: it consists of those homotopy n-spheres which bound a parallelizable manifold. This condition depends only on the h-cobordism class (This is clear if we use h-cobordism theorem). The main property we should know is that bP_{n+1} is a finite cyclic group and its members can be classfied by simple topological invariant. For simplesty we only consider $bP_{4m}, (m \geq 2)$, the collection of allparallelizable $4m$ manifolds with $\partial M = (4m-1)$-sphere. The corresponding signatures $\sigma(M)$ form a non trivial subgroup of $\mathbf{Z}$, denote it by $\sigma_m \mathbf{Z}$ where $\sigma_m \geq 0$. Then the following structure theorems are known:

(5.8) Let Σ_1, Σ_2 be two $4m-1$ homotopy spheres, $\partial M_i = \Sigma_i$, with M_i parallelizable. Then Σ_1 is h-cobordant to Σ_2 if and only if $\sigma(M_1) \equiv \sigma(M_2) (\mathrm{mod}\ \sigma_m)$. In another words, the signature (mod σ_m) classifies the smooth structures on S^{4m-1}.

So bP_{4m} is a subgroup of $\mathbf{Z}/\sigma_m\mathbf{Z}$, later we will see that all such parallelizable manifolds have signatures $\equiv 0 \pmod 8$, so the order of bP_{4m} divides $\sigma_m/8$. In fact it equals, and its value is also determined by Bernoulli numbers:

(5.9) The determination of bP_n is:

(1) $bP_{2k+1} = 0$

(2) $bP_{4m-2} = \mathbf{Z}/2\mathbf{Z}$ if $m \neq 1, 2, 4$

(3) bP_{4m} is cyclic of order $\sigma_m/8$, it equals

$$\epsilon_m 2^{2m-2}(2^{2m-1} - 1) \text{ numerator } \left(\frac{4B_m}{m}\right).$$

where $\epsilon_m = 1$ if m is odd, $= 2$ if m is even.