Global Geometry of Surfaces

(Manuscripts by Chin-Lung Wang)

These pages are selected from the second half of the course notes based on DoCarmo's book, with some supplementary results.

Contents

Gauss equation (Theorema Egregium)

"a most excellent theorem"

$$K = \frac{eg - f^2}{EG - F^2} \qquad \text{how to get } e, g, f$$

$$e = N\cdot X_{uu} = \frac{|X_u\ X_v\ X_{uu}|}{\sqrt{EG-F^2}} \qquad f = N\cdot X_{uv} = \frac{|X_u\ X_v\ X_{uv}|}{\sqrt{EG-F^2}}$$

$$g = N\cdot X_{vv} = \frac{|X_u\ X_v\ X_{vv}|}{\sqrt{EG-F^2}}$$

$$eg = \frac{1}{EG-F^2}\left(\begin{vmatrix} \boxed{\begin{matrix} X_u\cdot X_u & X_u\cdot X_v \\ X_v\cdot X_u & X_v\cdot X_v \end{matrix}} & \begin{matrix} X_u\cdot X_{vv} \\ X_v\cdot X_{vv} \end{matrix} \\ \begin{matrix} X_{uu}\cdot X_u & X_{uu}\cdot X_v \end{matrix} & \underline{X_{uu}\cdot X_{vv}} \end{vmatrix} \right)$$

$$= \begin{vmatrix} E & F & F_v - \frac{1}{2}G_u \\ F & G & \frac{1}{2}G_v \\ \frac{1}{2}E_u & F_u - \frac{1}{2}E_v & ? \end{vmatrix}$$

$$X_u\cdot X_{vv} = (X_u\cdot X_v)_v - X_{uv}\cdot X_v = F_v - \tfrac{1}{2}(X_v\cdot X_v)_u = F_v - \tfrac{1}{2}G_u$$

$$X_{uu}\cdot X_v = (X_u\cdot X_v)_u - X_u\cdot X_{uv} = F_u - \tfrac{1}{2}(X_u\cdot X_u)_v = F_u - \tfrac{1}{2}E_v$$

$$X_{uu}\cdot X_{vv} = (X_u\cdot X_{vv})_u - X_u\cdot X_{vvu}$$

$$= \left(F_v - \tfrac{1}{2}G_u\right)_u - \Big(\overbrace{X_u\cdot X_{uv}}^{\frac{1}{2}E_v}\Big)_v + X_{uv}\cdot X_{uv} \ \cdots\ ?$$

only get $\quad X_{uu}\cdot X_{vv} - X_{uv}\cdot X_{uv} = \left(F_v - \tfrac{1}{2}G_u\right)_u - \tfrac{1}{2}E_{vv}$

but $\quad f^2 = \dfrac{1}{EG-F^2}\begin{vmatrix} \boxed{\begin{matrix} X_u\cdot X_u & X_u\cdot X_v \\ X_v\cdot X_u & X_v\cdot X_v \end{matrix}} & \begin{matrix} X_u\cdot X_{uv} \\ X_v\cdot X_{uv} \end{matrix} \\ \begin{matrix} X_{uv}\cdot X_u & X_{uv}\cdot X_v \end{matrix} & \underline{X_{uv}\cdot X_{uv}} \end{vmatrix}$ $\quad$ By combining terms

get
$$K = \frac{1}{(EG-F^2)^2}\left\{ \begin{vmatrix} E & F & F_v - \frac{1}{2}G_u \\ F & G & \frac{1}{2}G_v \\ \frac{1}{2}E_u & F_u - \frac{1}{2}E_v & F_{uv} - \frac{1}{2}G_{uu} - \frac{1}{2}E_{vv} \end{vmatrix} - \begin{vmatrix} E & F & \frac{1}{2}E_v \\ F & G & \frac{1}{2}G_u \\ \frac{1}{2}E_v & \frac{1}{2}G_u & 0 \end{vmatrix} \right\}$$

conclusion: K depends only on ds^2 up to 2nd order.

Isometric Problem of Surface in $\mathbb{R}^3$ can be transformed into $\begin{cases} \alpha_v = p\beta \\ \beta_u = q\alpha \end{cases}$ $\qquad \alpha_{vu} = p_u\beta + p\beta_u$

$$= p_u\beta + pq\alpha = (\log p)_u\,\beta + pq\alpha$$

or, in graph can be of

" Monge-Ampere Equation ". possibly a degenerate hyp. equation!

* Codazzi 方程

compatibility equations

standard frame X_u, X_v, N $\qquad X_u = X_1$, $X_v = X_2$

Equation (运动方程式)

$$(X_u)_u =$$

$$\begin{cases} X_{11} = \Gamma_{11}^1 X_1 + \Gamma_{11}^2 X_2 + eN \\ X_{12} = X_{21} = \Gamma_{12}^1 X_1 + \Gamma_{12}^2 X_2 + fN \\ X_{22} = \Gamma_{22}^1 X_1 + \Gamma_{22}^2 X_2 + gN \end{cases}$$

observation : All Γ_{ij}^k can be determined by E, F, G and their 1st derivatives

$$N_u = \begin{cases} N_1 = a_{11} X_1 + a_{12} X_2 \\ N_2 = a_{21} X_1 + a_{22} X_2 \end{cases}$$

compatibility :

$$X_{112} = X_{121} : \quad \partial_2\Gamma_{11}^1 X_1 + \Gamma_{11}^1 X_{12} + \partial_2\Gamma_{11}^2 X_2 + \Gamma_{11}^2 X_{22} + e_2 N + e N_2$$

$$= \partial_1\Gamma_{12}^1 X_1 + \Gamma_{12}^1 X_{11} + \partial_1\Gamma_{12}^2 X_2 + \Gamma_{12}^2 X_{21} + f_1 N + f N_1$$

Gauss - Codazzi 方程式

① compare X_1 coeff.

$$\partial_2 \Gamma_{11}^1 + \Gamma_{11}^1 \Gamma_{12}^1 + \Gamma_{11}^2 \Gamma_{22}^1 + e\, a_{21}$$
$$= \partial_1 \Gamma_{12}^1 + \Gamma_{12}^1 \Gamma_{11}^1 + \Gamma_{12}^2 \Gamma_{12}^1 + f\, a_{11}$$

had seen: $a_{11} = \dfrac{fF - eG}{EG - F^2}$; $a_{21} = \dfrac{gF - fG}{EG - F^2}$

$$e\, a_{21} - f\, a_{11} = \frac{1}{EG - F^2}\left(egF - efG - f^2 F + feG \right)$$
$$= \frac{eg - f^2}{EG - F^2} F = F \cdot \underline{\underline{K}}$$

get $F \cdot K = \partial_1 \Gamma_{12}^1 - \partial_2 \Gamma_{11}^1 + \Gamma_{12}^1 \Gamma_{12}^2 - \Gamma_{11}^2 \Gamma_{22}^1$

This also proves the Theorema Egregium.

Rmk: cmp X_2 coeff get same thing. (no e, f, g is involved!)

② compare N coeff: (from $(X_{11})_2 = (X_{12})_1$:)

$$\Gamma_{11}^1 f + \Gamma_{11}^2 g + e_2 = \Gamma_{12}^1 e + \Gamma_{12}^2 f + f_1$$

i.e. $e_2 - f_1 = \Gamma_{12}^1 e + (\Gamma_{12}^2 - \Gamma_{11}^1) f - \Gamma_{11}^2 g$

compare N from $(X_{12})_2 = (X_{22})_1$:

$$f_2 - g_1 = \Gamma_{22}^1 e + (\Gamma_{22}^2 - \Gamma_{12}^1) f - \Gamma_{12}^2 g$$

(this totally symmetric via $1 \leftrightarrow 2$, $f \leftrightarrow f$, $e \leftrightarrow g$)

Conclusion:

① Gauss' equation gives intrinsic char of K

② Codazzi equation gives the necessary condition (given I) for a given II being possible for a surface $S \hookrightarrow \mathbb{R}^3$. (Compatibility equations)

• ~~Theorem~~ (~~Bonnet~~): Local existence & Uniqueness of Surfaces in $\mathbb{R}^3$.
student of Riemann. $\longleftrightarrow$ Gauss - Codazzi Equation

Proof of Bonnet's Theorem:

$$\frac{\partial x_k}{\partial u^\alpha} = U^k_\alpha(u, x) \qquad \begin{array}{l} k = 1 \dots n \text{ ; number of eq'n} \\ \alpha = 1 \dots m \qquad\qquad = n \cdot m \end{array}$$

number of functions = "k" m

compatibility conditions:

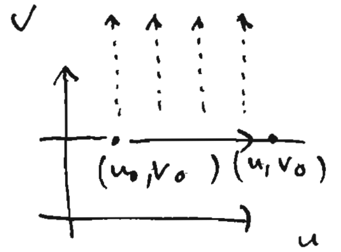
$$\frac{\partial^2 x_k}{\partial u^\alpha u^\beta} = \text{"} U^k_{\alpha;\beta} \text{"} = U^k_{\alpha,\beta} + U^k_{\alpha,j} \cdot \frac{\partial x_j}{\partial u_\beta} \quad \Bigg| \quad U^k_\alpha(\underbrace{u_1, \dots, u_m}_{\substack{\text{index} \\ \alpha, \beta \dots}}, \underbrace{x_1, \dots, x_n}_{\substack{\text{index} \\ i, j \dots}})$$

i.e. require

$$* \qquad U^k_{\alpha,\beta} + \sum_j U^k_{\alpha,j} \underbrace{\frac{\partial x_j}{\partial u_\beta}}_{\text{rep. by } U^j_\beta} = U^k_{\beta,\alpha} + \sum_j U^k_{\beta,j} \underbrace{\frac{\partial x_j}{\partial u_\alpha}}_{\text{rep by } U^j_\alpha}$$

pf: let $m = 2$. (then use induction)

$$\text{i.e.} \quad \star \begin{cases} \dfrac{\partial x^k}{\partial u} = U^k(u, v, x) \\[2ex] \dfrac{\partial x^k}{\partial v} = V^k(u, v, x) \end{cases}$$



① fix v_0, varies u

solve $\dfrac{dy_k}{du} = U^k(u, v_0, y)$

with initial condition $y_k(u_0, v_0) = x_{k0}$

this is just an ODE, always solvable.

② Take y_k as initial data, solve for each u

$$\frac{dx_k}{dv} = V^k(u, v, x) \qquad \text{initial data } x_k(u, v_0) = y_k(u, v_0)$$

Need to check that $\underbrace{\dfrac{\partial x_k}{\partial u}}_{\text{LHS}} = \underbrace{U^k(u, v, x)}_{\text{RHS}} \quad **$

$$\left(\frac{\partial x_k}{\partial u}\right)_{v = v_0} = \frac{dy_k}{du} = U^k(u, v_0, x) \quad \text{is ok.}$$

To check $**$, need only verify both LHS, RHS satisfies the same ODE. (in variable v)

For this purpose we need the solved $x(u, v)$ to be at least C^2, say, let U^k be C^1.

A. $\frac{\partial}{\partial v}\left(\frac{\partial X_k}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial X_k}{\partial v}\right) = \frac{\partial}{\partial u}\left(V^k(u,v,X)\right)$

one direction

B. $\frac{\partial}{\partial v}\left(u^k(u,v,X)\right)$ another direction

$$A: \frac{\partial}{\partial v}\left(\frac{\partial X_k}{\partial u}\right) = \frac{\partial V_k}{\partial u} + \sum_i \frac{\partial V_k}{\partial x_i}\left(\frac{\partial X_i}{\partial u}\right)$$

$$B: \frac{\partial}{\partial v} \underline{\underline{U_k}} = \frac{\partial U^k}{\partial v} + \sum_i \frac{\partial u_k}{\partial x_i} \cdot \frac{\partial x_i}{\partial v}$$

by construction this is already V_i by compatibility

done!

Applications to Bonnet's thm:

Frame $\{X_u, X_v, N\}$. get $\frac{\partial V_k}{\partial u} + \sum_i \frac{\partial V_k}{\partial x_i} \cdot \underline{\underline{U_i}}$ *.

fix an initial point and the compatibility condition turns out to be merely the Codazzi equation

Reason: View it as nine functions F^i, $i=1,\dots,9$

the PDE is then $\frac{\partial F^i}{\partial u} =$, $\frac{\partial F^i}{\partial v} =$ (here are all functions of X_u, X_v (not X) (indeed E,F,G) and e,f,g $\widehat{N}$ &)

So the compatibility condition is $\frac{\partial F^i}{\partial u \partial v} = \frac{\partial F^i}{\partial v \partial u}$. *

- To solve X, simply take

$$X(u,v) = \int_{u_0}^{u} X_1(\xi, v_0)\, d\xi + \int_{v_0}^{v} X_2(u_0,\eta)\, d\eta + X(u_0,v_0)$$

Then to check it really satisfies the Codazzi equation.

- The uniqueness may be reduced to the uniqueness of curves. eg. for $C = X(u(t), v(t))$ get

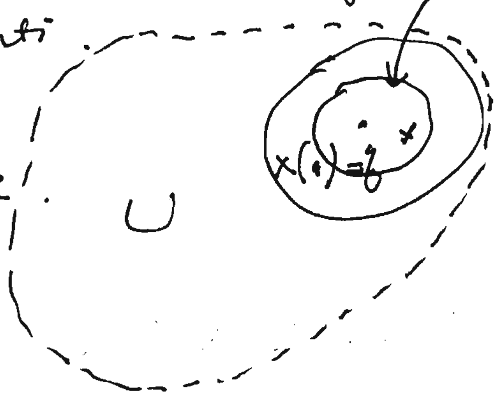
$$\begin{cases} X' = X_1 u' + X_2 v' = R(X_1, X_2, t) \\ X_1' = X_{11} u' + X_{12} v' \overset{\cdot}{=} S(X_1, X_2, t) \\ X_2' = X_{21} u' + X_{22} v' \overset{\cdot}{=} T(X_1, X_2, t) \end{cases}$$

via Codazzi eq'n.

So $(X(t), X_1(t), X_2(t))$ are uniquely det. by initial & E, F, G, e, f, g *.

(Global) Existence and conti of flow $\Phi(\xi, t)$ in the Lipchitz case.

$\xi \in \bar{V} = B_p(R/2)$



$U \supseteq B_p(R)$

$\bar{U}_1 = B_p(\tfrac{3}{4}R)$

Only need to prove the local case.

Set up: ODE on $\mathbb{R}^n$

$$\Phi' = F(\Phi) \qquad F : U \to \mathbb{R}^n \text{ Lipchitz}, \quad U \subset \mathbb{R}^n, \quad |F| \le M$$

$$\Leftrightarrow \quad x(t) = \underset{\xi}{\overset{\parallel}{x(a)}} + \int_a^t F(x(u))\,du$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\varphi_\xi(X)}$$

$C^0(J, U)$

$J = [a-\delta, a+\delta]$

$$|\varphi_\xi(X) - \varphi_\xi(Y)| \le \left| \int_a^t |F(x(u)) - F(Y(u))|\,du \right|$$

$$\underline{|t-a| \cdot M \cdot \sup_{u \in J} |x(u) - Y(u)|}$$

① make $\delta \cdot M \le \frac{1}{2}$ $\Rightarrow$ $|\varphi_\xi(X) - \varphi_\xi(Y)|_{C^0} \le \frac{1}{2}|X - Y|_{C^0}$

② make $\delta \cdot M \le \frac{R}{4}$ $\Rightarrow$ $|x(t)| \le |\xi| + \frac{R}{4} \le \frac{3}{4}R$

$\Rightarrow$ $\varphi_\xi : C^0(J, \bar{U}_1) \to C^0(J, \bar{U}_1)$ cont. mapping

$\quad$ ↳ complete space need $J, \bar{U}_1$ cpt

$\Rightarrow$ $\exists!$ fixed pt Φ_ξ in fact given by $\varphi_\xi^n(X)$ as $n \to \infty$ for any X.

Lipchitz conti of Φ_ξ in ξ:

$$|\varphi_{\xi_1}(\Phi_\xi) - \Phi_\xi| = |\xi_1 - \xi| \qquad \forall n$$

$$\Rightarrow |\varphi_{\xi_1}^n(\Phi_\xi) - \Phi_\xi| \le \sum_{i=1}^n |\varphi_{\xi_1}^i(\Phi_\xi) - \varphi_{\xi_1}^{i-1}(\Phi_\xi)| < 2|\xi_1 - \xi|$$

$n \to \infty$ $|\Phi_{\xi_1} - \Phi_\xi| \le 2|\xi_1 - \xi|$. $\quad |\Phi(\xi, t) - \Phi(\xi_1, t_1)|$ use Δ-ineq. ※

ODE: $x' = F(x)$ st. $F \in C^1(O, \mathbb{R}^n)$; $\phi(t, x)$ flow $\exists !$ C^0.

let $x(t)$ parti sol. on $J \ni 0$, $x_0 = x(0)$
closed int.

Variational equation:

let $A(t) = DF_{x(t)} \in C^0$, consider $U' = A(t)\,U$ (*)

~~idea~~: If $U_0 = U(0)$ small, $x(t) + U(t)$ approx. sol with initial data $x_0 + U_0$.

let $U(t, \xi)$ be the flow of VE, i.e. $U(0,\xi) = \xi$ $\forall \xi \in \mathbb{R}^n$ st. $x_0 + \xi \in O$

$y(t,\xi)$ " of $x' = F(x)$, $y(0,\xi) = x_0 + \xi$

Prop. $\lim_{\xi \to 0} \dfrac{|y(t,\xi) - x(t) - U(t,\xi)|}{|\xi|} = 0$ unif on $t \in J$.

Thm: Smoothness of the flow. (Hirsch - Smale): $\phi \in C^1$.

~~pf~~: To get $\frac{\partial \phi}{\partial x} \equiv D_2\phi$, for ξ small

$\phi(t, x_0 + \xi) - \phi(t, x_0) = y(t,\xi) - x(t) = \underbrace{U(t,\xi)}_{\text{linear in } \xi} + o(|\xi|)$

$\Rightarrow D_2\phi(t, x_0)\,\xi = U(t,\xi)$ conti in t, x_0 follows $\square$

So easy!: where is the dependence?

~~Rmk~~: Back to (*) get $\frac{d}{dt} D_2\phi(t,x_0) = DF_{\phi(t,x_0)} \cdot D_2\phi(t,x_0)$ with initial $D_2\phi(0,x_0) = id_{\mathbb{R}^n}$. Hence $F \in C^k \Rightarrow \phi \in C^k$ by induction.

~~pf of prop~~: Integral equations:

$x(t) = x_0 + \int_0^t F(x(s))\,ds$

$y(t,\xi) = x_0 + \xi + \int_0^t F(y(s,\xi))\,ds$

$U(t,\xi) = \xi + \int_0^t DF_{x(s)}(U(s,\xi))\,ds$

$\Rightarrow |y(t,\xi) - x(t) - U(t,\xi)| \le \int_0^t |F(\underline{y(s,\xi)}) - F(\underline{x(s)}) - DF_{\underline{x(s)}}\,U(s,\xi)|\,ds$

say $t \geq 0$

$\le \int_0^t |DF_{x(s)}(y(s,\xi) - x(s) - U(s,\xi))|\,ds + \int_0^t |R(\underset{a}{x(s)};\ \underset{h}{y(s,\xi) - x(s)})|\,ds$

~~Taylor~~

$\Rightarrow g(t) := \text{LHS} \le \underbrace{\max_{s \in J} |DF_{x(s)}|}_{= N} \cdot \int_0^t g(s)\,ds + \cdots$

ϵ: any > 0, $\ge \epsilon |y(s,\xi) - x(s)|$ (**) for $|h| \le \underline{\delta_0}$, $s \in J$.

such a $\delta_0 > 0$ exists by

Ex. Gronwall's inequality: If $u(t) \le \overset{\ge 0}{C} + \int_0^t \underset{\ge 0}{K}\, u(s)\,ds$ with $u \in C([0,a])$, $u \ge 0$, then $u(t) \le C e^{Kt}$

in (**), $C = |\xi|$ can be taken small: $|y(s,\xi) - x(s)| \le |\xi| e^{Ks} \le \delta_0$

$\Rightarrow g(t) \le N \int_0^t g(s)\,ds + \epsilon|\xi| \cdot C_1(K, |J|) \Rightarrow g(t) \le C_1 \cdot \epsilon \cdot |\xi|\, e^{Nt}$ ※

/ Covariant diff. Parallel transl. & geodesics
(intrinsic notion)

$$V = a\mathbf{x}_1 + b\mathbf{x}_2$$

$$\frac{DV}{dt} := a'\mathbf{x}_1 + a\mathbf{x}_{11}u' + a\mathbf{x}_{12}v' \quad \text{proj to } TS$$
$$+ b'\mathbf{x}_2 + b\mathbf{x}_{21}u' + b\mathbf{x}_{22}v'$$

$$= \left(a' + \Gamma_{11}^1 u'a + \Gamma_{12}^1 v'a + \Gamma_{21}^1 u'b + \Gamma_{22}^1 v'b\right)\mathbf{x}_1$$
$$+ \left(b' + \Gamma_{11}^2 u'a + \Gamma_{12}^2 v'a + \Gamma_{21}^2 u'b + \Gamma_{22}^2 v'b\right)\mathbf{x}_2$$

when $(u,v)(t)$ is given, this is simply linear ODE
system of

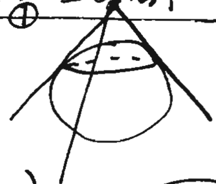
. when $V = \alpha' = u'\mathbf{x}_1 + v'\mathbf{x}_2$
not nec. by arc length, apriori

get

$$\frac{D\alpha'}{dt} = \left(u'' + \Gamma_{11}^1 u'^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1 v'^2\right)\mathbf{x}_1 + \left(v'' + \Gamma_{11}^2 u'^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2 v'^2\right)\mathbf{x}_2$$

geodesics $\equiv D\alpha'/dt = 0 \Rightarrow |\alpha'| = \text{const} \Rightarrow t \sim$ arc length.

examples of parallel transl.

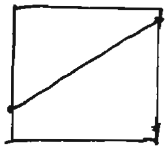
①  

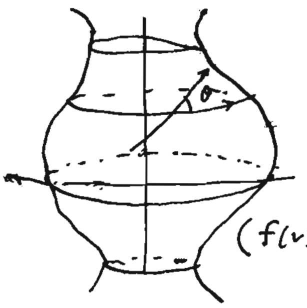
$$\frac{4\pi}{8} = \frac{\pi}{2}\,!$$

$$A = -\pi + (\angle A + \angle B + \angle C)$$

③ surface of revolution:

$$\mathbf{x}(u,v) = (f(v)\cos u,\ f(v)\sin u,\ g(v))$$

②  helix =  line


$(f(v), g(v))$

$$E = f^2,\ F = 0,\ G = f'^2 + g'^2$$

$$\begin{cases} E_1 = 0, & E_2 = 2ff'\ ; \quad F_i = 0 \\ G_1 = 0, & G_2 = 2(f'f'' + g'g'') \end{cases}$$

$$\boxed{\begin{array}{ll} \Gamma_{11}^1 = 0, & \Gamma_{11}^2 = -\dfrac{ff'}{f'^2+g'^2} \\ \Gamma_{12}^1 = \dfrac{ff'}{f^2}, & \Gamma_{12}^2 = 0 \\ \Gamma_{22}^1 = 0, & \Gamma_{22}^2 = \dfrac{f'f''+g'g''}{f'^2+g'^2} \end{array}}$$

ie. $$\begin{cases} u'' + \dfrac{2ff'}{f^2}u'v' = 0 \\ v'' - \dfrac{ff'}{f'^2+g'^2}u'^2 + \dfrac{f'f''+g'g''}{f'^2+g'^2}v'^2 = 0 \end{cases}$$

equiv. under I.

ie. $ds^2 = f^2du^2 + (f'^2+g'^2)\,dv^2$

or $1 = f^2\left(\frac{du}{ds}\right)^2 + (f'^2+g'^2)\left(\frac{dv}{ds}\right)^2$

$$\Rightarrow (f^2u')' = 2ff'u' + f^2u'' = 0$$
$\cdot v'$

ie. $f^2u' = \text{const } c$. ("$'$" is wrt arc length s in $\boxed{u,v}$)

Clairaut's relation

$f^2\frac{du}{ds} = c \Rightarrow$

$\frac{du}{dv} = \frac{c}{f}\sqrt{\frac{f'^2+g'^2}{f^2-c^2}}$ *

get $u = \int \cdots dv$

$$\underset{f}{r}\,\underset{fu'}{\cos\theta} = c \quad ; \quad \cos\theta = \frac{\langle \mathbf{x}_1, \mathbf{x}_1u' + \mathbf{x}_2v'\rangle}{|\mathbf{x}_1|} = fu'$$

$$f^2\frac{du}{dv}\sqrt{\frac{1 + [illegible]}{g'^2}}$$

$$\frac{DX_1}{\partial u} = (X_{11})^T = \left(\Gamma_{11}^1 X_1 + \Gamma_{11}^2 X_2 + eN\right)^T$$
$$= \Gamma_{11}^1 X_1 + \Gamma_{11}^2 X_2$$

$D_i X_j = X_{ij}^T = \sum_k \Gamma_{ij}^k X_k$ covariant differentiation.

or

$g_{kl} = X_k \cdot X_l$

$\partial_i g_{kl} = X_{ki}\cdot X_l + X_k \cdot X_{li}$

$= \Gamma_{ki}^j g_{jl} + \Gamma_{li}^j g_{jk}$

$X_l \cdot X_{ij}^T = \sum_k \Gamma_{ij}^k g_{kl}$

$X_l \cdot X_{ij} = (X_l\cdot X_i)_j - X_{lj}\cdot X_i$

$(X_l\cdot X_i)_j = \partial_j g_{li}$, $\quad X_{lj}\cdot X_i = \Gamma_{lj}^k g_{ki}$

General formula.

ie. $\partial_j g_{li} = \Gamma_{ij}^k g_{kl} + \Gamma_{lj}^k g_{ki} \xrightarrow[\text{cyclic trick}]{}$ formula for Γ_{ij}^k (see back).

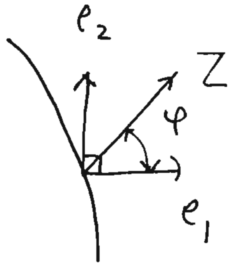
Holonomy point of view: ① on coor with $F=0$.

$$\frac{1}{2}\frac{1}{\sqrt{EG}}\left(-E_2\frac{du}{dt} + G_1\frac{dv}{dt}\right) = \left[\frac{D\vec{e}_1}{dt}\right]$$

Algebraic value $= \frac{D\vec{e}_1}{dt}\cdot\vec{e}_2$

what is it really?

"this is indeed a moving frame pf"



$\varphi := \angle(\vec{e}_1, Z)$

$\cos\varphi = Z\cdot e_1$

$-\varphi'\sin\varphi = Z'\cdot e_1 + Z\cdot e_1'$

$Z'\cdot e_1 = 0$ parallel

$Z\cdot e_1' = Z\cdot\lambda e_2 = \lambda\cos\left(\frac{\pi}{2}-\varphi\right) = \lambda\sin\varphi$

② ie. $\boxed{\lambda = -\varphi'}$

True for arbitrary unit v.f e_1.

cor: This $\Rightarrow$ for any 2 unit v.f w, v

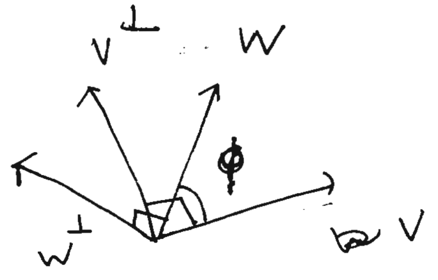
$$\left[\frac{Dw}{dt}\right] - \left[\frac{Dv}{dt}\right] = \frac{d\angle vw}{dt}.$$ (Z is cancelled out).

HW: chapter 4:

4.2: 4.6, 14, 17, 19

4.3: 1, 2, 3, 5, 7

4.4: 1, 2, 5, 12, 13, 14, 15, 17, 21, 23



Trivial Lemma

$$\cos\phi = w \cdot v$$

$$-\sin\phi \cdot \phi' = w' \cdot v + w \cdot v'$$

$w' = \lambda w^\perp$

$v' = \mu v^\perp$

$$= -\sin\phi \cdot \lambda + \sin\phi \cdot \mu$$

so $\underline{\phi' = \lambda - \mu}$

$$\text{ie. } \left[\frac{Dw}{dt}\right] - \left[\frac{Dv}{dt}\right] = \frac{d\phi}{dt}$$

---

now let $v = x_1/|x_1| = e_1$ Meaning of $\left[\frac{De_1}{dt}\right]$

<u>extrinsic way:</u>

pick coor st $F=0$. ie. $v = e_1 = \frac{x_1}{\sqrt{E}}$, $e_2 = \frac{x_2}{\sqrt{G}}$

$$\ast \quad \frac{Dv}{dt} = \frac{D}{dt}\left(\frac{x_1}{\sqrt{E}}\right) = \frac{-1}{2\sqrt{E}^3}\left(E_1 \frac{du}{dt} + E_2 \frac{dv}{dt}\right) x_1$$

$$+ \frac{1}{\sqrt{E}}\left(\text{“}x_{11}\text{”}\frac{du}{dt} + \text{“}x_{12}\text{”}\frac{dv}{dt}\right)$$

$$\left[\frac{De_1}{dt}\right] = \frac{D}{dt}\left(\frac{x_1}{\sqrt{E}}\right) \cdot \frac{x_2}{\sqrt{G}} = \frac{1}{\sqrt{EG}}\left(x_{11} \cdot x_2 \frac{du}{dt} + x_{12} \cdot x_2 \frac{dv}{dt}\right)$$

$$x_{11}\cdot x_2 = (x_1 \cdot x_2)_1 - x_1 \cdot x_{12} = 0 - \tfrac{1}{2}E_2, \qquad x_{12}\cdot x_2 = \tfrac{1}{2}G_1$$

$$= \underline{\frac{1}{2}\frac{1}{\sqrt{EG}}\left(G_1 \frac{dv}{dt} - E_2 \frac{du}{dt}\right)} \quad \ast$$

$\ast$: <u>better way (intrinsic)</u>

$$\frac{1}{2\sqrt{E}^3}\left(E_1 \frac{du}{dt} + E_2 \frac{dv}{dt}\right) x_1 + \frac{1}{\sqrt{E}}$$

$$\frac{d}{dt} = x_1 u' + x_2 v'$$

$$\nabla_{\partial_t} x_1 = u'\left(\Gamma_{11}^1 x_1 + \Gamma_{11}^2 x_2\right) + v'\left(\Gamma_{21}^1 x_1 + \Gamma_{21}^2 x_2\right)$$

want only x_2 component.

Finally $\cdot \frac{x_2}{\sqrt{G}}$

$$= \frac{1}{\sqrt{EG}}\left(\Gamma_{11}^2 G u' + \Gamma_{21}^2 G v'\right)$$

$$\Gamma_{11}^2 = \frac{1}{2} g^{22}\left(\cancel{\partial_1 g_{12}} + \cancel{\partial_1 g_{12}} - \partial_2 g_{11}\right)$$

$$= -\frac{1}{2} E_2 G^{-1}$$

$$\Gamma_{21}^2 = \frac{1}{2} g^{22}\left(\cancel{\partial_2 g_{12}} + \partial_1 g_{22} - \cancel{\partial_2 g_{21}}\right)$$

$$= \frac{1}{2} G_1 G^{-1}$$

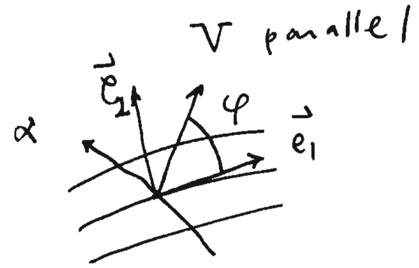
$$\begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix} \quad \frac{1}{EG}\begin{pmatrix} +G & 0 \\ 0 & +E \end{pmatrix} = \begin{pmatrix} \frac{+1}{E} & 0 \\ 0 & \frac{1}{G} \end{pmatrix}.$$

① $\int_\Omega k\,dA = \int_\Omega -\frac{1}{2}\cancel{\frac{1}{\sqrt{EG}}}\left(\left(\frac{E_2}{\sqrt{EG}}\right)_2 + \left(\frac{G_1}{\sqrt{EG}}\right)_1\right)\cancel{\sqrt{EG}}\,du\,dv$

$= \int_{\partial\Omega} \frac{1}{2}\left(\frac{E_2}{\sqrt{EG}}\,du + \frac{G_1}{\sqrt{EG}}\,dv\right)$

$= -\int_{\partial\Omega}\left[\frac{D\vec{e}_1}{dt}\right]dt$

$= \int_{\partial\Omega} \varphi'\,dt$ = holonomy angle

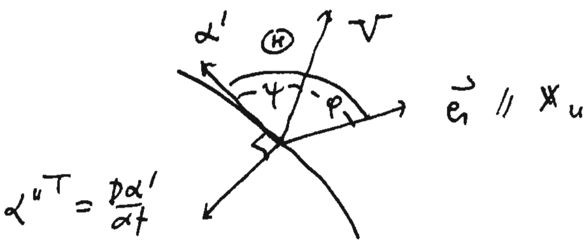


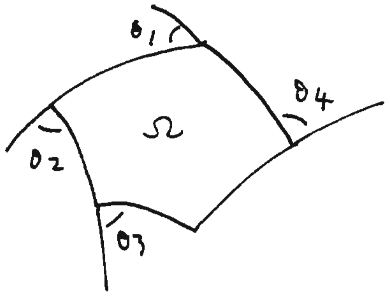
② $\varphi = \Theta - \psi$ ∠angle(α', V)
∠angle$(\alpha', \vec{e}_1)$

$\cos\varphi = V\cdot\vec{e}_1$
$-\sin\varphi\cdot\varphi' = V\cdot\lambda\,\vec{e}_2$
$= \sin\varphi\cdot\lambda$
$\Rightarrow \lambda = -\varphi'$

so $\int_{\partial\Omega} d\varphi = \int_{\partial\Omega} d\Theta - \int_{\partial\Omega} d\psi$

(Hopf) $2\pi - \sum_i \theta_i$ $\qquad \int_{\partial\Omega} k_g\,ds$





$\cos\psi = \alpha'\cdot Z$
$-\sin\psi\cdot\psi' = \alpha''\cdot Z$

here we pick t=s arc length for $\alpha = \partial\Omega$. $\rightarrow$ $= k_g\,\vec{n}\cdot Z$

$\cos(\psi + \frac{\pi}{2}) = -\sin\psi$

i.e. $\int_\Omega k\,dA + \int_{\partial\Omega} k_g\,ds + \sum\theta_i = 2\pi.$

(local Gauss Bonnet).

$\Rightarrow \psi' = k_g$

③ $R \subset S$ oriented, $\partial R = \cup C_i$

$R = \bigcup_{j=1}^{F} T_j$ triangulation, θ_{ji}, $i = 1,2,3$ outer angle

sum $\Rightarrow \int_R k\,dA + \int_{\partial R} k_g\,ds + \sum \theta_{jk} = 2F\pi$

• 2 ways to interprete 3F: $\underline{3\pi F} - \sum \varphi_{jk}$ - interior angle

∂R closed
$\Downarrow$
$E_e = V_e = V_{e,T} + V_{e,C}$

$= \pi\,(2E_i + E_e) - (2\pi V_i + \pi V_{e,T} + \sum_\ell (\pi - \theta_\ell))$

int angle $\quad$ ext. angle from T $\quad$ ext. angle from C_i

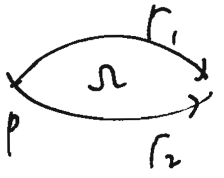
$= 2\pi(E_i + E_e) - 2\pi(V_i + V_e) + \sum_\ell \theta_\ell$

$= 2\pi\,(E - V) + \sum_\ell \theta_\ell$. $\Rightarrow$ global Gauss-Bonnet.

Applications of Gauss Bonnet:

The 1st one is #⑥ in next page.

① cpt surface with $K \geq 0$ and not all $= 0$ is homeo to S^2. (since $g = 0$: $\chi = 2 - 2g > 0$)

② 

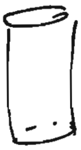
if $K \leq 0$ then no Γ_1, Γ_2 2 geodesics can bound a simple region Ω.

pf: $\int_\Omega K\, dA + \theta_1 + \theta_2 = 2\pi \Rightarrow \theta_1 = \theta_2 = \pi$ $*$.

$\underset{\leq 0}{\int_\Omega K\, dA}$

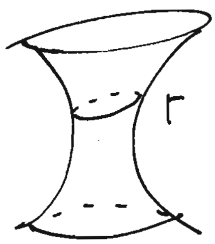
this also $\Rightarrow$ no simple closed geodesic.

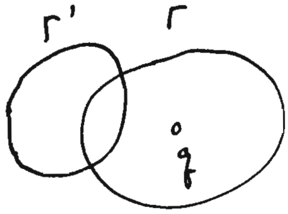
$= \partial\Omega$. Ω simple.

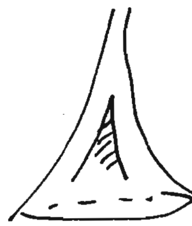
③ $S \underset{\text{homeo}}{\sim}$  cylinder with $K < 0$

not this model.

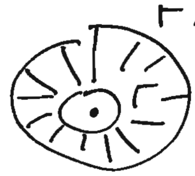
$\Rightarrow \exists$ at most one simple closed geodesic.



$\underset{\text{homeo}}{\sim}$ 



another Γ: if intersect Γ, must in 2 pts. $*$ to ②, it must be the case



Again $\int_\Omega K = 2\pi\chi(\Omega) = 0$

$\underset{\leq 0}{\int_\Omega K}$ $*$.

(No $*$ if $K \equiv 0$).

$\partial\Omega = \Gamma \cup \Gamma'$

if S is closed in $\mathbb{R}^3$ (i.e. complete surface, then will see $\exists!$ such Γ)

minimize generator of $\pi_1(S) \simeq \mathbb{Z}$.

④ S cpt, $K > 0$

Any 2 simple closed geodesics $\Gamma_1 \cap \Gamma_2 \neq \phi$.

S 

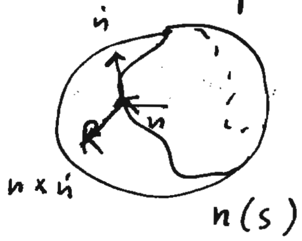
outside part Ω has $\chi(\Omega) = 0$

$0 < \int_\Omega K\, dA = 2\pi\chi(\Omega) = 0$ $*$.

this holds for any Ω orientable cpt surface with boundary

⑤ Jacobi's theorem:

$\alpha : I \to \mathbb{R}^3$, $\kappa > 0$ if $n(s) \in S^2$ is a simple curve then n separates S^2 into 2 regions of equal area.



pf: s arc length of α

$\bar{s}$ arc length of n

$$\text{Area}(R) = \int_R \underset{1}{\underset{\shortparallel}{K}}\, dA = 2\pi - \int_{\partial R} k_g \, d\bar{s}$$

so this is equiv to show that $\int_{\partial R} k_g \, d\bar{s} = 0$

$$k_g = \left[\frac{D\dot{n}}{d\bar{s}}\right] = \ddot{n} \cdot (n \times \dot{n})$$

indeed $\ddot{n} = k_g \, n \times \dot{n}$ + normal part

$$\dot{n} = n' \cdot \frac{ds}{d\bar{s}} = (-\kappa T - \tau B)\frac{ds}{d\bar{s}}$$

$$\ddot{n} = (-\kappa T - \tau B)\frac{d^2 s}{d\bar{s}^2} + \underbrace{(-\kappa' T - \tau' B)\left(\frac{ds}{d\bar{s}}\right)^2} + \left(-\kappa \kappa \underline{\underline{n}} - \tau \tau \underline{\underline{n}}\right)\left(\frac{ds}{d\bar{s}}\right)^2$$

by def of $\bar{s}$ get $|\dot{n}| = 1$ hence $\frac{ds}{d\bar{s}} = \frac{1}{\kappa^2 + \tau^2}$

$$\Rightarrow k_g = \ddot{n} \cdot (\kappa B - \tau T)\frac{ds}{d\bar{s}}$$

$$= \cancel{(\kappa\tau - \kappa\tau)} \frac{d^2 s}{d\bar{s}^2}\frac{ds}{d\bar{s}} + (\kappa'\tau - \kappa\tau')\left(\frac{ds}{d\bar{s}}\right)^{③} + 0$$

$$= \frac{\kappa'\tau - \kappa\tau'}{\kappa^2 + \tau^2} \cdot \frac{ds}{d\bar{s}} \qquad \left(\text{notice } \dot{\kappa} = \kappa' \cdot \frac{ds}{d\bar{s}}\right)$$

$$\text{so } k_g \, d\bar{s} = \frac{\kappa'\tau - \kappa\tau'}{\kappa^2 + \tau^2} ds = -\tan^{-1}\left(\frac{\tau}{\kappa}\right)' ds$$

This should be the 1st one: thus $\int_{\partial R} k_g \, d\bar{s} = -\tan^{-1}\frac{\tau}{\kappa}\Big|_{S} = 0$.

⑥ Geodesic triangle

$K = 0$ $K > 0$ $K < 0$

θ_i outer angles

$\varphi_i = \pi - \theta_i$ inner angle

$$\int_\Omega K \, dA + \sum_{i=1}^{3} \theta_i = 2\pi$$

$$\text{ie. } \underline{-\pi + \sum_{i=1}^{3} \varphi_i = \int_\Omega K \, dA}$$

$\uparrow$ the excess (angle) of Ω = area of $N(\Omega)$ (in S^2).

$\sum \varphi_i = \pi$ if $K = 0$

$> \pi$ if $K > 0$

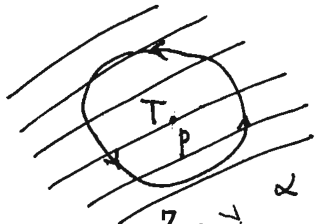
$< \pi$ if $K < 0$.

should also explain this in terms of holonomy angle.

Hopf-Poincaré index thm

S cpt surface, $\vec{v}$ sm tangent v.f. with isolated zeros
? orientable (or called sing. pts) $\Rightarrow$ finite #

Def$^{\text{n}}$: (rotation) index $I_p(\vec{v}) := \frac{1}{2\pi}\int_\alpha \varphi_1' \in \mathbb{Z}$

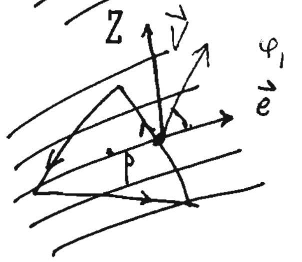


$\alpha = \partial T$

$\varphi_1 = \angle(\vec{e}, \vec{v})$,

independent of $\underline{\underline{\vec{e}}}$ and α.

or



Let $S = \cup T_i$, T_i contains at most one zero of $\vec{v}$.

$$\int_{T_i} k\, dA = \int_{\partial T_i} \varphi' \qquad \varphi = \angle(\vec{e}, Z)$$

Z parallel v.f along ∂T_i, indep of Z

$$2\pi\, I_{p_i}(\vec{v}) = \int_{\partial T_i} \varphi_1'$$

$$\Rightarrow \int_S k\, dA - 2\pi \underbrace{\sum I_{p_i}(\vec{v})}_{\text{Index}(\vec{v})} = \sum_i \int_{\partial T_i} (\varphi - \varphi_1)' = 0$$

$\varphi - \varphi_1 = \angle(\vec{v}, Z)$

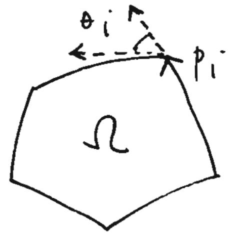
$\Rightarrow (\varphi - \varphi_1)'$ indep of Z ※

So $2\pi\, \text{Index}(\vec{v}) = \int_S k\, dA = 2\pi \chi(S)$

indep of $\vec{v}$!!

$\Rightarrow$ $\boxed{\text{Ind}(\vec{v}) = \chi(S)}$

Gauss Bonnet Theorem Do Carmo's Version:



$$\int_\Omega K\,dA$$

$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_2}{\sqrt{EG}}\right)_2 + \left(\frac{G_1}{\sqrt{EG}}\right)_1\right]$$

$$= -\frac{1}{2}\int_\Omega \left[\left(\frac{G_1}{\sqrt{EG}}\right)_1 - \left(-\left(\frac{E_2}{\sqrt{EG}}\right)_2\right)\right] du\,dv$$

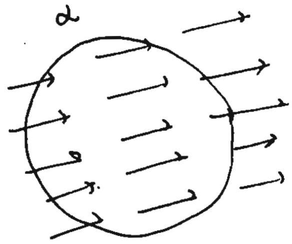
$$= -\frac{1}{2}\int_{\partial\Omega} -\frac{E_2}{\sqrt{EG}}\,du + \frac{G_1}{\sqrt{EG}}\,dv$$

$$= \int_{\partial\Omega}\left\{\frac{1}{2}\left(\frac{E_2}{\sqrt{EG}}\right)\cancel{du}\left(\frac{G_1}{\sqrt{EG}}\right)dv\right\}$$

$$= \int_{\partial\Omega} -\left[\frac{D\vec{e}_1}{dt}\right]dt$$

$$= \int_{\partial\Omega} d\varphi - \left[\frac{DW}{dt}\right]dt$$

geodesic curvature k_g if $W = \alpha'$.



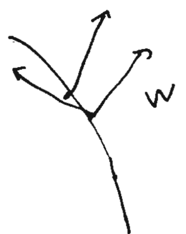
φ angle from X_1 to α'.
Hopf's theorem

$$\int_{\partial\Omega} d\varphi = 2\pi - \sum_i \theta_i$$

$$\Rightarrow \int_\Omega K\,dA + \int_{\partial\Omega} k_g\,ds + \sum_{p_i}\theta_i = 2\pi$$

Local Gauss-Bonnet

$D_{\partial t} W \equiv \frac{DW}{dt}$ covariant diff.



Under the restriction that $|W| = 1$.
then $W^\perp$ is uniquely defined.
so. $\frac{DW}{dt} = \lambda W^\perp$. i.e. $\lambda = \pm\left|\frac{DW}{dt}\right| = W'\cdot W^\perp$.
call it $\left[\frac{DW}{dt}\right]$ i.e. with sign "algebraic value of $\frac{DW}{dt}$".

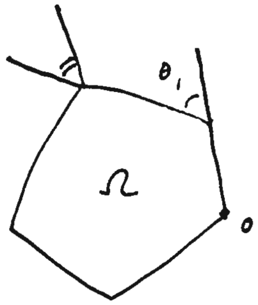
- when $W = \alpha'(s)$. s: arc length
call $k_g = \left[\frac{D\alpha'}{ds}\right]$ the geodesic curv.

Fix any C^∞ field $\vec{F}$: let $\varphi(t)$ = angle $(\vec{F}, W)$

Gauss–Bonnet Formula. (H. Hopf's version)

Thm: $\int_\Omega K\,dA + \int_{\partial\Omega} k_g\,ds + \sum_i \theta_i = 2\pi$

Remark:



$2\pi - \sum \theta_i$ is exactly the holonomy angle.

" local Gauss-Bonnet $\equiv$ holonomy def of K. "

Pf: $ds^2 = dr^2 + G^2 d\theta^2$

then $K = -\frac{G_{rr}}{G}$

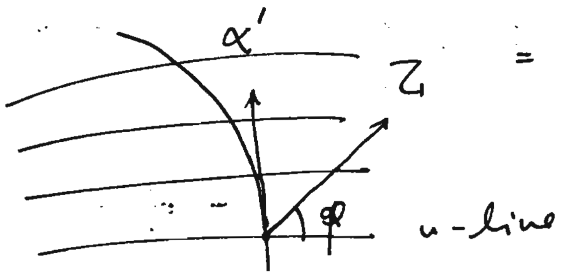
$$\int_\Omega K\,dA = \int_{\Omega} -\frac{G_{rr}}{G}\, G\, dr\, d\theta$$

(Ω ← U)

$$= -\int_U G_{rr}\, dr\, d\theta$$

$$= -\int_{\partial U} G_r\, d\theta \;\overset{\uparrow}{=}\; \int_{\partial U} d\varphi$$

" important formula for parallel translation "



φ = angle of Z to ∂_u.

$= \text{angle}(\alpha', \partial_u) - \text{angle}(\alpha', Z)$

Z, α' both unit vector fields

$$\cos\psi = \alpha' \cdot Z$$

$$-\sin\psi \cdot \psi' = \alpha'' \cdot Z + \underbrace{\alpha' \cdot Z'}_{=0}$$

$$= k_g \vec{n} \cdot Z \qquad \Rightarrow \psi' = k_g \text{ geodesic curvature}$$

$\vec{n}\cdot Z = \cos(\psi + \pi/2) = -\sin\psi$.

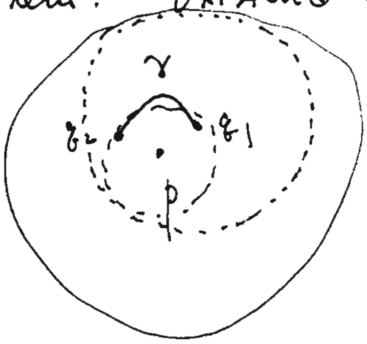
$$\Rightarrow \int_\Omega K\,dA = \int_{\partial\Omega} d\varphi = \int_{\partial\Omega} d\Theta - \int_{\partial\Omega} d\psi$$

holonomy | turning tangent via top. fact $= 2\pi - \sum_i \theta_i$ | geodesic curvature

Q.E.D.

Def^{n}: distance on S: $d(p,q) := \inf\limits_{\gamma \text{ joins } p,q} |\gamma|$

Theorem: Existence of convex neighborhood.



i.e. diff image of standard ball in T_pS via $\exp_p$

$B_r(p)$ geod ball

$q \in \overline{B}_{r/3}(p)$ $\quad B_{\delta_q}(q)$ $\underline{\text{geod ball}} \subset B$

$\delta_q = \frac{2}{3}r$ $\quad$ ① $\underline{\underline{\text{why diffeo?}}}$

$q' \in B_{\frac{2}{3}r}(q) \Rightarrow d(q',p) < \frac{r}{3} + \frac{2}{3}r \leq r.$

then any $q_1, q_2 \in B_{r/3}(p)$, $d(q_1,q_2) < \frac{r}{3} + \frac{r}{3} = \frac{2r}{3}$

i.e. $q_2 \in B_{2r/3}(q_1) \Rightarrow \exists!$ geod (shortest) γ joins q_1, q_2.

② $\underline{\underline{\text{Claim}}}$: $B_{r/3}(p)$ is geod. convex.

for ①: $\overline{B}_{r/3}(p)$ cpt $\overset{\text{Ex.}}{\Rightarrow}$ $\exists\, \delta > 0$ st. $B_\delta(q)$ diffeo-geod ball $\forall q \in \overline{B}_{r/3}(p)$

so if $\delta < \frac{2}{3}r$, replace r by $\frac{3}{2}\delta$.

for ②: To show $\gamma \subset B_{r/3}(p)$, using normal coor (u^1, u^2)

$$f(t) := d^2(p, \gamma(t)) = \sum_i (u^i)^2$$

$$f'(t) = \sum_i 2 u^i \dot{u}^i$$

$$f''(t) = 2\sum_i (\dot{u}^i)^2 + 2\sum_i u^i \ddot{u}^i$$

$$= 2\Big(\sum_i (\dot{u}^i)^2 - \sum_{ijk} u^i \Gamma^i_{jk} \dot{u}^j \dot{u}^k \Big)$$

Now pick r small enough such that $\sum_i |u^i \Gamma^i_{jk}| < \frac{1}{2}$ $\forall j,k$

conti, $=0$ at p

then $f''(t) \geq (\dot{u}^1 - \dot{u}^2)^2 \geq 0$

i.e. f is convex, max occur at q_1 or q_2 done $\square$.

Corollary (Ex). For S cpt, $\exists$ triangulation by geodesic triangles.

hint: If $B_\delta(p)$ is convex $\forall p \in S$

cover S by $B_{\delta/2}(p)$.

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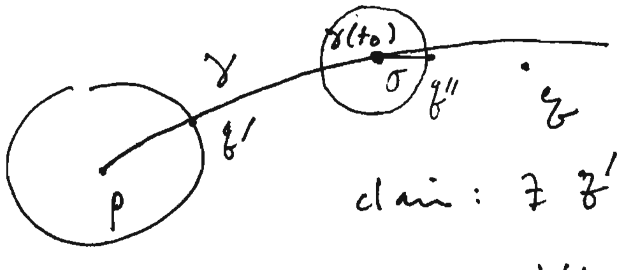
Lemma: $\exists p$, $\exp_p$ defined on T_pM

$\Rightarrow$ any q is conn to p by a shortest geod.

pf:



$B_p(r)$

normal conv ball

i.e. $\exp_p$ is diffeo.

$q \in B_p(r)$ done.

for $q \notin B_p(r)$,

claim: $\exists q' \in \partial B_p(r)$ st.

$d(p,q) = r + d(q',q)$

pf: Δ-ineq $\Rightarrow$ "$\leq$" $\forall q'$

now let $d(q',q) = \inf\limits_{q'' \in \partial B_p(r)} d(q'',q)$

def of $d \Rightarrow$ "$\geq$" $\divideontimes$

$\uparrow$ C^0 in q''

let $\gamma : [0,\infty) \to M$ be the unique geod $\ni q'$

Continuity Method: $Z := \{ t \mid d(p,\gamma(t)) + d(\gamma(t), q) = d(p,q) \}$

$Z \ni r$ ($\neq \phi$), closed, let $t_0 = \sup Z$, claim $\gamma(t_0) = q$.

otherwise, $\exists r_1 > 0$,

consider $\partial B_{\gamma(t_0)}(r_1) \ni q''$ st. $\underset{r_1}{\underset{\|}{d(\gamma(t_0), q'')}} + d(q'', q) = d(\gamma(t_0), q)$

let σ the unique geod $\overrightarrow{\gamma(t_0)\, q''}$.

$\Rightarrow$ $\underline{d(p,\gamma(t_0))} + d(\gamma(t_0), q) = d(p,q)$ $\quad *$

$\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad \underline{d(\gamma(t_0), q'')} + d(q'', q)$

$\Rightarrow$ $d(p,\gamma(t_0)) + d(\gamma(t_0), q'') = d(p, q'')$ $\quad$ ($\leq$ by $*$)

but then $\gamma|_{[0,t_0]} \cup \sigma|_{[0,r_1]}$ is a min. geod

$\Rightarrow$ $\sigma \equiv \gamma$ and so $t_0 + r_1 \in Z$ $\divideontimes$

Thm (Hopf-Rinow) 1) $\exists p$, $\exp_p$ $\Rightarrow$ 2) complete $\Rightarrow$ 3) $\forall p$, $\exp_p$.

pf: 1) $\Rightarrow$ 2) let q_i Cauchy, $\exists\, \gamma_i(t_i) = q_i$, $t_i \to t_0$

arc length, Cauchy

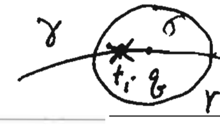
Also $\gamma'_{i_n}(0) \to v$ in subsequence i_n since $B_1(0)$ cpt in T_pM.

consider $\gamma(0) = p$. $\gamma'(0) = v$

ODE th $\Rightarrow$ $q_{i_n} = \gamma_{i_n}(t_{i_n}) \to \gamma(t_0)$, $\Rightarrow$ $q_i \to \gamma(t_0)$.

2) $\Rightarrow$ 3). If $\gamma(0) = p$. $\gamma'(0) = v$ only defined on $[0, t_0)$,

then $t_i \nearrow t_0 \Rightarrow \gamma(t_i)$ Cauchy $\Rightarrow \exists\, \gamma(t_0) = q$



extends $\divideontimes$

Thm : $S \subset \mathbb{R}^3$ cpt $K = \text{const} \Rightarrow S \equiv$ sphere.

let $k_1 \geq k_2$ C^∞ outside umbilical pts, C^0 on S.

lemma : If $K(p) > 0$, k_1 local max, k_2 local min at p

then p is umbilical.

pf: If NOT, $\exists$ coor (u,v) at p via lines of curvature

hence $F = f = 0$, and $k_1 = \frac{e}{E} > k_2 = \frac{g}{G}$ (say)

since $e = -dN(x_1)\cdot x_1 = k_1 |x_1|^2$

Recall (Mainardi-) Codazzi equations, in this coor :

$$e_2 - \cancel{f_1} = e\,\Gamma_{12}^1 + \cancel{f}(\Gamma_{12}^2 - \Gamma_{11}^1) - g\,\Gamma_{11}^2$$

$$\Gamma_{12}^1 = \frac{1}{2} g^{11}\left(\cancel{\partial_1 g_{21}} + \partial_2 g_{11} - \cancel{\partial_1 g_{21}}\right) = \frac{1}{2}\frac{G}{EG}E_2 = \frac{1}{2}\frac{E_2}{E}, \qquad \Gamma_{11}^2 = -\frac{1}{2}\frac{E_2}{G}$$

i.e.
$$\begin{cases} e_2 = \dfrac{E_2}{2}\left(\dfrac{e}{E} + \dfrac{g}{G}\right), \text{ and by symmetry,} \\ g_1 = \dfrac{G_1}{2}\left(\dfrac{e}{E} + \dfrac{g}{G}\right). \end{cases}$$

From $e = k_1 E \Rightarrow e_2 = (k_1)_2 E + k_1 E_2$

Also, $e_2 = \frac{E_2}{2}(k_1 + k_2)$

$$\Rightarrow (k_1)_2 E = \frac{E_2}{2}(k_2 - k_1), \quad \text{symmetry}$$

$$(k_2)_1 G = \frac{G_1}{2}(k_1 - k_2).$$

Now,
$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_2}{\sqrt{EG}}\right)_2 + \left(\frac{G_1}{\sqrt{EG}}\right)_1\right]$$

$$= -\frac{E_{22} + G_{11}}{2EG} + \frac{1}{4}\frac{(EG)_2}{(EG)^2}E_2 + \frac{1}{4}\frac{(EG)_1}{(EG)^2}G_1$$

Key Point :
$$E_2 = \frac{2E}{k_2 - k_1}(k_1)_2 = *\,(k_1)_2$$

$$E_{22} = \frac{2E}{k_2 - k_1}(k_1)_{22} + *\,(k_1)_2$$

$*$ means some C^∞ function $\neq 0$ at p.

Hence, by symmetry

$$-2EG \cdot K = \frac{2E}{k_2-k_1}(k_1)_{22} + \frac{2G}{k_1-k_2}(k_2)_{11} + *(k_1)_2 + *(k_2)_1$$

At p: $\quad \wedge \; 0 \qquad \geq 0 \qquad \geq 0 \qquad = 0 \qquad = 0$

get contradiction ! $\square$

pf of Thm: We must have $K > 0$ (since $\exists$ elliptic pt)

let p be the max of k_1 on S (p exists since S cpt)

then p is min of k_2 since $k_1 k_2 = K = $ const.

Lemma $\Rightarrow$ $k_1(p) = k_2(p)$

but then $\forall\, q \in S$, $k_1(q) \leq k_1(p) = k_2(p) \leq k_2(q)$

Hence $k_1(q) = k_2(q)$, ie. All pts on S are umbilical.

$\Rightarrow$ S is a portion of a sphere, hence a sphere. $\square$

Thm': $S \subset \mathbb{R}^3$ cpt, $K > 0$, $H = $ const $\Rightarrow$ $S = $ sphere.

this means $S = \partial\Omega$, Ω convex

(called "ovaloid")

[Same Proof].

Thm" (Alexander-Hopf), $K > 0$ is NOT needed.

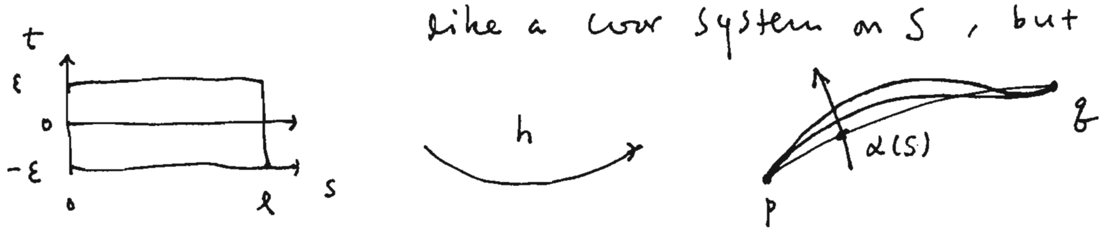
Def'n: $\alpha : [0, \ell] \to S$ by arc length s

variation $h : [0, \ell] \times (-\varepsilon, \varepsilon) \to S$ st $h_0 = \alpha$

$\qquad\qquad\qquad s \qquad\qquad t \qquad\qquad h_t(s) := h(s,t)$

proper (end pt fixed) if $h_t(0) = \alpha(0)$, $h_t(\ell) = \alpha(\ell)$.

like a coor system on S, but h is only C^∞.



h, dh may not be inj.

Fact: $V(s) := dh\begin{pmatrix}0\\1\end{pmatrix}_{t=0} \equiv \frac{\partial h}{\partial t}(s,0)$ is called the var. v.f.

$h_t'(s) \equiv T(s) := dh\begin{pmatrix}1\\0\end{pmatrix} \equiv \frac{\partial h}{\partial s}(s,t)$ is the tangent v.f. of h_t

Any V C^∞ on α, $\exists$ variation h,

if $V(0) = V(\ell) = 0$, h can be chosen to be proper

Pf: Consider $h(s,t) := \exp_{\alpha(s)}(t\,V(s))$. $\square$

Let $L(t) = \int_0^\ell \langle T, T\rangle^{1/2}\, ds$ = length for h_t

s is arc length only for $t=0$.

Thm: (1st Variation formula)

$$L'(0) = \langle V, T\rangle\Big|_0^\ell - \int_0^\ell \langle \nabla_T T, V\rangle\, ds \qquad (\text{Here } T = \alpha')$$

Hence $L'(0) = 0$ $\forall$ proper variation $\Leftrightarrow$ α is a geodesic.

Pf: $L'(t) = \int_0^\ell \frac{\partial}{\partial t}\langle T, T\rangle^{1/2}\, ds = \int_0^\ell \frac{\langle \frac{DT}{dt}, T\rangle}{\sqrt{\langle T, T\rangle}}\, ds$

Key point: $\frac{DT}{dt} = \nabla_V T = \nabla_T V$.

(true for coor system $\mathbf{x} = h$, $T = \mathbf{x}_1$, $V = \mathbf{x}_2$, Also for extrinsic reason $\frac{DT}{dt} = \left(\frac{\partial}{\partial t}\frac{\partial h}{\partial s}\right)^T = \left(\frac{\partial}{\partial s}\frac{\partial h}{\partial t}\right)^T = \frac{DV}{ds}$

$\Rightarrow\ = \langle \frac{DV}{ds}, T\rangle = \frac{d}{ds}\langle V, T\rangle - \langle V, \frac{DT}{ds}\rangle$

at $t=0$, $|T| = 1$, get Thm. $\square$

2nd variation of geodesics.

$$L'(t) = \int_0^\ell \frac{\langle \nabla_V T, T\rangle}{\langle T, T\rangle^{1/2}}\, ds$$

α : geodesic
$V \perp T$ normal variation
end-pt fixed

$$L''(t)\Big|_0 = \int_0^\ell \underbrace{\frac{\partial}{\partial t}\langle \nabla_V T, T\rangle}_{=\langle \nabla_V \nabla_T V, T\rangle + \langle \nabla_V T, \nabla_V T\rangle} - \langle \nabla_V T, T\rangle^2\, ds$$

$$= \int_0^\ell \left(|\nabla_T V|^2 + \langle (\nabla_V \nabla_T - \nabla_T \nabla_V) V, T\rangle \right) ds$$

$$+ \langle \nabla_T \nabla_V V, T\rangle \qquad - \langle \nabla_T V, T\rangle^2$$

$$\langle \nabla_T \nabla_V V, T\rangle = T\langle \nabla_V V, T\rangle - \cancel{\langle \nabla_V V, \nabla_T T\rangle} \qquad \langle \nabla_T V, T\rangle^2 = \left(T\langle V, T\rangle - \cancel{\langle V, \nabla_T T\rangle}\right)^2$$

$$= \langle \nabla_V V, T\rangle\Big|_0^\ell + \int_0^\ell \left(|\nabla_T V|^2 - \langle [\nabla_T, \nabla_V] V, T\rangle \right) ds$$

Lemma : $\langle [\nabla_T, \nabla_V] V, T\rangle = K \cdot |V|^2$. (in general $|V \wedge T|^2$)
the defect of commuting double covariant deri.

Pf : (How do we prove this in calculus on $\mathbb{R}^2$?)
A direct calculation by Gauss' equation. (uninteresting)
Actually it is ∞'tesimal version of G-B (1) via Parallel tr.

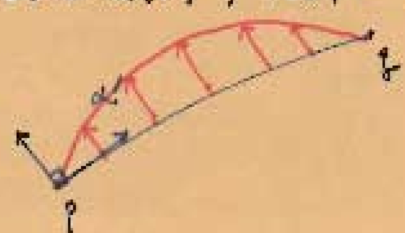
Def^n : A surface is geod complete if
every geod can be extended to ∞. i.e. $\exp_p$ is
defined on whole T_pS, $\forall p \in S$.

Thm (Hopf-Rinow) : geod complete $\Leftrightarrow$ complete
$\forall p \Leftrightarrow \exists$ one p

And all $\Rightarrow$ geod convex, any p, q are joined by shortest geod.

Thm (Bonnet) : S complete, $K \geq k > 0 \Rightarrow d \leq \frac{\pi}{\sqrt{k}}$, S cpt.

Pf : If $\exists\, p, q \in S$, $\ell = d(p,q) > \frac{\pi}{\sqrt{k}}$
let α be the geod joins p, q, $e_0 \perp \alpha'(0)$, $|e_0| = 1$
$e(s)$ parallel along α
Consider $V(s) = \sin\frac{\pi s}{\ell}\, e(s)$



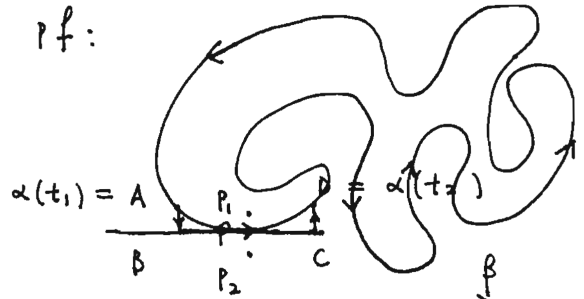
$$0 \leq L''(0) \leq \frac{\ell}{2}\left(\frac{\pi^2}{\ell^2} - k\right) < 0 \quad \longleftarrow$$

§ 5.7 Global Theory of Curves
(via degree of maps) $S \to S^1$ or $S \to S^2$

Thm 1 (C^∞ Jordan Curve Thm) $\quad \deg \dfrac{\alpha - p}{|\alpha - p|}$ winding #

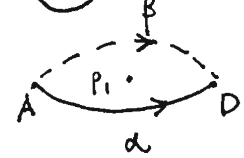
piece-wise C^2 is OK.

Pf:



- $w(p)$ is constant in each conn comp of $\mathbb{R}^2 \setminus \alpha[0,l]$
- At least 2 comp $\Omega(p_1)$, $\Omega(p_2)$: α is a graph near p $\Rightarrow w(p_2) - w(p_1) = \pm 1$ via β

β new curve



- α is oriented $\Rightarrow \exists$ tubular nbd U, may set $P_1, P_2 \in U$
- Ω any comp $\Rightarrow \partial_{top}\Omega \subset \alpha[0,l] \Rightarrow \Omega \cap \Omega(P_1) \neq \phi$ or $\Omega \cap \Omega(P_2) \neq \phi$.

Thus $\exists$ at most 2, hence exactly 2 conn. comp. $\square$

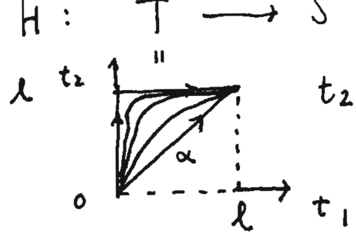
Thm 2 (Hopf's turning tangent)

α simple closed regular $\subset \mathbb{R}^2 \Rightarrow I_\alpha := \deg \alpha' = \pm 1$.

rotation index

Pf: construct homotopy via secant lines:

$H: T \to S^1$



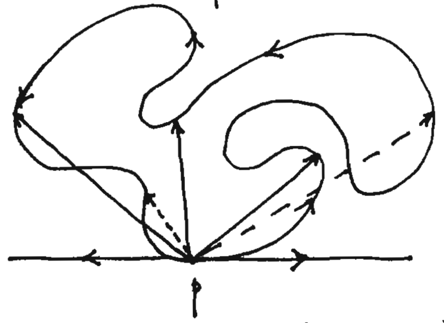
$t_2 \geq t_1$

$$H(t_1, t_2) := \frac{\alpha(t_2) - \alpha(t_1)}{|\alpha(t_2) - \alpha(t_1)|}, \quad t_1 \neq t_2$$

$$H(t,t) := \frac{\alpha'(t)}{|\alpha'(t)|}$$

$$H(0,l) := -\frac{\alpha'(0)}{|\alpha'(0)|}$$

β new curve



H conti $\Rightarrow \deg \alpha = \deg \beta = \pm 1$. $\square$

Cor: A regular (closed) plane curve is convex $\Leftrightarrow$ simple & k has sign.

2 equiv def'n, exterior vs interior

Pf (Ex + reading). $\quad \Leftrightarrow I = \pm 1$ and k has sign.

Space curves : (notice $k = |\kappa(s)|$ for plane curves)

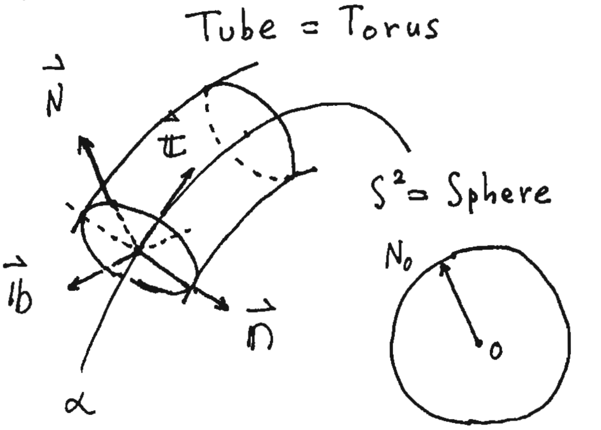
Thm 3 (Fenchel's thm)* $\int_\alpha k \geq 2\pi$, "=" $\Leftrightarrow$ α plane convex (simple) closed .

pf : Consider the tube of radius r : only need immersion

$$\mathbb{x}(s,v) = \alpha(s) + r(\cos v\, \vec{n} + \sin v\, \vec{b}) , \quad [0,\ell] \times [0,2\pi]$$

Had seen: $\mathbb{x}_s \times \mathbb{x}_v = r(1 - rk \cdot \cos v)\cdot N$, $\quad (1 - rk\cos v) > 0$

$$\vec{N} = -(\cos v\, \vec{n} + \sin v\, \vec{b})$$



$$N_s = k\cos v\, \vec{t} \ \underline{- \tau \sin v\, \vec{n} + \tau\cos v\, \vec{b}}$$

$$N_v = \underline{\sin v\, \vec{n} - \cos v\, \vec{b}}$$

$$\Rightarrow \quad N_s \times N_v = k\cos v\,(\sin v\, \vec{b} + \cos v\, \vec{n})$$

$$= -k\cos v\, \vec{N}$$

Hence $\quad K(s,v) = -\dfrac{k\cos v}{r(1 - rk\cos v)}$; $K \geq 0 \Leftrightarrow v \in [\frac{\pi}{2}, \frac{3\pi}{2}]$

let $R = [0,\ell] \times [\frac{\pi}{2}, \frac{3\pi}{2}]$

$\int_T K\, dA = 0$ (useless)

But $\int_R K\, dA = -\int_0^\ell k(s)\, ds \int_{\pi/2}^{3\pi/2} \cos v\, dv = 2\int_0^\ell k$

- Now any $N_0 \in S^2$ must be mapped by Gauss map over R (By the far / Near parallel planes $\perp N_0$ argument)
  hence $\quad 2\int_0^\ell k = \int_R K\, dA \geq 4\pi$.
- Moreover , for each s

$$\begin{array}{ccc} T|_s & \longrightarrow & \Gamma_s \subset S^2 \text{ great circle} \\ \cup & & \cup \\ R|_s & \longrightarrow & \Gamma_s^+ \end{array}$$

if α is a plane convex curve , $\partial\Gamma_s^+ = \{\pm P\}$ same $\forall s$ (north/south)

$\Gamma_{s_1}^+ \cap \Gamma_{s_2}^+$ only at $\pm P$ $\Rightarrow$ $\int_R K\, dA = 4\pi$.

* Rmk : For space curves $\int_\alpha k \in \mathbb{R}^+$ is in general not integer multiple of 2π .

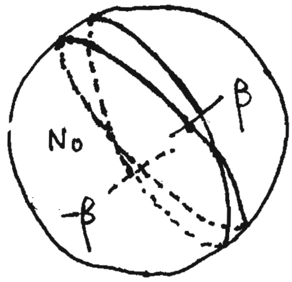
Claim: · Conversely, if $\int_\alpha k\, ds = 2\pi$ (so $\int_R K\, dA = 4\pi$)

$\Rightarrow$ $\exists\, \underline{p} \in S^2$ s.t. $\partial \Gamma_s^+ = \{\pm \underline{p}\}$ $\forall s$.

pf: If NOT, then $\partial \Gamma_s^+ = \{\pm \underline{p}(s)\}$ contains a curve $\pm\beta$ on S^2

Then $\exists\, s_1 \sim s_2$ with $\Gamma_{s_i} \cap \beta = \pm \underline{p}(s_i)$

in a nbd of $\partial \Gamma_{s_i}^+$. (ie. $\Gamma_{s_i} \neq \beta$)



Hence $\exists\, N_0 \in S^2$ being mapped by

2 points in $R^+|_{s_1}$ & $R^+|_{s_2}$ ($K > 0$)

The Gauss map is a local diffeomorphism there,

$\Rightarrow$ $\int_R K\, dA > 4\pi$ $\times$

why?

Now, notice that $\overrightarrow{(-p(s))p(s)} \parallel \vec{b}(s)$ $\Rightarrow$ $\vec{b} = $ const $\Rightarrow$ α plane curve.

$\overrightarrow{(-p)\,p}$ is a fixed direction

May assume $I_\alpha > 0$ (choose orientation) : (in fact $= 1$)

$$2\pi = \int_0^l |\kappa| \geq \int_0^l \kappa = 2\pi \cdot I_\alpha \geq 2\pi$$

Hence "=" holds, $\kappa = |\kappa| \geq 0$ $\Rightarrow$ convex. $\square$

"Simple" is not needed.

Thm 4 (Fary–Milnor) $\int_\alpha k\, ds < 4\pi \Rightarrow \alpha$ is unknotted.

pf: Choose a direction $v \in S^2$, $v \neq \vec{b}(s)$ $\forall s \in [0, l]$

Let $h_v(s) := \alpha(s) \cdot v$ the height fcn wrt v.

$h_v'(s) = \vec{t}(s) \cdot v$

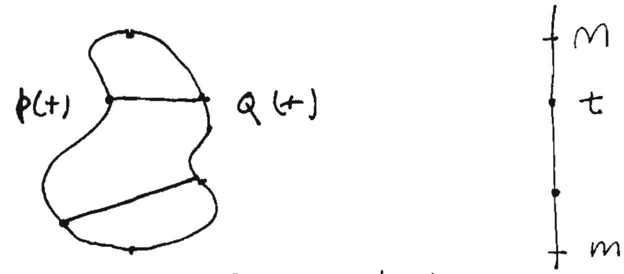
$h_v''(s) = k\, \vec{n}(s) \cdot v$

At any critical pt, we must have $\vec{n}(s) \cdot v \neq 0$

Since we may always assume $k > 0$,

the critical pt must be local maxima or minima.

We claim that h_v has exactly 2 critical pts for some v and then the theorem is proved since the line segment joins $h_v^{-1}(t) = \{P(t), Q(t)\}$

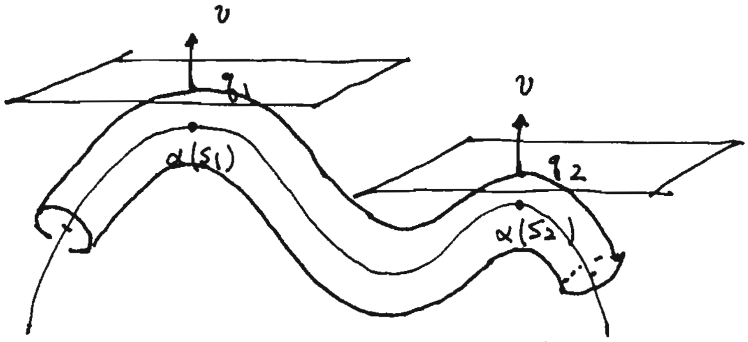


"Morse Theory"

generates a surface $S \underset{\text{homeo}}{\simeq}$ disk, with $\partial S = \alpha$.

So, suppose h_v has ≥ 3 critical pts $\forall v \notin \vec{b}([0,\ell])$

WLOG, assume $h_v(s_1)$ max, $h_v(s_2)$ (local) max.



then $\exists\, q_1, q_2 \in R$, T_{q_1}, T_{q_2} has normal vector v

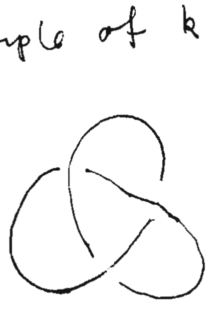
$v \neq \vec{b}(s_1), \vec{b}(s_2) \ni$ in fact $K(q_1) > 0$, $K(q_2) > 0$.

But this works $\forall v \notin \vec{b}([0,\ell])$, hence

$$\Rightarrow \int_R K\,dA \geq 8\pi, \text{ i.e. } \int_\alpha k\,ds \geq 4\pi \quad \times. \quad \square$$

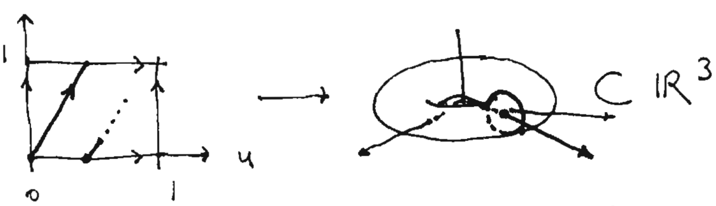
Example of knots:

(1)



Trefoil knot
(simplest)
(2,3)

(2) Torus knot $(p,q) = 1$



$t \in [0, 2\pi] \mapsto (u,v) \mapsto$ $x = (R + r\cos v)\cos u$

$(u,v) = (pt, qt)$ $\quad y = (R + r\cos v)\sin u$

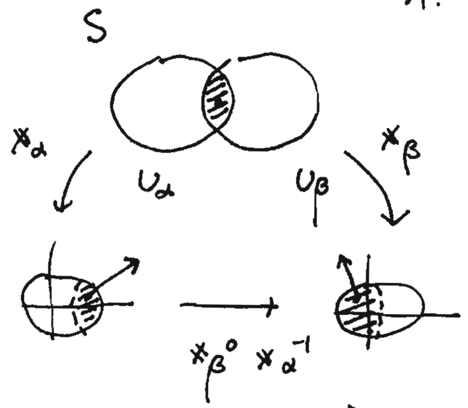
$z = r\sin v$

Can $\int k \underset{?}{=} 4\pi$ (why?) Ex. $\int k = ?$

minimize it over all (R, r).

Ans: Unless it approaches to the plane!

- An abstract surface (or 2-dim'l manifold) is a top space S covered by coor charts (U_α, x_α)



st. $\phi_{\beta\alpha} := x_\beta \circ x_\alpha^{-1}$ is C^k $\forall \alpha, \beta$, $k \geq 0$.

equiv. $S = (\coprod_\alpha U_\alpha)/\sim$

- For $k \geq 1$, the tangent space is defined as $T_p U_\alpha$, for $p \in U_\alpha$, under identification $d\phi_{\beta\alpha} : T_p U_\alpha \xrightarrow{\sim} T_p U_\beta$

denote a basis by $x_i \equiv \partial_i$ in $T_p U_\alpha$ (why use ∂_i ?)

- A Riemannian metric is a collection of 1st fund. form $\langle \cdot, \cdot \rangle_\alpha$ on TU_α st. $\langle v, w \rangle_\alpha = \langle d\phi_{\beta\alpha}(v), d\phi_{\beta\alpha}(w) \rangle_\beta$ $\forall \alpha, \beta$.

Denote as (S, g). g always exists by P.O.U.

denote $g_{ij} = \langle x_i, x_j \rangle$, $\alpha' = \sum x_i \, du^i \Rightarrow ds^2 = \langle \alpha', \alpha' \rangle = \sum g_{ij} \, du^i du^j$

- The covariant derivative (Levi-Civita connection)

D_v : Vect fields $\to$ Vect fields at $v \in T_p S$

is uniquely determined by, let $v = \alpha'$, $D_v \equiv \frac{D}{dt}$:

1) linear op : $\nabla_v (a w_1 + b w_2) = a \nabla_v w_1 + b \nabla_v w_2$, $a, b \in \mathbb{R}$

$\nabla_{a v_1 + b v_2} w = a \nabla_{v_1} w + b \nabla_{v_2} w$

2) Leibnitz rule : $\nabla_v (f w) = (vf) w + f \nabla_v w$

directional derivative

L.C. condition :

3) Metrical : $\frac{d}{dt} \langle w_1, w_2 \rangle = \langle \nabla_v w_1, w_2 \rangle + \langle w_1, \nabla_v w_2 \rangle$ $= \frac{d f \circ \alpha}{dt}$

4) Torsion free : For $\nabla_{x_i} x_j =: \sum \Gamma_{ij}^k x_k$

$\nabla_{x_i} x_j = \nabla_{x_j} x_i$, i.e. $\Gamma_{ij}^k = \Gamma_{ji}^k$

Then $\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$.

- K is determined by the Gauss eq'ns (any one of it)

$\Rightarrow$ parallel translation

Gauss-Bonnet Thm

exp map, geod. convex nbd

variation of geodesics (Bonnet's thm etc).

Rmk: For 2-dim'l reason, passing to the univ. cover

$(\tilde{S}, \tilde{g}) \longrightarrow (S, g) \quad \pi_1(\tilde{S}) = \{1\} \Rightarrow \begin{cases} \tilde{S} \simeq S^2 & \text{cpt} \\ \tilde{S} \simeq \mathbb{R}^2 & \text{non-cpt} \end{cases}$ homeo.

thus for $S \not\simeq S^2, \mathbb{RP}^2$, may assume there is only one chart (U, x).

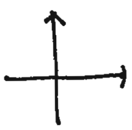
- Geometry of Space forms (S with $K = $ const, complete)

§5.5+§5.6+§5.8 (Cartan-Hadamard thm) Uniformization Thm:

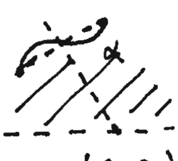
Elliptic	$K = 1$	$\Leftrightarrow$ S isometric to	S^2	$(S^2, \cancel{\mathbb{RP}^2})$ 2:1
Euclidean	$K = 0$	$\Leftrightarrow$ "	$\mathbb{R}^2$	$(T, \cancel{K}, \boxed{\ }, \underline{M})$ cover
Hyperbolic	$K = -1$	$\Leftrightarrow$	$\mathbb{H}$	$(\mathbb{H}/\Gamma)$ discrete sub gp of isometries

Rmk: Any cpt S can admit g with $K = $ const.
Need PDE to prove (Kazhdan-Warner, or Riem mapping)
by changing $g \mapsto fg$ $(f > 0, C^\infty)$ ie. ~~conformal~~.

- Poincaré model of $\mathbb{H}$ (4 models)

$\mathbb{R}^2$ 

- $ds^2 = e^{-2y}\, dx^2 + dy^2 = \dfrac{du^2 + dv^2}{v^2}$ let $u = x$, $v = e^y$ $\Rightarrow dv = v\,dy$ & $v \in (0, \infty)$.

$\mathbb{H}^2$ 

- $K = -\dfrac{1}{2\sqrt{EG}}\left[\left(\dfrac{E_2}{\sqrt{EG}}\right)_2 + \cancel{\left(\dfrac{G_1}{\sqrt{EG}}\right)_1}\right] \equiv -1$ (easy)

geodesics: $u = a + r\cos\theta$
if $\vec{PQ} \nparallel \vec{e_2}$: $v = r\sin\theta$

D^2 

$|z| < 1$

$|\alpha| = \displaystyle\int_{\theta_1}^{\theta_2} \frac{\sqrt{\cancel{dr^2} + r^2 d\theta^2}}{r\sin\theta} \geqslant \int_{\theta_1}^{\theta_2} \frac{d\theta}{\sin\theta}$

"=" $\Leftrightarrow$ $r = $ const. ie. α is a circle

All isometric.

Also, then $|\alpha| = -\ln(\csc\theta + \cot\theta)\Big|_{\theta_1}^{\theta_2} \longrightarrow \infty$ as $\theta_p \to \pi$
ie. $\mathbb{H}$ is complete.

if $\vec{PQ} \parallel \vec{e_2}$ get $d\theta = 0$.
ie. $\theta = \pi/2$

- $\mathbb{H} \to D$, $z = \dfrac{w - i}{w + i}$

$(\infty, i, 0) \mapsto (1, 0, -1)$

Ex. Determine geodesics in D & $\int_\Omega K\,dA$ 

CLAIM: indeed $ds^2 = \dfrac{4|dz|^2}{(1-|z|^2)^2}$ and $\mathrm{Isom}(D) = \mathrm{Aut}(D) = \left\{ e^{i\theta}\dfrac{z - \alpha}{1 - \bar{\alpha}z} \right\}$
by Schwartz lemma.

$$z = \frac{w-i}{w+i} \iff zw + zi = w - i$$
$$w(1-z) = i(1+z) \iff w = i\,\frac{1+z}{1-z}$$

$$y = \operatorname{Im} w = \operatorname{Re}\frac{1+z}{1-z} = \frac{1}{2}\left(\frac{1+z}{1-z} + \frac{1+\bar z}{1-\bar z}\right) = \cancel{\tfrac{1}{2}}\,\frac{\cancel{2}(1-|z|^2)}{|1-z|^2}$$

$$\begin{cases} w = u + iv \\ z = x + iy \end{cases} \qquad z = \frac{aw+b}{cw+d} \Rightarrow dz = \frac{a(cw+d)-(aw+b)c}{(cw+d)^2}\,dw$$
$$= \frac{ad-bc}{(cw+d)^2}\,dw$$

$$\Rightarrow |dz| = |ad-bc|\cdot|dw|\cdot\frac{1}{|cw+d|^2}$$
$$= 2\,\frac{|dw|}{|w+i|^2} \qquad w+i = i\left(\frac{1+z}{1-z}+1\right) = \frac{2i}{1-z}$$
$$= \frac{1}{2}|1-z|^2\,|dw|$$

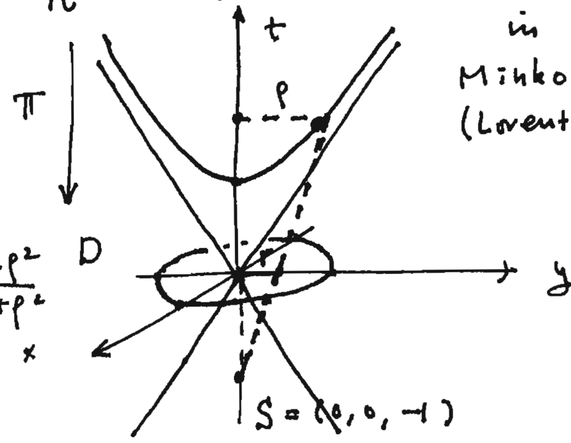
i.e. $ds^2 = \dfrac{4|dz|^2}{(1-|z|^2)^2}$ as expected.

- $\mathrm{Aut}(D) = \left\{ e^{i\theta}\dfrac{z-\alpha}{1-\bar\alpha z} \;\middle|\; \alpha \in D \right\}$ by Schwartz lemma in complex analysis.
- $\mathrm{Isom}(D) \supset \mathrm{Aut}(D)$ by direct check.
- $\subset$ since Isometric $\subset \mathrm{Conf}^+(D) = \mathrm{Aut}(D)$.

Hyperbolic projection:

$\mathcal{H}^+$ hyperboloid $\equiv$ pseudo-"sphere" in $\mathbb{R}^{1,2}$ with Minkowski / (Lorentz) metric

$$t^2 - \underbrace{(x^2+y^2)}_{\rho^2} = 1$$



$\mathbb{R}^{1,2}$

$\underline{t^2 = 1+\rho^2} \Rightarrow t\,dt = \rho\,d\rho$

$ds^2 = dt^2 - d\rho^2 = \dfrac{\rho^2}{1+\rho^2}d\rho^2 - d\rho^2 = \dfrac{d\rho^2}{1+\rho^2}$

Also, $\underline{r = \dfrac{\rho}{t+1}}$

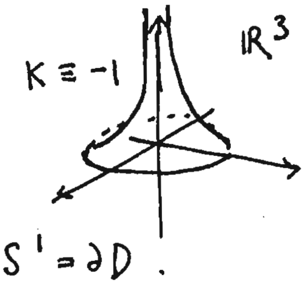
$\Rightarrow \rho = rt + r$

$\Rightarrow (\rho - r)^2 = r^2t^2 = r^2(1+\rho^2)$

i.e. $\rho - 2r = \rho r^2$

$\Rightarrow \boxed{\rho = \dfrac{2r}{1-r^2}}$

Notice that the pseudosphere in $\mathbb{R}^3$ is NOT complete. Yet it is in $\mathbb{R}^{1,2}$.



So $$ds^2 = \frac{\left[\dfrac{2(1-r^2)+4r^2}{(1-r^2)^2}\right]^2 dr^2}{1+\dfrac{4r^2}{(1-r^2)^2}} = \frac{4\,dr^2}{(1-r^2)^2} \equiv \frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2}.$$

Ex. $E \ni 0$ a plane $\ni C = E \cap \mathcal{H}^+$ geodesic $\iff \pi(C)$ a circle $\perp S^1 = \partial D$.

Ref: Dubrovin / Fomenko / Novikov : Modern Geometry I. Ch 2. (§10)

Thm (Hilbert) S complete, $K=-1$

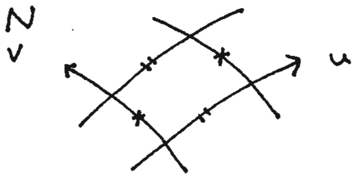
$\Rightarrow \nexists$ isometric immersion $S \hookrightarrow \mathbb{R}^3$.

Step 1: $K=-1 \Rightarrow \exists$ T-net

pf: if $S \hookrightarrow \mathbb{R}^3$, asymptotic curves form a ~~Tschebyshev net~~ : i.e. asymp coor (u,v)

$$e(u')^2 + 2f u'v' + g(v')^2 = 0 \Rightarrow e = 0 = g$$

$X(u,v)$ should have $\underline{E_2 = 0 = G_1}$ i.e. $X_{12} \parallel N$



opposite sides have equal length

$$N_1 \times N_2 = K\, X_1 \times X_2 = K D N, \quad D := \sqrt{EG-F^2} = |X_1 \times X_2|$$

$$-1 = K = \frac{eg-f^2}{EG-F^2} = -\left(\frac{f}{D}\right)^2 \Rightarrow \frac{f}{D} = \pm 1$$

$$N \times N_1 = \frac{1}{D}(X_1 \times X_2)\cdot N_1 = \frac{1}{D}\Big[\cancel{(\underset{-e=0}{X_1 \cdot N_1})}X_2 - (X_2 \cdot N_1)X_1\Big] = \frac{f}{D}X_1$$

Similarly $N \times N_2 = -\frac{f}{D}X_2$

$$(N \times N_1)_2 - (N \times N_2)_1 = \pm 2 X_{12} \quad \Rightarrow X_{12} = \pm N$$

$$= 2\cdot N_1 \times N_2 = 2N$$

$\left(\text{so } E_2 = (X_1 \cdot X_1)_2 = 2X_{12}\cdot X_1 = 0 \right.$
$\left. G_1 = 0 \text{ too.} \right)$

Step 2: for any T-net

<u>curvature in T-net</u>

$E_2 = 0 \Rightarrow$ set $\hat{u} = \int E\, du$

$G_1 = 0 \Rightarrow$ set $\hat{v} = \int G\, dv$

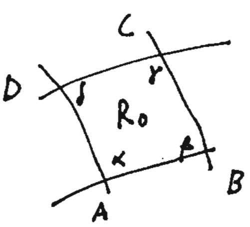
may assume (u,v) st $ds^2 = du^2 + 2\underbrace{\cos\omega}_{X_1\cdot X_2}\, du\, dv + dv^2$

§4.3 Ex.5 $\Rightarrow K = -\dfrac{w_{uv}}{\sin w}$

i.e. $\underline{w_{12} + K\sin w = 0}$.

Corollary: In any rectangular region R_0, $\underline{\left|\int_{R_0} K\, dA\right| < 2\pi}$

pf: $\int_{R_0} K\, dA = \int_{R_0} K |X_1 \times X_2|\, du\, dv = \int_{R_0} K \sin w\, du\, dv$



$$= -\int_{R_0} w_{uv}\, du\, dv$$

$$= -\int_B^C w_v\, dv + \int_A^D w_v\, dv$$

$$= -(\gamma - (\pi - \beta)) + (\pi - \delta) - \alpha = 2\pi - (\alpha+\beta+\gamma+\delta)$$

$$\in (-2\pi, 2\pi). \quad \square$$

Step 3 : Global T-net

By taking the universal covering $\hat{S} \to S$
(in fact, $\exp_p : T_pS \cong \mathbb{R}^2 \to S$ is a covering map)
may assume that $S = \mathbb{H}$ (or D), which has <u>∞ vol.</u>
isometric

by Minding's thm, since $K < 0 \Rightarrow$ $\exp_p$ is defined on the whole T_pS

Now the asymptotic coor system is also globally defined on S. (Again using completeness of S)

But then $S = \bigcup_{n=1}^{\infty} X\left([-n,n] \times [-n,n]\right) =: \bigcup_{n=1}^{\infty} Q_n$

$$|Q_n| = -\int_{Q_n} K \, dA < 2\pi \quad \text{in step 2} \quad (K \equiv -1)$$

$Q_n \subset Q_{n+1} \subset \cdots \Rightarrow |S| < \infty$ $\times$

Some Historical Remarks on isometric embeddings in $\mathbb{R}^n$:

Global

C^∞ topological : Whitney $M^n \hookrightarrow \mathbb{R}^{2n}$ so $S \hookrightarrow \mathbb{R}^4$

isometric : Nash (1956) Günther (1989)

$M^n \hookrightarrow \mathbb{R}^N$ $N = \frac{3}{2} n(n+1)(n+9)$; $\frac{1}{2} n(n+3) + 5$

in any ϵ-ball

so $S \hookrightarrow \mathbb{R}^{10}$

For $S = \mathbb{H}^2$: Blanuša (1955), $\exists$ explicit $\mathbb{H}^2 \hookrightarrow \mathbb{R}^6$ C^∞ isometric

Local

Conjecture (Yau) : $S \hookrightarrow \mathbb{R}^4$ isometric C^∞

for C^1 OK by Nash-Kuiper (1956).

~~Any local metric $S \hookrightarrow \mathbb{R}^4$~~, indeed into $x^2 - y^2 = z^2 + w^2$

~~For $\mathbb{R}^3$~~, OK near $p \in S$ if $K(p) \neq 0$

or if $K(p) = 0$ but $\nabla K(p) \neq 0$ (or finite order)

(C.S. Lin 1986).

for $\mathbb{H}$ certainly OK locally (Minding) ; Globally in $\mathbb{R}^{1,2}$.