of G-bundles

* characteristic class on cpx vector bundle

At the beginning, let $E \to X$ be a v.b. (real or complex). ∇ a ($\mathbb{R}$ or $\mathbb{C}$)-conn

let s_α local frame of $\Gamma(U, E)$

let $\nabla s_\alpha = \omega_\alpha^\beta s_\beta$

ie. $\nabla S = \omega S$

regard ω_α^β β

$$\omega = \begin{pmatrix} \omega_1^1 & \cdots & \omega_1^r \\ \vdots & \ddots & \vdots \\ \omega_r^1 & \cdots & \omega_r^r \end{pmatrix}$$

$$S = \begin{pmatrix} s_1 \\ \vdots \\ s_r \end{pmatrix}$$

∇ extend to (by leibnitz rule)

$$(*) \qquad \Lambda^0(E) \xrightarrow{\nabla} \Lambda^1(E) \xrightarrow{\nabla} \Lambda^2(E) \xrightarrow{\nabla} \cdots$$

$$\nabla^2 S = \nabla \nabla S = \nabla \omega S$$

$$= d\omega \cdot S - \omega \wedge \omega . S$$

$$= (d\omega - \omega \wedge \omega) \cdot S$$

ie. the curvature $\Omega \in \Lambda^2(\mathrm{End}\,E)$ is locally given by

$$\Omega = d\omega - \omega \wedge \omega$$

$$(\Omega_\alpha^\beta = d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta)$$

bec.

$$\nabla^2 fS = \nabla(df S + f \nabla S)$$

$$= -\cancel{df}\nabla S + \cancel{df}\nabla S + f \nabla^2 S$$

$$= f \nabla^2 S$$

is function linear

or equiv. for another frame $S' = A \cdot S$

get $\Omega' \cdot \underset{\underset{A \cdot S}{\parallel}}{S'} = A \cdot \Omega \cdot S$

ie. $\Omega' = A \, \Omega \, A^{-1}$.

(2nd) Bianchi identity :

$$\nabla \Omega = 0 \; ; \text{ or } d\Omega = [\omega, \Omega].$$

$$(**) \quad \text{as a map} : \Lambda^2 \,\mathrm{End}\,E \xrightarrow{\nabla} \Lambda^3 \,\mathrm{End}\,E$$

(∇ acts on $\mathrm{End}\,E = E \otimes E^*$ and extends by Leibnitz rule)

Let $\tilde{s}$ i be the dual basis of E^* wrt. s_i.

eg. $\nabla s_i \otimes \tilde{s}^j = (\nabla s_i) \otimes \tilde{s}^j + s_i \otimes \nabla \tilde{s}^j$

$\nabla \tilde{S} = -\tilde{S} \cdot \omega$

I think the case of Levi-Civita on 1-form

So for general $\Phi \in \Lambda^k \,\mathrm{End}\,E$:

$$\boxed{\nabla \Phi = d\Phi + (-1)^k \Phi \wedge \omega - \omega \wedge \Phi}$$

back side

$$= d\Phi + (-1)^k [\Phi, \omega]$$

so $\nabla \Omega = d\Omega + \Omega \wedge \omega - \omega \wedge \Omega$

but $d\Omega = -d\omega \wedge \omega + \omega \wedge d\omega = -\Omega \wedge \omega + \omega \wedge \Omega$

so. $\nabla \Omega = 0$. ∀

Notice that for any $\alpha \in \Lambda^k E$, say $\alpha = \varphi s$

$$\nabla^2\alpha = \nabla\nabla\varphi s = \nabla(d\varphi\cdot s + (-1)^k \varphi \nabla s)$$

$$= d\varphi\cdot\nabla s \cdot \cancel{(-1)}^{k+1} + (-1)^k \cancel{d\varphi}\cdot\nabla s + (-1)^{2k}\varphi\nabla^2 s$$

so always $\underline{\nabla^2\alpha = \Omega\alpha}$

Also. $\forall\ \Phi \in \Lambda^k \operatorname{End} E$:

$$\nabla^2\Phi = \nabla(d\Phi + (-1)^k \Phi\wedge\omega - \omega\wedge\Phi)$$

$$= (-1)^{k+1}\cancel{d\Phi}\wedge\omega - \omega\wedge d\Phi$$

$$+ (-1)^k \cancel{d\Phi}\wedge\omega + \Phi\wedge d\omega - d\omega\wedge\Phi + \omega\wedge d\Phi$$

$$+ (-1)^{k+1}(-1)^k\Phi\wedge\omega\wedge\omega - (-1)^k\omega\wedge\Phi\wedge\omega$$

$$- \omega\wedge(-1)^k\Phi\wedge\omega - \omega\wedge(-\omega\wedge\Phi)$$

$$= \Phi\wedge\Omega - \Omega\wedge\Phi = [\Phi, \Omega]$$

i.e. $\underline{\nabla^2\Phi = [\Phi, \Omega]}$ ~ Ω still acts as curvature but now as adjoint repr.

Conclusion: the curvature (operator) acts on the tensor bundle $\otimes^r E \otimes \otimes^s E^*$ as derivatives (Leibnitz rule) just like the case of tangent bundle (Levi-Civita).

characteristic (Chern) Forms: if $\mathbb{R}$, use 1 / if $\mathbb{C}$, use $\sqrt{-1}$ in order that c_i are $\mathbb{R}$-form

$$\det\left(I + \frac{\sqrt{-1}}{2\pi}\Omega\right) = 1 + c_1(E,\nabla) + c_2(E,\nabla) + \cdots + c_r(E,\nabla)$$

$c_i(E,\nabla) \in \Lambda^{2i}(X)$.

Proposition I. $d c_i(E,\nabla) = 0 \quad \forall i$.

Pf: c_i are a basis for sym. functions of eigenvalue of Ω

another set of basis is $\operatorname{tr}(\Omega^k)$ $k = 0, 1, \cdots, r$

It is equiv to show that $\underline{d \operatorname{tr}(\Omega^k) = 0}$.

Notice that $\operatorname{tr} AB = \operatorname{tr} BA$

for matrix AB with entries in a comm. ring.

Bianchi id. since $d\Omega = [\omega, \Omega]$

$$\begin{aligned} d\,\mathrm{tr}(\Omega^k) &= \mathrm{tr}(d(\Omega\wedge\cdots\wedge\Omega)) \\ &= k\cdot\mathrm{tr}((d\Omega)\wedge\Omega^{k-1}) \\ &= k[\mathrm{tr}(\omega\wedge\Omega^k) - \mathrm{tr}(\Omega\wedge\omega\wedge\Omega^{k-1})] \\ &= k(\mathrm{tr}(\omega\wedge\Omega^k) - \mathrm{tr}(\omega\wedge\Omega^k)) = 0 \quad \square. \end{aligned}$$

Proposition II. the de Rham cohomology class $[c_i(E,\nabla)] \in H^{2i}_{DR}(X;\mathbb{R})$ is indep. of choices of ∇.

pf: A näive approch does not work for $\mathrm{tr}(\Omega^k)$ when $k \geq 2$. (See next page).

Consider $P(A_1,\cdots,A_k)$ sym. function in matrix var. A_i which is G-inv. ie. $P(TA_1T^{-1},\cdots,TA_kT^{-1}) = P(A_1,\cdots,A_k)$

$\exists$ 1-1 correspondence (see back side)

$$\left\{ P : \begin{matrix}\text{sym. inv.}\\ \text{poly.}\end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} P(A) := P(A,\cdots,A) \\ G\text{-inv. poly.}\end{matrix} \right\}$$

the converse is called the complete polarization

Fact: $P([T,A_1],A_2,\cdots A_k) + P(A_1,[T,A_2],\cdots)$
$+\cdots+ P(A_1,A_2,\cdots,[T,A_k]) = 0$

pf: Bec. $\left.\frac{d}{dt}\right|_{t=0} P(e^{tT}A_1e^{-tT},\cdots,e^{tT}A_ke^{-tT}) \equiv 0 \quad \square.$

Given ∇_0, ∇_1 let $\nabla_t = (1-t)\nabla_0 + t\nabla_1 = \nabla_0 + t(\nabla_1 - \nabla_0)$

let $\eta = \nabla_1 - \nabla_0 \in \Lambda^1(\mathrm{End}\,E)$

Write $\nabla_t := \nabla + t\eta$

① $\Omega_t = (\nabla+t\eta)(\nabla+t\eta) = \nabla^2 + \cancel{t\eta\nabla} + \underset{t\nabla\eta - \cancel{t\eta\nabla}}{\underset{\shortparallel}{\nabla(t\eta\cdot)}} + t^2\eta\circ\eta$

$= \Omega + t\nabla\eta - t^2\eta\wedge\eta$

be very careful here

$$\boxed{\eta\circ\eta = -\eta\wedge\eta}$$

bec. the linearity now needs a sign

So, $\Omega_1 - \Omega_0 = \nabla\eta - \eta\wedge\eta$

$$\boxed{\frac{d\Omega_t}{dt} = \nabla\eta - 2t\cdot\eta\wedge\eta \equiv \nabla_t\eta}$$

② Or since $\omega_t = \omega + t\eta$

$\Omega_t = d\omega_t - \omega_t\wedge\omega_t \Rightarrow \underset{\frac{d\Omega_t}{dt}}{\shortparallel}\ \nabla\eta - 2t\,\eta\wedge\eta$ too.

$= d\omega + t\,d\eta - \omega_t\wedge\omega_t$

$$\operatorname{tr}\Omega_1 - \operatorname{tr}\Omega_0 = \operatorname{tr}(D\eta - \underline{\eta\wedge\eta})$$

$$\operatorname{tr}(\eta\wedge\eta) = -\operatorname{tr}(\eta\wedge\eta) = 0$$

$$= \operatorname{tr}(D\eta) = \operatorname{tr}(d\eta - \underline{\eta\wedge\omega} - \underline{\omega\wedge\eta})$$

$$= \operatorname{tr}(d\eta)$$

$$= d(\underline{\operatorname{tr}\eta})$$

Same reason since η, ω all $\operatorname{End}E$-1 forms

$\operatorname{tr}(\eta\wedge\omega) = -\operatorname{tr}(\omega\wedge\eta)$.

In general,

$$\operatorname{tr}(\Omega_1^k) - \operatorname{tr}(\Omega_0^k) = \int_0^1 \underline{\frac{d}{dt}\operatorname{tr}(\Omega_t^k)}\, dt$$

how to write this as dQ ?

$$= k\cdot\int_0^1 \operatorname{tr}\left(\frac{d\Omega_t}{dt}\wedge\Omega_t^{k-1}\right) dt$$

$$= k\int_0^1 \operatorname{tr}(\underline{D\eta}\wedge\Omega_t^{k-1}) - k\int_0^1 \operatorname{tr}(\underline{\eta}\wedge\underline{\eta}\wedge\Omega_t^{k-1})\, 2t\, dt$$

$$\|$$

$$d\eta - \eta\wedge\omega - \omega\wedge\eta$$

↑

Problem: in general $\operatorname{tr}(\underline{AB}C) \neq \operatorname{tr}(\underline{BA}C)$! so $\neq 0$

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let $Q_t = \operatorname{tr}(\eta\wedge\Omega_t^{k-1})$

then $dQ_t = \operatorname{tr}(d\eta\wedge\Omega_t^{k-1}) - \operatorname{tr}(\eta\wedge\underline{d\Omega_t\wedge\Omega_t^{k-2}})$

$- \cdots - \operatorname{tr}(\eta\wedge\underline{\Omega_t^{k-2}\wedge d\Omega_t})$

This works for $k=2$. but not for $k \geq 3$.

useless.

$$\operatorname{tr}\left(\eta\wedge\frac{d\Omega_t}{dt}\wedge\Omega_t^{k-2}\right)$$

$$= \operatorname{tr}(\eta\wedge(\nabla\eta - \eta\wedge\eta)\wedge\Omega_t^{k-2})$$

$$= \operatorname{tr}(\eta\wedge\nabla\eta\wedge\Omega_t^{k-2}) - \operatorname{tr}(\eta\wedge\eta\wedge\eta\wedge\Omega_t^{k-2})$$

?

But for sym. inv. poly $P(A_1, \cdots, A_k)$

if substitute A_i by some Λ^{d_i} element.

then always have commutation

$$P(A_1, \cdots, S, \cdots, T, \cdots A_k) = P(A_1, \cdots T, \cdots, S, \cdots A_k)$$

$$\cdot(-1)^*$$

some power.

$$\mathrm{tr}\,\Omega_1^k - \mathrm{tr}\,\Omega_0^k = \int_0^1 \left(\frac{d}{dt}\Omega_t^k\right) dt$$

<u>claim</u>: $\frac{d}{dt}\mathrm{tr}\,\Omega_t^k = dQ$ for $Q := \overset{Q(\eta,\Omega_t)}{=} P(\eta, \Omega_t, \cdots, \Omega_t)\cdot k$

In fact, for any inv. poly P : $\boxed{\frac{dP(\Omega_t)}{dt} = d\,\underset{\substack{\uparrow \\ \text{transgression}}}{\underline{Q(\eta,\Omega_t)}}}$

$$\frac{dP(\Omega_t)}{dt} = kP\left(\frac{d\Omega_t}{dt}, \Omega_t, \cdots, \Omega_t\right)$$

$Q(\nabla\eta,\Omega_t) -- Q(\eta\wedge\eta, \Omega_t)\cdot 2t$

$$= \underline{kP(\nabla\eta, \Omega_t, \cdots, \Omega_t)} + \overset{\shortparallel}{k P(\eta\wedge\eta, \Omega_t, \cdots, \Omega_t)\cdot \underline{2}t}$$

$$dQ = \underline{k\cdot P(\nabla\eta, \Omega_t, \cdots, \Omega_t)} + k(k-1)P(\eta, \underline{\nabla\Omega_t}, \Omega_t, \cdots, \Omega_t)$$

(both side are indep. of coor. then check at a special frame st $\omega(x_0) \equiv 0$. and frame of E).

Ex. do this existence.

Rmk: In Gilkey p.92, he uses a trick to set $\omega_{t_0}(x_0) \equiv 0$ at a fixed time $t = t_0$ then all 2nd terms disappear! But 1st term → $d\eta$. $\nabla\eta = d\eta - \eta\wedge\omega - \omega\wedge\eta$
$= d\eta - [\eta, \omega]$

$$\nabla\Omega_t = \nabla(\Omega + t\nabla\eta - t^2\eta\wedge\eta)$$
$$= t[\eta,\Omega] \underline{- t^2(\nabla\eta\wedge\eta - \eta\wedge\nabla\eta)} = t^2[\eta,\nabla\eta]$$
$$= t[\eta, \Omega + t\nabla\eta]$$
$$= t[\eta, \Omega_t] \quad \text{since } [\eta, \eta\wedge\eta]$$

Now $t P(\underset{\substack{\shortparallel \\ \eta\wedge\eta - (-1)^1\eta\wedge\eta \\ \shortparallel \\ 2\eta\wedge\eta}}{[\eta,\eta]}, \Omega_t, \cdots, \Omega_t) + \underbrace{tP(\eta, [\eta,\Omega_t], \Omega_t, \cdots) + \cdots}_{\substack{\text{all terms are equal} \\ = (k-1)P(\eta, \nabla\Omega_t, \Omega_t, \cdots, \Omega_t)}} = 0$

the claim is proved. □

So for any inv. poly. P eg. $\mathrm{tr}\,\Omega^k$ or $c_i(E,\nabla)$

$$\underset{\substack{\text{or} \ \Omega_1}}{P(\underline{\underline{\nabla_1}})} - \underset{\Omega_0}{P(\underline{\underline{\nabla_0}})} = \int_0^1 \frac{d}{dt} P(\Omega_t)\, dt$$
$$= \int_0^1 dQ(\eta, \Omega_t)\, dt$$
$$= d\left(\int_0^1 Q(\eta,\Omega_t)\, dt\right) \quad \#$$

Ex. Write out Q for $\underline{\mathrm{tr}\,\Omega^2}$. $\mathrm{tr}\,\Omega^3$. c_1, c_2, c_3 etc.

$P(A)$ inv. poly of degree 2.

then $P(A_1, A_2) = \frac{1}{2}\left(P(A_1 + A_2) - P(A_1) - P(A_2) \right)$

for $P(A) = \mathrm{tr}(A^2)$ get

$$P(A,B) = \frac{1}{2}\left(\mathrm{tr}(A+B)^2 - \mathrm{tr}\, A^2 - \mathrm{tr}\, B^2 \right)$$

$$= \frac{1}{2}\left(\mathrm{tr}\, AB + \mathrm{tr}\, BA\right) = \mathrm{tr}\, \underline{\underline{AB}}$$

- So $Q(\eta, \Omega_t) = \mathrm{tr}(\eta \wedge \Omega_t)$

$$= \mathrm{tr}\left(\eta \wedge (\Omega + t\nabla\eta - t^2 \eta\wedge\eta)\right)$$

$$= \mathrm{tr}\, \eta\wedge\Omega + t\, \mathrm{tr}\, \eta\wedge\nabla\eta - t^2\, \mathrm{tr}\, \eta\wedge\eta\wedge\eta$$

$$\int_0^1 q(\eta, \Omega_t)\, dt = \mathrm{tr}(\eta\wedge\Omega) + \frac{1}{2}\mathrm{tr}(\eta\wedge\nabla\eta) - \frac{1}{3}\mathrm{tr}(\eta\wedge\eta\wedge\eta)$$

important in chern-Simons theory

- Functoriality of characteristic classes (Axioms)
- Topological Aspect : Gauss Map to Grassmannian

Metrical: unitary condition:

$$d\langle \alpha, \beta\rangle = \langle \nabla\alpha, \beta\rangle + \langle \alpha, \nabla\beta\rangle$$

$\nabla s_i = \omega_i^j s_j$

$\Rightarrow \langle R\alpha, \beta\rangle + \langle \alpha, R\beta\rangle = 0$

$R \in \Lambda^2(\mathfrak{g})$

Lie algebra valued

$U(E)$; $O(E)$

under unitary frame:

real: $A^t = -A$

$\det(I + A) = \det(I + A^t)$

$= \det(I - A)$

$\operatorname{tr} A^k = (-1)^k \operatorname{tr} A^k$ k odd $\Rightarrow = 0$

complex: $\bar{A}^t = -A$ only get "c_k" $\in i\mathbb{R}$

$A + \bar{A}^t = 0$

$\lambda + \bar{\lambda} = 0$

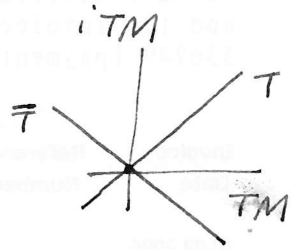
$\lambda \in i\mathbb{R}$

Torsion free condition ?

- cpx mfd M, $TM \otimes \mathbb{C} = T \oplus \bar{T}$

$$C^\infty(M)_{\mathbb{C}} \xrightarrow{d} T^*M \otimes \mathbb{C} = A^{1,0} \oplus A^{0,1}$$

$$d = \partial + \bar{\partial}$$



- holo. v.b. $E \to M$

Fund. Thm of Hermitian geom. $\nabla = \nabla + \bar{\nabla}_{=\bar{\partial}}$

$$\partial h_{ij} = \langle \nabla s_i, s_j\rangle = \omega_i^l h_{lj} \Rightarrow \omega_i^l = h^{jl} \partial h_{ij} = (\partial H) H^{-1}$$

$$0 = \partial^2 h_{ij} = \langle \overset{20}{R} s_i, s_j\rangle \quad \text{ie. } R^{2,0} = 0$$

$$R = R^{20} + R^{11} + R^{02} \quad \text{Also} \quad \bar{R}^t = -R, \Rightarrow R^{0,2} = 0$$

Hence (obvious ?) $\Omega = \bar{\partial}[(\partial H) H^{-1}] = (\bar{\partial}\partial H) H^{-1} + (\partial H) H^{-1} \wedge (\bar{\partial} H) H^{-1}$

in fact the ∂-part cancel out

$$\partial((\partial H) H^{-1}) - (\partial H) H^{-1} \wedge (\partial H) H^{-1} = 0 \quad \text{auto.}$$

Cor: for line bundle $\Omega = -\partial\bar{\partial} \log H$; $H = \langle e, e\rangle$ hol. sect.

in general $c_1(E, h) = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \det h_{ij}$

$\Rightarrow$ Normalization axiom (back page).

Remarks / Formulae in the case $E = T_M$.

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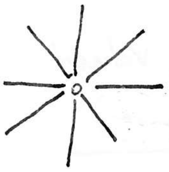
Fubini-Study

Hermitian metric = $\sum |Z_i|^2$

$\Omega = -\partial\bar\partial \log\left(1+\sum z_i \bar z_i\right)$ ← This metric is for $\mathcal{O}(-1)$, not $\mathcal{O}(+1)$.



$\mathbb{C}^{n+1}\setminus\{0\}$

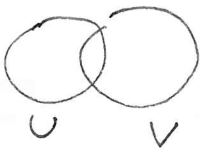
$$= -\partial\left(\frac{\sum z_i\, d\bar z_i}{1+|z|^2}\right) = -\frac{\left(\sum dz_i\wedge d\bar z_i\right)\left(1+|z|^2\right) + \sum z_i \bar z_j\, d\bar z_i\wedge dz_j}{\left(1+|z|^2\right)^2}$$

1 dim case: $\frac{\sqrt{-1}}{2\pi}\int_{\mathbb{C}} \frac{-dz\wedge d\bar z}{(1+|z|^2)^2} = +2\sqrt{-1}\,\frac{\sqrt{-1}}{2\pi}\int_0^{2\pi}\int_0^{\infty}\frac{r\,dr\,d\theta}{(1+r^2)^2}$

$$= -\frac{2\pi}{\pi}\cdot\frac{1}{2}\left(-\frac{1}{1+r^2}\right)\Big|_0^{\infty} = -1$$

$dz\wedge d\bar z = (dx+idy)\wedge(dx-idy) = -2i\,dx\wedge dy$

Mayer-Vietoris :



$$\begin{tikzcd}
 & U \arrow[dr, bend left] & \\
U\cap V \arrow[ur, bend left] \arrow[dr, bend right] & & U\cup V \\
 & V \arrow[ur, bend right] &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & A^p(U\cup V) \arrow[r] \arrow[ddd, "d"'] & A^p(U)\oplus A^p(V) \arrow[r] & A^p(U\cap V) \arrow[r] \arrow[ddd, "d"] & 0 \\
 & & (\rho_V\omega, \rho_U\omega) \arrow[r, mapsto] & \omega & \\
 & \eta \arrow[r, mapsto] & (\eta|_U, -\eta|_V) \arrow[d, "d"'] & & \\
 & A^{p+1}(U\cup V) \arrow[r] & A^{p+1}(U)\oplus A^{p+1}(V) \arrow[r] & A^{p+1}(U\cap V) & \\
 & & (d\rho_V\omega, d\rho_U\omega) \arrow[r, mapsto] & 0 &
\end{tikzcd}$$

- exact sequence
- chain map

ie. $H^p(U\cup V)\longrightarrow H^p(U)\oplus H^p(V)\longrightarrow H^p(U\cap V)\xrightarrow{\Delta} H^{p+1}(U\cup V)$

Exercise (1) exactness
(2) Five lemma

pf of Kunneth / Leray Hirsch / Poincaré duality / de Rham Thm :
All can be done by this method. #

Euler sequence :

$$\begin{tikzcd}
\text{called } \mathcal{O}_{\mathbb{P}(E)}(-1). & & \\
S \arrow[u] \arrow[r] \arrow[dr] & \pi^*E \arrow[r] \arrow[d] & E \arrow[d] \\
& \mathbb{P}(E) \arrow[r, "\pi"] & X
\end{tikzcd}$$

$$0 \to S \to \pi^*E \to Q \to 0$$

$$0 \to \mathbb{1} \to S^* \otimes \pi^*E \to S^* \otimes Q \to 0$$

$$c_r(S^* \otimes \pi^*E) = 0 \quad (i)$$

Fact : $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$. (ii)

Cor : $c_r(L \otimes V)$ cher roots

$$\prod_i (\xi + x_i)$$

$$= \xi^r + c_1(V)\xi^{r-1} + \cdots + c_r(V) \quad \divideontimes$$

< by splitting principle >

$$(i) \underset{\text{by Cor.}}{\Longrightarrow} \zeta^r + \pi^* c_1(E)\zeta^{r-1} + \cdots + \pi^* c_r(E) = 0$$

where $\zeta = c_1(S^*)$ ← called $\mathcal{O}_{\mathbb{P}(E)}(1)$.

Rmk : for (ii),
The pf is non-trivial

- Čech view pt + $\mathbb{Z} \to \mathcal{O} \to \mathcal{O}^\times$
- Curvature via Leibnitz rule
- univ. bundle (Hatcher Prop 3.10)

① Proof of Gauss-Bonnet Thm: $c_r(E) = e(E)$.

② Ex.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-1) \arrow[r] \arrow[d, equal] & p^*\underline{\mathbb{C}}^{n+1} \arrow[r] & Q \arrow[r] & 0 \\
 & S \equiv \xi^1 & & &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \underline{\mathbb{C}} \arrow[r] & \mathcal{O}(1)\otimes p^*\underline{\mathbb{C}}^{n+1} \arrow[r] \arrow[d, equal] & S^*\otimes Q \arrow[r] \arrow[d, equal] & 0 \\
 & & \mathcal{O}(1)^{\oplus n+1} & \mathrm{Hom}(S,Q) \arrow[d, equal] & \\
 & & & T\mathbb{P}^n &
\end{tikzcd}$$

$$c(\mathbb{P}^n) = (1+h)^{n+1}$$

③ Pontryagin classes: $p_i(E) = (-1)^i c_{2i}(E\otimes\mathbb{C}) \in H^{4i}(M,\mathbb{Z})$

(3'): $p(E\oplus F) = [1 - c_2(E_\mathbb{C}\oplus F_\mathbb{C}) + c_4(E_\mathbb{C}\oplus F_\mathbb{C}) \cdots]$

$= p(E)\cdot p(F)$

modulo 2-torsions.

the point is that $E\otimes\mathbb{C} \simeq \overline{E\otimes\mathbb{C}}$ (for E real) hence $c_i(E_\mathbb{C}) = (-1)^i c_i(E_\mathbb{C})$

(4'): For E complex. furthermore $E_\mathbb{R}\otimes\mathbb{C} \cong E\oplus\bar{E}$

so $c(\xi^1_\mathbb{R}\otimes\mathbb{C}) = c(\xi^1\oplus\bar{\xi}^1) = (1-h)(1+h) = 1-h^2$

i.e. $p(\xi^1_\mathbb{R}) = 1+h^2$

by (3'): $p(\mathbb{P}^n_\mathbb{C}) = (1+h^2)^{n+1}$ (notice that $p(E) = p(\bar{E})$ if E is cpx)