

* characteristic class on cpx vector bundle
 At the beginning, let $E \rightarrow X$ be a v.b. (real or complex). ∇ a $(\mathbb{R}$ or $\mathbb{C})$ -conn

let S_α local frame of $\Gamma(U, E)$

let $\nabla S_\alpha = \omega_\alpha^\beta S_\beta$

regard ω_α^β as

ie. $\nabla S = \omega S$

∇ extends to (by Leibnitz rule)

$$\omega = \begin{pmatrix} \omega_1^1 & \dots & \omega_1^r \\ \vdots & \ddots & \vdots \\ \omega_r^1 & \dots & \omega_r^r \end{pmatrix}$$

(*) $\Lambda^0(E) \xrightarrow{\nabla} \Lambda^1(E) \xrightarrow{\nabla} \Lambda^2(E) \xrightarrow{\nabla} \dots$

$$S = \begin{pmatrix} s_1 \\ \vdots \\ s_r \end{pmatrix}$$

$$\begin{aligned} \nabla^2 S &= \nabla \nabla S = \nabla \omega S \\ &= d\omega \cdot S - \omega \wedge \omega S \\ &= (d\omega - \omega \wedge \omega) \cdot S \end{aligned}$$

ie. the curvature $\Omega \in \Lambda^2(\text{End } E)$ is locally given by

$$\begin{aligned} \Omega &= d\omega - \omega \wedge \omega \\ (\Omega_\alpha^\beta &= d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta) \end{aligned}$$

bec.

$$\begin{aligned} \nabla^2 fS &= \nabla(dfS + f\nabla S) \\ &= -df\nabla S + df\nabla S + f\nabla^2 S \\ &= f\nabla^2 S \end{aligned}$$

(2nd) Bianchi identity:

$$\nabla \Omega = 0; \text{ or } d\Omega = [\omega, \Omega]$$

is function linear
 or equiv. for another
 frame $S' = A \cdot S$
 get $\Omega' \cdot S' = A \cdot \Omega \cdot S$
 $\Omega' = A \cdot \Omega \cdot A^{-1}$

(**) as a map: $\Lambda^2 \text{End } E \xrightarrow{\nabla} \Lambda^3 \text{End } E$

(∇ acts on $\text{End } E = E \otimes E^*$ and extends by Leibnitz rule)

Let \tilde{s}^i be the dual basis of E^* wrt. s_i .

eg. $\nabla s_i \otimes \tilde{s}^i = (\nabla s_i) \otimes \tilde{s}^i + s_i \otimes \nabla \tilde{s}^i$

So for general $\Phi \in \Lambda^k \text{End } E$:

$$\begin{aligned} \nabla \Phi &= d\Phi + (-1)^k \Phi \wedge \omega - \omega \wedge \Phi \\ &= d\Phi + (-1)^k [\Phi, \omega] \end{aligned}$$

back side

$$\nabla \tilde{s}^i = -\tilde{s}^j \omega^j_i$$

I think the case of Levi-Civita m 1-form

so $\nabla \Omega = d\Omega + \Omega \wedge \omega - \omega \wedge \Omega$

but $d\Omega = -d\omega \wedge \omega + \omega \wedge d\omega = -\Omega \wedge \omega + \omega \wedge \Omega$

so $\nabla \Omega = 0$.

Notice that for any $\alpha \in \Lambda^k E$, say $\alpha = \varphi S$

$$\begin{aligned} \nabla^2 \alpha &= \nabla \nabla \varphi S = \nabla (d\varphi \cdot S + (-1)^k \varphi \nabla S) \\ &= d\varphi \cdot \nabla S + (-1)^{k+1} \varphi \nabla^2 S + (-1)^{2k} \varphi \nabla^2 S \end{aligned}$$

so always $\nabla^2 \alpha = \Omega \alpha$

Also. $\forall \Phi \in \Lambda^k \text{End } E$:

$$\begin{aligned} \nabla^2 \Phi &= \nabla (d\Phi + (-1)^k \Phi \wedge \omega - \omega \wedge \Phi) \\ &= (-1)^{k+1} d\Phi \wedge \omega - \omega \wedge d\Phi \\ &\quad + (-1)^k d\Phi \wedge \omega + \Phi \wedge d\omega - d\omega \wedge \Phi + \omega \wedge d\Phi \\ &\quad + (-1)^{k+1} (-1)^k \Phi \wedge \omega \wedge \omega - (-1)^k \omega \wedge \Phi \wedge \omega \\ &\quad - \omega \wedge (-1)^k \Phi \wedge \omega - \omega \wedge (-\omega \wedge \Phi) \\ &= \Phi \wedge \Omega - \Omega \wedge \Phi = [\Phi, \Omega] \end{aligned}$$

ie. $\nabla^2 \Phi = [\Phi, \Omega]$ \searrow Ω still acts as curvature but now as adjoint repr.

Conclusion: the curvature (operator) acts on the tensor bundle $\otimes^r E \otimes \otimes^s E^*$ as derivatives (Leibnitz rule) just like the case of tangent bundle (Levi-Civita).

characteristic (Chern) forms: $\begin{cases} \text{if } \mathbb{R}, \text{ use } 1 \\ \text{if } \mathbb{C}, \text{ use } \sqrt{-1} \end{cases}$ in order that c_i are \mathbb{R} -form

$$\det \left(I + \frac{\sqrt{-1}}{2\pi} \Omega \right) = 1 + c_1(E, \nabla) + c_2(E, \nabla) + \dots + c_r(E, \nabla)$$

$$c_i(E, \nabla) \in \Lambda^{2i}(X).$$

Proposition I. $d c_i(E, \nabla) = 0 \quad \forall i.$

pf: c_i are a basis for sym. functions of eigenvalue of Ω
 another set of basis is $\text{tr}(\Omega^k)$ $k=0, 1, \dots, r$
 It is equiv to show that $d \text{tr}(\Omega^k) = 0$.

Notice that $\text{tr} AB = \text{tr} BA$
 for matrix AB with entries in a comm. ring.

Biauchi id.

$$\begin{aligned}
 d \operatorname{tr}(\Omega^k) &= \operatorname{tr}(d(\Omega \wedge \dots \wedge \Omega)) && \text{since } d\Omega = [\omega, \Omega] \\
 &= k \cdot \operatorname{tr}((d\Omega) \wedge \Omega^{k-1}) \\
 &= k [\operatorname{tr}(\omega \wedge \Omega^k) - \operatorname{tr}(\Omega \wedge \omega \wedge \Omega^{k-1})] \\
 &= k (\operatorname{tr}(\omega \wedge \Omega^k) - \operatorname{tr}(\omega \wedge \Omega^k)) = 0 \quad \square.
 \end{aligned}$$

Proposition II. the de Rham cohomology class $[\omega(E, \nabla)] \in H_{DR}^{2i}(X; \mathbb{R})$ is indep. of choices of ∇ .

pf: A naïve approach does not work for $\operatorname{tr}(\Omega^k)$ when $k \geq 2$. (See next page).

Consider $P(A_1, \dots, A_k)$ sym. function in matrix var. A_i which is G -inv. i.e. $P(TA_1T^{-1}, \dots, TA_kT^{-1}) = P(A_1, \dots, A_k)$
 $\exists (-1)$ correspondence (see back side)

$$\left\{ \begin{array}{l} P : \text{sym. inv.} \\ \text{poly.} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} P(A) := P(A, \dots, A) \\ G\text{-inv. poly.} \end{array} \right\}$$

the converse is called the complete polarization

Fact: $P([T, A_1], A_2, \dots, A_k) + P(A_1, [T, A_2], \dots) + \dots + P(A_1, A_2, \dots, [T, A_k]) = 0$

pf: Bec. $\left. \frac{d}{dt} \right|_{t=0} P(e^{tT} A_1 e^{-tT}, \dots, e^{tT} A_k e^{-tT}) \equiv 0 \quad \square.$

Given ∇_0, ∇_1 let $\nabla_t = (1-t)\nabla_0 + t\nabla_1 = \nabla_0 + t(\nabla_1 - \nabla_0)$

let $\eta = \nabla_1 - \nabla_0 \in \Lambda^1(\operatorname{End} E)$

write $\nabla_t := \nabla + t\eta$

$$\begin{aligned}
 \textcircled{1} \quad \Omega_t &= (\nabla + t\eta)(\nabla + t\eta) = \nabla^2 + t\cancel{\nabla\eta} + \nabla(t\eta) + t^2\eta \circ \eta \\
 &= \Omega + t\nabla\eta - t^2\eta \wedge \eta && \begin{array}{l} \text{"} \\ t\nabla\eta - t\cancel{\eta\nabla} \end{array}
 \end{aligned}$$

So, $\Omega_t - \Omega_0 = \nabla\eta - \eta \wedge \eta$

$$\boxed{\frac{d\Omega_t}{dt} = \nabla\eta - 2t\eta \wedge \eta \equiv \nabla t\eta}$$

be very careful here
 $\eta \circ \eta = -\eta \wedge \eta$
 bec. the linearity now needs a sign.

$$\begin{aligned}
 \textcircled{2} \quad \text{Or since } \omega_t &= \omega + t\eta && \frac{d\omega_t}{dt} \equiv \\
 \Omega_t &= d\omega_t - \omega_t \wedge \omega_t && \Rightarrow \nabla\eta - 2t\eta \wedge \eta \text{ too.} \\
 &= d\omega + t d\eta - \omega_t \wedge \omega_t
 \end{aligned}$$

$$\begin{aligned} \text{tr } \Omega_1 - \text{tr } \Omega_0 &= \text{tr}(\nabla \eta - \underline{\eta \wedge \eta}) & \text{tr}(\eta \wedge \eta) \\ &= \text{tr}(\nabla \eta) = \text{tr}(d\eta - \underline{\eta \wedge \omega} - \underline{\omega \wedge \eta}) & = -\text{tr}(\eta \wedge \eta) = 0 \\ &= \text{tr}(d\eta) & \text{Same reason since } \eta, \omega \text{ all} \\ &= d(\underline{\text{tr } \eta}) & \text{end E - form} \end{aligned}$$

Same reason since η, ω all end E - form
 $\text{tr}(\eta \wedge \omega) = -\text{tr}(\omega \wedge \eta)$.

In general,

$$\begin{aligned} \text{tr}(\Omega_t^k) - \text{tr}(\Omega_0^k) &= \int_0^1 \frac{d}{dt} \text{tr}(\Omega_t^k) dt \\ &= k \int_0^1 \text{tr} \left(\frac{d\Omega_t}{dt} \wedge \Omega_t^{k-1} \right) dt & \text{how to write this as } dQ? \\ &= k \int_0^1 \text{tr} \left(\underline{\nabla \eta} \wedge \Omega_t^{k-1} \right) - k \int_0^1 \text{tr} \left(\underline{\eta \wedge \eta} \wedge \Omega_t^{k-1} \right) dt \end{aligned}$$

$$\text{d}\eta - \eta \wedge \omega - \omega \wedge \eta$$

Problem: in general
 $\text{tr}(ABC) \neq \text{tr}(BAC)!$ so $\neq 0$

Let $Q_t = \text{tr}(\eta \wedge \Omega_t^{k-1})$

then $dQ_t = \text{tr}(d\eta \wedge \Omega_t^{k-1})$

This works for $k=2$. but not for $k \geq 3$.
 useless.

$$\begin{aligned} &\text{tr}(\eta \wedge d\Omega_t \wedge \Omega_t^{k-2}) \\ &\dots \text{tr}(\eta \wedge \Omega_t^{k-2} \wedge d\Omega_t) \end{aligned}$$

$$\begin{aligned} &\text{tr} \left(\eta \wedge \frac{d\Omega_t}{dt} \wedge \Omega_t^{k-2} \right) \\ &= \text{tr} \left(\eta \wedge (\nabla \eta - \eta \wedge \eta) \wedge \Omega_t^{k-2} \right) \\ &= \text{tr} \left(\eta \wedge \nabla \eta \wedge \Omega_t^{k-2} \right) - \text{tr} \left(\eta \wedge \eta \wedge \eta \wedge \Omega_t^{k-2} \right) \end{aligned}$$

But for sym. inv. poly $P(A_1, \dots, A_k)$
 if substitute A_i by some Λ^{d_i} element.
 then always have commutation

$$P(A_1, \dots, S, \dots, T, \dots, A_k) = P(A_1, \dots, T, \dots, S, \dots, A_k)$$

• (-1)^{*}
 some power.

$$\text{tr } \Omega_1^k - \text{tr } \Omega_0^k = \int_0^1 \left(\frac{d}{dt} \text{tr } \Omega_t^k \right) dt$$

claim: $\frac{d}{dt} \text{tr } \Omega_t^k = dQ$ for $Q := P(\eta, \Omega_t, \dots, \Omega_t) \cdot k$

In fact, for any inv. poly P :

$$\frac{dP(\Omega_t)}{dt} = \frac{dQ(\eta, \Omega_t)}{dt}$$

↑
transgression

$$\begin{aligned} \frac{dP(\Omega_t)}{dt} &= k P \left(\frac{d\Omega_t}{dt}, \Omega_t, \dots, \Omega_t \right) \\ &= k P(\nabla \eta, \Omega_t, \dots, \Omega_t) + k P(\eta \wedge \eta, \Omega_t, \dots, \Omega_t) \cdot 2t \end{aligned}$$

$$dQ = k \cdot P(\nabla \eta, \Omega_t, \dots, \Omega_t) + k(k-1) P(\eta, \nabla \Omega_t, \Omega_t, \dots, \Omega_t)$$

(both side are indep. of cov. then check at a special frame st $\omega(x_0) \equiv 0$ and frame of E)

Ex. do this existence.

Remark: In Gilkey p.92, he uses a trick to set $\omega_{t_0}(x_0) \equiv 0$ at a fixed time $t = t_0$ then all 2nd terms disappear! But 1st term $\rightarrow d\eta$. $\nabla \eta = d\eta - \eta \wedge \omega - \omega \wedge \eta$

$$\begin{aligned} \nabla \Omega_t &= \nabla(\Omega + t \nabla \eta - t^2 \eta \wedge \eta) \\ &= t [\eta, \Omega] - t^2 (\nabla \eta \wedge \eta - \eta \wedge \nabla \eta) = t^2 [\eta, \nabla \eta] \end{aligned}$$

$$\begin{aligned} &= t [\eta, \Omega + t \nabla \eta] \\ &= t [\eta, \Omega_t] \quad \text{since } [\eta, \eta \wedge \eta] = 0 \end{aligned}$$

Now $t P([\eta, \eta], \Omega_t, \dots, \Omega_t) + t P(\eta, [\eta, \Omega_t], \Omega_t, \dots) + \dots = 0$

$\eta \wedge \eta - (t)^1 \eta \wedge \eta$
 $2 \eta \wedge \eta$
all terms are equal
 $= (k-1) P(\eta, \nabla \Omega_t, \Omega_t, \dots, \Omega_t)$

The claim is proved. \square

So for any inv. poly. P eg. $\text{tr } \Omega^k$ or $\zeta_i(E, \nabla)$

$$\begin{aligned} P(\underline{\Omega}_1) - P(\underline{\Omega}_0) &= \int_0^1 \frac{d}{dt} P(\Omega_t) dt \\ \text{or } \Omega_1 &= \int_0^1 dQ(\eta, \Omega_t) dt \\ &= d \left(\int_0^1 Q(\eta, \Omega_t) dt \right) \quad \# \end{aligned}$$

Ex. Write out Q for $\text{tr } \Omega^2$, $\text{tr } \Omega^3$, $\zeta_1, \zeta_2, \zeta_3$ etc.

$P(A)$ inv. poly of degree 2.

$$\text{then } P(A_1, A_2) = \frac{1}{2} (P(A_1 + A_2) - P(A_1) - P(A_2))$$

for $P(A) = \text{tr}(A^2)$, get

$$P(A, B) = \frac{1}{2} (\text{tr}(A+B)^2 - \text{tr} A^2 - \text{tr} B^2)$$

$$= \frac{1}{2} (\text{tr} AB + \text{tr} BA) = \text{tr} \underline{AB}$$

$$\text{So } Q(\eta, \Omega_t) = \text{tr}(\eta \wedge \Omega_t)$$

$$= \text{tr}(\eta \wedge (\Omega + t \nabla \eta - t^2 \eta \wedge \eta))$$

$$= \text{tr} \eta \wedge \Omega + t \text{tr} \eta \wedge \nabla \eta - t^2 \text{tr} \eta \wedge \eta \wedge \eta$$

$$\int_0^1 Q(\eta, \Omega_t) dt = \text{tr}(\eta \wedge \Omega) + \frac{1}{2} \text{tr}(\eta \wedge \nabla \eta) - \frac{1}{3} \text{tr}(\eta \wedge \eta \wedge \eta)$$

important in Chern-Simons theory

- Functoriality of characteristic classes (Axism)
- Topological Aspect: Gauss Map to Grassmannian