Basics of Asymptotic series (W-W, or Wasow)
(Poincaré theory) also Stieltjes

$$E_i(x) = \int_{-\infty}^{x} e^t \frac{dt}{t} = \frac{1}{t}\, d(e^t) \quad (x<0)$$ then analytic conti to $z \in \mathbb{C}\setminus\{0\}$ with 0 a ∞-branch pt

$$= \frac{e^t}{t}\Big|_{-\infty}^{x} + \int_{-\infty}^{x} e^t \frac{dt}{t^2}$$

$$= \frac{e^x}{x}\left(1 + \frac{1}{x} + \frac{2!}{x^2} + \cdots + \frac{m!}{x^m} + \underbrace{(m+1)!\, x \int_{-\infty}^{x} e^{t-x} \frac{dt}{t^{m+2}}}_{=\,R_m(x)}\right)$$

for $|z|$ large and

Ex. $|R_m(z)| \le 2\frac{(m+1)!}{|z|^{m+1}}$ in $S = \{\delta < \arg z < 2\pi - \delta\}$

The partial sum $S_m(x)$ approx $x e^{-x} E_i(x)$ in the sense for m fixed, the error $\to 0$ in the strong sense:

$$\lim_{S \ni |z| \to \infty} |z|^m |R_m(z)| = 0$$

even though $S_m(z)$ diverges as $m \to \infty$ for any $z \neq 0$!

Definition: Let $z_0 \in \bar{S}$, $S \subset \mathbb{C}$, f defined on S. Then

$f \sim \sum_{j=0}^{\infty} a_j (z-z_0)^j$ as $z \to z_0$ in S if for any $m \ge 0$

$$\lim_{\substack{z \to z_0 \\ \text{in } S}} |z-z_0|^{-m}\left(f(z) - \sum_{j=0}^{m} a_j (z-z_0)^j\right) = 0$$ ($\Rightarrow$ $a_0, a_1, a_2, \cdots$ solved inductively, hence uniqueness of the asymp series)

Ex. (i) Algebra: If $f \sim \sum a_j z^j$, $g \sim \sum b_j z^j$ as $z \to 0$ in S then $f \pm g$, fg, f/g and $f \circ g$ (with $g(0)=0$) have asymp

(ii) if S contains $B_0^{\times}(\varepsilon)$ then $\sum a_j z^j$ conv and $= f$

Thus S is in general only a sector $\arg z \in (\theta_0, \theta_1)$ or $[\ ,\ ]$ etc.

Analysis: Integration/diff of asymp. expansions

Theorem If f holo $\sim \sum_{r=0}^{\infty} a_r z^r$ in a sector S at 0

then (a) $\int_{0^+}^{z} f(t)\,dt \sim \sum_{r=0}^{\infty} \frac{a_r}{r+1} z^{r+1}$ along any path in S

(b) $f' \sim \sum_{r=0}^{\infty} r a_r z^{r-1}$ in S^*, any proper sector of S

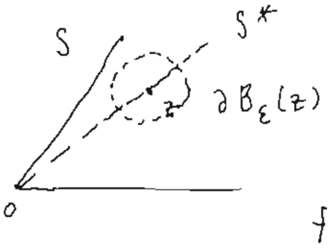
pf (a) $f(z) = S_m(z) + z^m E_m(z)$ with $\lim_{S \ni z \to 0} E_m(z) = 0$

f holo in S $\Rightarrow$ E_m holo in S

$$\Rightarrow \int_{0^+}^{z} f(t)\,dt = \int_{0^+}^{z} S_m(t)\,dt + z^{m+1}\int_0^1 E_m(sz)\, s^m\, ds$$ $\xrightarrow{z \to 0 \text{ in } S} 0$

along a straight line $t = sz$, $s \in [0,1]$ *

(b) $f'(z) = S_m'(z) + m z^{m-1} E_m(z) + z^m E_m'(z)$ holds



pick $\varepsilon = |z|\delta$ st $B_\varepsilon(z) \subset S$ $\forall z \in S^*$

Then $|E_m'(z)| \leq \frac{1}{|z|\delta} \max_{\partial B_{|z|\delta}} |E_m(z)|$

$$f'(z) = S_m'(z) + z^{m-1}\left(m E_m(z) + z E_m'(z) \right)$$

as $z \to 0$ in S $\quad \to 0$. $\square$

Rmk. Not every holo fcn in S has asymp expansion

e.g. $f(z) = z^2 \log z$ in $\arg z < \frac{\pi}{2}$ $\quad (a_0 = 0,\ a_1 = 0,$ but $\nexists\ a_2)$

Nevertheless, any power series and any S, $\exists$ holo f in S with such an asymptotic in S. (Wasow: Thm 9.3)

Asymp expan with parameter

$$f(z,t) \qquad (z,t) \in S \times T \subset \mathbb{C}^2$$

bdd sector, T = bdd domain

$$(*) \quad f(z,t) \sim \sum_{j=0}^{\infty} a_j(t) z^j \quad \text{as } z \to 0 \text{ in } S, \ \forall t \in T$$

Easy to see $(*)$ is uniform in $T \Leftrightarrow (\star)$ $f(z,t) = S_m + z^{m+1} E_m(z,t)$

Lemma with E_m bounded on $S \times T$ $\forall m$

Corollary If f holo on $S \times T$ and has unif asymp as $(*)$ then $a_j(t)$ is holo in T and $\frac{\partial f}{\partial t} \sim \sum_{j=0}^{\infty} \frac{\partial a_j(t)}{\partial t} z^j$

unif in cpt proper $T' \subset T$

Pf. $a_0(t) = \lim_{S \ni z \to 0} f(z,t)$ is a uniform limit, hence hol

inductively, $a_m(t) = \lim_{S \ni z \to 0} z^{-m}\left(f(z,t) - \sum_{j=0}^{m-1} a_j(t) z^j \right)$

is a unif limit of holo fcn, hence holo

Now in $(\star)$, E_m is holo in $t \in T$ (also in z)

$$\Rightarrow \quad \frac{\partial f}{\partial t} = \frac{\partial S_m}{\partial t} + z^{m+1} \frac{\partial E_m}{\partial t}$$

Lemma $\Rightarrow$ $|E_m| \leq M_m$ on $S \times T$. Let $r < d(T', \partial T)$

Then Cauchy $\Rightarrow$ $|\partial E_m / \partial t| \leq \frac{1}{r} M_m$ on $S \times T'$, done $\square$

Ex. Corollary: Suppose $(*)$ is unif on $T \supset B_\varepsilon(t_0)$

then the Taylor coefficients on both sides are $\sim$ as $z \to 0$ in S

Thm: For any formal series $p = \sum_{j=0}^{\infty} a_j(t) z^j$ with $a_j(t)$ holo in $T = \overline{B_\varepsilon}$, and S any sector, $\exists$ f hol in $S \times T$ $\sim p(z,t)$ unif in T

Introduction to ODE with irreg sing. pt ⟨ Birkhoff Theory ⟩

$h \geq 2$, $y' = \frac{a(z)}{z^h} y$ a holo in z, $a(0) \neq 0$ $a \in M_{n\times n}(U, \mathcal{O})$

or an "asymptotic expansion".

The case $n=1$. Scalar equation (i.e. a_0 $n\times n$ scalar matrix)

$$y_0' = \frac{a_0}{z^h} y_0 \Rightarrow y_0 = \tilde{c}\, e^{\frac{-a_0}{h-1}\frac{1}{z^{h-1}}}$$

Hence
$$y' = \frac{1}{z^h}\left(a_0 + a_1 z + \cdots + a_{h-2} z^{h-2} + a_{h-1} z^{h-1} + \sum_{k=h}^{\infty} a_k z^k\right) y$$

$$= \left(I_{rr} + \frac{a_{h-1}}{z} + b(z)\right) y \quad (*)$$

If $b(z) \equiv 0$, has sol. $y_{h-1}(z) = \tilde{C}_{h-1}\, e^{-\left(\frac{a_0}{h-1}\frac{1}{z^{h-1}} + \cdots + a_{h-2}\frac{1}{z}\right) + a_{h-1}\log z}$?

see * below

Expect to have a solution like
$$y(z) = c(z)\cdot z^{a_{h-1}} \cdot e^{-\sum_{k=1}^{h-1}\frac{a_{h-1-k}}{k} z^{-k}}$$

with $c(z) = c_0 + c_1 z + c_2 z^2 + \cdots$

Main Question: holo, or at least "asymp expan", of some actual function?

*Observation: The only requirement for the calculations to be valid for $n\times n$ matrix is the commutativity, e.g. $a_0, \cdots, a_{h-1}$ diagonal

$$y' = c' y_{h-1} + c\, y_{h-1}' = c' y_{h-1} + c\left(I_{rr} + \frac{a_{h-1}}{z}\right) y_{h-1}$$

$$= c' c^{-1} y + c\left(I_{rr} + \frac{a_{h-1}}{z}\right) c^{-1} y$$ can be further simplified via (*) ~~only if commutativity holds~~ !

in that case, say for $n=1$ case, get

$c'(z) = b(z)\, c(z)$ If $b(z)$ holo, get a regular eq'n at $z=0$.

$\Rightarrow$ sol $c(z)$ holo. If b is only asymp then c is too.

(via $\int$ and take exp).

Quiz How to make sense of

$$c(z)\cdot y_{h-1}(z) = \underbrace{\left(\sum_{j=0}^{\infty} c_j z^j\right)}_{\text{positive } z \text{ power}} \cdot z^{a_{h-1}} \cdot \underbrace{e^{-\sum_{k=1}^{h-1}\frac{a_{h-1-k}}{k} z^{-k}}}_{\text{negative } z \text{ power}} \; ?$$

This is an actual function

A basic example: $x^3 u'' + (x^2 + x) u' - u = 0$

initial equation: set $u = x^\lambda$ get

$$\lambda(\lambda-1) x^{\lambda+1} + \lambda(x^2+x) x^{\lambda-1} - x^\lambda \rightsquigarrow (\lambda-1) x^\lambda = 0$$

So should start with $\lambda = 1$:

$u = \sum_{j=1}^{\infty} a_j x^j$ then

x^j $(j\geq 1)$: $+a_j j x^j - a_j x^j$ $\Rightarrow$ Recursive relation :

$a_{j-1}(j-1)(j-2)\, x^j + a_{j-1}(j-1)\, x^j$ $\qquad (j-1)a_j = -(j-1)^2 a_{j-1}$

$\Rightarrow$ $a_0 = 0$, a_1 is free, say $=1$ $\qquad a_j = -(j-1)a_{j-1} = \cdots = (-1)^{j-1}(j-1)!$

i.e. $u = x - x^2 + 2!\,x^3 - 3!\,x^4 + \cdots$

divergent $\forall\, x \neq 0$. Is this of any use ?!

In general, is that true $u = x^\lambda g_1(x)$, g_1 a "formal series" ?
(as in reg sing)

Answer: NO Need also exponential corrections

In matrix form: $\begin{pmatrix} u \\ u' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \frac{1}{x^3} & -\left(\frac{1}{x}+\frac{1}{x^2}\right) \end{pmatrix}\begin{pmatrix} u \\ u' \end{pmatrix} = \frac{1}{x^3}\begin{pmatrix} 0 & x^3 \\ 1 & -(x^2+x) \end{pmatrix}\begin{pmatrix} u \\ u' \end{pmatrix}$

$u'' = \frac{u}{x^3} - \left(\frac{1}{x}+\frac{1}{x^2}\right)u'$ No canonical way to write this indeed !

Except the reg. sing. case use $(u, xu')^t$. φ: From $n=1$ to $n\in\mathbb{N}$??

Basic Idea: $\exists$ sectors $\{\Omega_i\}$ cover a nbd of 0, so that:

in each Ω_i, u is an asymp. expan. of an actual sol.

In matrix case, $\begin{pmatrix} u \\ u' \end{pmatrix}$ " of the fund. sol. matrix in Ω_i.
which is holo

Set $y = PZ$ $\quad P$ hol $n\times n$ $\quad \det P(0)\neq 0$ (change z to x)

$y' = P'Z + PZ' = \frac{a(x)}{x^h}PZ \Rightarrow Z' = \frac{b(x)}{x^h}Z$, $Pb := aP - x^h P'$

Set $P = \sum_0^\infty P_j x^j$, $b = \sum_0^\infty b_j x^j$ $\qquad$ From $a \sim \sum_0^\infty a_j x^j$ get
change index !

$$\begin{cases} P_0 b_0 - a_0 P_0 = 0 \\ P_k b_0 - a_0 P_k = \sum_{j=0}^{k-1}\left(a_{k-j}P_j - P_j b_{k-j}\right) - (h-k+1)P_{h-k+1} \qquad (*) \end{cases}$$

Now assume a_0 has 2 sets of eigenvalues $\lambda_1,\dots,\lambda_p$; $\lambda_{p+1},\dots,\lambda_n$

s.t. $\lambda_i \neq \lambda_j$ for all $1\leq i\leq p$; $p+1\leq j\leq n$. Linear algebra $\Rightarrow$

May assume $a_0 = \begin{pmatrix} a_0^{11} & 0 \\ 0 & a_0^{22} \end{pmatrix}$ in block-wise form (via P_0)

Now set $P_0 = I$, $b_0 = a_0$, get $(*)'$: $[P_k, a_0] = -b_k + H_k$

Since $h > 1$, $\Rightarrow$ H_k dep only on P_j, b_j with $j<k$

$(*)'$ is now a singular linear eq'n in P_k !

Rmk: The case $h=1$ (reg. sing case) get $k P_k$ term $\Rightarrow$ non-sing.

and then all P_k, b_k can be solved uniquely and P, b conv.

Ansatz: $(h>1)$ Set for $k \geq 1$ $\quad b_k = \begin{pmatrix} b_k^{11} & 0 \\ 0 & b_k^{22} \end{pmatrix}$, $\quad p_k = \begin{pmatrix} 0 & p_k^{12} \\ p_k^{21} & 0 \end{pmatrix}$

Then $(*)'$ becomes
$$\begin{pmatrix} 0 & p_k^{12} a_0^{22} - a_0^{11} p_k^{12} \\ p_k^{21} a_0^{11} - a_0^{22} p_k^{21} & 0 \end{pmatrix} = \begin{pmatrix} -b_k^{11} + H_k^{11} & H_k^{12} \\ H_k^{21} & -b_k^{22} + H_k^{22} \end{pmatrix}$$

Since a_0^{11}, a_0^{22} has no common eigenvalues, p_k^{12} and p_k^{21} are uniquely solved, also $b_k^{11} = H_k^{11}$, $b_k^{22} = H_k^{22}$ is forced.

Theorem Up to possibly formal change $y = Pz$ may assume a is blockwise. In particular, if a has distinct eigenvalues, then may assume a diagonal.

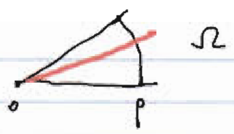
Rmk: If p is a convergent series, then the problem for distinct eigenvalues is reduced to the $n=1$ case, which is done. However, p is in general divergent.

Cor: The formal fund sol matrix $Y_f = PCe^{\Lambda(z)}$ for distinct eigen val. is $P_0\left(\sum_{j=0}^{\infty} \psi_j z^j\right) e^{\Lambda(z)}$, with $\psi_0 = I$, $\Lambda(z) = -\sum_{k=1}^{h-1} \frac{b_{h-1-k}}{k} \frac{1}{z^k} + b_{h-1} \log z$ is a diagonal matrix

$b_0 = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$

Asymptotic structure: Sector $\Omega = \{z \mid 0 < |z| < \rho,\ \arg z \in (\theta_1, \theta_2)\}$

If holo fund sol matrix Y exists on Ω with Y_f being asymp of Y in Ω as $z \to 0$.



For another such $\tilde{Y}$, must have $S := Y^{-1}\tilde{Y}$ being constant matrix

$$\Rightarrow S = \lim_{z \to 0} e^{-\Lambda(z)} (I + O(z)) e^{\Lambda(z)}$$

i.e. $S_{ij} = \lim_{z \to 0} e^{\Lambda(z)_{jj} - \Lambda(z)_{ii}} (\delta_{ij} + O(z)) \Rightarrow S_{ii} = 1 \quad \forall i$

To get $S = I$, we must assume that $\forall\, i<j$, $\Omega \supset \ell = (\mathrm{Re}\,(\lambda_i - \lambda_j) z^{-(h-1)} = 0)$ since we can make $\mathrm{Re}(\Lambda(z)_{jj} - \Lambda(z)_{ii}) \to -\infty \Rightarrow S_{ij} = 0$.

Let $r = h-1$ (Poincaré rank), $\exists\ 2r$ such rays ℓ's $\forall\, i<j$ called the Stokes rays.

Def^n: Ω is a Stokes sector if Ω contains exactly one Stokes ray for each $i<j$.

Ex: $\Omega = \angle(\theta - \delta, \theta + \frac{2\pi}{2r})$ with δ small is a Stokes sector, others are $\Omega_j = e^{\frac{\pi i}{r}(j-1)}\Omega_1$; $j = 1, \cdots, 2r$. Denote $\Omega_{2r+1} = \Omega_1$. Then $\Omega_j \cap \Omega_{j+1}$ contains no ray, $\Rightarrow S_j := Y_j^{-1} Y_{j+1}$ is triangular (after reordering)

Def^{n}. S_1 is called the Stokes matrix, hence $S_1 \cdots S_{2n} = T$ (monodromy)

Stokes data (phenomenon): $Sph = \{ b_0, \dots, b_r; S_1, \dots S_{2r} \}$

Ex. Two ODE are equivalent if they have the same Sph

[Hint: $y' = Ay$, $\tilde{y}' = \tilde{A}\tilde{y}$, let $g(z) := \tilde{y}_1 y_1^{-1}$ on Ω_1. Analytic conti g to $\Omega_2, \dots, \Omega_{2r}$ and show it is holo, bdd on $B_\rho^\times(0)$.]

Final Remarks and Statement for the General Case :

may assume a has only one λ, and $\lambda = 0$.

$y = y_0 e^{-\lambda \frac{1}{r} \frac{1}{z^r}} \Rightarrow \frac{a}{z^{r+1}} y = y' = y_0' e^{-\lambda \frac{1}{r}\frac{1}{z^r}} + \frac{\lambda}{z^{r+1}} y_0 e^{-\lambda \frac{1}{r}\frac{1}{z^r}}$

$\Rightarrow y_0' = \frac{1}{z^{r+1}} (a - \lambda I) y_0'$

Also set a in Jordan form, with at one block of dim ≥ 2.

Linear Algebra If H, K are shifting matrices of dim h, k. let

$$M = h \begin{pmatrix} \overbrace{\boxed{\text{const}}}^{k} \\ \alpha_1 \ \cdots, \alpha_k \end{pmatrix} \leftarrow \text{any given}$$

Then $\exists ! \ (\alpha_1, \dots, \alpha_k)$ st $PK - HP = M$ is solvable with $P \in M_{h \times k}$.

The equation $[P_k, a_0] = -b_k + H_k$ can be solved by further partition of b_k, H_k wrt $a_0 = h_1 \oplus \cdots \oplus h_s$ into s^2 eq^{n}s

$P_k^{ij} h_j - h_i P_k^{ij} = -b_k^{ij} + H_k^{ij}$ simply set $\boxed{\text{const}} = 0$

dep only on lower P, b in k.

$\Rightarrow$ P_f is solved as a formal series

which is also the asymp. expansion of a holo P_Ω in sector Ω.

$y = P y_0 \Rightarrow y_0' = \frac{b}{z^{r+1}} y_0$ st $b \sim b_f = \sum_{j=0}^{\infty} b_j z^j$

$b_0 = h_1 \oplus \cdots \oplus h_s$ (shifting)

$j \geq 1 \Rightarrow b_j$ has entries $= 0$ except last rows corr to h_k's.

Shearing Transformation : $y_0 = S y_1$, $S(z) = \begin{pmatrix} 1 & & 0 \\ & z^g & \\ 0 & \ddots & z^{(n-1)g} \end{pmatrix}$

will finally bring the eq^n to the case of distinct eigenvalues, in $t = z^{1/p}$, $p \in \mathbb{N}$.

All our discussion also works for asymp for $s = 1/z \to \infty$

Ex Solve $\frac{d^2u}{ds^2} = su$ near ∞, ie. $y' = \begin{pmatrix} 0 & 1 \\ s & 0 \end{pmatrix} y$ for $y = \begin{pmatrix} u \\ u' \end{pmatrix}$

This is the Airy function $u = Ai(s)$. (cf. Stein app.)

Eg 1 $\quad y' = \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} y \qquad$ Set $\quad y = \begin{pmatrix} 1 & 0 \\ 0 & x^{\delta} \end{pmatrix} y_1 , \qquad y' = \begin{pmatrix} 0 & 0 \\ 0 & \delta x^{\delta-1} \end{pmatrix} y_1 + \begin{pmatrix} 1 & 0 \\ 0 & x^{\delta} \end{pmatrix} y_1'$

Airy equation, Poincaré rank $= \delta+1 < 2$. $\qquad \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & x^{\delta} \end{pmatrix} y_1 = \begin{pmatrix} 0 & x^{\delta} \\ x & 0 \end{pmatrix} y_1$

$$\begin{pmatrix} 1 & 0 \\ 0 & x^{\delta} \end{pmatrix} y_1' = \begin{pmatrix} 0 & x^{\delta} \\ x & -\delta x^{\delta-1} \end{pmatrix} y_1$$

$t = x^{\delta}$

$$y_1' = \begin{pmatrix} 1 & 0 \\ 0 & x^{-\delta} \end{pmatrix}\begin{pmatrix} 0 & x^{\delta} \\ x & -\delta x^{\delta-1} \end{pmatrix} y_1 = \begin{pmatrix} 0 & x^{\delta} \\ x^{1-\delta} & -\delta x^{-1} \end{pmatrix} y_1 \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial t}\, \delta x^{\delta-1}$$

$$\Rightarrow \quad \frac{\partial y_1}{\partial t} = \frac{1}{\delta}\begin{pmatrix} 0 & x \\ x^{2-2\delta} & -\delta x^{-\delta} \end{pmatrix} y_1 \qquad \text{let } \delta = \tfrac{1}{2},\ x = t^2$$

$$= 2\begin{pmatrix} 0 & t^2 \\ t^2 & -\frac{1}{2} t^{-1} \end{pmatrix} y_1$$

$$= \left[2t^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - t^{-1}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] y_1$$

$$= t^2 \left[\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} - t^{-3}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] y_1$$

$\lambda^2 - 4 = 0 \quad \lambda = \pm 2$ dist. eigenvalues

> $z = \frac{1}{w} \qquad \frac{\partial}{\partial w} = \frac{-1}{w^2}\frac{\partial}{\partial z}$
>
> Corr. changes between $0 \leftrightarrow \infty$
>
> $y' = \frac{a(z)}{z^h} y$
>
> $-w^2 \frac{dy}{dw} = w^h\, a(1/w)\, y$
>
> $\frac{dy}{dw} = -w^{h-2}\, a(1/w)\, y$
>
> rank $= h-1 = \delta+1$

$\lambda = 2, \quad \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} v = 0 \ \rightarrow\ v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad P_0 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \ \Rightarrow\ P_0^{-1}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} P_0 = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

$\lambda = -2, \quad \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} v = 0 \ \rightarrow\ v = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad P_0^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad P_0^{-1}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} P_0 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\Rightarrow$ for $y_1 = P_0 y_2$, get:

$$\frac{\partial y_2}{\partial t} = t^2\left[\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} - \frac{1}{2} t^{-3}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right] y_2 \qquad \text{rank} = \delta+1 = 3\ !$$

Ex (Laplace Ansatz)

Choose Γ, v in $y = \int_\Gamma v\, e^{zt}\, dt$ to solve $y'' = zy$ ($\exists$ 3 choices of Γ at ∞.)

This can be reduced to 1 variable case by finding P, b

Asymp expansion requires other methods.

Eg 2 Confluence of ODE **From reg sing to irregular sing pts**

Example on $E = \mathbb{C}/\Lambda$

$$y'' = \big(A(\wp(z-(a+h)) + \wp(z-a)) + B(\zeta(z-(a+h)) - \zeta(z-a)) + c \big)\, y$$

Q: $h \to 0$? Set $B = B_1/h$ get $2A\wp(z-a) - B_1 \partial(z-a)$ still regular

Another way: $\quad y'' = \big(A(\wp(z-(a+h)) - \wp(z-a)) + c \big)\, y \quad$ sign changes

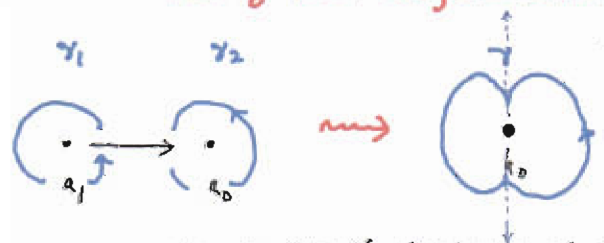
Now need scaling $\longrightarrow$ set $A = A_1/h$ get $A_1 \wp'(z-a)$ irreg sing pt

[Rmk: On $\mathbb{CP}^1$ the most basic example is HG to CHG eq'n (cf. Whittaker-Watson)

reg sing pt at $0, a, \infty \quad \lim_{a \to \infty} \rightarrow \underset{\text{reg}}{0} \quad \underset{\text{irreg}}{\infty}$

To conclude this course on complex analysis II, where we emphasize "Analytic Continuations / monodromy" (either via study on global "periods integrals" or local "monodromy / Stokes' phenomenon" at sing pts), we propose the following research problem

Problem: Study the monodromy effects vs. Stokes' phenomenon during the degeneration / confluence procedure



- Is this the origin how Stokes' rays appear ?
- When we collapse more points, do we get higher rank, and more rays ?

- Compare with H. Weyl's treatment of 3rd → 2nd diff

Dragon 5/23, 2015