Riemann Theta functions

$\Omega \in M_{g\times g}(\mathbb{C})$, $\Omega^t = \Omega$ with $\operatorname{Im}\Omega > 0$ $\quad \Omega \in \mathcal{H}_g$

Siegal Upper Half Space

$$\vartheta(z,\Omega) := \sum_{n\in\mathbb{Z}^g} e^{\pi i n^t \Omega n} e^{2\pi i n^t z}, \quad z \in \mathbb{C}^g \quad \text{"even function"}$$

Ex Converges exponentially, hence holo on $\mathbb{C}^g \times \mathcal{H}_g$

Lattice $\Lambda = \Lambda_\Omega := \mathbb{Z}^g + \Omega\mathbb{Z}^g$

- $\vartheta(z+m,\Omega) = \vartheta(z,\Omega) \quad \forall\, m \in \mathbb{Z}^g$
- $\vartheta(z+\Omega m,\Omega) = \sum_{n\in\mathbb{Z}^g} e^{\pi i n^t \Omega n + 2\pi i n^t \Omega m} e^{2\pi i n^t z}$

sym

$$e^{\pi i n^t \Omega n + 2\pi i n^t \Omega m} = e^{\pi i (n+m)^t \Omega (n+m) - \pi i m^t \Omega m}$$

$$= e^{-\pi i m^t \Omega m - 2\pi i m^t z}\, \vartheta(z,\Omega) \quad \forall\, m \in \mathbb{Z}^g$$

ϑ is Λ_Ω quasi-periodic (of wt $\ell = 1$)

Def'n $f \in R^\Omega_\ell$ is Λ_Ω-q p of wt $= \ell$ if $f(z+m) = f(z)$ &

Supplementary on translates of theta functions

$$f(z+\Omega m) = e^{\ell(-\pi i m^t \Omega m - 2\pi i m^t z)} f(z) \quad \forall\, m \in \mathbb{Z}^g$$

To write out a basis of R^Ω_ℓ, we use for $a, b \in \mathbb{Q}^g$:

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z,\Omega) := e^{\pi i a^t \Omega b + 2\pi i a^t (z+b)}\, \vartheta(z+\Omega a + b, \Omega)$$

$$= \sum_{n\in\mathbb{Z}^g} e^{\pi i (n+a)^t \Omega (n+a)} e^{2\pi i (n+a)^t (z+b)}$$

$\Rightarrow$

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z+m,\Omega) = \underline{e^{2\pi i a^t m}}\, \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z,\Omega)$$

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z+\Omega m,\Omega) = \underline{e^{-2\pi i b^t m}}\, e^{-\pi i m^t \Omega m - 2\pi i m^t z}\, \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z,\Omega)$$

Exercise $\dim R^\Omega_\ell = \ell^g$. A basis is given by

$$f_a(z) := \vartheta\begin{bmatrix} a/\ell \\ 0 \end{bmatrix}(\ell z, \ell\Omega), \quad a = (a_i)_1^g \quad 0 \le a_i < \ell \quad \text{or}$$

$$g_b(z) := \vartheta\begin{bmatrix} 0 \\ b/\ell \end{bmatrix}(z, \ell^{-1}\Omega) \quad b = (b_i)_1^g \quad 0 \le b_i < \ell \quad \text{or}$$

$$h_{ab}(z) := \vartheta\begin{bmatrix} a/k \\ b/k \end{bmatrix}(k^2 z, \Omega), \quad \text{if } \ell = k^2, \quad 0 \le a_i, b_i < k$$

Remark: Lefschetz proved $\mathbb{C}^g/\Lambda_\Omega \hookrightarrow \mathbb{P}^N$ using these functions

GOAL: Use ϑ to construct all meromorphic functions on a cpt Riem surface under $X \hookrightarrow \mathbb{C}^g/\Lambda_\Omega$

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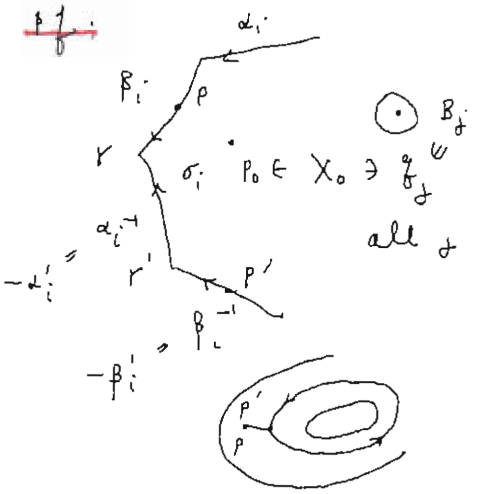
Explicit Inversion by ϑ

Let $z \in \mathbb{C}^g$

$f(p) := \vartheta\left(z + \int_{p_0}^{p} \vec{\omega}, \Omega\right)$ locally single-valued on $p \in X$

Theorem (Riemann) Either $f \equiv 0$, or f has zero divisor $D = g_1 + \cdots + g_g$. In that case, $\varphi(D) + z \equiv \Delta \mod \Lambda_\Omega$ where $\Delta \in \mathbb{C}^g$ is independent of z

pf.



$$0 = \int_{X_0 \setminus \cup B_j} d\left(\frac{df}{f}\right) = \int_{\partial(X_0 \setminus \cup B_j)} \frac{df}{f}$$

$$= \sum_{i=1}^{g} \int_{\alpha_i - \alpha_i' + \underbrace{\beta_i - \beta_i'}} \frac{df}{f} - \sum_j \int_{\partial B_j} \frac{df}{f}$$

difference is a α_i^{-1} shift under analytic conti

α_i : $f \longmapsto f$ inv under α_i perio

β_i : $f\, e^{-\pi i \Omega_{ii} - 2\pi i\left(\int_{p_0}^{p} \omega_i + z_i\right)}$ under β_i

ie under β_i $\quad \frac{df}{f} \longmapsto \frac{df}{f} - 2\pi i\, \omega_i$

so the 1st integral $= \sum_{i=1}^{g} 2\pi\sqrt{-1} \int_{\alpha_i} \omega_i = 2\pi\sqrt{-1}\, g$

2nd integral $= 0$ $\quad \Rightarrow f$ has g zeros (with mult.)

Now write $\omega_k = d g_k$ on X_0 with $g_k(p_0) = 0$

Repeat the argument for $g_k \frac{df}{f}$:

$$0 = \int_{X_0 \setminus \cup B_j} d\left(g_k \frac{df}{f}\right) = \sum_{i=1}^{g} \int_{\alpha_i - \alpha_i' + \beta_i - \beta_i'} g_k \frac{df}{f} - \sum_{\ell=1}^{g} \int_{\partial B_\ell} g_k \frac{df}{f}$$

The last term $= -2\pi\sqrt{-1} \cdot g_k(g_j) = -2\pi\sqrt{-1} \int_{p_0}^{g_j} \omega_k$

$$g_k\big|_{\beta_i} = g_k\big|_{\beta_i'} + \int_{\alpha_i} \omega_k = g_k\big|_{\beta_i'} + \delta_{ki}$$

$$\Rightarrow \int_{\beta_i - \beta_i'} g_k \frac{df}{f} = \delta_{ki} \int_{\beta_i} \frac{df}{f} = \delta_{ki}\left(-\pi\sqrt{-1}\, \Omega_{ii} - 2\pi\sqrt{-1}\left(\int_{p_0}^{p_i} \omega_k + z_i\right)\right)$$

here $p_i = \alpha_i \cap \beta_i$ in X_0 ; change all of them to say p_1 will diff only by $2\pi\sqrt{-1}\, m_i$, $m_i \in \mathbb{Z}$ *

Ex. Work out the remaining part

Let $\mathcal{E} \subset J(X)$ be the loci of z st $\vartheta\left(z + \int_{P_0}^{P} \vec{\omega}\right) \equiv 0$ $\quad U = \mathcal{E}^c$

Corollary Let $D = \sum_{i=1}^{g} P_i$, and let $\varphi(D - g P_0) \equiv -z + \Delta$, Then $\vartheta\left(z + \int_{P_0}^{P_i} \vec{\omega}\right) = 0 \quad \forall i$ Thus if $z \in U$ then z det D uniquely (Ex)
(solution in p 41)

$E_e(x,y) := \vartheta\left(e + \int_x^y \vec{\omega}\right)$ Riemann's prime form, where $\vartheta(e) = 0$
st $E_e(x,y) \not\equiv 0$ $\quad$ why $\exists e$? Ans. $\dim \mathcal{E} \leq g-2$ (P.41)

Claim: $\exists\ r_1, \ldots r_{g-1}, s_1, \ldots s_{g-1}$ st $E_e(x,y) = 0 \iff$
$x = y$ or $x = r_i \ (\forall y)$ or $y = s_i \ (\forall x)$

pf choose R st $E_e(R, y) \not\equiv 0$, then $\exists 1 \ \sum_{i=1}^{g} s_i$ zero div
set $s_g = R$ $\quad (s_1, \ldots s_g)$ is characterized uniquely by

$$\sum_{i=1}^{g} \int_{P_0}^{s_i} \vec{\omega} \equiv -\left(e - \int_{P_0}^{R} \vec{\omega}\right) + \Delta \mod \Lambda_\Omega$$

ie $$\sum_{i=1}^{g-1} \int_{P_0}^{s_i} \vec{\omega} \equiv -e + \Delta \mod \Lambda_\Omega$$

Notice that : $s_1 + \ldots + s_{g-1}$ dep only on e, NOT on R !
ie the dependency only lies in $s_g = R$.
on R

There are only finite number of R's st $E_e(R, y) \equiv 0$
for $s_0 \neq s_1, \ldots, s_{g-1}$, $E_e(x, s_0) = 0$ iff $x = s_0$ or
x is a value R st $E_e(x,y) \equiv 0$ $(\forall y)$
So $\vartheta\left(e + \int_x^{s_0} \vec{\omega}\right) = \vartheta\left(-e + \int_{s_0}^{x} \vec{\omega}\right)$ has g zeros (ϑ is even
$x = r_1, \ldots, r_{g-1}$ and $r_g = s_0$ $\quad$ This proves the claim $\square$

"Explicit Construction" of mero function f with
Ansatz : $$(f) = \sum_{i=1}^{d} q_i - \sum_{i=1}^{d} P_i$$
choose $e \in \mathbb{C}^g$ st $\vartheta(e) = 0$, $E_e(P_i, y) \not\equiv 0$, $E_e(q_i, y) \not\equiv 0$

Set $$f(y) := \prod_{i=1}^{d} \frac{E_e(q_i, y)}{E_e(P_i, y)} = \prod_{i=1}^{d} \frac{\vartheta\left(e + \int_{q_i}^{y} \vec{\omega}\right)}{\vartheta\left(e + \int_{P_i}^{y} \vec{\omega}\right)}$$

since $\varphi(\sum q_i - \sum P_i) \in \Lambda_\Omega$, may adjust "a path" to make it $= 0$
Then f is a well-defined single valued function f gives the solution since the extra zeros $s_1, \ldots, s_{g-1}$ cancel out !

The remaing pf requires some notions in Alg geom

It remains to (i) understand the choice of e (why $\exists$)

(ii) and even the pf of the earlier Cor

For (ii): Consider $I_g : \mathrm{Sym}^g X \to J(X)$ by $\Phi(D - gP_0)$:

[Sol. to Ex] i.e. $(P_1, \cdots, P_g) \mapsto \sum_{i=1}^{g} \int_{P_0}^{P_i} \vec{\omega}$

$$\begin{tikzcd} & W \subset \mathrm{Sym}^g X \times J(X) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathrm{Sym}^g X \text{ smooth} & & J(X) \end{tikzcd}$$

$\ni (\Sigma P_i, z)$ s.t.

(a): $\Sigma \int_{P_0}^{P_i} \vec{\omega} \equiv -z + \Delta$ &

(b): $\vartheta\left(z + \int_{P_0}^{P} \vec{\omega}\right)$ has ΣP_i as zeros

The Q is whether (a) $\Rightarrow$ (b)?

Since $p_2^{-1}(U) \xrightarrow{\sim} U$ dense in $J(X)$, $\Rightarrow$ p_2 surj & $\dim W \geq g$

Now p_1 is clearly inj on W, $\underline{\dim W \geq g}$ $\Rightarrow$ $\underline{p_1 \text{ surj}}$, hence $\simeq$

ie ΣP_i determines (a), and (b) holds automatically

while diff ΣP_i may give same z, but for $z \in U$

we get $U \simeq p_2^{-1}(U) \simeq p_1 p_2^{-1}(U)$ $\square$

(i) will follow from (ii) via "more alg geom"

Lemma: for any $p \in X$, the analytic subset

$D_p := \{ e \in \mathbb{C}^g/\Lambda_\Omega : \vartheta(e + \int_p^y \vec{\omega}) \equiv 0 \}$ has codim ≥ 2

pf. Let $D \subset D_p$ be an irred comp

$X_p \subset J(X)$ the locus $\{ \int_p^y \vec{\omega} : y \in X \}$

Then $D \subset D + X_p \subset (\vartheta)_0$, the "theta divisor"

$\Rightarrow \dim(D + X_p) \leq g - 1$. (Also $D + X_p$ is irred.)

If $\dim D = g-1$, then

$D = D + X_p = \cdots = D + X_p + X_p + \cdots + X_p$

but I_g is surjective by (ii), get $\times$ Hence $\dim D \leq g - 2$ $\square$

Remarks:

(a) The theta div $\Theta := (\vartheta)_0$ is very important further study on it leads to the Torelli's thm

$X_1 \simeq X_2 \iff J(X_1) \simeq J(X_2)$ as complex tori

Ex. In fact, $\Theta = \Delta - I_{g-1}(\mathrm{Sym}^{g-1} X)$

(b) The actual choice of e can be more explicit (Mumford