Abel-Jacobi Map 5/19

for any complex basis of $\Gamma(X, \Omega^1_X)$, $\eta_1, \ldots, \eta_g$, and $p_0 \in X$,

$$p \in X \longmapsto \int_{p_0}^{p} \vec{\eta} = \left[\int_{p_0}^{p} \eta_j \right]_{j=1}^{g} \in \mathbb{C}^g$$

ambiguity given by periods $\int_\alpha \vec{\eta} \in \mathbb{C}^g$, form Λ lattice

full rank. otherwise $\exists\, \alpha$ $\int_\alpha \eta_j = 0$ $\forall j \Rightarrow \int_\alpha \bar{\eta}_j = 0$ too $\times$

The map $X \xrightarrow{\varphi} \mathbb{C}^g/\Lambda =: \mathrm{Jac}(X)$ extends to any $D \in \mathrm{Div}(X)$

$$D = \sum_{i=1}^{r} n_i P_i \Rightarrow \varphi(D) := \sum_{i=1}^{r} n_i \int_{P_0}^{P_i} \vec{\eta} \quad \text{an additive map}$$

The full Abel-Jacobi Theory asserts that $\varphi : \underset{\mathrm{Div}^0(X)/\sim}{\mathrm{Pic}^0(X)} \xrightarrow{\sim} \mathrm{Jac}(X)$.

and need only $r = g$

Rmk. Fix a symp basis $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ and $\int_{\alpha_i} \eta_j = \delta_{ij}$

then $\mathrm{Re}\, \eta_j$, $\mathrm{Im}\, \eta_j$ is a basis of har forms

Thm (Riemann): $\int_{\beta_i} \eta_j$ is Symm with Imaginary part positive def

Remark

in fact, if we choose $\mathrm{Re}\,\eta_j = dU'_{\beta_j}$ then $\int_{\alpha_i} \mathrm{Re}\,\eta_j = \delta_{ij}$

but $\mathrm{Im}\,\eta_j = dV'_{\beta_j}$ gives $\int_{\alpha_i} dV'_{\beta_j} = \int dU'_{\alpha_i} \wedge dV'_{\beta_j}$ not zero

In any case, $\int_{\alpha_i} d\omega'_{\beta_j} \left(= \lambda_{ij} = \delta_{ij} + \sqrt{-1}\, h_{ij} \right)$ is invertible $/\mathbb{C}$

Write γ_j the basis $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$, then $(\alpha_i, \beta_j) \sim \left[\begin{array}{c|c} 0 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \\ \hline \begin{matrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{matrix} & 0 \end{array}\right]$

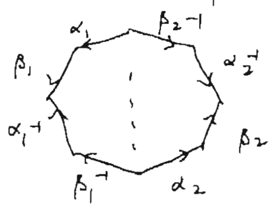
Let $\eta_j = \sum_k \lambda^{jk} d\omega'_{\beta_k}$

then $\int_{\alpha_i} \eta_j = \int_{\alpha_i} \lambda^{jk} d\omega'_{\beta_k} = \sum_k \lambda^{jk} \lambda_{ik} = \delta_{ij}$

$$\int_{\beta_i} \eta_j = \sum_k \lambda^{jk} \int_{\beta_i} d\omega'_{\beta_k} = \sum_k \lambda^{jk} b_k$$

Q: Sym in i, j? pos def in Im part?

This is standard using polygon development



But H. Weyl said that the major advantage of his method is to avoid it!

Ex. Prove Riemann's thm (bi-linear relations) from Weyl's basis

<u>Abel's thm</u>: A divisor D is principal iff

$\deg D = 0$ and $\varphi(D) \in \Lambda$ $\quad (\exists\ \varphi : \mathrm{Pic}^0(X) \hookrightarrow J(X))$

<u>pf</u>: $\Rightarrow$ let $(f) = D$, then $f : X \longrightarrow \mathbb{P}$

the divisor $f^{-1}(t)$ is defined for $t \in \mathbb{P}^1$

hence $t \mapsto \varphi\left(f^{-1}(t)\right)$ is a holo map $\mathbb{P}^1 \to J(X) \cong \mathbb{C}^g/\Lambda$

but $\mathbb{P}$ is simply connected

$$\begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & J(X) \\ & \searrow^{\psi} & \uparrow \\ & & \mathbb{C}^g \end{array}$$

$\Rightarrow$ the map lifts to $\psi : \mathbb{P}^1 \to \mathbb{C}^g$

hence is constant ie $\psi(f^{-1}(0)) = \varphi(f^{-1}(\infty))$

ie $\varphi(D) = \varphi\left(f^{-1}(0) - f^{-1}(\infty)\right) = 0$ in $J(X)$ #

$\Leftarrow$: Write $D = \sum_{j=1}^{k} a_j p_j - \sum_{j=1}^{r} b_j q_j$; $\sum a_i = \sum b_j$

To construct the desired function, set (~~this is~~ Weyl's mult functions)

$$f(p) = \exp 2\pi\left(\sum_{j=1}^{k} a_j \int_{q_0}^{p} d\omega_{p_0 p_j} - \sum_{j=1}^{r} b_j \int_{q_0}^{p} d\omega_{p_0 q_j}\right)$$

$=: \exp \int_{q_0}^{p} T$ for a mero 1-form T

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If f is single-valued, then clearly $(f) = D$

~~check periods of T~~: real part $= 0$, apply sym law to Im part

for $\partial\alpha = 0$, $2\pi \int_\alpha d\omega_{p_0 p_j} = 2\pi i \left(\mathrm{Re} \int_{p_0}^{p_j} d\omega'_\alpha + (\alpha, \beta)\right)$ etc

$\Rightarrow \int_{\gamma_j} T = 2\pi i\, \mathrm{Re} \int^{D} d\omega'_{\gamma_j}$ $\quad j = 1, \ldots, 2g$ $\qquad \hat{\mathbb{Z}}$

Since Λ is generated by $\int_{\gamma_i} (\eta_j)_{j=1}^{g}$ in $\mathbb{C}^g$

$\varphi(D) \in \Lambda \Rightarrow \int^D \vec{\eta} = \sum_{i=1}^{2g} n_i \int_{\gamma_i} \vec{\eta}$ $\Rightarrow$ same relation holds.

$\int^D d\omega'_{\gamma_j} = \sum_{i=1}^{2g} n_i \int_{\gamma_i} d\omega'_{\gamma_j}$ for each j. Hence

$\mathrm{Re} \int_{\gamma_i} d\omega'_j = \int_{\gamma_i} dU'_j = (\gamma_i, \gamma_j) \in \mathbb{Z}$ hence ok $\square$

<u>Theorem (Jacobi Inversion)</u> φ is surj with $r = g$ in parti, $\varphi \cong$

<u>pf.</u> Ex. $\int_{a_1}^{p_1} \vec{\eta} + \cdots + \int_{a_g}^{p_g} \vec{\eta}$ surj a nbd of $0 \in \mathbb{C}^g$ Set $D = \sum_{i=1}^{g} p_i - g a$

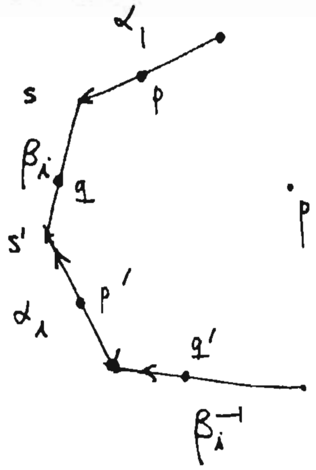
then "nD" surj any given pt Since $D' := nD + ga$ has

$\deg D' = g$ $\quad RR \Rightarrow D' \sim q_1 + \cdots + q_g \geq 0_g$

Abel's thm $\Rightarrow \varphi(nD) = \varphi(D' - g a) = \sum_{i=1}^{g} \int_a^{q_i} \vec{\eta}$ $\square$

Pf X R S genus $= g$ Abel-Jacobi $X \to J(X) = \mathbb{C}^g/\Gamma$

(of Riemann's Thm)



ω holo 1-form

Δ simply conn

$\omega = d\varphi$ with $\varphi = \int_{p_0}^{z} \omega$ holo fun

$$\underline{\varphi(p') - \varphi(p)} = \int_p^{p'} d\varphi = \int_p^{p'} \omega$$

$$= \cancel{\int_p^{s} \omega} + \int_s^{s'} \omega + \cancel{\int_{s'}^{p'} \omega}$$

$$= \underline{\int_{\beta_i} \omega} \quad \text{indep of the position of } p!$$

Same reason: $\underline{\varphi(q') - \varphi(q) = -\int_{\alpha_i} \omega}$

Basic integration identity

$$\int_X \omega \wedge \eta = \int_{\partial\Delta} \varphi\eta = \sum_{i=1}^{g} \int_{\alpha_i + \alpha_i^{-1}} \varphi\eta + \int_{\beta_i + \beta_i^{-1}} \varphi\eta$$

$$= \sum_{i=1}^{g} \left(-\int_{\beta_i} \omega \int_{\alpha_i} \eta + \int_{\alpha_i} \omega \int_{\beta_i} \eta \right)$$

for ω holo η any 1-form

From now on, we change notation by using $\vec{\omega}$ (instead of $\vec{\eta}$) by

normalize $\omega_1 \cdots \omega_g$ st $\int_{\alpha_i} \omega_j = \delta_{ij}$

then the period matrix $\underline{(I, \Omega)}$ $\quad \Omega_{ij} = \int_{\beta_i} \omega_j$

$$0 = \int_X \omega_i \wedge \omega_j = \sum_k \left(-\int_{\beta_k} \omega_i \int_{\alpha_k} \omega_j + \int_{\alpha_k} \omega_i \int_{\beta_k} \omega_j \right)$$

$$= -\Omega_{ji} + \Omega_{ij} \Rightarrow \underline{\Omega \text{ s symmetric}}$$

$$\text{"}0 < \text{"}\ i \int \omega_i \wedge \overline{\omega_j} = i \sum_k \left(-\Omega_{ji} + \overline{\Omega_{ij}} \right) = 2\,\mathrm{Im}\,\Omega_{ij}$$

$\Rightarrow$ Im Ω is positive definite in the sense of quadratic (Hermitian) form ※

Rmk hence $\mathbb{C}^g/\Gamma$ is an principally polarized abelian variety