$d\omega_\alpha := d\omega_{pp'} := dU_{pp'} + i * dU_{pp'} = dU_{pp'} + i\, dV_{pp'} = \frac{1}{2\pi}\left(\frac{dz'}{z'} - \frac{dz}{z} + \quad\right)$

$d\omega'_\alpha = dU'_\alpha + i * dU'_\alpha = dU'_\alpha + i\, dV'_\alpha = \frac{1}{2\pi i}\left(\frac{dz'}{z'} - \frac{dz}{z} + \cdot\right)$

these are abelian diff of 3rd kind

if $\partial\alpha = 0$, the 1st $\equiv 0$, $dU'_\alpha \sim dS_\alpha$ harm. repr.

Denote $dW_\alpha := d\omega'_\alpha$ holomorphic, ab diff of 1st kind

$dT_n = d z^{-n} + \cdots = \frac{-1}{n}\frac{dz}{z^{n+1}} + \quad = dU_n + i\, dV_n$ $\qquad n \geq 1$

$dT'_n = d(i z^{-n}) + \cdots = \frac{-i}{n}\frac{dz}{z^{n+1}} + \cdots = dU'_n + i\, dV'_n$

called ab diff of 2nd kind (one pole only)

Corollary For any prescribed principal part on S, there exist such ab diff It is unique up to diff of 1st kind

$\Leftrightarrow$ sum of residues $= 0$

Ex. Elementary Symmetry (1) For closed curves, basis γ_i

$d\omega_i \equiv 0$

$\int_{\gamma_i} dW_j =: s_{ij} + \sqrt{-1}\, t_{ij} = \int_{\gamma_i} dU'_{\gamma_j} + \sqrt{-1}\int_{\gamma_i} dV'_{\gamma_j}$

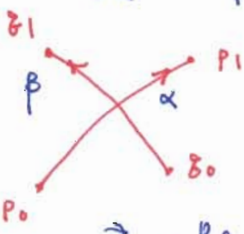
$\int_X dU'_{\gamma_i} \wedge dU'_{\gamma_j} \qquad \int_X dU'_{\gamma_i} \wedge dV'_{\gamma_j}$

Re skew-sym: (γ_i, γ_j) $\qquad$ Im symmetric $\langle dU'_{\gamma_i}, dU'_{\gamma_j}\rangle$

$(t_{ij}) > 0$ pos def quad form

(2) For open curves

$\int_\alpha d\omega_\beta = \int_\alpha dU_\beta + i\, dV_\beta$ $\qquad$ compare with



$\| \leftarrow$ indep of path $\qquad 0 = \int_X dV_\alpha \wedge dV_\beta$

$\int_X dV_\alpha \wedge dU_\beta \qquad \Sigma\,\mathrm{res} = 0 \quad \|$ by Ex

$= -\langle dU_\alpha, dU_\beta\rangle \qquad \int_\alpha dV_\beta - \left(\int_\beta dU'_\alpha + (\alpha,\beta)\right)$

$\Rightarrow$ $\mathrm{Re}\int_\alpha d\omega_\beta$ is sym. $\qquad \Rightarrow \mathrm{Im}\int_\alpha d\omega_\beta - \mathrm{Re}\int_\beta d\omega'_\alpha = (\alpha,\beta)$.

Finally, $\mathrm{Im}\int_\alpha d\omega'_\beta = \int_\alpha dV'_\beta = \int_X dU'_\alpha \wedge dV'_\beta = \langle dU'_\alpha, dU'_\beta\rangle$

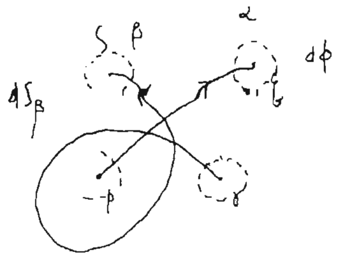
is sym. $\qquad$ residues at γ_0, γ_1 are both $= 0$

Rmk: If one curve is closed, then it reduces to

$\mathrm{Re}\int_\alpha d\omega'_\beta = (\alpha,\beta)$ (already seen) $= \mathrm{Im}\int_\alpha d\omega_\beta - \mathrm{Re}\int_\beta d\omega'_\alpha$

$\mathrm{Im}\int_\alpha d\omega'_\beta$ is symmetric (new)

Solution to the Ex used in Symmetry (2):



$\frac{1}{2\pi}\int_S d\phi \wedge df$ eg $df = \lambda f_1 + a\, dS_\beta$, f_1 smooth

$= \frac{1}{2\pi}\int_{\partial B_p} f\,d\phi - \frac{1}{2\pi}\int_{\partial B_q} f\,d\phi + \frac{1}{2\pi}\int_{\partial B_r} f\,d\phi - \frac{1}{2\pi}\int_{\partial B_s} f\,d\phi$

$f\,d\phi = d(f\phi) - \phi\, df$

$= \int_\alpha df - a(\phi(s) - \phi(r))$

$\phi(s)-\phi(r) = \int_\beta d\phi + (\alpha, \beta)$

this could be too "winding" and the angle is negatively counted wrt pt "p"

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Function Theory of Riemann Surface

Riemann-Roch & Abel-Jacobi

Divisor $D = \sum_{i=1}^r m_i P_i$, $m_i \in \mathbb{Z}$ (or $P_1^{m_1} \cdots P_r^{m_r}$ classically)

Linear system $L(D) := \{ f \in \mathcal{M}(X) \mid \underbrace{(f) + D}_{D'} \geq 0 \}$ ie eff. div $D' \sim D$

where $(f) := \sum_{p \in X} \mathrm{ord}_p(f)\, p$

ie f has pole of order $\leq m_i$ at P_i if $m_i > 0$,

f has zero of P_i of order $\geq -m_i$ if $m_i < 0$.

$L(D)$ is a vector space. Let $\ell(D) := \dim L(D)$

Thm (Riemann-Roch) $\ell(D) - \ell(K-D) = \deg D + 1 - g$

Rmk: (1) $K := (\omega)$ for any meromorphic 1-form ω

diff choice leads to equiv. "canonical divisor"

$\deg K = 2g - 2$ by Hurwitz or Gauss-Bonnet

(2) $D \sim D' \Rightarrow L(D) \cong L(D')$

Ex Hurwitz formula for $\varphi\ X \to Y$ for Euler # & for $\deg K$

From 3rd kind to 2nd kind: Reciprocity Laws

$z_0 = 0 \xrightarrow{\beta} \varepsilon = z_1$ ~~let $\varepsilon \to 0$ collapsing~~

For $\omega_{z_0 z_1}$ we use local model

$$2\pi\, \Phi = \mathrm{Re} \log \frac{z - z_2}{z - z_1} \cdot \frac{1 - \bar{z}_1 z}{1 - \bar{z}_1 z}$$

$$= \mathrm{Re} \log \frac{z - \varepsilon}{z}(1 - \varepsilon z)$$

$$= \mathrm{Re}\left[\log(1 - \varepsilon z^{-1}) + \log(1 - \varepsilon z)\right]$$

$$= -\mathrm{Re} \sum_{n=1}^\infty \frac{\varepsilon^n}{n}(z^{-n} + z^n)$$

$$\mathrm{Re}\left(z^{-n} + \frac{z^n}{a^{2n}}\right) = \mathrm{Re}(z^{-n} + z^n)$$

Set radius $a = 1$ in construction of U_{n, z_0}

< prescribing asymp. behavior determine the harm solution >

$\Rightarrow$

Some convergence argument

$$U_{z_0 z_1}(p) = -\frac{1}{2\pi} \sum_{n=1}^\infty \frac{\varepsilon^n}{n} U_{n\, z_0}(p)$$

$z_1 = \varepsilon$

i.e. $\omega_\beta(p) = -\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\varepsilon^n}{h}\tau_n(p)$ $\quad\beta$ is the curve connecting ξ_0 & ξ_1

$$\int_\alpha dU_{\xi_0\xi_1}(p) \overset{*}{=} \int_{\xi_0}^{\xi_1} dU_\alpha(\xi) = U_\alpha(\xi_1) - U_\alpha(\xi_0) = U_\alpha(\varepsilon) - U_\alpha(0) = \sum_{n=1}^{\infty}\frac{\varepsilon^n}{n!}\frac{d^n U_\alpha(0)}{d\varepsilon^n}$$

α is ANY curve away from ξ_0, ξ_1

$$= -\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\varepsilon^n}{h}\int_\alpha dU_{n\,\xi_0}(p)$$

$\Rightarrow$ $Re\int_\alpha d\tau_{n\,\xi_0}(p) = \frac{-2\pi}{(n-1)!} Re\frac{d^n\omega_\alpha}{d\xi^n}(\xi_0)$

now ξ_0 is also allow to change

$$\int_\alpha dV_{\xi_0\xi_1}(p) \overset{**}{=} \int_{\xi_0}^{\xi_1} dU_\alpha(p)$$

$\Rightarrow$ $Im\int_\alpha d\tau_{n,\xi_0}(p) = \frac{-2\pi}{(n-1)!} Re\frac{d^n\omega_\alpha'}{d\xi^n}(\xi_0)$

in the case $\beta = \widehat{\xi_0\xi_1}$ is too small so that $(\alpha,\beta)=0$

Q should we call them infinitesimal period relations?

Thm of Riemann-Roch:

$$\ell(D) - \ell(K-D) = \deg D + 1 - g$$

pf: (I) Riemann inequality $\quad \ell(D) \geq \deg D + 1 - g$

Let $D = \sum_{i=1}^{r} m_i P_i$

Set $f = \sum_{i \text{ with } m_i>0}\sum_{j=1}^{m_i}\left(a_{ji}\tau_{jP_i} + a_{ji}'\tau_{jP_i}'\right) + b + \sqrt{-1}\,b'$

real combinations

Period constraints $\int_{\gamma_k} df = 0$ for all curves $\gamma_1, \dots \gamma_{2g}$

Ex Prove the similar reciprocity laws for $d\tau'$

$$Re\int_\alpha d\tau_{n,\xi_0}'(p) = \frac{-2\pi}{(n-1)!} Im\frac{d^n\omega_\alpha}{d\xi^n}(\xi_0),$$

$$Im\int_\alpha d\tau_{n,\xi_0}'(p) = \frac{-2\pi}{(n-1)!} Im\frac{d^n\omega_\alpha'}{d\xi^n}(\xi_0)$$

for $m_i<0$ require vanishing constraint at most $\sum_{m_i<0}(-m_i)$ $/\mathbb{C}$

hence $\ell(D) \geq \frac{1}{2}\left[2\sum_{m_i>0} m_i + 2 - 2g\right] - \sum_{m_i<0}(-m_i) = \deg D + 1 - g$

complex dim

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(II) Roch's additional term $\ell(K-D)$:

First assume $D \geq 0$ (which is the case if $\deg D \geq g$

We need to know exactly how many linearly indep conditions are there in period constraint?

i.e. $0 = \sum_{i,j}\left(a_{ji}\int_{\gamma_k} d\tau_{jP_i} + a_{ji}'\int_{\gamma_k} d\tau_{jP_i}'\right)$ $\quad$ iff there is a holo 1-form η with $(\eta) - D \geq 0$,

ker $\int_{\gamma_k}$ $\forall k$ $\leftrightarrow$ $L(D)$

Let γ_η be the corr curve class

Say $\eta = \sum_{i=1}^{2g}\lambda_i\, dw_i$

$\int_{\gamma_\eta}\sum a_{ji}\,d\tau_{jp_i} + a_{ji}'\,d\tau_{jp_i}'$ $\quad$ Claim: it provides a relation automatically

$$= \int_X \sum \eta\wedge\left(a_{jp_i}\,d\tau_{jp_i} + a_{ji}'\,d\tau_{jp_i}'\right)$$

No use!

a diff 2 form with only simple poles

(well-defined : hence $\int_{B_\varepsilon}\frac{1}{|z|}\, r\,dr\,d\theta = 2\pi\varepsilon \to 0$)

Better way : use reciprocity law :

$$\int_\alpha dT_{jp_i} = \frac{-2\pi}{(n-1)!}\left(\mathrm{Re}\,\cancel{\frac{d^j\omega_\alpha}{dg^j}}(p_i) + \sqrt{-1}\,\mathrm{Re}\frac{d^j\omega_\alpha'}{dg^j}(p_i)\right) = 0 \quad \text{for } \alpha = \gamma_\eta$$

$$d w_\alpha = \frac{\partial \omega_\alpha'}{\partial g}\, dg \quad \text{has order } m_i \text{ at } p_i$$

Also $$\int_\alpha d\tau_{jp_i}' = \frac{-2\pi}{(n-1)!}\left(\mathrm{Im}\,\cancel{\frac{d^j\omega_\alpha}{dg^j}}(p_i) + \sqrt{-1}\,\mathrm{Im}\frac{d^j\omega_\alpha'}{dg^j}(p_i)\right) = 0$$

A Linear Algebra rank argument $\Rightarrow$ ℓ, constraints has $\dim_{\mathbb{R}}$ $2g - 2\ell(K-D)$

Hence $\ell(D) = \deg D + 1 - (g - \ell(K-D))$ $*$

If $\deg D \geq g$ then $D \sim$ ef $\Rightarrow$ R R
if $\deg D \leq g-2$ then $D' = K - D \sim$ ef since $\deg D' \geq 2g-2-(g-2) \geq g$

so $$\underset{\ell(K-D)}{\ell(D')} - \underset{\ell(D)}{\ell(K-D')} = \underset{2g-2-\deg D}{\deg D'} + 1 - g \quad \Rightarrow \text{ ok}$$

Finally, the remaining case is $\deg D = g-1$ and both D, $K-D$ non-ef
but then R.R reduces to $0 - 0 = 0$ which is trivial $\square$

Applications (1) Weierstrass gap theorem : for $D = kp$ $k = 0, 1, \cdots 2g-1$ ($2g$ interval)
get $\ell((2g-1)p) = 2g-1+1-g = g$, ie get only g functions

Def'n : $p \in X$ is a Weierstrass pt if $\ell(kp) \neq 1, 1, \cdots, 1, 2, 3, \cdots g$

Ex (a) p is a W-pt $\iff$ ord_p Wronskin $(w_1, \cdots, w_g) =: d_p > 0$
(b) Show that the Wronskian W is a holo $(\Omega^1_X)^{\otimes \frac{g(g+1)}{2}}$ form
(c) compute d_p in terms of the gap sequence hence $\Sigma d_p = g(g^2-1)$

Also, Once $\deg D \geq 2g-1$, any further adding pt get one more function

(2) $\ell(K) = g$ set $D = 0$), hence $g - 1 = \deg K + 1 - g$ $\Rightarrow$ $\deg K = 2g-2$
So R.R already includes both statement

Consider $\Phi_{|K|} : X \dashrightarrow \mathbb{P}^{g-1}$ "canonical map"
$p \mapsto (w_1(p) : \cdots \; w_g(p))$

can we find W st $w(p) \neq 0$? ie if $\ell(K) > \ell(K-p)$?

$\ell(K-p) - \underset{1}{\ell(p)} = 2g-2-1+1-g = g-2$ ie $\ell(K-p) = g-1 < g$
no function with only one simple pole (except constant)

$\Rightarrow$ $\Phi_{|K|} : X \longrightarrow \mathbb{P}^{g-1}$

Q. Is $\Phi_{|K|}$ an embedding ? If not, then $\exists\, q$ st $\ell(K-p-q) = \ell(K-p) = g-1$ (may $= p$)
but then $\ell(p+q) - (g-1) = 2+1-g$ ie $\dim L(p+q) = 2$
ie $\exists$ nonconstant mero function f with simple poles at p & q
$\Rightarrow$ $\Phi_{|p+q|} : X \longrightarrow \mathbb{P}^1$ a 2:1 branch cover
$(f : 1)$ or just f Hurwitz $\Rightarrow$ # branch pt $= 2(g+1)$

There is called an hyper-elliptic structure on X
with $g \geq 2$,

Ex For hyp-elliptic curves, W-pts = branch pts ($g = 1$ no W-pts)
In the model $y^2 = f(x)$, write down a basis of hol 1-forms