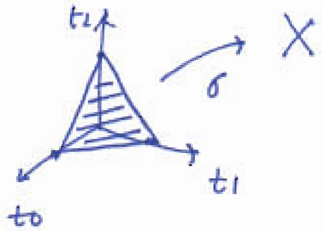
Topology of surfaces (oriented, cpt)

Homology of a top space X



$\Delta_p := \{ (t_i) \in \mathbb{R}^{p+1}, \ \sum_{i=0}^{p} t_i = 1, \ t_i \geq 0 \}$

singular p-simplex $\sigma : \Delta_p \to X$ conti

$C_p(X)$ = free ab. gp gen by all σ's

face (boundary) map $\partial_i \sigma : \Delta_{p-1} \to X$

$(t_0, \cdots, t_{p-1}) \mapsto \sigma(t_0, \cdots, \overset{i\text{th}}{0}, \ , t_{p-1})$

$\partial \sigma := \sum_{i=0}^{p} (-1)^i \partial_i \sigma \in C_{p-1}(X)$

Lemma: $\partial^2 = 0$

I **Singular homology th.** (cf. Vick. Bott-Tu)

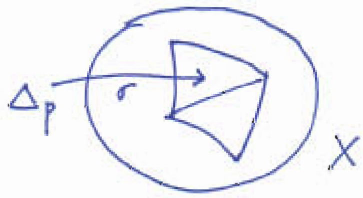
$\xrightarrow{\partial} C_p(X) \xrightarrow{\partial} C_{p-1}(X) \longrightarrow \cdots \qquad H_p(X,\mathbb{Z}) := \ker \partial_p / \mathrm{Im}\, \partial_{p-1} = Z_p / B_p$

II **Simplicial homology th.** (cf Munkres)

If X has a decomposition into "non-degenerate" simplexes σ's, $C_p(X)$:= free ab gp gen by these σ's of dim p

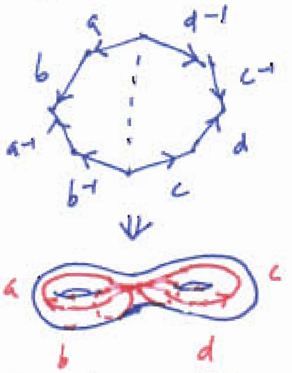
III **CW complex str.** (Cell decomp, Vick)

Theorem: All give the same $H_p(X,\mathbb{Z})$



Example 1 Surface of genus g

canonical form (cf Massey)



Using CW complex

$$\begin{array}{ccccc} C_2(X) & \xrightarrow{\partial} & C_1(X) & \xrightarrow{\partial} & C_0(X) \\ \| & & \| & & \| \\ \mathbb{Z} & & \mathbb{Z}^{2g} & & \mathbb{Z} \end{array}$$

All maps are "zero", so

$H_0 = H_2 = \mathbb{Z}, \quad H_1(X,\mathbb{Z}) \simeq \mathbb{Z}^{2g}$

Ex Prove this using sing homology

The intersection pairing (α, β) has matrix
$\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g$ symplectic basis:

$$\left(\begin{array}{c|c|c} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & & 0 \\ \hline & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & \\ \hline 0 & & \ddots \end{array} \right)$$

Rmk Cohomology $C^p(X,\mathbb{Z}) := \mathrm{Hom}_{\mathbb{Z}}(C_p(X,\mathbb{Z}), \mathbb{Z})$

with δ_p = adjoint map of ∂_{p+1} $\quad \delta^2 = 0 \Rightarrow H^p(X,\mathbb{Z})$

Poincaré duality: linear functional $\longleftrightarrow$ int functional $(\cdot, \beta)$
ie "cohomology" ie "homology"

PD holds for any orientable cpt mfd of dim n

$H^{n-p}(X,\mathbb{Z}) \simeq H_p(X,\mathbb{Z})$ For $n=2$ this is clear as above

~~de Rham cohomology~~ of a differentiable mfd X

$$\cdots \xrightarrow{d} \Omega^p(X) \xrightarrow{d} \Omega^{p+1}(X) \to \cdots$$

locally on U with basis $a_I(x)\, dx^I = a_I(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$

Cartan d operator : $d(a_I\, dx^I) := da_I \wedge dx^I$

Ex Lemma : well-defined, $d^2 = 0$ total diff $= \sum_{i=1}^n \frac{\partial a_I}{\partial x_i} dx^i$

$\Rightarrow H^p_{dR}(X) := \ker d_p / \operatorname{Im} d_{p-1} \equiv$ closed forms / exact forms

~~Integration of p forms~~ over a p-simplex (differentiable) hence to all p-chains

$$\int_\sigma \omega := \int_{\Delta_p} \sigma^* \omega \quad \text{as a Riemann integral}$$

Stokes' theorem $T \in C_{p+1}(X)$, $\omega \in \Omega^p(X) \Rightarrow \int_{\partial T} \omega = \int_T d\omega$

Thm (de Rham) : The map $H^p_{dR}(X) \xrightarrow{\int} H_p(X, \mathbb{R})^* \cong H^p(X, \mathbb{R})$ is isom.

Moreover, it preserves the "product structure" ($\wedge$ v.s $\cap$)

This is a hard thm, but we will give an explicit proof of it when $n = \dim X = 2$, following ~~H. Weyl (§11)~~

Fact : Poincaré lemma : locally, closed $\Rightarrow$ exact ($p=1$ by Stokes')

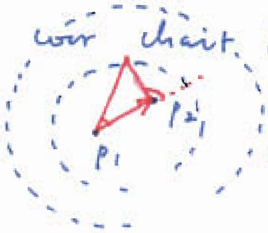
Caution : in Weyl "df" is a local notation, "grad f" is our "df" for a closed 1-form

same reason $\Rightarrow \int$ is injective for $p=1$ (All periods $=0$)

~~Angle 1 form~~ : $d\phi := \frac{1}{2\pi} \underbrace{d\theta}_{= * d \log r} = \frac{1}{2\pi} \frac{-y\,dx + x\,dy}{x^2+y^2}$ in $\mathbb{R}^2$

source-sink :

in a coor chart $B_a \ni p_1, p_2$



$\supset B_{\tilde a}$, $\tilde a < 1$

λ C^∞ cut-off

$\psi := \phi_2 - \phi_1$

$\tilde\psi := \lambda\psi$ global in X

$d\psi_{p_1 p_2} := d(\lambda\psi)$; $= d\psi$ in B_a

has sing at p_1, p_2

For a curve α with division pts $p_1, p_2, \ldots, p_n$, $\underline{ds_\alpha := \sum_{i=1}^{n-1} d\psi_{p_i p_{i+1}}}$

Prop $\quad \underline{\int_X ds_\alpha \wedge \eta} = \underline{\int_\alpha \eta}$ for any closed 1-form

~~Pf~~ : Locally, $\int_X d\psi_{p_1p_2} \wedge df = -\lim_{\varepsilon \to 0} \int_{X \setminus B_\varepsilon(p_1) \setminus B_\varepsilon(p_2)} d(f\, d\psi_{p_1p_2})$

$= \int_{\partial B_\varepsilon(p_1) + \partial B_\varepsilon(p_2)} f\, d\psi_{p_1p_2} = f(p_2) - f(p_1) = \int_{\alpha_{12}} df \quad \square$

~~Rmk~~   If $\partial\alpha = 0$, then $\int_\alpha ds_\beta = (\alpha, \beta)$ (Weyl took this as def)

Ex. Thm   If η has "poles" at $b, b' \notin \alpha$, $\Rightarrow$ de Rham Thm

with residues $-A, A$ resp Then $\underline{\int_X ds_\alpha \wedge \eta} = \underline{\int_\alpha \eta} - \underline{A\left(\int_\beta ds_\alpha + (\alpha, \beta)\right)}$

for any choice of β connecting b to b'

Geometry of surface (C^∞)

Dirichlet integrals on forms

hodge $*$ operator $\quad \eta = \eta_1\,dx + \eta_2\,dy \Rightarrow *\eta := -\eta_2\,dx + \eta_1\,dy$

Fact (1) $*^2 = -1$ $\quad$ (2) the def'n is indep of conformal coor change

inner product $\quad (\eta, \xi) := \eta \wedge *\xi = (\eta_1\xi_1 + \eta_2\xi_2)\,dx\wedge dy$

$\langle \eta, \xi \rangle := \int_S (\eta, \xi)$ $\quad$ Q minize $\|\eta\|^2 := D_1(\eta) := \langle \eta, \eta \rangle$
inside $[\eta] \in H^1_{dR}(S)$?

if the minimizer η_0 exists (and C^∞), then $\forall \varepsilon \in \mathbb{R}$:

$$D_1(\eta_0 + \varepsilon\,df) = D_1(\eta_0) + 2\varepsilon\, D_1(\eta_0, df) + \varepsilon^2 D_1(df) \geqslant D_1(\eta_0)$$

$$\Rightarrow 0 = \int_S \eta_0 \wedge *df = \int_S df \wedge *\eta_0 \underset{\uparrow}{=} \int_S f\, d*\eta_0 \quad \forall f \in C^\infty(S)$$

Leibniz rule of d & Stokes' thm

But, in local (U, x) with conformal transitions.

$$d\eta_0 = 0 \Rightarrow \eta_0 = du \Rightarrow d*\eta_0 = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) dx\wedge dy$$

Thus $\Delta u = 0$ $\quad$ Call η_0 a harmonic form, u harm potential

Thm (Hodge) ~1940 Every $\eta_0 \in H^1_{dR}(X)$ has a unique harm. repr.

Weyl: for $X = S$ a cpt on surface, (This is due to Hilbert ~1900)

S cpt $\Rightarrow$ No global harmonic function
in fact, no Green function (by Stokes')

Let $g_{p,q}$ = bi-polar Green, then

$dg = -d\log|z-p| + d\log|z-q| + \cdots$ har 1-form with 2 "poles"

$\Rightarrow dg + i*dg = -d\log(z-p) + d\log(z-q) + \cdots = -\frac{dz}{z-p} + \frac{dz}{z-q} + \cdots$
is a meromorphic 1-form η_{pq}

$\Rightarrow f := \frac{\eta_{pq}}{\eta_{rs}}$ is a global meromorphic function on S

ie $f : S \longrightarrow \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ is a branched covering map

Corollary: S has a triangulation induced from $S^2 = \mathbb{P}^1$

Notice that $\frac{1}{2\pi} *dg_{pq} = d\phi_q - d\phi_p + \cdots$

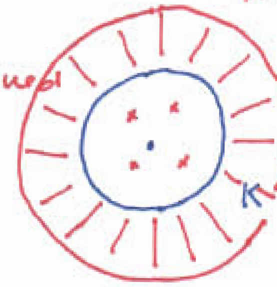
Q can we use this to represent $d\phi_{pq}$?
If so then we get har repr. of ds_α
for closed curve α by $\frac{1}{2\pi}\sum_{i=1}^{n-1} *dg_{p_i p_{i+1}}$ as before.

Answer : No! dg_{pq} dep on p, q only, not the curve p to q $\Rightarrow$ $\rightarrow 0$

H. Weyl's procedure for solving harmonic function
on a surface S with prescribed singularities

Local model of singularities: z coor near $p \in S$, $z(p)=0$

Φ har
- single-valued in $K' \setminus K$
- but not nec in K



$K' := B_b(0)$

$K := B_a(0)$

$S \setminus K$ punched surface
K : hole $\subset K'$: lid
$K' \setminus K$: the lock-ring
Φ harmonic in K', even not single valued!
but possibily with sing. in K
st $\frac{\partial \Phi}{\partial n} = 0$ along ∂K

competition functions on S: $v \in C^1(S \setminus \text{sing}\,\Phi)$ st

on K: $v = \Phi + \tilde{v}$ with $\tilde{v} \in C^1(S)$ (reg part)

Renormalized Dirichlet integral wrt Φ:

$$D(v) := D_{1, S\setminus K}(dv) + D_{1,K}(d\tilde{v})$$

Fact: A minimizer u, if exists, must satisfies

$\Delta u = 0$ on $S \setminus K$, also $= 0$ on K in the sense $\Delta \tilde{u} = 0$

pf: $D(u + \varepsilon \cdot f) = \int_{S\setminus K} du \wedge * du + 2\varepsilon \int_{S \setminus K} df \wedge * du + \varepsilon^2 \int_{S\setminus K} df \wedge * df$

smooth on S $\quad + \int_K d\tilde{u} \wedge * d\tilde{u} + 2\varepsilon \int_K df \wedge * d\tilde{u} + \varepsilon^2 \int_K df \wedge * df$

$$\Rightarrow 0 = \underline{\int_{S\setminus K} d(f * du)} - \int_{S\setminus K} f \, d * du + \underline{\int_K d(f * d\tilde{u})} - \int_K f \, d * d\tilde{u}$$

$$= - \int_{\partial K} f \, \underline{* d\Phi} \qquad (d * du = \Delta u \, dx \wedge dy).$$

$= \nabla \Phi \, \vec{n} \, ds$

Thm (Weyl, §14) The minimizer exists uniquely

- Examples 1. $\operatorname{Re} z^{-n}$: $r^{-n} \cos n\theta + \frac{r^n}{a^{2n}} \cos n\theta$ $\qquad \frac{\partial}{\partial n} = \frac{\partial}{\partial r}$

  $- \operatorname{Im} z^{-n}$: $r^{-n} \sin n\theta + \frac{r^n}{a^{2n}} \sin n\theta$

  solutions $U_n \mapsto$ analytic diff $\xi_n = d\tau_n := dU_n + i \, \underline{* dU_n}$

  multi-valued (locally) $= \underline{dV_n}$

  $U_n' \mapsto \xi_n' = d\tau_n' := dU_n' + i \, \underline{dV_n'}$

- Examples 2. Two sing pts $z_1, z_2 \in K = B_1(0)$ $\qquad$ Q: why not 1 pt?

$\Phi = \frac{1}{2\pi} \operatorname{Re} \log \frac{z - z_2}{z - z_1} \cdot \frac{1 - \bar{z}_2 z}{1 - \bar{z}_1 z}$ $\quad$ on ∂K $\frac{1}{z} = \bar{z}$ $\Rightarrow \operatorname{Im} \log(\;) = 0 \Rightarrow \frac{\partial \Phi}{\partial n} = 0$

$\Phi' = \frac{1}{2\pi} \operatorname{Im} \log \frac{z - z_2}{z - z_1} \Big/ \frac{1 - \bar{z}_2 z}{1 - \bar{z}_1 z}$ $\quad$ on ∂K, $\operatorname{Re} \log 1 = 0 \Rightarrow \frac{\partial \Phi'}{\partial n} = 0$ $\quad$ single valued?

Solutions U_{12} is NOT new But U_{12}' is! It solves Hodge Thm