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Solving Polynomials via Series: $t = p(x) = x + \cdots \Rightarrow x(t) = \frac{1}{2\pi i}\int x \frac{p'(x)\,dx}{p(x)-t}$.

1) Key Example: $x^{m+1} - x + t = 0$, $m \geq 1$. Solve $x(t)$ in series?

Recall Lagrange's thm (1770): ϕ analytic $|t\phi(z)| < |z-a|$ on C

then $\forall f$ $\quad f(\zeta) = \frac{1}{2\pi i}\int_C f(z)\frac{1-t\phi'(z)}{z-a-t\phi(z)}\,dz$ for the unique root ζ of $\zeta - a = t\phi(\zeta)$ inside C

$$= f(a) + \sum_{n=1}^{\infty}\frac{t^n}{n!}D_a^{n-1}\left(f'(a)\phi(a)^n\right).$$ (Exercise) W-W. p.149, no.25

Now set $x = \zeta^{-1/m}$ get $1 - x^{-m} + t\,x^{-(m+1)} = 0$ i.e. $\zeta - 1 = t\,\zeta^{\frac{m+1}{m}} = t\phi(\zeta)$

Let $f(\zeta) = x(\zeta)$, for $k = 1, 2, \cdots, m$, get root (write $N = m+1$)

$$x_k = e^{-2\pi i\,k/m} + \sum_{n=1}^{\infty}\frac{t^n}{n!}\cdot\frac{-1}{m}\frac{\Gamma\left(\frac{N(n-1)}{m}+1\right)}{\Gamma\left(\frac{N(n-1)}{m}+1-(n-1)\right)}e^{2\pi i k\frac{n-1}{m}}$$

$f'(a)\phi(a)^n = \frac{-1}{N-1}a^{\frac{-N}{N-1}}a^{\frac{nN}{N-1}} = \frac{-1}{N-1}a^{\frac{N(n-1)}{N-1}}$, substitute $a = e^{2\pi i}$

$D_a^{n-1}(\cdots) = \frac{-1}{m}\Gamma\left(\frac{N(n-1)}{m}+1\right)/\Gamma\left(\frac{N(n-1)}{m}+1-(n-1)\right)$

$$\Rightarrow x_k(t) = e^{-2\pi i\frac{k}{m}} - \frac{t}{m}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{(m+1)n}{m}+1\right)}{\Gamma(n+2)\,\Gamma\left(\frac{n}{m}+1\right)}t^n e^{2\pi i\frac{kn}{m}};\quad \sum_{j=1}^{m+1}x_j = 0.$$

Q: Is this a satisfactory answer?

- Equation with 3 terms?
- One unknown coefficient?
- Recursive (ODE) str on n-th term?

Remark: For quintic equations, using quadric & cubic eq'ns can reduce it to $x^5 + px + q = 0$ (Bring 1796, even $p = -1$)

The above applies to ANY tri-nomial equations

The ratio with gap m $\quad \frac{a_{n+m}}{a_n} = \frac{\Gamma(n+2)}{\Gamma(n+m+2)}\cdot\frac{\Gamma\left(\frac{m+1}{m}n+m+2\right)}{\Gamma\left(\frac{m+1}{m}n+1\right)}\cdot\frac{\Gamma\left(\frac{n}{m}+1\right)}{\Gamma\left(\frac{n}{m}+2\right)}$

$$= \frac{\left(\frac{m+1}{m}n+m+1\right)\cdots\left(\frac{m+1}{m}n+1\right)}{(n+m+1)\cdots(n+2)\cdot\left(\frac{n}{m}+1\right)}$$ is a rational function in n $\quad Q(n)$

$\Rightarrow$ Each partial sum with gap $= m$ is a HG series.

Def'n: HG series ${}_pF_q(a_1,\cdots,a_p; b_1,\cdots,b_q; z) = F\left(\begin{matrix}a_1 \cdots a_p\\ b_1 \cdots b_q\end{matrix}\middle| z\right) := 1 + \sum_{n=1}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}$

where $(a)_n := a(a+1)\cdots(a+(n-1)) = \frac{\Gamma(a+n)}{\Gamma(a)}$ for all $n \in \mathbb{C}$

for $n \in \mathbb{N}$

Exercise: F is a sol of $\left[\delta\prod_{i=1}^{q}(\delta+b_i-1) - z\prod_{i=1}^{p}(\delta+a_i)\right]f = 0$

Here $\delta = z\frac{d}{dz}$. This is a Fuchsian eq'n on $\mathbb{C}P^1 \Leftrightarrow p = q+1$

set $n = j + m\ell$ $(\ell \geq 0)$, $j = 0, \cdots, m-1$; $Q_j(\ell) = Q(j+m\ell)$ is rat'l in ℓ

Conclusion: Each root $x_k(t)$ is a linear comb. of m HG series, with Fuchsian HG eq'ns

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2) Differential Resolvent : Transform Alg Eq'n to Differential Eq'n

Example 1 : $x^2 + x - t = 0$ find $x(t)$

$2x\,x' + x' - 1 = 0 \Rightarrow x' = \frac{1}{1+2x}$

Expect $a x'' + b x' + c = 0$ i.e. $a + b(x + 2x^2) + c(1+2x) = 0$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{-2x+2t}$

$(-b+2c)x + (a + 2bt + c) = 0$

$\Rightarrow b = 2c$, $a + (4t+1)c = 0$. Set $a = 4t+1$, $c = -1$, $b = -2$, get

$(4t+1)\,x'(t) - 2x(t) - 1 = 0$

$\frac{dx}{2x+1} = \frac{dt}{4t+1} \Rightarrow (x+\frac12)^{\frac12} = c(t+\frac14)^{\frac14}$

$x=0 \Rightarrow t=0 \Rightarrow c = \pm 1$

so $x = -\frac12 \pm \sqrt{t+\frac14}$.

Q : Is this the most stupid method to get the quadratic root formula ?

Example 2 : $x^5 - 5x^3 + 5x - t = 0$ Find $x(t)$

$5x^4x' - 15x^2x' + 5x' - 1 = 0 \Rightarrow x' = \frac{1}{5(x^4 - 3x^2 + 1)}$

$20x^3(x')^2 + 5x^4x'' - 30x(x')^2 - 15x^2x'' + 5x'' = 0$

$\Rightarrow x''(5x^4 - 15x^2 + 5) = -(20x^3 - 30x)(x')^2 = \frac{-2x(2x^2-3)}{5(x^4-3x^2+1)^2}$

$\Rightarrow x'' \cdot 25(x^4 - 3x^2 + 1)^3 = -2x(2x^2-3)$

- In general, we need to compute up to x', x'', x''', x'''' inductively in terms of rational expressions in x. Then plug in $a_4 x^{(4)} + a_3 x^{(3)} + \cdots + a_0 x + a_\infty = 0$. Division $\Rightarrow$ coeff of $x^0, x^1, \cdots, x^4 = 0 \Rightarrow a_i(t)$ up to a scale $\Rightarrow$ ODE
- In this lucky example we only need 2nd order ODE !

```
    1 0 -3 0 1
 x  1 0 -3 0 1
 ------------------
    1 0 -3 0 1
       -3 0 9 0 -3
            1 0 -3 0 1
 ------------------------
    1 0 -6 0 11 0 -6 0 1
```

```
1 0 -5 0 5 -t | 1 0 -6 0 11 0 -6 0 1
              | 1 0 -5 0 5 -t
              ------------------------
               -1 0 6 t -6 0 1
               -1 0 5 0 -5 t
               ------------------
                  1 t -1 -t 1
                x 1 0 -3 0 1
                ------------------
                  1 t -1 -t 1
                    -3 -3t 3 3t -3
                           1 0 -1 -t 1
                  ----------------------------
                 | 1 t -4 -4t 5 4t -4 -t 1
                 | 1 0 -5 0 5 -t
                  ----------------------------
                   t 1 -4t 0 5t -4 -t 1
                   t 0 -5t 0 5t -t^2
                   ----------------------
                     1 t 0 0 t^2-4 -t 1
                     1 0 -5 0 5 -t
                     ----------------------
                       t 5 0 t^2-9 0 1
                       t 0 -5t 0 5t -t^2
                       ----------------------
                      | 5 5t t^2-9 -5t t^2+1 0
                      | 5 0 -25 0 25 -5t
                       ----------------------
                        5t t^2+16 -5t t^2-24 5t
```

claim: $a x'' + b x' + c x = 0$

i.e. $\frac{-a \cdot 2x(2x^2-3)}{25(x^4-3x^2+1)^3} + \frac{b}{5(x^4-3x^2+1)} + cx = 0$

$-a(4x^3 - 6x) + 5b(x^4 + tx^3 - x^2 - tx + 1)$
$\quad + 25c\,(5tx^4 + (t^2+16)x^3 - 5tx^2$
$\quad\quad + (t^2-24)x + 5t\,) = 0$

Solution : $a = 25(4 - t^2)$, $b = -25t$, $c = 1$.

b, c part $= 25[-5t^2x^3 + 5t^2x + (t^2+16)x^3 + (t^2-24)x]$
$\quad = 25[(-4t^2+16)x^3 + (6t^2-24)x]$
$\quad = -25(t^2-4)(4x^3-6x) \Rightarrow a$ ✱

ODE : $25(4 - t^2)\,x'' - 25t\,x' + x = 0$

Ex. Solve $x_j(t)$, $j = 1, \cdots, 5$ corr. to $x_j(0)$ being roots of $x(x^4 - 5x + 5) = 0$
And show that eq'n in Example 2 is solvable by radicals.

3) Search for the underlying "geometric structures" ?

Rmk: Although 2) in principle works for higher degree poly eq'ns, the resulting ODE (or PDE if coefficients have more freedom) has no obvious str. in terms of the eq'n, thus an "algorithm", not a "formula".

Hermite gave such a formula (1858). Also, Kronecker & Brioschi.

$u = \varphi(\tau) := k^{1/4}$, $\psi(\tau) := k'^{1/4}$. For n prime, consider $v = \varphi(n\tau)$.

v and u are related by modular equation of degree $n+1$

Thm 1: For $n=5$: $u^6 - v^6 + 5u^2v^2(u^2-v^2) + 4uv(1-u^4v^4) = 0$.

For $v = \varphi(n\tau)$ given, all roots are $u = \varphi(\tau)$, $\varphi(\frac{1}{n}(\tau+16m))$, $0 \le m \le n-1$.

For $n=5$, denote the roots by $r_\infty, r_0, r_1, r_2, r_3, r_4$.

Set $\Phi(\tau) := (r_\infty + r_0)(r_1 - r_4)(r_2 - r_3)$

Thm 2: $\Phi(\tau + 16j)$, $0 \le j \le 4$, are 5 roots of the quintic eq'n:

$$\Phi^5 - 2000\,\varphi^4\psi^{16}\,\Phi - 1600\sqrt{5}\,\varphi^3\psi^{16}(1+\varphi^8) = 0$$

Write $\Phi = 2\cdot 5^{3/4}\varphi\psi^4 x \Rightarrow \underline{x^5 - x + a = 0}$; $a = \frac{2}{5}5^{-1/4}\frac{1+\varphi^8}{\varphi^2\psi^4}$.

Rmk: The modular equations (correspondences) have many applications in number theory (cf. McKean & Moll). The general derivation is non-trivial. For $n=5$, a simpler way was found by Hermite. A theoretic reason for Thm 2 via alg. curves was given later by M. Green 1978.

Solution to $x^5 - x + a = 0$:

(1) Let $A = \frac{5}{2}5^{1/4}a \underset{\text{expect}}{=} \frac{1+\varphi^8}{\varphi^2\psi^4} = \frac{1+k^2}{k^{1/2}k'}$

i.e. Solve k in $A^2\cdot k(1-k^2) = (1+k^2)^2$, i.e. $k^4 + A^2k^3 + 2k^2 - A^2k + 1 = 0$

(2) Get K, K' from k, and then $\tau = i\frac{K'}{K}$ $q = e^{\pi i\tau}$

(3) Get formula $x_j = \frac{\Phi(\tau+16j)}{2\cdot 5^{3/4}\varphi(\tau)\psi^4(\tau)}$ $j = 0, 1, 2, 3, 4$.

In (1), k has explicit solution: $\tan\left(\frac{1}{4}\sin^{-1}\frac{4}{A^2}\right)$, also radical in A.

But, how did they discover it? Think "$\sqrt[n]{a} = \exp\frac{1}{n}\int^a \frac{dx}{x}$! (*)

Modern Theory (Thomae's formula, Umemura 1984, in Mumford's Tata Lect, II)

polynomial $f(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$

→ hyperelliptic curve $y^2 = f(x)$ $\underline{\text{ or } f(x)\cdot x(x-1)}$ n odd; $f(x)\cdot x(x-1)(x-2)$ n even

→ period matrix Ω of hyp. elliptic integral, generalizing (*).

→ Theta functions $\vartheta[\vec{\eta}](\vec{z}, \Omega)$ with characteristic $\eta_i \in \frac{1}{2}\mathbb{Z}$ ⇒ Roots !