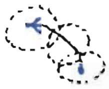
$\mathfrak{f} = \{ (f, \Omega) \mid \Omega \subset \mathbb{C}$ region, $f \in \mathcal{O}(\Omega) \}$ equiv class of function element / or germ under analytic conti

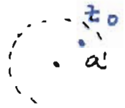
All such (Ω, z)'s defines a Riemann Surface S, the natural domain of def'n of $\mathfrak{f}$ to make it single valued

$$S \xrightarrow{\pi} z \in \mathbb{C}$$



pre-branch points of π over $a \in \mathbb{C}$

if $a \notin \pi(S)$ but $\pi(S) \supset B_\rho(a)^\times$



Fact: $\pi_1(B_\rho(a)^\times, z_0) \simeq \mathbb{Z}$ (Ex)

$0 < |z-a| < \rho$

We assume that $\mathfrak{f}$ can be analytic conti along all arcs in $B_\rho(a)^\times$

Q: why do we need to assume this? ie Is π a covering map?

Start with a fixed germ $\mathfrak{f}_{z_0} = (f, \Omega \ni z_0)$

Let $C = \partial B_r(a)$ where $r = |z_0 - a|$, generator of π_1

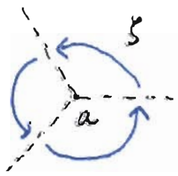
if $h := \min\{ m \in \mathbb{N} :$ conti along C^m returns to $\mathfrak{f}_{z_0} \} < \infty$

introduce (add) a new chart $(B_{\rho^{1/h}}(a), \zeta)$ to $S \cup \{a\}$

and extend π by $z - a = (\zeta - a)^h$

$F(\zeta) := f(a + (\zeta - a)^h)$ is then single valued

eg $h = 3$



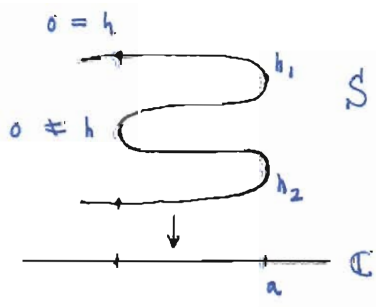
Notice that $a \in \mathbb{C}$ may have many pre-branch pts above it, and may have different h

Laurent expansion $F(\zeta) = \sum_{n=-\infty}^{\infty} A_n (\zeta - a)^n$

Def'n: $\zeta = a$ is called an algebraic sing or branch pt if $F(\zeta)$ has at most a pole at $\zeta = a$, ordinary alg sing if smooth

in the ν-th branch, get $f_\nu(z) = \sum_{n=n_0}^{\infty} A_n e^{2\pi i \frac{\nu n}{h}} (z-a)^{n/h}$

"fractional power series"



S of $\mathfrak{f}$

Def'n: f is an <u>alg. fcn</u> if $\exists\, P(w,z) \not\equiv 0$ st $P(f(z),z) = 0$ for one element (f, Ω) (hence for all)

- May assume P irred, hence unique up to const (via resultant)

Conversely, for irred. $P(w,z) = a_0(z)\, w^n + \quad + a_n(z)$

z_0 not a zero of $a_0(z)$ and $R(z) := R(P, \partial P/\partial w)$ discriminant

<u>claim</u>: $\exists$ disk $\Delta \ni z_0$, $(f_1, \Delta), \dots, (f_n, \Delta)$ st $P(f_i(z), z) = 0\ \forall i$ given by exactly the n roots for given $z \in \Delta$

pf let $w_1, \dots, w_n$ be simple zeros of $P(w, z_0) = 0$

let $\overline{B}_\varepsilon(w_i)$ be disjoint & $P(w, z_0) \neq 0$ on $C_i = \partial B_\varepsilon(w_i)\ \forall i$

$$\Rightarrow\ 1 = \frac{1}{2\pi i}\int_{C_i} \frac{P_w(w,z)}{P(w,z)}\, dw \quad \text{at } z = z_0, \text{ hence for } z \in \Delta$$

some small disk

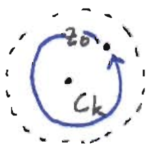
On Δ, the i-th "root function $f_i(z)$"

is holomorphic since
$$f_i(z) = \frac{1}{2\pi i}\int_{C_i} w\, \frac{P_w(w,z)}{P(w,z)}\, dw$$

Cor Any (f, Ω) with $P(f(z), z) = 0$ can be conti along any arcs away from the excluded points $c_1, \dots, c_m$

Q: <u>How to see these "root germs" are conti. to each other</u>? ie define only ONE global analytic fcn?

At c_k, consider $z_0 \in B_\delta(c_k)^\times$, δ small



start with any root germ (f_i, z_0), $\exists$ finite # of conti (f_j, z_0)'s around c_k, hence $\exists\, h \in \mathbb{N}$
$$f_i(z) = \sum_{n=-\infty}^{\infty} A_n (z - c_k)^{n/h}$$

If $a_0(c_k) \neq 0$ then $f_i(z)$ is bounded near c_k $\Rightarrow$ <u>ord. alg. sing.</u>

If $a_0(c_k) = 0$ of order m, then $f_i(z)(z - c_k)^m$ is bounded
$$a_0(z)(z - c_k)^{-m} + a_1(z)(z - c_k)^{-m} f_i(z)^{-1} + \quad + a_n(z)(z - c_k)^{-m} f_i(z)^{-n} = 0$$
$\downarrow z \to c_k$

$\neq 0$ $\Rightarrow f(\zeta)\, \zeta^{mh}$ bounded $\Rightarrow$ <u>alg. pole</u>

Ex. show that $z = \infty$ is also the case (alg pole at most)

Let f be one global analytic fcn obtained, with $\ell < n$ branches $\Rightarrow (w - f_1(z)) \cdots (w - f_\ell(z))$ has rat'l coefficients (mero on $\mathbb{CP}^1$) in z $\Rightarrow$ irred poly of lower degree $\times$

$a_0(z)w'' + a_1(z)w' + a_0(z)w = 0$ $\quad a_i(z)$ entire, no common zero

Fact: If $a_0(z_0) \neq 0$ (~~ordinary pt.~~)
the sol germ (f, z_0) is uniq det by $f(z_0)$ and $f'(z_0)$

Def$'^n$: In $w'' = p(z)w' + q(z)w$, say z_0 is a ~~regular sing pt~~ if p has at most simple, q has at most double, pole

claim $\exists$ solution at z_0 of the form $w = z^\alpha g(z)$
g analytic and $\neq 0$ at z_0

Set $z_0 = 0$ Get $\alpha(\alpha-1)z^{\alpha-2}g + 2\alpha z^{\alpha-1}g' + z^\alpha g'' = p\left(\alpha z^{\alpha-1}g + z^\alpha g'\right) + q z^\alpha g$

i.e. $g'' = \left(p - \frac{2\alpha}{z}\right)g' + \left(q + \frac{\alpha}{z}p - \frac{\alpha(\alpha-1)}{z^2}\right)g$

Now put $g = b_0 + b_1 z + b_2 z^2 + \cdots,\quad b_0 \neq 0$

$z^{-2} \Rightarrow -b_0\left(\underline{\alpha(\alpha-1) - p_{-1}\alpha - q_{-2}}\right) = 0$ the indicial eq$'^n$

$z^{-1} \Rightarrow \underline{p_{-1}b_1 - 2\alpha b_1} + q_{-1}b_0 + \alpha p_0 b_0 = 0$ $\quad p_{-1} - 2\alpha$

$z^{n-2} \Rightarrow \left(\underline{-n(n-1) + p_{-1}n - 2\alpha n}\right)b_n +$ in lower b_j's $= 0$
$n \geq 2$ $\quad$ " $n\,[\,p_{-1} - 2\alpha - (n-1)\,]$

Let α_1, α_2 be roots of $\alpha^2 - (p_{-1}+1)\alpha - q_{-2} = 0$

$\underline{\alpha_1 + \alpha_2 = p_{-1} + 1} \Rightarrow p_{-1} - 2\alpha_1 = (\alpha_2 - \alpha_1) - 1$

Thm: If $\alpha_2 - \alpha_1 \notin \mathbb{Z}$, get 2 sol's $(z-z_0)^{\alpha_i}g_i(z)$, $i = 1, 2$
if $\alpha_2 - \alpha_1 \in \mathbb{Z}_{\leq 0}$, still get $(z-z_0)^{\alpha_1}g_1(z) =: w_1(z)$

Ex when $\alpha_2 - \alpha_1 \in \mathbb{Z}_{\leq 0}$, $w_2(z) = (z-z_0)^{\alpha_2}g_2(z) + A\,w_1(z)\log(z-z_0)$
for some $g_2(z)$, $A \in \mathbb{C}$ If $\alpha_2 = \alpha_1$ then $A \neq 0$ $\quad (g_2(0) \neq 0)$
(Hint: try $w_2 = h\,w_1$) Determine the monodromy matrix

Now assume $a_i(z)$ are polynomials, ie $p, q \in \mathbb{C}(z)$

~~At $z = \infty$~~, put $u = \frac{1}{z}$, $w' = -\frac{dw}{du}u^2$, $w'' = \frac{d^2w}{du^2}u^4 + 2u^3\frac{dw}{du}$

get $\frac{d^2w}{du^2} = -\left(2u^{-1} + u^{-2}p\right)\frac{dw}{du} + u^{-4}q\,w$

* ordinary pt at $\infty \Leftrightarrow 2z + z^2p(z)$, $z^4q(z)$ <u>bounded</u> at ∞

** reg sing at $\infty \Leftrightarrow 2 + zp(z)$, $z^2q(z)$ "

Cor The most general 2nd order ODE with regular sing pts

$$w'' + \sum_{i=1}^{n} \frac{1-(\alpha_i+\alpha_i')}{z-a_i} w' + \left(\sum_{i=1}^{n} \frac{\alpha_i \alpha_i'}{(z-a_i)^2} + \frac{D_i}{z-a_i} \right) w = 0$$

with $\Sigma(\alpha_i+\alpha_i') = n-2$, $\Sigma D_i = 0$. & $\Sigma\left(a_i D_i + \alpha_i \alpha_i'\right) = 0 = \Sigma\left(a_i^2 D_i + 2a_i\alpha_i\alpha_i'\right)$.
if ∞ is an ordinary pt

$n=1$: may set $a_1=0$, get $w'' + \frac{2}{z}w' = 0 \Rightarrow w = \frac{a}{z}+b$

$n=2$: may set $a_1=0$, $a_2=\infty$. $** \Rightarrow w'' + \frac{1-(\alpha+\alpha')}{z} w' + \frac{\alpha\alpha'}{z^2} w = 0$

$\Rightarrow w = Az^{\alpha}+Bz^{\alpha'}$ if $\alpha \neq \alpha'$; or $w = Az^{\alpha} + Bz^{\alpha}\log z$ $(\alpha=\alpha')$

The 1st interesting case is $n=3$: Hypergeometric Eq'n.

A moment thought using $*$ leads to

$$0 = w'' + \left(\frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c}\right) w' + \left(\frac{\alpha\alpha'(a-b)(a-c)}{z-a} + \frac{\beta\beta'(b-c)(b-a)}{z-b} + \frac{\gamma\gamma'(c-a)(c-b)}{z-c}\right) \frac{w}{(z-a)(z-b)(z-c)}$$

with single relation $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma' = 1$
specializing to $(a,b,c) = (0,1,\infty)$, get

no other freedom besides exponents ie local monodromy

$$HG: w'' + \left(\frac{1-\alpha-\alpha'}{z} + \frac{1-\beta-\beta'}{z-1}\right) w' + \left(\frac{\alpha\alpha'}{z^2} + \frac{\beta\beta'}{(z-1)^2} - \frac{\alpha\alpha'+\beta\beta'-\gamma\gamma'}{z(z-1)}\right) w = 0$$

Multiplying the function elements w by $z^{-\alpha}(z-1)^{-\beta}$, get
(via Riemann symbol)

$$z^{-\alpha}(z-1)^{-\beta}\, P\begin{Bmatrix} 0 & 1 & \infty & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & \gamma+\alpha+\beta & z \\ \alpha'-\alpha & \beta'-\beta & \gamma'+\alpha+\beta & \end{Bmatrix}$$

Denote $a = \alpha+\beta+\gamma$, $b = \alpha+\beta+\gamma'$, $\alpha'-\alpha = 1-c \Rightarrow \beta'-\beta = c-a-b$
we get the classical (Gauss) HG eq'n:

$$z(1-z)w'' + (c-(a+b+1)z)w' - ab\,w = 0$$

with one sol $F(a,b,c,z) := 1 + \frac{a\cdot b}{1\cdot c}z + \frac{a(a+1)\cdot b(b+1)}{1\;2\;\; c(c+1)} z^2 + \cdots$

if $c \notin \mathbb{Z}_{\leq 0}$ $F \in \mathbb{C}[z]$ if a or $b \in \mathbb{Z}_{\leq 0}$, otherwise conv. radius $=1$

Theorem (Riemann) Any 2 dim space of global analytic functions with prescribed exponents at $0, 1, \infty$ is given precisely by the solution elements of HG
st. $\alpha_i - \alpha_i' \notin \mathbb{Z}$, $\Sigma\, \alpha_i+\alpha_i' = 1$

idea of pf: Given (f_1, Ω), (f_2, Ω) and (f, Ω) get

$cf + c_1 f_1 + c_2 f_2 = 0$ and also f', f'' $\Rightarrow \begin{vmatrix} f & f_1 & f_2 \\ f' & f_1' & f_2' \\ f'' & f_1'' & f_2'' \end{vmatrix} = 0 \Rightarrow f'' = pf' + q$
analyze str of p, q ..

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