Weierstrass Elliptic functions (revisited) 3/17

$\zeta(z)$ "$:=$" $-\int^z \wp(w)\,dw$ well-defined since $\oint \frac{dw}{w^2} = 0$

$$:= \frac{1}{z} + \sum_{w\in\Lambda}{}' \left(\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2} \right)$$

$\wp(z;\Lambda)$ on $\mathbb{C}/\Lambda$

today next time

$$= \frac{1}{z} - E_4 z^3 - E_6 z^5 - \cdots$$ odd function (no const term)

but $\zeta(z+w_i) =: \zeta(z) + \eta_i$ not periodic $i = 1, 2$

quasi-periods $\eta_i = 2\zeta\left(\frac{w_i}{2}\right)$ (say, put $z = -\frac{w_i}{2}$)

$$\sigma(z) \text{ "}{:=}\text{" } e^{\int^z \zeta(w)\,dw} = e^{\log z}\, e^{-\frac{1}{4}E_4 z^4 - \frac{1}{6}E_6 z^6 - \cdots}$$

$$\text{or} \quad = e^{\log z} \prod_{w\in\Lambda}{}' e^{\log(z-w) - \log(-w) + \frac{z}{w} + \frac{1}{2}\frac{z^2}{w^2}}$$

$$= z \prod_{w\in\Lambda}{}' \left(1 - \frac{z}{w}\right) e^{\frac{z}{w} + \frac{1}{2}\frac{z^2}{w^2}}$$

Hadamard product
entire fcn order 2
simple zero at Λ

ie $(\log\sigma)' = \zeta$, so

$$\left(\log \frac{\sigma(z+w_i)}{\sigma(z)}\right)' = \zeta(z+w_i) - \zeta(z) = \eta_i$$

$$\Rightarrow \sigma(z+w_i) = \sigma(z)\, e^{\eta_i z + c_i}, \quad z = \frac{-w_i}{2} \Rightarrow e^{c_i} = e^{\pi i + \frac{1}{2}\eta_i w_i}$$

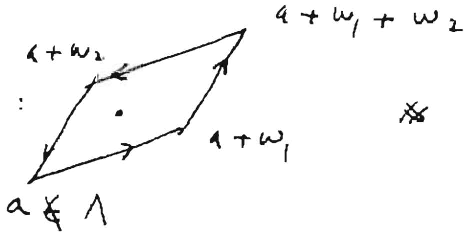
Hence get "theta fcn" like transformation law

$$\sigma(z+w_i) = -\sigma(z)\, e^{\eta_i\left(z + \frac{w_i}{2}\right)}$$

Theorem (Legendre period relation)

$$\begin{vmatrix} \eta_1 & \eta_2 \\ w_1 & w_2 \end{vmatrix} = \eta_1 w_2 - \eta_2 w_1 = 2\pi i$$

pf $2\pi i = \int_{\partial P} \zeta(w)\,dw$ on P:



Ex Important role of $\sigma(z)$:

Any elliptic function on $\mathbb{C}/\Lambda$ takes the form $f(z) = c\,\frac{\prod_{i=1}^{n}\sigma(z-a_i)}{\prod_{i=1}^{n}\sigma(z-b_i)}$

$a_1, \dots, a_n \in \mathbb{C}$, $[a_i]$ may be repeated, $b_1, \dots, b_n \in \mathbb{C}$, $[b_i]$ may be repeated, such that $\sum a_i = \sum b_i$ in $\mathbb{C}$ (not $\mathbb{C}/\Lambda$)

3/19 Modular functions (for $\Gamma(2)$)

Recall $\omega_3 = \omega_1 + \omega_2$, $e_i := \wp\left(\frac{\omega_i}{2}\right)$, $e_1 + e_2 + e_3 = 0$

changing basis $\omega_2' = a\omega_2 + b\omega_1$, $\omega_1' = c\omega_2 + d\omega_1$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$

permutes e_i's. In fact fixed for $\left\{ g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2 \right\} = \Gamma(2)$

Let $\lambda(\tau) := \dfrac{e_3 - e_2}{e_1 - e_2}$ Set $\omega_1 = 1$, $\omega_2 = \tau$

$\neq 0, 1, \infty$ since e_i's differ

On generators $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, get

$$(*) \quad \lambda(-1/\tau) = 1 - \lambda(\tau), \quad \lambda(\tau+1) = \frac{\lambda(\tau)}{\lambda(\tau) - 1}, \quad \text{periodic } \tau \mapsto \tau + 2$$

$$e_3 - e_2 = \sum_{m,n} \frac{1}{\left((m-\frac{1}{2}) + (n-\frac{1}{2})\tau\right)^2} - \frac{1}{\left(m + (n-\frac{1}{2})\tau\right)^2} = \pi^2 \sum_n \frac{1}{\cos^2\pi(n-\frac{1}{2})\tau} - \frac{1}{\sin^2\pi(n-\frac{1}{2})\tau}$$

$$e_1 - e_2 = \sum_{m,n} \frac{1}{\left((m-\frac{1}{2}) + n\tau\right)^2} - \frac{1}{\left(m + (n-\frac{1}{2})\tau\right)^2} = \pi^2 \sum_n \frac{1}{\cos^2 \pi n\tau} - \frac{1}{\sin^2\pi(n-\frac{1}{2})\tau}$$

leading terms in $e^{\pi i \tau} =: q$ for $\Gamma(2)$, coordinate at ∞ — unif conv for $\operatorname{Im}\tau \geq \delta > 0$

for $\operatorname{Im}\tau \to \infty$: $\lambda(\tau) \sim \dfrac{2\pi^2}{\pi^2} \cdot 4 e^{\pi i \frac{\tau}{2} 2} \sim 16 e^{\pi i \tau} + \cdots \to 0$

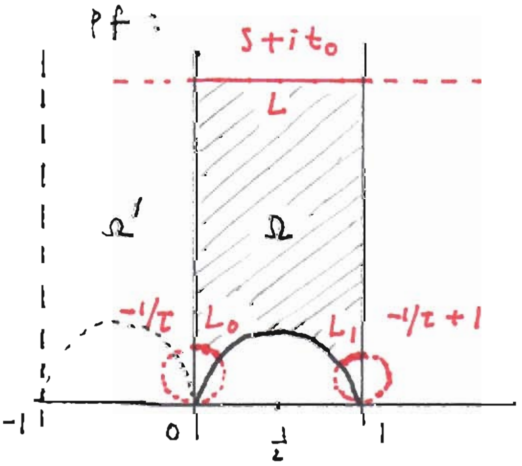
$e_1 - e_2 \to \pi^2$ $(n=0)$ $\qquad$ $(n=0,1)$ $\qquad$ rational coefficients power series in q

$(*) \Rightarrow \lambda(\tau) \to 1$ as $\tau \to 0$ from above, $\lambda(\tau) \to \infty$ as $\tau \to 1$

Theorem: Let $\Omega = \{ \tau \in \mathbb{H} \mid 0 < \operatorname{Re}\tau < 1, \ |\tau - \frac{1}{2}| > \frac{1}{2} \}$

Then $\lambda : \Omega \xrightarrow{\sim} \mathbb{H}$. Moreover λ extends $\bar{\Omega}$ st λ is real on $\partial\Omega$, $(0, 1, \infty) \mapsto (1, \infty, 0)$

Pf:



Map L to a portion of circle near 1 by $-1/\tau + 1$, say L_1

Then λ maps L_1 to

$$\lambda(-1/\tau + 1) = \frac{\lambda(-1/\tau)}{\lambda(-1/\tau) - 1} = \frac{1 - \lambda(\tau)}{-\lambda(\tau)}$$

$$= 1 - 1/\lambda(\tau) \sim 1 - \frac{1}{16} e^{-\pi i \tau}$$

i.e. half circle in $\mathbb{H}$ with large radius

Hence any point in $\mathbb{H}$ has winding number 1 wrt $\lambda(\partial\Omega)$. Argument principle $\Rightarrow$ Thm $\square$

Denote by $\Gamma(N) = \{ g \in SL(2,\mathbb{Z}) \mid g \equiv I_2 \mod N \}$

Theorem : $\bar{\Omega} \cup \Omega'$ is a fund domain of $\Gamma(2)$ in $\mathbb{H}$,

Also, λ induces modular curve $X(2) := \Gamma(2)\backslash\mathbb{H} \xrightarrow{\sim} \mathbb{C}\setminus\{0,1\}$

By adding 3 "cusps", get $\overline{X(2)} \simeq \mathbb{C}\mathbb{P}^1$

For the pf, one needs $SL(2,\mathbb{Z})/\Gamma(2)$ has representatives

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\-1&0\end{pmatrix}, \begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}1&-1\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\-1&1\end{pmatrix}, \begin{pmatrix}1&0\\1&1\end{pmatrix}$$

and the old easier fact on the fund domain of $SL(2,\mathbb{Z})$ □

Def'n A (mero) fcn on $\mathbb{H}$ is automorphic (modular) if it is invariant under a subgroup of $SL(2,\mathbb{R})$ ($SL(2,\mathbb{Z})$)

Ex (Ahlfors) $j(\tau) := 1728 \dfrac{g_2^3}{g_2^3 - 27 g_3^2}$ gives $SL(2,\mathbb{Z})\backslash\mathbb{H} \xrightarrow{\sim} \mathbb{C}$

disc Δ in $\wp'^2 = 4\wp^3 - g_2\wp - g_3$

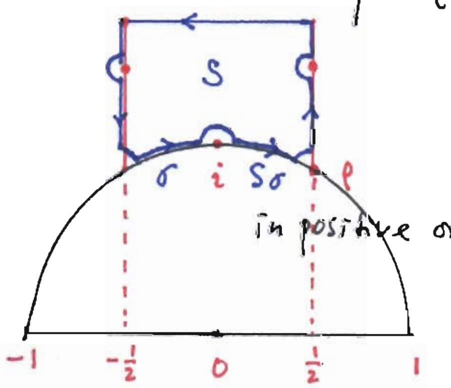
3/24 Modular Forms (for $SL(2,\mathbb{Z})$)

Theorem Let f be modular of wt k wrt $SL(2,\mathbb{Z})$ (k must be even)

Then $v_\infty(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_\rho(f) + \Sigma^* v_p(f) = \frac{k}{12}$

in $q = e^{2\pi i \tau}$ coordinate; in a fund domain

Pf



$$\Sigma^* v_p(f) = \frac{1}{2\pi i}\int_{\partial S} \frac{df}{f}$$

$$\to -\frac{1}{2} v_i(f) - 2\cdot\frac{1}{6} v_\rho(f) - v_\infty(f) + \frac{1}{2\pi i}\int_{\sigma - S\sigma}\frac{df}{f} \quad ★$$

$\frac{1}{2}(-\frac{1}{2})$ form a ⊙ around $q=0$

$$f(S\tau) = \tau^k f(\tau) \Rightarrow \frac{df(S\tau)}{f(S\tau)} = k\frac{d\tau}{\tau} + \frac{df(\tau)}{f(\tau)}, \text{ get } ★ \to \frac{k}{12} \quad □$$

Def'n: M_k = space of modular forms (ie. holo) of wt. k

M_k^0 = cusp forms, ie vanishes at (all) cusps (here ∞).

Recall Einsenstein series: $E_k(\tau) = \sum' \dfrac{1}{(a+b\tau)^k} = 2\zeta(k) + 2\dfrac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$

for $k = 4, 6, 8, \ldots$ In terms of $\dfrac{t}{e^t - 1} =: \sum_{k\geq 0} B_k \dfrac{t^k}{k!}$,

$E_k(\tau) = 2\zeta(k)\left(1 - \dfrac{2k}{B_k}\sum_{n>0}\sigma_{k-1}(n) q^n\right) \in (2\pi i)^k M_{k,\mathbb{Q}}$, call $\tilde{E}_k$

Hence $M_k = M_k^0 \oplus \mathbb{C}E_k$, $k \geq 4$ Clearly $M_k^0 = 0$ for $k < 12$

$\Rightarrow M_k = \mathbb{C}E_k$ for $k = 4, 6, 8, 10$ (eg $\tilde{E}_4^2 = \tilde{E}_8$, $\tilde{E}_4\tilde{E}_6 = \tilde{E}_{10}$)

Theorem $M^{\circ}_{12,\mathbb{Z}} = \mathbb{Z}\,\Delta$ & $\Delta = \underset{\uparrow}{}\,\eta(\tau)^{24} = q \prod_{n=1}^{\infty} (1-q^n)^{24}$

$(2\pi)^{12}$ $\quad$ since wt $\eta = 1/2$

Indeed, $\Delta := g_2^3 - 27 g_3^2 = \frac{(2\pi)^{12}}{2^6 3^3}(\tilde{E}_4^3 - \tilde{E}_6^2) = (2\pi)^{12}(q - 24q^2 + 252q^3 \ldots)$.

- The statement on $\eta(\tau)$ is proved in CA-I, last lecture

Cor 1 $\quad \Delta\, M_{k-12} \simeq M_k^{\circ}$ Also,

The only zero of E_4, E_6, Δ is at ρ, i, ∞ resp and simple

Cor 2 : M_k has a basis $E_4^m E_6^n$ with $4m+6n = k$, $m, n \geq 0$

pf
- Let $k \geq 8$, Always $\exists$ st (m,n)

  $f \in M_k \Rightarrow f - \lambda E_4^m E_6^n \in M_k^{\circ}$ for some λ,

  by Cor 1, this is done by decending induction on k

- If these monomials are linearly dependent,

  say $E_4^{m_1} E_6^{n_1}$, $E_4^m E_6^n$ then $2(m_1 - m) + 3(n_1 - n) = 0$

  ie $n_1 = n + 2\ell$, $m_1 = m - 3\ell$

$\Rightarrow f = E_6^2/E_4^3$ satisfies an alg equation $/\mathbb{C}$

$*$ by looking at the highest order pole at $\tau = i$ $\square$

ie $M = \bigoplus_k M_k \simeq \mathbb{C}[E_4, E_6]$ is a graded poly algebra

Back to $j = 1728 \frac{g_2^3}{\Delta}$ modular of wt. $= 0$

holomaphic on $\mathbb{H}$, simple pole at ∞ $(q = 0)$

Thm $\quad j \quad SL(2,\mathbb{Z}) \backslash \mathbb{H} \xrightarrow{\sim} \mathbb{C}$ (Ex prove by dim formula)

Cor $\quad$ Any modular function is a rat'l function of $j(\tau)$,

hence quotient of 2 modular forms of the same wt

Similarly, any modular function wrt $\Gamma(2)$ is rat'l in $\lambda(\tau)$

Summary : First Interview with modular curves (defined $/\mathbb{Q}$)

1) $\overline{X} = \overline{SL(2,\mathbb{Z}) \backslash \mathbb{H}} \overset{j}{\simeq} \mathbb{C}P^1$ has one cusp $[\infty]$, 2 orbifold points

$[i]$ with stabilizer $\simeq \mathbb{Z}/2$, $[\rho]$ with $\mathbb{Z}/3$

2) $\overline{X(2)} = \overline{\Gamma(2) \backslash \mathbb{H}} \overset{\lambda}{\simeq} \mathbb{C}P^1$ has 3 cusps $[0], [1], [\infty]$, 2 orbi. pts $[i], [-i]$

3) $\overline{X_0(2)} = \overline{\Gamma_0(2) \backslash \mathbb{H}} \simeq \mathbb{C}P^1$ has one cusp $[\infty]$, one orbi pt $[i]$

This is discussed in CA-I $\quad$ But in general $\overline{X(N)}, \overline{X_0(N)} \not\simeq \mathbb{C}P^1$