3/10, 2015

Green functions & Uniformizations of Riemann Surfaces

starting pt $\Omega \Subset \mathbb{C}$, u solves $u = \log|z-q|$ in $\partial\Omega$, $\Delta u = 0$ in Ω then $g(p,q) = -\log|p-q| + u(p)$

2nd semester of complex analysis (2015 spring) RMT is simply $e^{-(g(p,0)+i\tilde{g})}$, harmonic conjugate

Q: $e^{\pi} - \pi = \underline{19.999}\,0\,\underline{999}\,79189$ (e^{π} Gelfand constant) Q: Reason for this ?

Solve $e^x - x - 20 = 0$ $\quad f(x) = e^x - x - 20$

$$F(x) = x - \frac{f(x)}{f'(x)} = x - \frac{e^x - x - 20}{e^x - 1} = \frac{xe^x - e^x + 20}{e^x - 1}$$ why should $x \sim \pi$?

Green function on S : a Riemann surface $\nearrow$ in Gamelin's def (p.427)

let $q \in S$, $\mathcal{F}_q := \{ u$ subharmonic fcn on $S \setminus q$, cpt sup in S, and $u + \log|z(p)|$ subharmonic on some coor chart z of $q \}$

Let $g(p,q) := \sup\{ u(p) \mid v \in \mathcal{F}_q \}$ be the (pointwise) upper envelope

Thm If $g(p,q)$ exists (ie not $+\infty$ everywhere), then

1) $g(p,q) > 0$, harmonic for $p \neq q$
2) $g(p,q) + \log|z(p)|$ is harmonic at q
3) if $h(p)$ satisfies both 1) and 2), then $h \geq g$ on $S \setminus q$

1) the pf is identical to the case $S \subset \mathbb{C}$ via Perron method

2 let $(z, B_\rho(q)) \subset S$ $\quad v := \begin{cases} -\log|z(p)| + \log\rho, & p \in B_\rho(q) \\ 0 & p \notin B_\rho(q) \end{cases} \Rightarrow v \in \mathcal{F}_q \Rightarrow g(p,q) \geq v(p) \to +\infty$ as $p \to q$

hence $g(p,q) > 0$ for $p \in S \setminus \{q\}$ (min p

Let $M = \max\limits_{|z(p)|=\rho} g(p,q$, then $u + \log|z| \leq M + \log\rho$ on $\partial B_\rho(q$, $\forall u \in \mathcal{F}_q$

sh on $B_\rho(q$ $\Rightarrow$ $\leq$ holds on $p \in B_\rho(q$ too

Take sup over $u \in \mathcal{F}_q \Rightarrow 2)$, since isolated sing of bdd har is removable

3) $u \in \mathcal{F}_q \Rightarrow u - h$ sh on S, < 0 outside a cpt set MP $\Rightarrow u - h < 0$ on S *

Thm $g(p,q)$ defined for one q $\Rightarrow$ for all $q \in S$ (assume S is connected)

Moreover, $g(p,q) = g(q,p)$

Need a lemma fix $q_0 \in S$, $z, B_r = \{|z| \leq r\}$ defined, $z(q_0) = 0$

let $\mathcal{F} := \{ u$ subharmonic fcn on $S \setminus B_r$ st $u \leq 1$ & supp u cpt $\}$

be a Perron family, $w(p) := \sup\limits_{u \in \mathcal{F}} u(p)$ for $p \in S \setminus B_r$ $\;\in [0,1]$

ie solving Dirichlet problem $= 1$ on ∂B_r, $= 0$ at ∞ since barrier exists on each pt in ∂B_r ($\Rightarrow \mathcal{F} \neq \phi$)

Lemma (i) If $g(p,q)$ exists for some $q \in B_r^\circ$, then $0 < w < 1$ on $S \setminus B_r$

(ii) Conversely, $0 < w < 1$ on $S \setminus B_r \Rightarrow g(p,q)$ exists $\forall q \in B_r^\circ$

Pf : (i) let $g(p,q) \geq c > 0$ on ∂B_r

then $u \in \mathcal{F} \Rightarrow u(p) \leq g(p,q)/c$

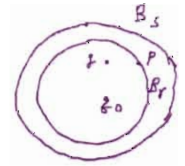
Now $\inf\limits_{p \in S} g(p,q) = 0$ (if $= a > 0$ then $g(p,q) - a < g(p,q)$ satisfies 1),2) but $\not\to$ to 3))

$\Rightarrow \inf\limits_{p \in S} u(p) = 0$ too and we get $0 < w < 1$ (not $\equiv 1$)

(ii) let $s > r$ st B_s is defined, and let $w(p) \leq k < 1$ on ∂B_s

let $|\log|z(p) - z(q)|| \leq C$ on $p \in \partial B_r$

for $u \in \mathcal{F}_q$, let $M = \max\limits_{p \in \partial B_s} u(p)$



$u(p) + \log|z(p) - z(q)|$ is subharmonic on B_s (sh is a local condition ind of charts)

$\leq M + C$ on ∂B_s (hence on B_s)

$\Rightarrow$ $u \leq M + 2C$ on ∂B_r i.e. $u(p) \leq (M+2C)\,w(p)$ on ∂B_r since $w \equiv$ on ∂B_r

$\Rightarrow$ $M \leq (M+2C)\,k$ on $\partial B_s \subset S \setminus B_r$ hence on $p \in S \setminus B_r$

i.e. $M \leq \dfrac{2Ck}{1-k}$ and so $u(p) \leq \dfrac{2Ck}{1-k}$ for $p \in \partial B_s$

for All $u \in \mathcal{F}_q$, hence $g(p,q)$ exists ✲

Cor The sets q st $g(p,q)$ exists or does not exist are both open,
hence if S is connected, then 1st part of Thm follows

Cor If $R \subset S$ is open with $\partial R \supset$ "an arc", then g for R exists
"Pf" subharmonic barrier exists for every point in the sm locus of the arc Γ,
hence $w(p) \to 0$ as $p \to \Gamma$, so we are in the case $0 < w < 1$ ✲ by piecewise analytic arcs

Striking Theorem of Rado S any RS $\Rightarrow$ $S = \bigcup_{i=1}^{\infty} S_i$, S_i finite bordered R.S (open)
($S_i \cup \partial S_i$ cpt hence countable top) *

Pf: Suppose $g(p,q)$ exists. Fix $p_0 \neq q$ $u_n \in \mathcal{F}_q$ st $u_n(p_0) \longrightarrow g(p_0, q)$
Let S_n be a f b R S $\subset S$ st $u_n \equiv 0$ outside a cpt subset of S_n $S_n \ni p_0, q$
Also, $S_{n-1} \cup \partial S_{n-1} \subset S_n$ Also assume $S \setminus S_n$ has no cpt component
(simply join it to S_n)

* Remark (1) Every 2-dim rea surface has a Riemann surface structure
by the existence of isothermal coordinates, once a Riemannian
metric exists However, this requires countable topology
(2) In general, $\exists$ of countable basis of topology is imposed as an axiom
of manifolds, either real or complex It is automatic only for R S

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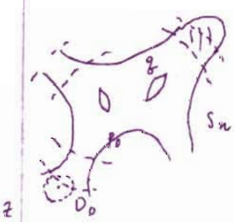


cpt component join to S_n

$g_n(p,q)$ on S_n also exists, $\nearrow$ in n

$u_n(p_0 \leq g_n(p_0, q) \leq g(p_0, q)$

Harnack $\Rightarrow$ $g_n(p,q) \longrightarrow u(p)$ har, unif on cpt Subset of $S \setminus \{q\}$

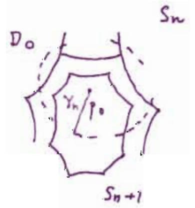
but $u(p) - g(p,q) \leq 0$ and $= 0$ at $p = p_0$

max p $\Rightarrow$ $u(p) \equiv g(p\,q)$ $\Rightarrow$ $g_n(p\,q) \longrightarrow g(p\,q)$ unif on cpt subset of $\bigcup S_n$

Claim $\bigcup S_n = S$ If not, take $p_0 \in S \setminus \bigcup S_n$, and coor chart (z, D_0) with $z(p_0) = 0$
if $\partial S_n \subset D_0$ then $\exists$ a cpt comp of $S \setminus S_n$ in D_0 ✱ This applies to all n!

⟨ Can't follow the argument in Gamelin p.433! ⟩

Idea to consider the $\bigcap_n (\overline{D}_0 \setminus S_n) \neq \phi$ and more than 1 pt p_0
take E to be a continuum $\ni p_0$



$\Rightarrow$ $\exists$ subharmonic barrier at each pt $\in E \cap \partial(S - E)$

$\Rightarrow$ $g_E(p,q) \longrightarrow 0$ as $p \to E$

Green function for $S \setminus E$, which exists & $g_n(p,q) \leq g_E(p,q)$

$\Rightarrow$ $g(p,q) \longrightarrow 0$ as $p \to E$ ✱ hence $E = \phi$

Now suppose $g(p,q)$ does not exist Then find a disk $D_0 \subset S$
and $g(p\,q)$ exists on $S \setminus D_0$, then Rado's thm follows ✲

↑

we may actualy take ∂S_n to be analytic (hence smooth)
by approximating it using leve set of some harmomc function

Symmetry of Green's function : $g(p,q) = g(q,p)$

By Radó's thm, only need to prove the case S is a finite bordered R S

The case $D \subset \mathbb{C}$ a bdd domain with "analytic ∂D"

$U = B_\varepsilon(\zeta_i)$, $D_\varepsilon := D \setminus (U_0 \cup U_1)$, $u_1(z) := g(z, \zeta_1)$ is harmonic for $z \neq \zeta_1$

$$\int_{\partial D} - \int_{\partial U_0} - \int_{\partial U_1} \left(u_0 \frac{\partial u_1}{\partial n} - u_1 \frac{\partial u_0}{\partial n} \right) ds = \int_{D_\varepsilon} \left(u_0 \Delta u_1 - u_1 \Delta u_0 \right) = 0$$

$\overset{\shortparallel}{0}$ since $u_1 = u_0 = 0$ on ∂D

since up to a harm fcn $u(z) \sim -\log|z - \zeta_i|$, for $\varepsilon \to 0$, get $2\pi \left(u_1(\zeta_0) - u_0(\zeta_1) \right) = 0$ done

for $D = S$ a finite bordered R.S simply use the diff form version of Green's identity, or by using triangulations and patching together (omitted) ✻

Thm : Existence of bi-polar Green's functions, $\forall\, q_1 \neq q_2 \in S$, $\exists\, G(p; q_1, q_2)$ harmonic on $S \setminus \{q_1, q_2\}$ and locally $G \sim -\log|z - q_1|$, $G \sim +\log|z - q_2|$

Lemma : $q_1 \neq q_2 \in S$, a finite bordered R.S (B_i, z_i) coor chart at q_i

Then $\forall$ R S $R \supset \bar{S}$ st g_R exists, $\left| g_R(p, q_1) - g_R(p, q_2) \right| \le C$ on $p \in R \setminus (B_1 \cup B_2)$

eg R' is also a finite bordered R.S

pf of Thm : $S = \bigcup S_n$; get $G_n(p; q_1, q_2) := g_n(p, q_1) - g_n(p, q_2)$ on S_n Lemma $\overset{*}{\Rightarrow}$ $\exists$ conv subsequence ✻

Remark (1) $g(p,q)$ does not exist for any compact R.S Max. P to $-g$)

(2) For $S = \mathbb{C}^*$, $-\log z$ gives $G(z; 0, \infty)$

The Lemma can be vividly seen in this specia case (Exercise)

*(3) Any unif bounded sequence of har fcn's are equi-continuous (by applying derivative to Poisson formula, or Cauchy integral formula), hence $\exists$ conv subsequence by Arzela-Ascoli

pf: Let $A_i = B_\rho(q_i) \subset B_i = B_\sigma(q_i)$, $\rho < \sigma$ $\quad M_j = \max\limits_{\partial A_j} g_R(p, q_j)$

$p \in \partial B_j \Rightarrow g_R(p, q_j) + \log|z_j(p)| \le \max\limits_{q \in \partial B_j} g_R(q, q_j) + \log\sigma$ trivially

$\Rightarrow$ holds also for $p \in B$

Take $\sup\limits_{p \in \partial A}$ $\Rightarrow$ $M_j + \log\rho \le \max\limits_{q \in \partial B_j} g_R(q, q_j) + \log\sigma = g_R(p_j, q_j) + \log\sigma$ for some $p_j \in \partial B_j$

ie $M_j - g_R(p_j, q_j) \le \log \sigma/\rho$ for some $p_j \in \partial B_j$

$M_j - g_R(p, q_j) > 0$ and harmonic on $S \setminus (A_1 \cup A_2)$

Harnack inequality $\Rightarrow$ for all $p \in \partial B_1 \cup \partial B_2$ $\quad M_j - g_R(p, q_j) \le C_0$ indep of R

ie $M_j - C_0 \le g_R(p, q) \le M_j$

$g_R(p, q_1)$ harmonic on B_2 hence the above holds for $p \in B_2$ too

in particular, $M_1 - C_0 \le g_R(q_2, q_1) \le M_1$ $\quad \Rightarrow |M_1 - M_2| \le C_0$

$M_2 - C_0 \le \overset{\shortparallel}{g_R(q_1, q_2)} \le M_2$ $\quad \Rightarrow \left| g_R(p, q_1) - g_R(p, q_2) \right| \le 2C_0$, $p \in \partial B_1 \cup \partial B_2$

Since $g_R = 0$ on ∂R, maximal principle $\Rightarrow$ Lemma $\forall\, p \in R \setminus (B_1 \cup B_2)$ with $C = 2C_0$ ✻

proof of the Uniformization theorem (of Koebe and Poincaré)

if $g(p, z)$ for S exists then $g(p, z_0)$ has a log pole at z_0

$\Rightarrow \exists$ analytic $\varphi(p)$ near z_0 st $\quad \varphi(p) = e^{-g(p, z_0)}$

Key point: S is simply connected $\Rightarrow$ φ can be analytically continued to all S — the monodromy theorem

in particular, $|\varphi| < 1$, ie $\varphi : S \to D$, $z_0 \to 0$ simple zero.

It's enough to show that φ is 1-1

for any $z_1 \in S$, let $\psi = \dfrac{\varphi - \varphi(z_1)}{1 - \overline{\varphi(z_1)}\,\varphi}$, then $\psi(z_1) = 0$, $\psi : S \to D$

$\forall u \in \mathcal{F}_{z_1}$, $u + \log|\psi|$ is sh on S (by def of $\mathcal{F}_{z_1}$ & ψ)

and < 0 outside a cpt subset of S, hence on S by max P

Take sup over $u \in \mathcal{F}_{z_1}$ $\Rightarrow$ $g(p, z_1) + \log|\psi(p)| \le 0$ for $p \in S$

$\Rightarrow g(z_0, z_1) + \log|\psi(z_0)| = g(z_0, z_1) + \log|\varphi(z_1)| = g(z_0, z_1) - g(z_1, z_0) = 0$ — by def of ψ & φ

Again, max P $\Rightarrow$ $\log|\psi(p)| = -g(p, z_1)$

in particular, $\psi(p) \neq 0$ for $p \neq z_1$, ie $\varphi(p) = \varphi(z_1) \iff p = z_1$ & φ is 1-1

This finishes the "hyperbolic case", namely when the Green function exists *

Remark $\quad g(p, z)$ exists $\iff S \simeq D = \{|z| < 1\}$, called the hyperbolic case

Notice "~~$\exists$ a lot bounded analytic functions~~" on D, but none on $\mathbb{C}$, S^2

In fact, this also characterize hyperbolicity (in 2 dimensions)

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Similar idea leads to

Lemma: $\exists$ bounded non-const analytic function $\Rightarrow$ Green function exists

Pf: To construct $g(p, z)$ for given $z \in S$, consider analytic f with $f(z) = 0$ & $|f| < 1$ (we choose z st this is a simple zero)

$u \in \mathcal{F}_z \Rightarrow u + \log|f|$ is sh on S & < 0 outside a cpt set $\;$ MP $\Rightarrow < 0$ on S

ie $\;u(p) \le -\log|f(p)|\;$ hence $g(p, z)$ exists *

Pf of Uniformization Thm (remaining case)

Suppose $g(p, z)$ does not exists, hence $\nexists$ bdd analytic function

S simply connected $\Rightarrow \exists$ meromorphic function φ on S st

$\varphi(p) = e^{-G(p; z_1, z_2)}$ which has a simple zero at z_1, a simple pole at z_2

in particular $\frac{1}{C} \le |\varphi(p)| \le C$, $p \in S \setminus (B_1 \cup B_2)$

Claim: φ is 1-1

For $z_0 \neq z_1, z_2$, set $\varphi_0(p) = e^{-G(p; z_0, z_2)}$

$\Rightarrow \psi(p) = \dfrac{\varphi(p) - \varphi(q_0)}{\varphi_0(p)}$ has no pole, bdd analytic $\Rightarrow \equiv$ const $C \neq 0$

ie $\varphi(p) = \varphi(q_0) \iff p = q_0$ *

Finally $\quad \varphi(S) \subset \mathbb{C} \cup \{\infty\} =: \mathbb{C}^* \simeq S^2$ is a simply connected domain containing ∞

if $S^2 \setminus \varphi(S)$ has more than 1 pt, RMT $\Rightarrow \varphi(S) \sim$ disk $\Rightarrow$ Green exists ⨯

Hence $\varphi(S) \simeq S^2$ or $S^2 \setminus \{pt\} \simeq \mathbb{C}$ $\quad$ The pf of UT is complete $\square$

EX Determine $Aut(D)$, $Aut(\mathbb{C})$ and $Aut(\mathbb{C}^*)$, and fixed-point-free subgroup

$\overset{\text{ST}}{Aut(\mathbb{H})}$