Complex Analysis II (2015 Spring at NTU)

a course by Chin-Lung Wang

§ Dirichlet problem / Riemann mapping (2/26)

Recall the Poisson integral formula

(Stein CA p.67, p.109 ; Ahlfors Thm 22, p.168, or G-T)

$u \in C(\overline{B_R(0)})$ and $\Delta u = 0$ in $B_R(0)$, then

$$(*) \quad u(z) \equiv P_{u|_{\partial B_R}}(z) := \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}\, u(Re^{i\theta})\, d\theta$$

conversely any $U \in C(\partial B_R)$, $u := P_U$ solves the Dirichlet problem

Ex. (Schwartz) why $\lim_{z\to \zeta \in \partial B_R} P_U(z) = U(\zeta)$?

Also, deduce $(*)$ from MVT & Mobius transform.

e.g.

if $z = \rho e^{i\varphi}$

Since
$$\frac{R-\rho}{R+\rho} \le \frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}\left(= \frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\theta-\varphi)}\right) \le \frac{R+\rho}{R-\rho}$$

if $U \ge 0$ on ∂B_R, then $* \Rightarrow$ Harnack inequality

$$\frac{R-\rho}{R+\rho}\, u(0) \le u(z) \le \frac{R+\rho}{R-\rho}\, u(0)$$

Cor. let u_n non-decreasing, harmonic on Ω

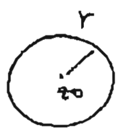
then either $u_n(z) \nearrow \infty$ unif on cpt subset,

or $u_n \nearrow u$ a harmonic function, unif on cpt subset.

In fact, a conti fcn is harmonic $\Leftrightarrow$ MVT holds

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta \qquad \forall z_0, r$$

say,



construct harmonic v in $B_r(z_0)$ using Poisson

then max/min $\Rightarrow$ $v \equiv u$ in $B_r(z_0)$.

Pf. Harnack $\Rightarrow$ $u_n - u_{n-1} \ge 0$ has unif conv sum u on cpt set

if $u \ne \infty$, pass to $\frac{1}{2\pi}\int$ get MVT for u $\Rightarrow$ u harmonic ※

Def. S.h : $v \in C(\Omega)$ is subharmonic if $\forall$ harmonic u in $\Omega' \subset \Omega$, $v - u$ st. max principle holds : i.e. no max in Ω' unless const.

Fact : enough to check locally.

Ex. $\Delta v > 0 \Rightarrow v$: s.h. (for C^2, $\Leftarrow$)

Th. v sh $\Leftrightarrow$ $v(z_0) \le \frac{1}{2\pi}\int_0^{2\pi} v(z_0 + re^{i\theta})\, d\theta \quad \forall r$.

pf $\Leftarrow$ trivial (since u st. $=$)

$\Rightarrow$ consider $P_v(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}\, v(re^{i\theta})\, d\theta$ ($z - z_0$, $z_0 +$) Poisson integral

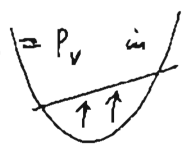
$v - P_v = 0$ on $B_r(z_0)$

if max $> 0 \Rightarrow$ in interior, $\ne$ const ✗

hence $v \le P_v$ in $B_r(z_0)$, in particular for $z = z_0$ ✗

Facts : 1. v s.h. $k \ge 0 \Rightarrow kv$ s.h.
2. v_1, v_2 s.h $\Rightarrow v_1 + v_2$ s.h.) $\Leftarrow$ Th.
3. $\max(v_1, v_2)$ s.h (by def)
4. harmonic lift $P_{\Delta,v}$ is s.h. for any disk $\bar\Delta \subset \Omega$

Perron family, $\Omega \subset \mathbb{C}$ bdd open & connected. $P_{\Delta,v} = P_v$ in Δ
f bdd fun on $\Gamma = \partial\Omega$



$\mathcal{F}_f = \{ v \in C(\Omega) \mid v$ s.h. & $\overline{\lim}_{z \to \zeta \in \Gamma} v(z) \le f(\zeta) \}$ $= v$ in $\Omega \setminus \Delta$

$\ne \emptyset$ since if $|f| \le M$ then $-M \in \mathcal{F}_f$.

Lemma 1 : $u(z) := \sup_{v \in \mathcal{F}_f} v(z)$ is harmonic in Ω.

pf : $v \in \mathcal{F}_f \Rightarrow v \le M$ in Ω : let $E_\varepsilon = \{ z \mid v(z) \ge M + \varepsilon \}$

$\Rightarrow E_\varepsilon$ is closed, bdd (cpt) then E_ε^c is open

if $E_\varepsilon \ne \emptyset$, take max of v in E_ε, hence in Ω ✗

take $\varepsilon \to 0$ get $v \le M$.

let $z_0 \in \Delta$, $\bar\Delta \subset \Omega$, by def, $\exists\, v_n \in \mathcal{F}_f$ st $v_n(z_0) \longrightarrow u(z_0)$.

set $V_n = \max(v_1, \dots, v_n) \in \mathcal{F}_f$ and $V_n \nearrow$ in n

$V_n' := P_{\Delta, V_n} \in \mathcal{F}_f \nearrow$ in n non-decreasing

$v_n(z_0) \le V_n(z_0) \le V_n'(z_0) \le u(z_0) \Rightarrow V_n'(z_0) \longrightarrow u(z_0)$

Harnack $\Rightarrow V_n' \longrightarrow$ har. U in Δ.

~~for another $z_1 \in \Delta$~~, let $w_n(z_1) \to u(z_1)$

do one more step: $\bar{w}_n = \max(v_n, w_n)$ first

$W_n = \max(\bar{w}_1, \cdots, \bar{w}_n)$

and then W_n' via Poisson integral, then har. limit U_1.

then $U \le U_1 \le u$ and $U_1(z_1) = u(z_1)$

$U - U_1$ has max $= 0$ at z_0, hence $U \equiv U_1$

and so $u(z_1) = U(z_1)$ $\forall z_1 \in \Delta$.

Hence, U is har. in any disk Δ, hence Ω ※

- if U solves the Dirichlet problem with ∂-value f then $U \in \mathcal{F}_f$ and so $u \ge U$. But $U - u \le 0$ by def (max. p.) $\Rightarrow$ any sol (! if $\exists$) must coincide with Perron sol.

3/3. ~~Lemma 2~~: Suppose $\exists$ har. ω in Ω, cont. on $\bar{\Omega}$ st $\omega(\zeta_0) = 0$, $\zeta_0 \in \Gamma$ and $\omega(\zeta) > 0$ $\forall \zeta \in \Gamma \setminus \{\zeta_0\}$ (~~Barrier at ζ_0~~)

if f is cont. at ζ_0, then $\lim_{z \to \zeta_0} u(z) = f(\zeta_0)$.

>0 or <0 is NOT important for har fcn.

Pf: $\overline{\lim}_{z \to \zeta_0} u(z) \le f(\zeta_0) + \varepsilon$ $\forall \varepsilon > 0$:

take Δ nbd of ζ_0 st. $|f(\zeta) - f(\zeta_0)| < \varepsilon$ for $\zeta \in \Delta$

ω has pos. min $\omega_0 > 0$ in $\Omega \setminus (\Omega \cap \Delta)$ (max. p. of ω)

$W(z) := f(\zeta_0) \underline{+} \varepsilon \underline{+} \frac{\omega(z)}{\omega_0}(M \mp f(\zeta_0))$; ∂-values:

$\zeta \in \Delta$: $W(\zeta) \underset{\le}{\ge} f(\zeta_0) \underline{+} \varepsilon \underset{<}{\ge} f(\zeta)$

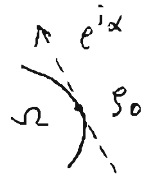
$\zeta \notin \Delta$: $W(\zeta) \underset{\le}{\ge} f(\zeta_0) \underline{+} \varepsilon \underline{+} M - f(\zeta_0) \underset{<}{\ge} f(\zeta)$

$\Rightarrow$ $v(z) < W(z)$ $\forall v \in \mathcal{F}_f$ $\Rightarrow$ $u \underset{\ge}{\le} W$ ← Red: Use W har. $\Rightarrow W \in \mathcal{F}_f \Rightarrow W \le u$

$\Rightarrow$ $\overline{\lim}_{z \to \zeta_0} u(z) \underset{\ge}{\le} W(\zeta_0) = f(\zeta_0) \underline{+} \varepsilon$ ※

for $\underline{\lim}_{z \to \zeta_0} u(z) \ge f(\zeta_0) - \varepsilon$, simply use red color ※

~~Examples~~: (1)



$\omega(z) := \mathrm{Im}\, e^{-i\alpha}(z - \zeta_0)$

Rmk: for $\mathbb{R}^n$ (1) works, but not (2).

(2)

$\mathbb{C} \cup \{\infty\} \setminus \overline{\zeta_0 \zeta_1}$ is simply connected

$\Rightarrow \sqrt{\frac{z - \zeta_0}{z - \zeta_1}}$ has a single-valued $f(z)$, a branch on it, valued in a half plane $\Rightarrow$ $\exists \alpha$ st. $\omega(z) := \mathrm{Im}\, e^{-i\alpha} f(z)$.

~~Rmk~~: Barrier can be defined locally using ~~s.h. fcn~~ in $\Omega \cap \Delta$, $\underline{\le 0 \text{ in } \partial}$ $\underline{\text{outside } \zeta_0}$.

3|3 cont. ~~Riemann's pf of RMT~~

$E \subset \mathbb{C}^* \equiv S^2$ a continuum if E is cpt & simply conn & $\neq$ one pt

<u>lemma</u>: let $\zeta_0 \in \Gamma = \partial\Omega$, if ζ_0 is in a continuum E in $\mathbb{C} \setminus \Omega$, then ζ_0 is regular.

pf: let $\zeta_0 \neq \zeta_1 \in E$. $S^2 \setminus E$ is simply conn.

$\Rightarrow \log \dfrac{z-\zeta_0}{z-\zeta_1}$ has a branch $f(z)$ on $\mathbb{C} \setminus E$

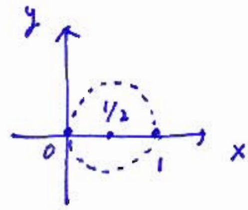
$$\zeta_0 \in D_0 = \left\{ \left| \frac{z-\zeta_0}{z-\zeta_1} \right| < \frac{1}{e} \right\} \text{ is a disk } \xrightarrow[w = f(z)]{} \{ \mathrm{Re}\, w < -1 \}$$

$$\Big\downarrow \; u = \frac{-1}{w}$$

$\Rightarrow -\mathrm{Re}\,\dfrac{1}{f}$ is a <u>harmonic</u> <u>barrier</u> at $\zeta_0 \longrightarrow 0$ ✱

ps. weaker than Ahlfors' def'n

<u>Now, in the pf of RMT</u>, may assume Ω is bonded (step 1), $0 \in \Omega$ & 1-connected. so all pts on $\partial\Omega$ are regular.



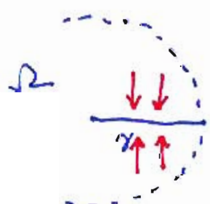
Solve $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u(\zeta) = \log|\zeta| & \text{in } \partial\Omega \end{cases}$ with v harmonic conjugate in Ω

let $\varphi(z) = z\, e^{-(u(z) + i v(z))}$ hol in Ω

- $|\varphi(z)| \to 1$ as $z \to \partial\Omega$ by construction
  $\Rightarrow |\varphi(z)| < 1$ for $z \in \Omega$
- $\varphi(z)$ has only a zero at $z = 0$, which is simple
  argument principle $\Rightarrow \varphi(z) - w$ has simple zero $\forall\, w \in B_1$ ✱

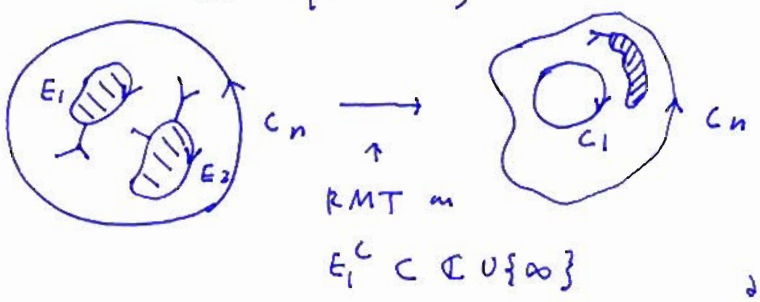
<u>Rmk</u> (cf. Gamelin p. 404-405, Gilbarg-Trudinger p. 26)

subharmonic barrier w is required to have "limit" < 0 on $\partial\Omega \cap \Delta \setminus \{\zeta_0\}$ only, hence the ∂-value in solving the Dirichlet problem can be assigned with <u>different values</u> if γ is a "common ∂" as in LHS picture.



let $\Omega \subset \mathbb{C}$ bdd conn. open of connectivity $n > 1$:
$\mathbb{C} \setminus \Omega = E_1 \cup \dots \cup E_n$, E_n unbdd component
(each E_i is simply conn.)

$E_n^c \xrightarrow{\sim} D$ by RMT st.
in $\mathbb{C}$ $\quad E_n = \{|z| \geq 1\}$



key point: C_n is mapped to an analytic curve
keep going $E_2^c \subset \mathbb{C} \cup \{\infty\}$ etc. get all C_i's are analytic.
$\partial \Omega =: C = C_1 + \dots + C_n$.

Def^n: Harmonic measure $\omega_k(z)$ of C_k wrt Ω:

$$\begin{cases} \Delta \omega_k = 0 & \text{in } \Omega \\ \omega_k = 1 \text{ on } C_k & \text{and } = 0 \text{ on } C_i,\ i \neq k \end{cases} \Rightarrow \begin{array}{l} 0 < \omega_k < 1 \text{ in } \Omega \\ \omega_1 + \dots + \omega_k \equiv 1. \end{array}$$

Moreover, ω_k is harmonic in a larger region $\Omega' \supset \Omega$ by reflection principle on each C_i.

$C_1, \dots, C_{n-1}$ is a basis of $H_1(\bar{\Omega}, \mathbb{Z})$.

Def^n: conjugate (harmonic) differential $\quad {}^*d\omega_k := -\frac{\partial \omega_k}{\partial y} dx + \frac{\partial \omega_k}{\partial x} dy$

or wrt. outer normal n: $\quad \equiv \frac{\partial \omega_k}{\partial n} ds$

Fact: If u has conj har fcn v, then ${}^*du = dv$.
in general, v may not be singled valued, then use ${}^*du$.

3/5 Claim: No $\lambda_1 \omega_1 + \dots + \lambda_{n-1} \omega_{n-1}$ has singled-valued conjugate unless $\lambda_1 = \dots = \lambda_{n-1} = 0$. $(\lambda_i \in \mathbb{R})$

Pf: If $\lambda_1 \omega_1 + \dots + \lambda_{n-1} \omega_{n-1} = \mathrm{Re}(f)$, then f extends to $\Omega' \supset \Omega$.
but $\mathrm{Re}(f)|_{C_i} = \lambda_i$, $1 \leq i \leq n-1$ and $\mathrm{Re}(f)|_{C_n} = 0$
i.e. each C_i is mapped to a line segment
let $w_0 \notin$ any line segment above, so $\arg(f - w_0)$ is defined (singled-valued) on each C_i.
Argument. P. $\Rightarrow$ $f(z) \neq w_0$ in Ω for all such w_0 $\times$
unless $f \equiv$ const and $\mathrm{Re}(f) \equiv 0$, i.e. $\lambda_i = 0 \ \forall i$.

Cor. Let $\alpha_{kj} := \int_{C_j} {}^*d\omega_k$ be the "periods". Then
$\lambda_1 \alpha_{1j} + \dots + \lambda_{n-1} \alpha_{n-1\,j} = 0$ for $j = 1, \dots, n-1$ $\Rightarrow \lambda_i = 0 \ \forall i = 1, \dots, n-1$.

Pf: Any non-trivial sol $\Rightarrow \lambda_1 \omega_1 + \dots + \lambda_{n-1} \omega_{n-1}$ has single-value conj $\times$

In particular, $\exists !$ Sol to

$$
(*) \quad
\begin{aligned}
\lambda_1 \alpha_{11} + \cdots + \lambda_{n-1} \alpha_{n-1,1} &= 2\pi \\
\lambda_1 \alpha_{12} + \cdots + \lambda_{n-1} \alpha_{n-1,2} &= 0 \\
\vdots \qquad\qquad & \quad \vdots \\
\lambda_1 \alpha_{1,n-1} + \cdots + \lambda_{n-1} \alpha_{n-1,n-1} &= 0
\end{aligned}
$$

and hence $\Rightarrow \lambda_1 \alpha_{1,n} + \cdots + \lambda_{n-1}\alpha_{n-1,n} = -2\pi$ since $\alpha_{k1} + \cdots + \alpha_{kn} = 0$.

i.e. int. f is multiple valued with period $2\pi i$ on C_1, $-2\pi i$ on C_n.

$= 0$ on other C_i's. $\mathrm{Re}(f) = \lambda_k$ on C_k, $\lambda_n = 0$.

$\Rightarrow F(z) := e^{f(z)}$ is single valued.

Thm: $F : \Omega \xrightarrow{\sim} \{ 1 < |w| < e^{\lambda_1} \} \setminus \bigcup_{i=2}^{n-1} (\text{arc in } |w| = e^{\lambda_i})$.

Pf: # of roots of $F(z) = w_0$ is $(w_0 \notin C_i)$

$$(**) \quad \frac{1}{2\pi i} \int_{C_1} \frac{F'(z)\,dz}{F(z) - w_0} + \cdots + \frac{1}{2\pi i} \int_{C_n} \frac{F'(z)\,dz}{F(z)-w_0}$$

Fig 1.

for $w_0 = 0$, get $1, 0, \cdots, 0, -1$ by def in $(*)$

since $(\log F)' = f'$.

imagine each slit inside has "2-sides".

- The picture is for $\lambda_1 > 0$, but this needs to be proved!

$I_1 = 1$ as long as $|w_0| < e^{\lambda_1}$
$\quad\; 0$ for $|w_0| > e^{\lambda_1}$

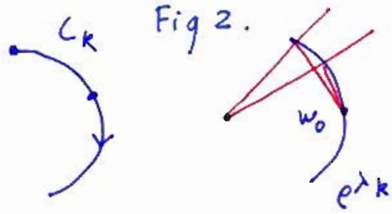
$I_n = -1$ as long as $|w_0| < 1 = e^{\lambda_n}$
$\quad\; 0$ for $|w_0| > 1$

$I_i = 0 \;\; \forall i \neq 1, n$ and $|w_0| \neq e^{\lambda_i}$.

choose $|w_0| \neq e^{\lambda_i} \;\; \forall i = 1, \cdots, n \Rightarrow 1 < |w_0| < e^{\lambda_1} > 0$.

Now, if $|w_0| = e^{\lambda_k}$, then in the residue thm we should use Cauchy principle value pr.v. and the multiplicity is counted by 1/2.

Fig 2.



$$\text{pr.v.} \int_{C_k} \frac{F'(z)\,dz}{F(z) - w_0} = \text{pr.v.} \int_{C_k} d \arg (F(z) - w_0) \underset{\uparrow}{=} \frac{1}{2} \int_{C_k} d \arg F(z) = \frac{1}{2} \int_{C_k} \frac{F'(z)\,dz}{F(z)}$$

junior high school geometry in Fig 2. $\qquad$ the case $w_0 = 0$

$(**) = \frac{1}{2}, 0, \cdots 0, -\frac{1}{2}$ resp.

$\Rightarrow F$ is 1-1 on C_1 and $C_n \Rightarrow 0 < \lambda_i < \lambda_1 \quad \forall i \neq 1, n$.

Now if $1 < |w_0| < e^{\lambda_1}$, then $(**) = 1$, hence $F(z)$ maps to w_0:

1) once in the interior
2) twice on boundary, or
3) once on boundary, mult 2

$\Rightarrow$ Thm. as in Fig 1. $\square$

eg. end pts of arc corr. to max and min.