2/19 Local Fourier Transform and Stationary Phase Principle

<u>Picture</u> There exists an equivalence of categories

$$\left(\begin{array}{l}\text{Holonomic } \mathbb{C}\langle x, \partial_x\rangle\text{-module } M \text{ with } \operatorname{Sing} M \subseteq \{0, \infty\} \\ \qquad\qquad\qquad\qquad \text{and } \infty : \text{regular}\end{array}\right)$$

$$\Big\downarrow\ \mathcal{F} : \text{Fourier Transform} \begin{cases} x \longmapsto -\partial_\xi \\ \partial_x \longmapsto \xi \end{cases}$$

$$\left(\begin{array}{l}\text{Holonomic } \mathbb{C}\langle \xi, \partial_\xi\rangle\text{-module } N \text{ with } \operatorname{Sing} N \subseteq \{0, \infty\}, \\ 0 : \text{regular singularity, and slope of } \infty < 1.\end{array}\right)$$

$$\text{slope}: \quad \begin{array}{ccc} N(a) & \xrightarrow{\ \mathcal{F}\ } & N(\mathcal{F}a) \\ (u, v) & \longmapsto & (u+v, -v) \end{array}$$

$E = \mathbb{C}$, $E' =$ dual of E.

$\bar{E}$: completion of E

$$\begin{tikzcd} & \bar{E} \times E' \arrow[dl, "p"'] \arrow[dr, "p'"] & \supseteq \underset{(x,\xi)}{E \times E'} \supseteq \{\langle x, \xi\rangle \leq 0\} \\ \bar{E} & & E' \end{tikzcd}$$

$$L^+ = (E \times E') \cup \underbrace{\{\langle x, \xi\rangle > 0\}}_{\text{complement of } \{\langle x,\xi\rangle \leq 0\} \text{ in } \bar{E} \times E'} : \text{open in } \bar{E} \times E'$$

$\mathbb{C}_{L^+}$: sheaf on $\bar{E} \times E'$, $\mathbb{C}$ on L^+, 0 otherwise;

For a complex $G \in D^+(B, \mathbb{C})$, set

$$\mathcal{F}^+ = Rp'_*(p^* G \otimes \mathbb{C}_{L^+}) \in D^+(B', \mathbb{C})$$

Similarly, we can define L^- and $\mathcal{F}^-$.

<u>Theorem</u> Suppose M is holonomic D-module with slope < 1 at ∞.

Then, we have isomorphism $\mathrm{Sol}\, \mathcal{F}M \simeq \mathcal{F}^{+} \mathrm{Sol}(M)[1]$.

proof: Write $G = \mathrm{Sol}(M)$.

$$\xrightarrow{\text{Yesterday}} \mathrm{Sol}(\mathcal{F}M) = Rp'_{*}\left(p^{*}G \otimes e^{-x\xi}\right)^{<0}[1].$$

Also, we have

$$\begin{tikzcd}[column sep=tiny] & \left(p^{*}G \otimes e^{-x\xi}\right)^{<0} \otimes \mathbb{C}_{L^+} \arrow[dl] \arrow[dr] & \\ \left(p^{*}G \otimes e^{-x\xi}\right)^{<0} & & p^{*}G \otimes \mathbb{C}_{L^+} \end{tikzcd}$$

Now, it suffices to show this two maps are isomorphism.

Check locally on $(\bar{E} \setminus E) \times E'$.

- At (θ_0, ξ_0) with $\mathrm{Re}(e^{i\theta_0}\xi_0) \le 0$, since slope at $\infty < 1$,

$$\rightsquigarrow \left(\underbrace{p^{*}G \otimes e^{-x\xi}}_{}\right)^{<0} = 0.$$

function exponentially decay!

- At (θ_0, ξ_0) with $\mathrm{Re}(e^{i\theta_0}\xi_0) > 0$:

$$\rightsquigarrow \left(p^{*}G \otimes e^{-x\xi}\right)^{<0} = p^{*}G.$$

$\square$

• Stokes structure of $\mathcal{F}M$ at ∞.

M: holonomic D-module, $N = \mathcal{F}M$.

$\underset{x}{\frown}$ $E = \mathbb{C}$ $\qquad$ $\tilde{E} \xrightarrow{\pi} E$: real blow up

$S = \pi^{-1}(0)$

$\underset{\smile}{\xi}$ E' : dual to E $\qquad$ $\overline{E'}$: completion of E'

$S' = \overline{E'} - E'$.

Write V, V' : local system corresponding to $\underline{H}^{-1}Sol(M)$ and $\underline{H}^{-1}Sol(N)$.

Stokes structure : V is filtrated by Ω

local system of meromorphic differential

$$\alpha = \sum_{\substack{k \in \mathbb{Q} \\ (\text{finite})}} \frac{\alpha_k}{x^k}\, dx \quad \text{modulo} \quad k \le 1 \quad (\text{log pole}).$$

V' is filtrated by Ω' (restrict to slope < 1)

local system of meromorphic differential

$$\alpha = \sum_{k \in \mathbb{Q}} \beta_k \xi^k d\xi \quad \text{with } k < 0 \quad \text{modulo} \quad k \le -1.$$

order on Ω : $\alpha \underset{\theta}{<} \beta \iff e^{\int \alpha - \beta}$ moderate growth i.e. $O(|x|^{-N})$ on a small sector near θ.

$\{V^\alpha\}_{\alpha \in \Omega}$: filtration of V, V^α_θ = solution near θ s.t. $e^{\int \alpha}$ is moderate growth.

$\{V_{\alpha,\theta}\}_{\alpha \in \Omega}$: locally constant sheaf on S near an open set of $e^{i\theta}$

$$V_\theta = \bigoplus_\alpha V_{\alpha,\theta}$$

$V^{<0}$: subsheaf of V by replacing V_α by 0 when $\alpha \not< 0$.

Similarly, we define $V'^{<0}$.

<u>Theorem</u> $(V')^{<0} = \mathcal{F}^{+} V^{<0} [1]$.

$V^{<0} = \mathcal{F}^{-} (V')^{<0} [1]$.

<u>Proposition</u> Suppose $h(x)$ is holomorphic on $\Sigma_\delta = \{ |\arg x - \theta_c| < \delta \}$, modrate growth at ∞, $h(x) = e^{f(x)} \varphi(x)$ near 0 with φ dominated by $a x^\lambda (\log x)^p$. D: half line in Σ_δ from 0 to ∞.

Then, $\hat{h}(\xi) = \int_D h(x) e^{-x\xi} dx$ exists on some small $\Sigma'_\varepsilon = \{ |\arg \xi + \theta_c| < \varepsilon \}$

Also, as $\xi \to \infty$, $\hat{h}(\xi)$ has expansion $e^{g(\xi)} \hat{\varphi}(\xi)$,

$\hat{\varphi}$ dominated by $a \left[\frac{2\pi}{f''(u)} \right]^{1/2} u^\lambda (\log u)^p \Big|_{\xi = f'(u)}$.

arg close to $-\theta_c$

proof: Take a half-line passing through u.

We may only take integral in $\{ |x| \le b \} \cap D = D'$ since

$$\left| \int_{D-D'} h(x) e^{-x\xi} dx \right| \le C \cdot e^{-d|\xi|} \quad \text{as } \xi \in \Sigma'_\varepsilon,\ \xi \to \infty$$

for some constant $C, d > 0$.

Enough to show $\varphi(x) = a x^\lambda (\log x)^p (1 + \sum_{k \ge 1} a_k x^k)$.

Case 1: $f(x) = \dfrac{\alpha_r}{x^r}$.

Let $c \in (0,1)$, independent of ξ. We claim that it suffice to integrate over $\{x \in D \mid |x-u| \leq c \cdot |u|\}$.

First, we have $g(\xi) = f(u) - u f'(u) = \dfrac{(r+1)\alpha_r}{u^r}$.

$$\xrightarrow{x=tu} \quad f(x) - x\xi = f(x) - x f'(u) = \frac{\alpha_r}{u^r} \underbrace{\left[\frac{1}{t^r} + rt\right]}$$

minimal $(r+1)$ at $t=1$

goes ∞ as $t \to 0$ and ∞.

Then, for $t>0$, $|t-1| \geq c$, we have

$$\left|\frac{1}{t^r} + rt\right| \geq k(r+1) \quad \text{for some } k>1.$$

Make k smaller (still >1), we have

$$\frac{1}{t^r} + rt \geq k(r+1) + \frac{\ell}{t^r} \quad \text{for some } \ell > 0.$$

Thus, for $x \in D$ with $|x-u| \geq c \cdot |u|$, we have

$$\operatorname{Re}(f(x) - x\xi) \leq k \operatorname{Re}(g(\xi)) + \ell \operatorname{Re}(f(x)).$$

Now, the integral on $D \cap \{|x| \leq (1-c)|u|\}$ and on $D \cap \{(1+c)|u| \leq |x| \leq b\}$

are bounded by $C \cdot e^{k \cdot \operatorname{Re}(g(\xi))}$. claim done!

$$\xrightarrow{x=tu} \quad e^{-g(\xi)} \hat{h}(\xi) = \int_{1-c}^{1+c} \exp\left[\frac{\alpha_r}{u^r}\left(\frac{1}{t^r} + rt - r - 1\right)\right] \varphi(tu)\, u\, dt$$

$$+ C.$$

$$\xrightarrow{t=1+s} \int_{-c}^{c} \exp\Big[\frac{\alpha_r}{u^r}\underbrace{g(s)}_{= \frac{1}{(s+1)^r}+rs-1}\Big]\varphi((1+s)u)\,u\,ds$$

On the other hand, on $|s| \le c$, the expansion of $\varphi((1+s)u)$

is uniform in s when $u \in \Sigma_\delta$, $u \to 0$, i.e. $u^\lambda \sum u^k (\log u)^\ell \varphi_{k,\ell}(s)$

with $k \geq 0$, $0 \le \ell \le p$, $\varphi_{k,\ell}$: holomorphic.

Dominant term : $u^\lambda (\log u)^p \varphi_{0,p}(s)$ with $\varphi_{0,p}(0) = a$.

$g(s)$ attain 0 as minimum at 0. $g''(0) > 0$.

Now, to estimate this integral, we apply the following lemma :

<u>Lemma</u> (Laplace method)

Suppose $f \in C^2([a,b])$ and there is a unique point $x_0 \in (a,b)$ s.t.

f attains minimum at x_0 with $f''(x_0) > 0$.

Then, we have $$\lim_{M\to\infty} \frac{\int_a^b e^{-Mf(x)}\,dx}{e^{-Mf(x_0)}\sqrt{\frac{2\pi}{Mf''(x_0)}}} = 1$$

$\rightsquigarrow$ The integral is just in $O(|x|^N)$, i.e. moderate growth!

done!

$\square$

For general case $f(x) = \sum_{\substack{0<k\le r \\ \text{(finite)}}} \frac{\alpha_r}{x^k}$, similarly, we get

$$e^{-g(\xi)}\hat{h}(\xi) = \int_{-c}^{c} \exp\Big[\frac{\alpha_r}{u^r}\underbrace{g'(s.u)}_{\substack{\| \\ g(s) + \sum_{k<r} \frac{\alpha_k}{\alpha_r} u^{r-k}\left[\frac{1}{(s+1)^k} - ks - 1\right]}}\Big]\varphi((1+s)u)\,u\,ds + C.$$

Change of variable $s = \psi(y,u)$, ψ : polynomial in $y^n u^r$ s.t. $n \in \mathbb{Z}$, $r \in \mathbb{Q}$.

$g'(\psi(y.u),u) = y^2$, $\psi(y.0) = \psi(y)$.

$[-c.c] \xrightarrow{\psi} \gamma_u$

Cauchy theorem $\rightsquigarrow$ $\int_{\gamma_u} = \underbrace{\int_{\gamma_0}}_{\gamma_0 = (d', d'')} + \underbrace{\int_{\gamma_u'} + \int_{\gamma_u''}}_{\text{away from } 0 \;\rightsquigarrow\; \text{negligible!!}}$

$\rightsquigarrow \int_{\gamma_0}$: use asymontic expansion and the above lemma again!

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