2/18 § Fourier Transform

Suppose M is a $\underbrace{\mathbb{C}\langle x, \partial_x\rangle}_{\text{Weyl algebra}}$-module.

The Fourier transform of M, $\mathcal{F}M$, is a $\mathbb{C}\langle \xi, \partial_\xi\rangle$-module defined

by $x \longmapsto -\partial_\xi$, $\partial_x \longmapsto \xi$.

The inverse transform is $x \longmapsto \partial_\xi$, $\partial_x \longmapsto -\xi$.

For a holonomic D-module, we define

$$\mathrm{DR}\, M = R\underline{\mathrm{Hom}}_{D_X}(\mathcal{O}_X, M)[1] \quad \text{de Rham complex} \quad \text{in } D^b(\mathbb{C}_X).$$
$$\mathrm{Sol}\, M = R\underline{\mathrm{Hom}}_{D_X}(M, \mathcal{O}_X)[1] \quad \text{solution complex}$$

Today's goal is to prove the following theorem:

<u>Theorem</u> M: holonomic $\mathbb{C}\langle x, \partial_x\rangle$-module, then we have

$$\mathrm{Sol}\, \mathcal{F}M = R\varrho_{2*}(\underbrace{\varrho_1^* G}_{\text{containing the Stokes structure at } \infty} \otimes e^{-x\xi})^{<0}[1], \text{ where } G = \mathrm{Sol}(M).$$

To prove this, we need to introduce Fourier integral on $\mathbb{C}$:

$\mathcal{S}$ = Schwartz space = C^∞-function on $\mathbb{C} = \mathbb{R}^2$ decay rapidly near ∞ (as well as their derivatives.)

Write $\mathcal{S}\hat{\otimes}\mathcal{O}$ to be the sheaf on $\mathbb{C}$ of holomorphic functions with value in $\mathcal{S}$.

For $g \in S \hat{\otimes} \mathcal{O}$, define $\operatorname{tr} g \in \mathcal{O}$ by $\quad g(\cdot, \xi) \in \mathcal{O}$.

$$(\operatorname{tr} g)(\xi) = \frac{1}{2\pi i} \int_{\mathbb{C}} g(x, \xi)\, dx \wedge d\bar{x}.$$

Consider the complex

$$(*) \quad \Big[\underset{\bullet}{S \hat{\otimes} \mathcal{O}} \longrightarrow (S \hat{\otimes} \mathcal{O})^2 \longrightarrow (S \hat{\otimes} \mathcal{O}) \Big]$$

$$g \longmapsto \left(\frac{\partial g}{\partial \xi} + x g, \frac{\partial g}{\partial \bar{x}} \right)$$

$$(g_1, g_2) \longmapsto -\frac{\partial g_1}{\partial \bar{x}} + \left(\frac{\partial g_2}{\partial \xi} + x g_2 \right)$$

$$\rightsquigarrow \quad (*) \xrightarrow{\theta} \mathcal{O}[-1]$$

$$(g_1, g_2) \longmapsto \operatorname{tr} g_2 .$$

<u>Theorem</u> θ is a quasi-isomorphism.

sketch of proof: Using double complex

$$\begin{tikzcd} S \hat{\otimes} \mathcal{O} \arrow[r, "\frac{\partial}{\partial \xi} + x"] \arrow[d, "\frac{\partial}{\partial \bar{x}}"'] & S \hat{\otimes} \mathcal{O} \arrow[d, "\frac{\partial}{\partial \bar{x}}"] \\ S \hat{\otimes} \mathcal{O} \arrow[r, "\frac{\partial}{\partial \xi} + x"] & S \hat{\otimes} \mathcal{O} \end{tikzcd}$$

and short exact sequence $\quad 0 \longrightarrow S \xrightarrow{\frac{\partial}{\partial \bar{x}}} S \xrightarrow{\varepsilon} \mathbb{C}[[x^{-1}]][x] / \mathbb{C}[x] \longrightarrow 0$.

Write $\overline{\mathbb{C}}$ to be the real blow-up at ∞ on $\mathbb{P}^1$. $S = \overline{\mathbb{C}} \setminus \mathbb{C}$.

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, hook, "j"] & \overline{\mathbb{C}} \times \mathbb{C} & S \times \mathbb{C} \arrow[l, hook', "i"'] \end{tikzcd}$$

$$\begin{tikzcd} & \mathbb{C}^2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{C} & & \mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} & \overline{\mathbb{C}} \times \mathbb{C} \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ \overline{\mathbb{C}} & & \mathbb{C} \end{tikzcd} \quad \text{(projections)}$$

On $\underset{(x,\,\xi)}{\bar{\mathbb{C}} \times \mathbb{C}}$, we define a sheaf $C^{<0}$ of smooth functions, flat on $S \times \mathbb{C}$ and holomorphic in ξ.

<u>Fact</u>: $C^{<0} \xrightarrow{\frac{\partial}{\partial \bar{x}}} C^{<0}$: surjective.

Write $\ker\left(C^{<0} \xrightarrow{\frac{\partial}{\partial \bar{x}}} C^{<0}\right) := \mathcal{O}^{<0}$.

Now, $q_{2*} C^{<0} = S \hat{\otimes} \mathcal{O}$, $R^i q_{2*} C^{<0} = 0$ for $i \geq 1$.

Then, $R q_{2*} \mathcal{O}^{<0} \simeq [\underset{\bullet}{S \hat{\otimes} \mathcal{O}} \xrightarrow{\frac{\partial}{\partial \bar{x}}} S \hat{\otimes} \mathcal{O}]$

Use $\Theta \rightsquigarrow R q_{2*} [\mathcal{O}^{<0} \xrightarrow{\frac{\partial}{\partial \xi} + x} \mathcal{O}^{<0}] = \mathcal{O}_{\mathbb{C}}[1]$.

<u>Lemma</u> $\frac{\partial}{\partial \xi} + x : \mathcal{O}^{<0} \longrightarrow \mathcal{O}^{<0}$ is surjective.

proof: On $z \in \mathbb{C}^2$, we juse use $\mathcal{O}^{<0}_z = \mathcal{O}_z = \mathcal{O}_{\mathbb{C}^2}$ and $\frac{\partial}{\partial \xi} + x = e^{-x\xi} \frac{\partial}{\partial \xi}(e^{x\xi} \cdot)$.

For $z = (\theta_0, \xi_0) \in S \times \mathbb{C}$ and $f \in \mathcal{O}^{<0}_z$, f is defined on open set $\{ |x| > \frac{1}{\varepsilon},\ |\arg x - \theta_0| < \varepsilon,\ |\xi - \xi_0| < \varepsilon \}$.

Choose ξ_1 s.t. $|\xi_0 - \xi_1| < \varepsilon$ and $\mathrm{Re}\left(e^{i\theta_0}(\xi_1 - \xi_0)\right) < 0$.

Set $g(x,\xi) = \underbrace{\int_{\xi_1}^{\xi}}_{\text{line integral on } [\xi_1, \xi]} e^{x(\eta - \xi)} f(x,\eta)\, d\eta \rightsquigarrow \frac{\partial g}{\partial \xi} + xg = f$.

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Now, we analysis its kernel:

At $(x_0, \xi_0) \in \mathbb{C}^2$, the kernel is the set of $f(x)e^{-x\xi}$ for $f \in \mathcal{O}_{\mathbb{C}, x_0}$.

At $z = (\theta_0, \xi_0) \in S \times \mathbb{C}$, such germ is represented by $f(x)e^{-x\xi}$ with

f : holomorphic on $\{ |x| > \frac{1}{\varepsilon}, |\arg x - \theta_0| < \varepsilon \}$ and $\underbrace{f(x)e^{-x\xi}}_{\text{exponentially decay on a sector}} \in \mathcal{O}_z^{<0}$

Then, the kernel is $\mathcal{O}^{<0} \cap (j_* p_1^* \mathcal{O}_{\mathbb{C}} \otimes e^{-x\xi}) =: (j_* p_1^* \mathcal{O}_{\mathbb{C}} \otimes e^{-x\xi})^{<0}$.

$\xrightarrow{\theta} R\mathfrak{z}_{z*}(j_* p_1^* \mathcal{O}_{\mathbb{C}} \otimes e^{-x\xi})^{<0}[1] \cong \mathcal{O}_{\mathbb{C}}$.

Now, $\mathbb{C}\langle x, \partial_x \rangle$ action on it and $(*)$ are compatible.

Take a free resolution of M : $L^\bullet \longrightarrow M$

$\rightsquigarrow \mathcal{F}L^\bullet$: free resolution of M

$\rightsquigarrow \mathrm{Sol}\,\mathcal{F}M = R\mathfrak{z}_{z*} \underline{\mathrm{Hom}}_{\mathbb{C}\langle x, \partial_x \rangle}\left(L^\bullet, (j_* p_1^* \mathcal{O}_{\mathbb{C}} \otimes e^{-x\xi})^{<0}\right)[2]$. $(\star)$

(Here, view $L^\bullet$ on $\overline{\mathbb{C}} \times \mathbb{C}$)

Suppose M is holonomic, write $G = \mathrm{Sol}(M)$.

$\rightsquigarrow G = \underline{\mathrm{Hom}}_{\mathbb{C}\langle x, \partial_x \rangle}(L^\bullet, \mathcal{O}_{\mathbb{C}})[1]$.

G only has cohomology at degree $-1, 0$; outside the singularity set Σ of

M, we have $\underline{H}^0 G = 0$, $\underline{H}^{-1} G$: local system.

(Here, we regard G as direct image of $\mathrm{Sol}(M)$ by Rj_*.
Thus, $\underline{H}^{-1} G$ carrys the Stokes structure at ∞.)

Finally, we want to compute (✩).

(1) On $\mathbb{C}^2$, the action $\mathbb{C}\langle x, \partial_x\rangle$ on $p_1^* \mathcal{O}_{\mathbb{C}} \otimes e^{-x\xi}$ is given by

$$x(f\cdot e^{-x\xi}) = (xf)e^{-x\xi}, \quad \partial_x(f\cdot e^{-x\xi}) = \frac{\partial f}{\partial x} e^{-x\xi}.$$

$\rightsquigarrow \underline{\mathrm{Hom}}_{\mathbb{C}\langle x, \partial_x\rangle}(L^\bullet, p_1^*\mathcal{O}_{\mathbb{C}} \otimes e^{-x\xi})[1] = p_1^* G.$

(2) For $z = (\theta_0, \xi_0) \in S \times \mathbb{C}$, $(j_* p_1^* \mathcal{O}_{\mathbb{C}} \otimes e^{-x\xi})^{<0} \ni f(x) e^{-x\xi}$ with $f(x) e^{-x\xi}$

decay exponentally, so we may replace ∂_x by $\partial_x + \xi_0$ to assume $\xi_0 = 0$.

Then, if we write $(\underline{H}^{-1} G \otimes e^{-x\xi})^{<0}_z$ the elements of $(\underline{H}^{-1}G)_{\theta_0}$,

its filtration satisfies $e^{\int \alpha - \xi_0 x}$ is exponentially small.

$\longrightarrow \underline{\mathrm{Hom}}_{\mathbb{C}\langle x, \partial_x\rangle}(L^\bullet, (j_* p_1^* \mathcal{O}_{\mathbb{C}} \otimes e^{-x\xi})^{<0})_z = (\underline{H}^{-1} G \otimes e^{-x\xi})^{<0}_z.$

This is a sheaf on $S \times \mathbb{C}$ which stalk is the same as $\underline{H}^{-1} p_1^* G$.

$\rightsquigarrow$ glue them to get $(\varrho_1^* G \otimes e^{-x\xi})^{<0}$.

Therefore, we get $\mathrm{Sol}(\mathcal{F}M) = R\varrho_{2*}(\varrho_1^* G \otimes e^{-x\xi})^{<0}[1]$.

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Similarly, use $D\mathcal{F}M = \overline{\mathcal{F}}DM$ and $DR\, M = \mathrm{Sol}\, DM$, we can obtain

$$DR(\mathcal{F}M) = R\varrho_{2*}(\varrho_1^* G \otimes e^{x\xi})^{<0}[1].$$

(Here $DM := {}^{\vee}\underline{\mathrm{Ext}}^n_{D_X}(M, D_X)$ dual of right D-module $\underline{\mathrm{Ext}}^n_{D_X}(M, D_X)$).