Main Reference : Équations Différentielles à Coefficients Polynomiaux by B. Malgrange.

2/17 § Singularities of holonomic D-module.

X : complex analytic variety of dimension 1.
$\mathcal{O}_X$: sheaf of holomorphic functions on X
D_X : sheaf of linear differential operators with coefficient in $\mathcal{O}_X$.

M: (left) D_X-module i.e. a sheaf of D_X-module.
coherent D_X-module i.e. locally of finite presentation.

A good filtration on M is a filtration $\{M_k\}_{k\geq 0}$ of M s.t.

- $M_k \subseteq M$: coherent $\mathcal{O}_X$-submodule
- $\underbrace{D_\ell}_{\text{differential operator of order } \leq \ell} M_k \subseteq M_{k+\ell}$ for all $k, \ell \geq 0$, "=" holds for $k \geq k_0$.

$\rightarrow$ $\mathrm{gr}\, M := \bigoplus_{k\geq 0} M_k / M_{k+1}$ $\rightsquigarrow$ $\dim M := \dim(\underbrace{\overbrace{\mathrm{supp}\ \mathrm{gr}\, M}^{\mathrm{car}\, M \,:\, \text{characteristic variety}}}_{\text{support on } T^*X})$

(view it on T^*X)

M is said holonomic if $\dim M = 1 = \dim X$.

Examples $\mathcal{E}$: vector bundle on X with integrable connection ∇.

$\rightarrow$ $\mathcal{E}_k = \mathcal{E}$ for all $k \geq 0$ $\rightsquigarrow$ $\mathrm{car}\, M = 0$-section of T^*X

$\rightsquigarrow$ $\mathcal{E}$: holonomic.

Nonexample $M = D_X$, $\mathrm{car}\, M = T^*X$ $\rightsquigarrow$ $\dim(D_X) = 2$.

Now, we recall the local theory of ODE :

$X \underset{\text{open}}{\subseteq} \mathbb{C}$, $0 \in X$, M: holonomic D_X-module s.t. $\operatorname{sing} M \subseteq \{0\}$, and

$$M \simeq \underbrace{\mathcal{O}_X(*\{0\})}_{\text{sheaf of meromorphic functions with possible poles at } 0.} \otimes_{\mathcal{O}_X} M$$

i.e. locally, we have $M \simeq \mathcal{O}_X(*\{0\})^r$ as a free $\mathcal{O}_X(*\{0\})$-module.

$\vec{u}$: vector with generators as entries $\rightsquigarrow$ $\frac{d}{dz}\vec{u} = \underbrace{A(z)}_{\in M_{r\times r}(\mathcal{O}_X(*\{0\}))}\vec{u}$

- Regular singularities

If we can choose system of generators $\{u_1, \dots, u_r\}$ of M s.t. $zA(z)$ has no pole at 0, then we say 0 is a regular singularity.

In this case, there are r linearly independent solutions

$$\vec{u}_j = z^{\lambda_j} \sum_{s=0}^{r-1} \vec{a}_{j,s}(z)(\log z)^s \quad \text{for } j = 1, 2, \dots, r$$

$\vec{a}_{j,s}(z)$: vector of holomorphic functions defined on neighborhood of 0.

$\xrightarrow[\text{generators}]{\text{change}}$ $\vec{v} = \underbrace{D(z)}_{\in GL_r(\mathcal{O}_X(*\{0\}))}\vec{u}$ to get $z\frac{d}{dz}\vec{v} = C \cdot \vec{v}$ for some $C \in M_{r\times r}(\mathbb{C})$.

May assume C: Jordan form. Then, we can read out the structure

of $M \simeq \bigoplus_{\substack{\text{some } \lambda \\ \text{some } m}} D_X(*\{0\}) \Big/ D_X(*\{0\})\left(z\frac{d}{dz} - \lambda\right)^{m+1}$ as D_X-module.

Also, $\displaystyle\bigoplus_{\substack{\text{some } \lambda \\ \text{some } m}} \mathcal{D}_X(*\{0\}) \Big/ \mathcal{D}_X(*\{0\})\left(z\frac{d}{dz}-\lambda\right)^{m+1}$

$\cong$ multiply z^k

$$\bigoplus_{\substack{\text{some } \lambda \\ \text{some } m}} \mathcal{D}_X \Big/ \mathcal{D}_X\left(z\frac{d}{dz}-\lambda-k\right)^{m+1}$$

Also, the local system on $X \setminus \{0\}$ is

$$L = \underbrace{Sol_X(M)\Big|_{X\setminus\{0\}}}_{\text{solution sheaf}} = \left\{ \vec{u} \in (\mathcal{O}_{X\setminus\{0\}})^r \;\middle|\; \frac{d}{dz}\vec{u} = A(z)\vec{u} \right\}$$

$Sol_X(M) = R\underline{Hom}_{\mathcal{D}_X}(M, \mathcal{O}_X)$

L has monodromy $\exp(2\pi\sqrt{-1}\cdot C)$

$\rightsquigarrow$ L completely determines M.

- Irregular singularities

In the irregular case, we have the r linearly independent formal solution

$$\hat{u}_j = e^{\varphi_j(z)} z^{\lambda_j} \sum_{s=0}^{r-1} \vec{a}_{j,s}(z)(\log z)^s, \text{ where } \varphi_j(z) \in z^{-1/m}\mathbb{C}[z^{-1/m}]$$

for some $m \in \mathbb{Z}_{>0}$, $\lambda_j \in \mathbb{C}$, and $\displaystyle\vec{a}_{j,s}(z) = \sum_{n \in \frac{1}{m}\mathbb{Z}_{\geq 0}} \underbrace{\vec{a}_{j,s,n}}_{\in\, \mathbb{C}^r} z^n \in \mathbb{C}[[z^{1/m}]]^r$

These formal solutions have the following property:

For any $\theta_0 \in \mathbb{R}$, $j = 1, 2, \ldots, r$, there exists a neighborhood

$\Sigma_{\theta_0} = \{ z = re^{i\theta} \mid |\theta - \theta_0| < \varepsilon \text{ and } 0 < r < \delta \}$ for some small ε, δ and a

holomorphic solution $u_j \in \mathcal{O}_X(*\{0\})^r$ such that $u_j \sim \hat{u}_j$, i.e.

for any $N > 0$, there exists $C > 0$ s.t. $|u_j - \hat{u}_j^N| \leq C \cdot |z|^{N + \lambda_j}$,

where $\displaystyle \hat{u}_j^N = e^{\varphi_j(z)} z^{\lambda_j} \sum_{s=0}^{r-1} \sum_{\substack{n \in \frac{1}{m}\mathbb{Z}_{\geq 0} \\ n \leq N}} \vec{a}_{j,s,n} z^n (\log z)^s$ is the truncated

formal solution taken branch on Σ_{θ_0}.

In D-module language, we have the following theorem:

<u>Theorem</u> M: connection on $\widehat{\mathbb{C}\{x\}[x^{-1}]} = \hat{K}$. Then, after a change of

coordinate $\tilde{K} = K[t]$, $t^p = x$, $\tilde{M} = \tilde{K} \otimes_K M$, we have a decomposition

$\tilde{M} = \bigoplus \underline{L_\omega} \otimes M_\omega$, where $\omega \in (\widehat{\tilde{K}})\,dt$ and M_ω: regular singular.

rank one connection $\langle e_\omega \rangle$ with $\partial e_\omega + \alpha e_\omega = 0$.

corresponding to $\omega = \alpha\, dt$

To prove this, we need to introduce the notion of Newton

polygon.

- Newton Polygon.

$K = \underbrace{\mathbb{C}\{x\}}_{\text{convergent power series}}[x^{-1}] \qquad K\langle\partial\rangle = D[x^{-1}]$.

<u>Proposition</u> M: connection on $\hat{K}$ of rank m. Then, $\exists\, e \in M$ s.t.

$$\{e, \partial e, \partial^2 e, \dots, \partial^{m-1} e\} \text{ is a basis of } M \text{ on } \hat{K}.$$

Then, e has minimal polynomial $\left(\partial^m + \sum a_i \partial^{m-i}\right) =: a$.

This gives $M \simeq \hat{K}\langle\partial\rangle / \hat{K}\langle\partial\rangle a$.

$\underset{\neq 0}{a} \in K\langle\partial\rangle$, the Newton polygon of $a = \sum a_{k\ell}\, x^\ell \partial^k$ is

$N(a)$ = convex hall of $\displaystyle\bigcup_{a_{k\ell}\neq 0} \{u \le k,\ v \ge \ell - k\}$ in uv-plane.

The slopes of a are the slope of non-vertical sides.

- if the horzontal side is $\{u \le 0\}$, then we exclude slope 0!

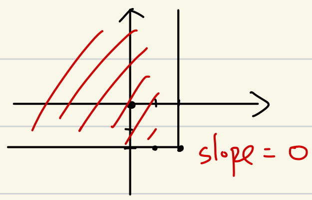
Sometimes, we will say slope of a to be the great slope.

<u>Example</u> (modified Bessel operator) $(z\partial_z)^2 - z^2 = z^2\partial_z^2 + z\partial_z - z^2$

Near 0: $\partial_z^2 + \frac{1}{z}\partial_z - 1$

$(\ell, k) = (0,2), (-1,1), (0,0)$

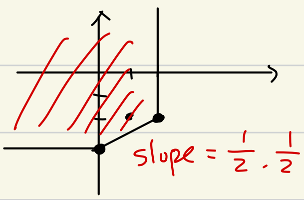
$(k, \ell-k) = (2,-2), (1,-2), (0,0)$



Near ∞: $z = \frac{1}{w}$, $\partial_z = -w^2\partial_w \rightsquigarrow \partial_w^2 + \frac{1}{w}\partial_w - \frac{1}{w^3}$

$(\ell, k) = (0,2), (-1,1), (-3,0)$

$(k, \ell-k) = (2,-2), (1,-2), (0,-3)$



• Reduce to single slope:

<u>Lemma</u> $a \in \hat{K}\langle\partial\rangle$ with slopes $0 \le p_1 < p_2 < \cdots < p_s < p_{s+1} < \cdots < p_r$.

Then, $\exists!\ b, c \in \hat{K}\langle\partial\rangle$ s.t. ① $a = bc$

② slopes of b are $p_1 \cdots p_s$, slopes of c are $p_{s+1}, \cdots, p_r$

③ constant term of c is 1.

Use this to get

$$0 \longrightarrow \underbrace{\hat{K}\langle\partial\rangle / \hat{K}\langle\partial\rangle b}_{=N} \longrightarrow \underbrace{\hat{K}\langle\partial\rangle / \hat{K}\langle\partial\rangle a}_{=M} \longrightarrow \underbrace{\hat{K}\langle\partial\rangle / \hat{K}\langle\partial\rangle c}_{=P} \longrightarrow 0$$

Also, use $\mathrm{Ext}^i_{\hat{K}\langle\partial\rangle}(N, P) = \mathrm{Ext}^i_{\hat{K}\langle\partial\rangle}(P, N) = 0$ for $i = 0, 1$.

(b acts on P) and (c acts on N) are bijective.

$\rightsquigarrow M = P \oplus N$.

Now, we begin to prove our theorem:

proof: M: connection on $\hat{K}$. $m = \operatorname{rank} M$. Find a cyclic vector of M and draw the Newton polygon of the minimal polynomial of it.

Then, use induction on slopes p and m, we may assume p is constant.

Case 0: $p=0$, then this is already regular singular.

Case 1: $p\in\mathbb{N}$ say the cyclic vector $e\in M$ with $\partial^m e+\sum a_i\partial^{m-i}=0$.

Since the Newton polygon has slope p, we have

$$a_i=\frac{\alpha_i}{x^{(p+1)i}}+(\text{higher order terms})$$

pole at most $(p+1)i$

$\mu^m+\sum\alpha_i\mu^{m-i}=0$: characteristic polynomial $\longrightarrow$ find a root λ with multiplicity g.

Let $\omega=\dfrac{\lambda\,dx}{x^{p+1}}$. Then, $L_\omega\underset{\hat{K}}{\otimes}M$ has cyclic vector $e_\omega\otimes e$ with minimal polynomial $a'=\left(\partial+\dfrac{\lambda}{x^{p+1}}\right)^m+\sum a_i\left(\partial+\dfrac{\lambda}{x^{p+1}}\right)^{m-i}$

$\Downarrow$ characteristic polynomial

$$\xi^m+\sum\frac{\beta_i}{x^{(p+1)i}}\xi^{m-i}.$$

- If $g=m$, then a' has slope $<p$ $\rightsquigarrow$ induction done!
- If $g<m$, then $\beta_m=\cdots=\beta_{m-g+1}=0$. $\beta_{m-g}\neq0$

$\rightsquigarrow$ a' has slope $<p-1$ $\rightsquigarrow$ induction done!

Case 2 $p\in\mathbb{Q}^+$. Write $p=\dfrac{g}{r}$. Then, use $t^r=x$ and reduce to the first case.

#

• Stokes phenomena

This phenomena is to describe how the solution change on different sectors.

Choose another small sector Σ_{θ_1} s.t. $\underset{u_j}{\Sigma_{\theta_0}} \cap \underset{u_j'}{\Sigma_{\theta_1}} \neq \phi$. u_j' : solutions

Then, we write $u_j' = \sum_k \underbrace{a_{j.k}}_{\cap\ \mathbb{C}} u_k$ on $\Sigma_{\theta_0} \cap \Sigma_{\theta_1}$.

$\rightsquigarrow$ $(a_{j.k})$: Stokes matrix.

In formal, we need to introduce the "real blow-up" to explain what happen to this phenomena.

• <u>Real blow-up</u> (blow-up of 0 on $\underset{\{|z|<r\}}{D}$)

$$\underset{[0.r)\times S^1}{\widetilde{D}} \xrightarrow{\pi} D \quad , \quad S = \pi^{-1}(0) \quad , \quad j : D\setminus\{0\} \hookrightarrow \widetilde{D}$$

$$(\rho, e^{i\theta}) \longmapsto \rho e^{i\theta}$$

$L = Sol_D(M)|_{D\setminus\{0\}}$, $\widetilde{L} = (j_* L)|_S$: local system on S of rank r.

From the formal decomposition, we have $\hat{M} \underset{K}{\otimes} \widetilde{K} \longrightarrow \hat{M}' = \bigoplus_\alpha L_\alpha \otimes M_\alpha$.

Then, the horizontal section of M' is of the form $\sum e^{\int \alpha} \underbrace{f_\alpha}_{\text{solution to } M_\alpha}$ on a sector near θ.

Ω: local system on S defined as follows:

On a small sector, it has the from $\sum_{-n}^{\infty} a_k x^{k/p} dx$ (mod pole of order ≤ 1)

On Ω, we define a partial order: for $e^{i\theta} \in S$, we write $\alpha \leq_\theta \beta$

if $e^{\int \alpha - \beta}$ is modrate growth (i.e. $O(|x|^{-N})$) on small sector near θ.

<u>Definition</u> Let V be a local system on S. A <u>Stokes structure</u>

(or Ω-filtration) on V is a family of subsheaves $\{V^\alpha\}$ s.t.

(1) for any $e^{i\theta} \in S$, we have $V_\theta = \bigoplus_\alpha \underbrace{V_{\alpha.\theta}}$

locally constant sheaf constant near a small neighborhood of $e^{i\theta}$ with solution $e^{\int \alpha} f_\alpha$.

(2) for $e^{i\theta'}$ near $e^{i\theta}$, we have $V^\alpha_{\theta'} = \bigoplus_{\beta \leq_{\theta'} \alpha} V_{\beta.\theta}$

<u>Theorem</u> (Deligne)

$$\left(\text{holonomic } D\text{-module } M \text{ s.t. } \mathrm{Sing}(M) \subseteq \{0\},\ M \simeq M(*\{0\})\right)$$

$$\Big\downarrow \Sigma : \text{is an equivalence of categories}$$

$$\left(\text{local system on } S \text{ together with } \Omega\text{-filtrations}\right)$$