Recall. $\Theta_x = \Theta + x$, $\Theta = (\theta)$, $W_d = \alpha(X^{(d)}) \subseteq J(X)$.

$\alpha(X) \not\subseteq \Theta_x \rightsquigarrow \alpha^*\Theta_x = Z_1(x) + \cdots + Z_g(x)$.

Lemma. $K := x - \sum \alpha(Z_i(x)) \in J(X)$ is indep to x.

Thm. (Riemann). We have $\Theta_{-K} = W_{g-1}$.

pf. $D := p_1 + \cdots + p_g \in X^{(g)}$ generic. Let $x = \alpha(D) + K$.

If $\alpha(X) \not\subseteq \Theta_x$, then $\Theta_x \cap \alpha(X) = \alpha(p_1) + \cdots + \alpha(p_g)$ if $\alpha^{-1}(\alpha(D)) = \{D\}$. ($p_i$ distinct)

$$\theta(\alpha(p_1) + \cdots + \alpha(p_{g-1}) + K) = \theta(x - \alpha(p_{g-1})) = \theta(\alpha(p_{g-1}) - x)$$
$$= \theta_x(\alpha(p_{g-1})) = 0$$

$\Rightarrow \theta_{-K}(\alpha(p_1 + \cdots + p_{g-1})) = 0 \Rightarrow \theta_{-K}$ vanishes in an open set of W_{g-1}

$$\Rightarrow W_{g-1} \subseteq \Theta_{-K}$$

Write $\Theta_K = a \cdot W_{g-1} + E$, $a \in \mathbb{N}$. $E \geq 0$, $W_{g-1} \not\subseteq E$.

- $a = 1$. Note that $[\eta \mapsto -\eta]$. acts trivially on $H_2(J(X))$

$$\Rightarrow \alpha(X) \sim -\alpha(X).$$

Let $A = \alpha(p_1) + \cdots + \alpha(p_g) \in J(X)$ generic s.t $-\alpha(X) \not\subseteq W_{g-1} - A$

If $-\alpha(X) \not\subseteq W_{g-1} - A$, $-\alpha(X) \cap (W_{g-1} - A) =$ some isolated points.

$$\boxed{\begin{array}{ccc} -\alpha(p_i) & = \displaystyle\sum_{j \neq i} & \alpha(p_j) - A \\ \cap & & \cap \\ -\alpha(X) & & W_{g-1} - A \end{array}} \Rightarrow \deg \alpha^* W_{g-1} \geq g \Rightarrow \begin{cases} a = 1, \\ \deg \alpha^* E = 0. \end{cases}$$

• $E = 0$. Let $E_\lambda = E + \lambda$, $\lambda \in J(X)$.

Then $\deg \alpha^* E_\lambda = 0 \Rightarrow$ "$\alpha(X) \cap E_\lambda \neq \emptyset \Rightarrow \alpha(X) \subseteq E_\lambda$".

Claim. $W_n \cap E_\lambda \neq \emptyset \Rightarrow W_n \subseteq E_\lambda$.

pf of Claim. Induction on n, $n=1$. done. For $n>1$,

$$\alpha(p_1) + \cdots + \alpha(p_n) \in E_\lambda \Rightarrow \alpha(p_1) + \cdots + \alpha(p_{n-1}) \in E_\lambda - \alpha(p_n) = E_{\lambda - \alpha(p_n)}.$$

ind hyp $\Rightarrow \alpha(p_1') + \cdots + \alpha(p_{n-1}') \in E_{\lambda - \alpha(p_n)} \quad \forall p_1', \dots, p_{n-1}' \in X.$

$$\Rightarrow \alpha(p_1') + \cdots + \alpha(p_{n-2}') + \alpha(p_n) \in E_{\lambda - \alpha(p_{n-1}')}$$

$$\Rightarrow \alpha(p_1') + \cdots + \alpha(p_{n-2}') + \alpha(p_n') \in E_{\lambda - \alpha(p_{n-1}')} \quad \forall p_i' \in X$$

$$\Rightarrow W_n \subseteq E_\lambda.$$

Take $n = g$, $W_g = J(X)$, but $J(X) \not\subseteq E$

$$\Rightarrow J(X) \cap E = \emptyset. \Rightarrow E = 0 \qquad \square$$

For X, we have defined $\begin{cases} J(X) = H^0(X, \Omega^1)^\vee / H_1(X, \mathbb{Z}), \\ [\omega_X](\alpha \wedge \beta) = \langle \alpha . \beta \rangle. \end{cases}$

$\rightsquigarrow (J(X), [\omega_X])$ principally polarized abelian variety.

Thm. (Torelli). If X, X' are cpt Riemann surfaces s.t.

$$(J(X), [\omega_X]) \cong (J(X'), [\omega_{X'}]).$$

i.e. $\exists f : J(X) \xrightarrow{\sim} J(X')$, $f^*[\omega_{X'}] = [\omega_X]$, then $X \cong X'$.

pf. $g \le 1$. trivial. Assume $g \ge 2$

Riemann Thm $\Rightarrow$ $\Theta_{-k} = W_{g-1} \subseteq J(X)$.

Gauss map:

$$\mathcal{G} : (\Theta_{-k})_{reg} = (W_{g-1})_{reg} \to Gr(g-1, g) = \check{\mathbb{P}}^{g-1}$$

$$x \mapsto T_x \Theta_{-k} \subseteq T_x J(X) \cong \mathbb{C}^g$$

$\alpha(D) \in (\Theta_{-k})_{reg} \Leftrightarrow \dim |D| = 0$. In this case, $T_{\alpha(D)} \Theta_{-k} = \overline{D}$

$B \subset \check{\mathbb{P}}^{g-1}$ be the branch locus of $\mathcal{G}$.

$\check{C} = \{ H \in \check{\mathbb{P}}^{g-1} \mid H \text{ tangent to } C \}$.

$I = \{ (p, H) \mid T_p C \subseteq H \} \subset C \times \check{\mathbb{P}}^{g-1}$

We have

$$\begin{array}{ccc} I & \twoheadrightarrow & \check{C} \subseteq \check{\mathbb{P}}^{g-1} \\ \pi \downarrow & & \\ C & & \end{array}$$

$\pi^{-1}(p)$ is irred, C is irred $\Rightarrow$ I irred

$\Rightarrow$ $\check{C}$ irred.

Claim 1. If X non-hyperelliptic, then $\overline{B} = \check{C}$.

pf. $\# C \cap H = 2g-2$. $\forall H \in \check{\mathbb{P}}$ $\Rightarrow$ $\mathcal{G} : (\Theta_{-k})_{reg} \to \check{\mathbb{P}}^{g-1}$ finite.

(generically $\binom{2g-2}{g-1}$ fiber.)

If $H \notin \check{C}$, then $C \cap H = p_1 + \cdots + p_{2g-2}$ $\leftarrow$ all distinct.

$\{ \alpha(D_I) = \sum_{i \in I} p_i \}_{|I| = g-1}$ are fibers of H

If $\alpha(D_I)$ distinct, then $\# \mathcal{G}^{-1}(H) = \binom{2g-2}{g-1}$ $\Rightarrow H \notin B$.

If $\alpha(D_I) = \alpha(D_J)$ for some $I \neq J$,

Abel Thm $\Rightarrow$ $D_I \sim D_J \Rightarrow \dim|D_I| = \dim|D_J| > 0$.

$\Rightarrow \alpha(D_I), \alpha(D_J) \notin (\Theta_{-k})_{reg} \Rightarrow H \notin B$.

Suppose $H \in \check{C}$, say $C \cap H = p_1 + p_2 + \cdots + p_{2g-2}$, $p_1 = p_2$

Assume that $p_1, p_3, \ldots, p_g$ indep in $\mathbb{P}^{g-1} \Rightarrow D = p_1 + p_3 + \cdots + p_g \in (\Theta_{-k})_{reg}$.

Let $H(t)$ be a path in $\check{\mathbb{P}}^{g-1}$ s.t. $H(0) = H$, $H(t) \notin \check{C}$ for $t \neq 0$.

$C \cap H(t) = p_1(t) + p_2(t) + \cdots + p_{2g-2}(t)$.

$D_i(t) = p_i(t) + p_3(t) + \cdots + p_g(t)$, $i = 1, 2$.

$\Rightarrow \mathcal{G}$ branched over H $\Rightarrow H \in B. \Rightarrow \overline{B} = C$ $\square$

Claim 2 If X hyperelliptic, $p_1 \sim p_{2g+2}$ ramification pts of $X \to C \cong \mathbb{P}^1$.

Then $\overline{B} = \check{C} \cup \left(\bigcup_{i=1}^{2g+2} \check{p}_i \right)$.

pf. $\# C \cap H = g-1$. $\mathcal{G}^{-1}(H) = \{\alpha(D) \mid \dim|D| = 0, \varphi_k(D) \subseteq C \cap H\}$

$C \cap H = q_1 + \cdots + q_{g-1}$, $\varphi_k^* q_i = r_i + \sigma(r_i)$. $\sigma: X \to X$

If $D \geq r_i + \sigma(r_i)$ for some i, then $\ell(D) \geq h^0(C, \mathcal{O}(q_i)) = 2$

$\Rightarrow D \notin (\Theta_{-k})_{reg}$.

Conversely, if $D \not\geq r_i + \sigma(r_i)$ $\forall i$, then $\forall E \in |K - D|$

$\Rightarrow D + E \in |K| = (g-1)g_2^1 \Rightarrow D + E = \sum (r_i + \sigma(r_i))$

$\Rightarrow E = \sigma D \Rightarrow \dim |D| \underset{\substack{\uparrow \\ RR}}{=} \dim |K-D| = 0$.

If $H \notin \check{C}$ and $p_i \notin H$ $\forall i$, then $r_i, \sigma(r_i)$ $i = 1 \sim g$ all distinct

$\Rightarrow H \notin B$.

If $H \in \check{C}$, then we do the same thing as in Claim 1.

If $H \ni p_i$ for some i, say $D = p_1 + x_2 + \cdots + x_{g-1}$. ↙ all distinct.

Take $H(t) \in \check{\mathbb{P}}^{g-1}$, $C \cap H(t) = \varphi_K(x_1(t)) + \cdots + \varphi_K(x_{g-1}(t))$

$D(t) = x_1(t) + \cdots + x_{g-1}(t)$, $D'(t) = \sigma(x_1(t)) + x_2(t) + \cdots + x_{g-1}(t)$.

$\Rightarrow H \in B$. $\square$

$(J(X), [\omega_X]) \rightsquigarrow \Theta$ (up to translation) $\rightsquigarrow \bar{B}$ (up to automorphism).

So, we need to show that $\bar{B} = \bar{B}' \Rightarrow X \cong X'$!

By Claim 1 and Claim 2, X, X' are the same type.

- X, X' non-hyperelliptic $\Rightarrow \check{C} = \check{C}'$

For $p \in C$, $(T_pC)^\vee = \{H \in \check{\mathbb{P}}^{g-1} \mid T_pC \subseteq H\} \subseteq \check{C} = \check{C}'$

Bertini $\Rightarrow$ generic element of $\{H.C'\}_{H \in (T_pC)^\vee}$ is smooth.

outside the base locus of $T_pC \cap C'$

But H tangents to C' $\forall H \in (T_pC)^\vee$, so T_pC tangents to C'.

If line L tangent to C' at $q \neq q'$, then

$h^0(2q + 2q') = 4 - 1 = 3$

$\Rightarrow 2q+2q'$ special. Clifford $\Rightarrow 2q+2q' = K$ or $\cancel{\underline{X \text{ hyperelliptic}}}$

Then $2g-2=4 \Rightarrow g=3$.

For $g>3$, $\exists!\, p'$ s.t. $T_pC = T_{p'}C'$ then $[p \mapsto p'] \Rightarrow C \cong C'$.

For $g=3$. $C, C' \subseteq \mathbb{P}^2 \rightsquigarrow$ only finite bitangents to C, C' in $\mathbb{P}^2$.

$\Rightarrow [p \mapsto p']$ birational $\Rightarrow C \cong C'$.

- X, X' hyperelliptic $\Rightarrow \check{C} \cup (\bigcup \check{p}_i) = \check{C}' \cup (\bigcup \check{p}_i')$.

$\rightsquigarrow C = C'$ $\left(h^0(2q+2q') = 5 \right)$.

and $\{p_i\} = \{p_i'\}$. $\rightsquigarrow X \cong X'$.

$\square$