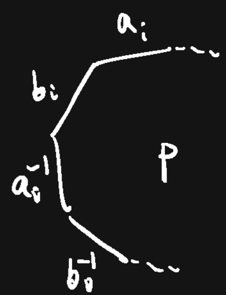
X : cpt Riemann surface of genus $g \geq 1$.
(proj curve / $\mathbb{C}$).

$$\begin{tikzcd}
H^1(X,\mathbb{Z}) \arrow[r] \arrow[d, "\text{P.D.}"', "\wr"] & H^1(X,\mathcal{O}) \arrow[r] \arrow[d, "\text{S.D.}"', "\wr"] & H^1(X,\mathcal{O}^\times) \arrow[r, "c_1"] \arrow[d, equal] & H^2(X,\mathbb{Z}) \arrow[d, "\text{PD}"', "\wr"] \\
H_1(X,\mathbb{Z}) \arrow[r] & H^0(X,\Omega^1)^\vee & \operatorname{Pic} X \arrow[r, "\deg"] & \mathbb{Z}
\end{tikzcd}$$

$$J(X) := \operatorname{Pic}^0 X \cong H^0(X,\Omega^1)^\vee / H_1(X,\mathbb{Z})$$. Jacobian var of X.

$X \rightsquigarrow$ 

$\{a_i, b_i\}$ basis of $H_1(X,\mathbb{Z})$

s.t. $\langle a_i, a_j\rangle = \langle b_i, b_j\rangle = 0$
$\langle a_i, b_j\rangle = \delta_{ij}$.

$\langle\ ,\ \rangle$: Poincaré pairing.

$w^1, \dots, w^g \in H^0(X,\Omega^1)$, dual basis of $a_1, \dots, a_g$, i.e. $\int_{a_i} w^j = \delta^j_i$.

Let $\lambda_i = \left(\int_{a_i} w^1, \dots, \int_{a_i} w^g\right) = e_i$.

$$\lambda_{n+i} = \left(\int_{b_i} w^1 \dots \int_{b_i} w^g\right) =: (Z_i^1, \dots, Z_i^g) = Z_i .$$

$\Rightarrow J(X) = \mathbb{C}^g / \Lambda$. Riemann bilinear relation $\rightsquigarrow$ Z symm and $\operatorname{Im} Z > 0$.

Fix $p_0 \in X$. $\alpha : X \longrightarrow J(X)$

$$p \longmapsto \left(\int_{p_0}^{p} w^1, \dots, \int_{p_0}^{p} w^g\right) \bmod \Lambda .$$

$\rightsquigarrow$ $\alpha : \operatorname{Div} X \longrightarrow J(X)$

$\qquad \sum n_i p_i \longmapsto \sum n_i \alpha(p_i)$.

Thm. (Abel). For $D \in \mathrm{Div}X$ with $\deg D = 0$,

$$D \sim 0 \iff \alpha(D) = 0.$$

Cor. If $g \geq 1$. $\alpha : X \to J(X)$ inj.

pf. $\alpha(p) = \alpha(q) \Rightarrow \alpha(p-q) = 0 \Rightarrow p - q = (f) \Rightarrow f : X \xrightarrow{\sim} \mathbb{P}^1$

Let $X^{(d)} = X^d / S_d = \{ p_1 + \cdots + p_d \in \mathrm{Div}\,X \mid p_i \in X \}$.

$\Rightarrow \alpha : X^{(d)} \to J(X)$, $\alpha(D) = \alpha(E) \rightsquigarrow D \sim E$

So $\alpha(D) = x \Rightarrow \alpha^{-1}(x) = |D| \leftarrow$ proj. linear space.

Let x^i real coor on $\mathbb{C}^g$ dual to λ_i.

$\omega = \omega_X = \sum dx^i \wedge dx^{g+i}$ rep. a principal polarization of $J(X)$.

Yesterday $\overset{n=g}{\rightsquigarrow}$ L : line bundle on $J(X)$; (determined by ω)

$\theta \in H^0(J(X), L)$ theta-function

$\Theta = (\theta)$ theta-divisor.

Prop. The degree of $\alpha^* \Theta \in \mathrm{Div}\,X$ is g.

pf. $\deg \alpha^* \Theta = \int_X c_1(\alpha^* \Theta) = \int_X \alpha^* [\omega]$.

$\int_{a_i} \alpha^* dx^j = \int_0^{\lambda_i} dx^j = \delta_i^j$, $\int_{b_i} \alpha^* dx^j = \int_0^{\lambda_{n+i}} dx^j = \delta_{n+i}^j$

$\Rightarrow$ $\alpha^* dx^i$ Poincaré dual to $-b_i$

$\alpha^* dx^{g+i}$ " a_i

So $\sum \int_X (\alpha^* dx^i) \wedge (\alpha^* dx^j) = \sum \langle -b_i, a_i \rangle = g$. $\square$

For $x \in J(X)$. $\Theta_x := \Theta + x = (\theta_x)$. $\theta_x(z) = \theta(z-x)$. divisor of $L_x = \tau_x^* L$.

$c_1(L_x) = c_1(L) \Rightarrow \alpha(X) \subseteq \Theta_x$ or $\#(\alpha(X) \cap \Theta_x) = g$.

For $p \in X$, consider $d\alpha_p = \frac{d}{dz}\left(\int_{p_0}^{p} \omega^1, \dots, \int_{p_0}^{p} \omega^g\right)$

$= (f^1, \dots, f^g)$ where $\omega^i = f^i dz$.

Gauss map. $\mathcal{G}_\alpha(p) = [f^1 : \dots : f^g]$ is the canonical map.

$$\varphi_K : X \to \mathbb{P}^{g-1} \quad (g \geq 2)$$

For $D = p_1 + \dots + p_d$. If p_i distinct, take z_i: local coor near p_i

$\Rightarrow z_1, \dots, z_d$ local coor near D in $X^{(d)}$.

$d\alpha_D = (f^j(p_i))$. $\underset{g \geq 2}{\Rightarrow}$ "rank $d\alpha_D = d \Leftrightarrow \{\varphi_K(p_i)\}$ lin. ind in $\mathbb{P}^{g-1}$"

$g=1$. $d \leq g$: always true. generically true when $d \leq g$.

So. $W_d := \alpha(X^{(d)}) \subseteq J(X)$ is a subvar. of dim d.

Fibers of α are proj. lin spaces (which are irreducible).

Generically, α has finite fiber $\Rightarrow$ α generically 1-1.

For $D = p_1 + \dots + p_d$, $\overline{D} = \overline{p_1 \cdots p_d} :=$ proj. lin space spanned by $\varphi_K(p_i)$ in $\mathbb{P}^{g-1}$

($2p_i$: tangent line, $3p_i$: tangent plane, ...)

R.R $\Rightarrow$ $\ell(D) - \ell(K-D) = d - g + 1$.

$|K-D| = \{H \in \check{\mathbb{P}}^{g-1} \mid D \leq H\}$. $\Rightarrow \dim \overline{D} = (g-2) - \dim|K-D|$ lift to affine.

So $\ell(D) + \dim \bar{D} = \deg D$ (geometric form of R.R).

Suppose $\dim |D| = r$, then R.R. $\Rightarrow$ $\dim \bar{D} = d - r - 1$.

Prop. The tangent cone $T_{\alpha(D)} W_d = \bigcup_{E \in |D|} \bar{E}$.

pf. Suppose $E = n_1 p_1 + \cdots + n_k p_k$ with p_i distinct, z_i local coor near p_i.

Since $X^{(d)} = X^d / S_d$, we parametrize $X^{(d)}$ at E by.

$$u_i^1 := z_{i1} + \cdots + z_{in_i}, \ \ldots, \ u_i^{n_i} := z_{i1}^{n_i} + \cdots + z_{in_i}^{n_i}, \quad i = 1 \sim k.$$

$$F_i^j(u_i^1, \ldots, u_i^{n_i}) = \int_{p_0}^{z_{i1}} \omega^j + \cdots + \int_{p_0}^{z_{in_i}} \omega^j, \quad F_i = (F_i^1, \ldots, F_i^g).$$

$$\Rightarrow \ dF_i = \frac{\partial F_i}{\partial u_i^1} du_i^1 + \cdots + \frac{\partial F_i}{\partial u_i^{n_i}} du_i^{n_i} = f(z_{i1}) dz_{i1} + \cdots + f(z_{in_i}) dz_{in_i}.$$

$$du_i^j = j(z_{i1}^{j-1} dz_{i1} + \cdots + z_{in_i}^{j-1} dz_{in_i}).$$

coeff of dz_{ik}: $\displaystyle \frac{\partial F_i}{\partial u_i^1} + \frac{\partial F_i}{\partial u_i^2} \cdot 2 z_{ik} + \cdots + \frac{\partial F_i}{\partial u_i^{n_i}} n_i z_{ik}^{n_i - 1} = f(z_{ik})$

$$= f(p_i) + f'(p_i) z_{ik} + \cdots + \frac{f^{(n_i-1)}(p_i)}{(n_i-1)!} z_{ik}^{n_i-1} + O(z_{ik}^{n_i})$$

$\forall k$

$$\rightsquigarrow \ \frac{\partial F_i}{\partial u_i^1}(0) = f(p_i), \ \frac{\partial F_i}{\partial u_i^2}(0) = \frac{f'(p_i)}{2}, \ \ldots, \ \frac{\partial F_i}{\partial u_i^{n_i}}(0) = \frac{f^{(n_i-1)}(p_i)}{n_i!}.$$

$$\Rightarrow d\alpha_E = \begin{pmatrix} f(p_1) \\ f'(p_1)/2 \\ \vdots \\ f^{(n_1-1)}(p_1)/n_1! \\ \vdots \\ \vdots \\ f^{(n_k-1)}(p_k)/n_k! \end{pmatrix} \Rightarrow \ \operatorname{Im} d\alpha_E = \bar{E} \Rightarrow T_{\alpha(D)} W_d = \bigcup_{E \in |D|} \bar{E}.$$

If $r=0$, $T_{\alpha(D)} W_d = \overline{D} \leftarrow \dim d-1$ $\Rightarrow$ W_d smooth at $\alpha(D)$.

If $r>0$, for $p_0 \cdots p_r \in C := \varphi_K(X)$,

$$\dim |D - p_0 - \cdots - p_r| \geq \dim |D| - r = 0.$$

$\Rightarrow \exists\, E \in |D - p_0 - \cdots - p_r| \Rightarrow T_{\alpha(D)} W_d \supseteq \overline{E + p_0 + \cdots + p_r} \supseteq \overline{p_0 \cdots p_r}$.

So $T_{\alpha(D)} W_d \supseteq Sec_r(C) := \bigcup_{p_i \in C} \overline{p_0 \cdots p_r}$: r^{th} secant variety.

$C \subseteq Sec_r(C)$, no hyperplane H of $\mathbb{P}^{g-1}$ contains $C \Rightarrow W_d$ singular at $\alpha(D)$.

So, "W_d smooth at $\alpha(D) \Leftrightarrow \dim |D| = 0$".

$\deg \alpha^* \Theta_x = g \rightsquigarrow \alpha^* \Theta_x = z_1(x) + \cdots + z_g(x)$. (for $\alpha(X) \not\subseteq \Theta_x$).

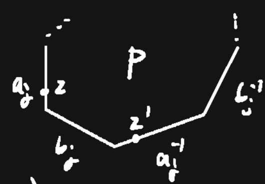
Lemma. $K := x - \sum \alpha(z_i(x)) \in J(X)$ is indep to x.

pf. Represent X as a polygon P.

$$\begin{tikzcd} P \arrow[r, "\tilde{\alpha}"] \arrow[d, "\pi"] & \mathbb{C}^g \arrow[d] \\ X \arrow[r, "\alpha"] & J(X) \end{tikzcd}$$

$\theta_x \circ \tilde{\alpha}$ vanishes exactly on $z_i(x)$.

$$\Rightarrow \sum \alpha(z_i(x)) = \frac{1}{2\pi i} \int_{\partial P} \tilde{\alpha}(z)\, d \log \theta_x(\tilde{\alpha}(z))$$



For $z \in a_j$, $z' \in a_j^{-1}$ s.t. $\pi(z) = \pi(z')$,

$$\tilde{\alpha}(z') = \tilde{\alpha}(z) + \int_{b_j} w = \tilde{\alpha}(z) + Z_j = \tilde{\alpha}(z) + \lambda_{n+j}.$$

$$\theta_x(\tilde{\alpha}(z')) = e^{-2\pi i (\tilde{\alpha}(z)^j - x^j + Z_j^j/2)}\, \theta_x(\tilde{\alpha}(z)).$$

$$\Rightarrow d\log \theta_x(\tilde{\alpha}(z')) = d \log \theta_x(\tilde{\alpha}(z)) - 2\pi i \boxed{\frac{d\tilde{\alpha}(z)^j}{dz} dz} = w_j$$

$$\frac{1}{2\pi i}\Big(\int_{a_j}\tilde{\alpha}(z)\,d\log\theta_x(\tilde{\alpha}(z)) + \int_{a_j^{-1}} \underset{\substack{\| \\ (\tilde{\alpha}(z)+Z_j)}}{\tilde{\alpha}(z')}\ \underset{\substack{\| \\ (d\log\theta_x(\tilde{\alpha}(z)) - 2\pi i\, w^j)}}{d\log\theta_x(\tilde{\alpha}(z'))}\Big)$$

$$= \frac{-Z_j}{2\pi i}\int_{a_j} d\log\theta_x(\tilde{\alpha}(z)) + \boxed{\int_{a_j}(\tilde{\alpha}(z)+Z_j)\,w^j} \text{ — indep to } x.$$

If z_1, z_2 endpoints of a_j, then $\tilde{\alpha}(z_2) = \tilde{\alpha}(z_1) + \lambda_j$

$\Rightarrow \theta_x(\tilde{\alpha}(z_2)) = \theta_x(\tilde{\alpha}(z_1)) \Rightarrow \frac{1}{2\pi i}\int_{a_j} d\log\theta_x(\tilde{\alpha}(z)) \in \mathbb{Z} \Rightarrow$ const.

For $z \in b_j$, $z' \in b_j^{-1}$ s.t. $\pi(z) = \pi(z')$.

$$\tilde{\alpha}(z') = \tilde{\alpha}(z) - \lambda_j \Rightarrow \theta_x(\tilde{\alpha}(z')) = \theta_x(\tilde{\alpha}(z))$$

$$\Rightarrow \frac{1}{2\pi i}\Big(\int_{b_j}\tilde{\alpha}(z)\,d\log\theta_x(\tilde{\alpha}(z)) + \int_{b_j^{-1}} \underset{\substack{\| \\ \tilde{\alpha}(z) - \lambda_j = e_j}}{\tilde{\alpha}(z')}\, d\log\theta_x(\tilde{\alpha}(z'))\Big)$$

$$= \frac{e_j}{2\pi i}\int_{b_j} d\log\theta_x(\tilde{\alpha}(z'))$$

If z_1, z_2 endpoints of b_j, then $\tilde{\alpha}(z_2) = \tilde{\alpha}(z_1) + \lambda_{n+j}$.

$$\Rightarrow \theta_x(\tilde{\alpha}(z_2)) = e^{-2\pi i\left(\tilde{\alpha}(z)^j - x^j + \frac{Z_j^j}{2}\right)}\theta_x(\tilde{\alpha}(z_1)).$$

$$\Rightarrow \frac{1}{2\pi i}\int_{b_j} d\log\theta_x(\tilde{\alpha}(z')) \equiv \underbrace{-\tilde{\alpha}(z)^j + x^j - \frac{Z_j^j}{2}}_{\text{indep to } x} \pmod{\mathbb{Z}}.$$

$$\Rightarrow \frac{1}{2\pi i}\int_{b_j} d\log\theta_x(\tilde{\alpha}(z')) - x^j \quad \text{const.}$$

$$\sum_j \tilde{\alpha}(z_j(x)) - \underset{\substack{\| \\ x}}{x^j e_j} \quad \text{const.} \qquad \square$$