n-torus $M = V/\Lambda$. $\dim_{\mathbb{C}} V = n$. $\Lambda = \langle \lambda, \dots \lambda_{2n} \rangle$. lattice

$z^1, \dots z^n$ coordinates on $V \Rightarrow \mathcal{H}^{p,q}(M) = \langle dz^I \wedge d\bar{z}^J \rangle_{|I|=p, |J|=q}$.

V covering space of $M \rightsquigarrow H_1(M, \mathbb{Z}) = \pi_1(M)^{ab} = \Lambda$.

Let $x^1, \dots, x^{2n}$ be dual coor. of $\lambda_1, \dots \lambda_{2n}$ on $V \Rightarrow \int_{\lambda_i} dx^j = \delta_i^j$.

$\Rightarrow H^p(M, \mathbb{Z}) = \langle dx^I \rangle_{|I|=p}$

Thm (Kodaira). A complex cpt mfd M is alg.

$\Leftrightarrow \exists [w] \in H^2(M, \mathbb{Z}) \cap H^{1,1}(M) \subseteq H^2(M, \mathbb{C})$, w positive.

If $\tilde{w}$ is such form, then average trick $\rightsquigarrow$ inv. $w = i \sum h_{ij}\, dz^i \wedge d\bar{z}^j$.

$H = (h_{ij}) > 0$.

w integral. we can write $w = \frac{1}{2} \sum q_{ij}\, dx^i \wedge dx^j$.

$\xrightarrow{\text{some alg.}}$ we can choose $\lambda_1, \dots, \lambda_{2n}$ so that $Q = (q_{ij}) = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}$, $\Delta = \begin{pmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_n \end{pmatrix}$. $\delta_i \geq 0$

w positive $\Rightarrow w$ non-degenerate $\Rightarrow \delta_i > 0$.

Let $e_i = \delta_i^{-1} \lambda_i$, $i = 1 \sim n$. $\lambda_{n+i} = Z_i^j e_j$.

z^i dual coor of e_i on V.

$\Rightarrow dz^i = \delta_i\, dx^i + Z_j^i\, dx^{n+j}$

$d\bar{z}^i = \delta_i\, dx^i + \bar{Z}_j^i\, dx^{n+j}$

$$\begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} = \begin{pmatrix} \Delta & \Delta \\ Z^T & \bar{Z}^T \end{pmatrix} \begin{pmatrix} 0 & iH \\ -iH^T & 0 \end{pmatrix} \begin{pmatrix} \Delta & Z \\ \Delta & \bar{Z} \end{pmatrix} = i \begin{pmatrix} \Delta H \Delta - \Delta H^T \Delta & \Delta H \bar{Z} - \Delta H^T Z \\ \bar{Z}^T H^T \Delta - Z^T H \Delta & Z^T H \bar{Z} - \bar{Z}^T H^T Z \end{pmatrix}$$

$$\rightsquigarrow \begin{cases} \Delta H \Delta = \Delta H^T \Delta & \rightsquigarrow \quad H = H^T \\ i\Delta(H\bar{Z} - H^T Z) = \Delta & \rightsquigarrow \quad \bar{Z} = Z - iH^{-1},\ \mathrm{Im}\, Z = 2H^{-1} > 0. \\ Z^T H \bar{Z} = \bar{Z}^T H^T Z & \rightsquigarrow \quad Z^T H (Z - iH^{-1}) = (Z - iH^{-1})^T H Z \iff Z^T = Z. \end{cases}$$

Riemann condition. $M = V/\Lambda$ alg. $\Leftrightarrow$ $\exists\ \lambda_1, \dots, \lambda_{2n}$ basis for Λ.

$e_1, \dots, e_n$ basis for V

$$dz^i = \delta_i\, dx^i + Z^i_j\, dx^{n+j}$$ with Z symmetric and $\mathrm{Im}\, Z > 0$.

$[\omega]$ is called a polarization of M.

$[\omega]$ is principal if $\delta_i = 1\ \forall i$.

From now on, $[\omega]$ is a principal polarization of M.

Line bundles on M

Suppose
$$\begin{array}{c} L \\ \downarrow \\ M \end{array} \rightsquigarrow \begin{array}{ccc} \pi^* L & \longrightarrow & L \\ \downarrow & & \downarrow \\ V & \xrightarrow{\pi} & M \end{array}$$

Since $\underset{\substack{\parallel \\ 0}}{H^1(V, \mathcal{O})} \to H^1(V, \mathcal{O}^\times) \to \underset{\substack{\parallel \\ 0}}{H^2(V, \mathbb{Z})}$, $\exists\ \varphi : \pi^* L \to V \times \mathbb{C}$.

For $z \in V$, $\lambda \in \Lambda$, we get
$$\begin{array}{ccc} \pi^* L_z & \longrightarrow & \mathbb{C} \\ \parallel & & \\ L_{\pi(z)} & & \Big\downarrow\, e_\lambda(z) \in \mathbb{C}^\times \\ \parallel & & \\ \pi^* L_{z+\lambda} & \longrightarrow & \mathbb{C} \end{array}$$

$\{e_\lambda \in \mathcal{O}^*(V)\}_{\lambda \in \Lambda}$ satisfies:

$$e_{\lambda'}(z+\lambda)\, e_\lambda(z) = e_{\lambda+\lambda'}(z).$$

Conversely, these e_λ gives a line bundle.

Prop. $L_0 \to M$ defined by $e_{\lambda_i} \equiv 1$, $e_{\lambda_{n+i}}(z) = e^{-2\pi i z^i}$ $i = 1 \sim n$.

satisfies $c_1(L_0) = [\omega]$. $\quad (H^1(M, \mathcal{O}^\times) \xrightarrow{c_1} H^2(M, \mathbb{Z}))$

pf. • e_{λ_0} defines a line bundle:

$$e_{\lambda_i}(z+\lambda_j)\, e_{\lambda_j}(z) = e_{\lambda_j}(z+\lambda_i)\, e_{\lambda_i}(z) \quad \forall i,j.$$

If $i \le n$ or $j \le n$. trivial. So we only need to check

$$e_{\lambda_{n+i}}(z+\lambda_{n+j})\, e_{\lambda_{n+j}}(z) = e^{-2\pi i (z^i + Z^i_j)} \cdot e^{-2\pi i z^j}$$

$$= e^{-2\pi i (z^i + z^j + Z^i_j)}, \text{ symmetric in } i,j.$$

• $c_1(L) = [\omega]$.

θ_0 section of $L_0|_U \to U \subseteq M$ $\Leftrightarrow$ $\theta = \varphi^*(\pi^*\theta_0)$ on $\pi^{-1}(U)$

s.t. $\theta(z+\lambda_i) = \theta(z)$

$\theta(z+\lambda_{n+i}) = e^{-2\pi i z^i}\theta(z)$.

Fix a metric $\|\cdot\|$ on L_0, $\Rightarrow$ $\|\theta_0\|^2 = h \cdot |\theta|^2$ for some $h > 0$.

and $h(z+\lambda_i) \cdot |\theta(z+\lambda_i)|^2 = h(z) \cdot |\theta(z)|^2 \quad \forall i$.

$\Leftrightarrow$ $h(z+\lambda_i) = h(z)$, $h(z+\lambda_{n+i}) = |e^{2\pi i z^i}|^2 h(z)$. $(*)$.

Let $Y = (Y^i_j) = \operatorname{Im} Z$, $W = (W^i_j) = Y^{-1}$. (Y, W symmetric, >0)

Claim. $h(z) = \exp\left(-2\pi \sum W^i_j \operatorname{Im} z^i (\operatorname{Im} z^j - Y^i_j)\right)$ satisfies $(*)$.

pf of Claim. For $k = 1 \sim n$, $\operatorname{Im}(z+\lambda^k) = \operatorname{Im} z$

$\Rightarrow h(z+\lambda_k) = h(z)$.

$$\log \frac{h(z+\lambda_{n+k})}{h(z)} = -2\pi \sum W_j^i \left((\operatorname{Im} z^i + Y_k^i)(\operatorname{Im} z^j - Y_j^j + Y_k^j) - \operatorname{Im} z^i (\operatorname{Im} z^j - Y_j^j) \right)$$

$$= -2\pi \sum W_j^i \left(Y_k^i (\operatorname{Im} z^j - Y_j^j) + Y_k^j (\operatorname{Im} z^i + Y_k^i) \right)$$

$$= -4\pi \operatorname{Im} z^k = 2 \log \left| e^{2\pi i z^k} \right| . \qquad \square .$$

Curvature form. $\Theta = -\partial \bar\partial \log h.$

$$= 2\pi \cdot \partial \sum W_j^i \left((\operatorname{Im} z^j - Y_j^j)\left(-\frac{d\bar z^i}{2i}\right) + \operatorname{Im} z^i \left(-\frac{d\bar z^j}{2i}\right) \right)$$

$$= \pi i \sum W_j^i \left(\frac{dz^j}{2i} \wedge d\bar z^i + \frac{dz^i}{2i} \wedge d\bar z^j \right)$$

$$= \pi \sum W_j^i \, dz^i \wedge d\bar z^j$$

$$= \pi \sum W_j^i \, (dx^i + Z_k^i dx^{n+k}) \wedge (dx^j + \bar Z_\ell^j dx^{n+\ell})$$

$$= \pi \sum \left(W_j^i \, dx^i \wedge dx^j + W_k^i (\bar Z_j^k - Z_j^k) dx^i \wedge dx^{n+j} \right.$$
$$\left. + Z_i^k W_j^i \bar Z_\ell^j \, dx^{n+k} \wedge dx^{n+\ell} \right).$$

$$= \pi \sum W_k^i (-2i \, Y_j^k) \, dx^i \wedge dx^{n+j}$$

$$= -2\pi i \sum dx^i \wedge dx^{n+i}$$

$\rightsquigarrow c_1(L_0) = \left[\frac{i}{2\pi} \cdot \Theta \right] = [\omega] \in H^2(M, \mathbb{Z}). \quad \square .$

Thm. Let $L \to M$ be a line bundle s.t. $c_1(L) = [\omega]$. Then

(1) $L = \tau_x^* L_0$ for some $x \in M$.

(2) $h^0(M, \mathcal{O}(L)) = 1$.

pf. (1). Let $V \longrightarrow M' = V / \langle \lambda_1, \dots, \lambda_n \rangle \cong (\mathbb{C}^*)^n \xrightarrow{\pi'} M.$

$$\left(\begin{array}{c} \mathbb{C}/\mathbb{Z} \to \mathbb{C}^* \\ z \mapsto e^{2\pi i z} \end{array} \right)$$

x^{n+i} is a well-defined function on M' $\rightsquigarrow$ $[dx^{n+i}] = 0 \in H^1_{dR}(M')$.

$= c_1((\pi')^*L) = (\pi')^* c_1(L) = (\pi')^*\left(\sum dx^i \wedge dx^{n+i}\right) = 0$.

$$\begin{array}{ccccccc} H^1(M', \mathcal{O}) & \to & H^1(M', \mathcal{O}^\times) & \xrightarrow{c_1} & H^2(M', \mathbb{Z}) & \to & H^2(M', \mathcal{O}) \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

$\rightsquigarrow$ $(\pi')^*L$ trivial.

$\Rightarrow$ $e_{\lambda_i}(z) \equiv 1$, $i = 1 \sim n$.

Consider
$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \mathbb{C} \arrow[r] \arrow[d] & \mathbb{C}^\times \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathcal{O} \arrow[r] & \mathcal{O}^\times \arrow[r] & 0 \end{tikzcd}$$

$$\rightsquigarrow \begin{tikzcd} H^1(M, \mathbb{C}) \arrow[r] \arrow[d, "\pi^{0,1}"] & H^1(M, \mathbb{C}^\times) \arrow[r] \arrow[d] & H^2(M, \mathbb{Z}) \arrow[d, equal] \\ H^1(M, \mathcal{O}) \arrow[r] & H^1(M, \mathcal{O}^\times) \arrow[r] & H^2(M, \mathbb{Z}) \end{tikzcd}$$

$*$

$\tilde{L} := L \otimes L_0^{-1} \longmapsto 0$

$\Rightarrow$ $\tilde{e}_\lambda \in \mathbb{C}^\times(V) \hookrightarrow \mathcal{O}^\times(V)$ $\Rightarrow$ $e_{\lambda_{n+i}}(z) = u_i e^{-2\pi i z^i}$, $u_i \in \mathbb{C}^\times$

Let $x = \sum \frac{i}{2\pi} \log u_i \cdot e_i$, then $e_{\lambda_{n+i}}(z - x) = e^{-2\pi i z^i}$ $\rightsquigarrow$ $L = T_x^* L_0$.

(2). $h^0(L) = h^0(T_y^* L_0)$ $\forall y$. Take $y = \frac{1}{2} \sum Z_i^i e_i$.

$\theta_0 \in H^0(L)$ $\Leftrightarrow$ $\theta : V \to \mathbb{C}$ s.t. $\theta(z + \lambda_i) = \theta(z)$ $(*)$

$$\theta(z + \lambda_{n+i}) = e^{-2\pi i z^i - \pi i Z_i^i}\,\theta(z).$$

$(*) \Leftrightarrow$
$$\begin{tikzcd} V \arrow[r] \arrow[d] & \mathbb{C} \\ M' \cong (\mathbb{C}^\times)^n \arrow[ur] & \end{tikzcd}$$

$z^{i*} = e^{2\pi i z^i}$ coor of $(\mathbb{C}^\times)^n$ $\rightsquigarrow$ $\theta(z) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha z^{*\alpha} = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha e^{2\pi i \alpha^T z}$

$$\theta(z+\lambda_{n+i}) = e^{-2\pi i z^i - \pi i Z_i^i}\,\theta(z)$$

$$\Big\| \quad = \sum_{\alpha\in\mathbb{Z}^n} a_\alpha\, e^{2\pi i (\alpha - e^i)^T z}\, e^{-\pi i Z_i^i}$$

$$= \sum_{\alpha\in\mathbb{Z}^n} a_{\alpha+e^i}\, e^{-\pi i Z_i^i}\, e^{2\pi i \alpha^T z}$$

$$\sum_{\alpha\in\mathbb{Z}} a_\alpha\, e^{2\pi i \alpha^T \lambda_{n+i}}\, e^{2\pi i \alpha^T z}$$

$$\Rightarrow a_{\alpha+e^i} = e^{2\pi i \alpha^T \lambda_{n+i} + \pi i Z_i^i}\, a_\alpha \rightsquigarrow a_\alpha = e^{\pi i(\alpha^T Z \alpha)}\, a_0.$$

So $\theta(z) = a_0 \sum_{\alpha\in\mathbb{Z}^n} e^{\pi i(\alpha^T Z\alpha) + 2\pi i \alpha^T z}$ determined by a_0

$\operatorname{Im} Z > 0 \rightsquigarrow \left| e^{\pi i(\alpha^T Z\alpha)} \right| \leq e^{-\pi c|\alpha|^2}$ for some $c > 0$

$\Rightarrow \theta$ conv. and $H^0(L) = \langle \theta \rangle$ $\square$

$\uparrow$

Riemann theta-function of $[M, [\omega]]$

$\Theta = (\theta)$: Riemann theta-divisor of $(M, [\omega])$.

(uniquely determined up to translation by $[\omega]$).