§. Thaddeus' pair on curves

Let X be a smooth projective curve of genus $g \geq 2$. (E, ϕ) is a pair of rank 2 vector bundle $/X$ and a nonzero section $\phi \in H^0(E)$. $\sigma \in \mathbb{Q}^+$, $\Lambda \to X$: line bundle, $\deg = d$

Def. (σ-stability) The pair (E, ϕ) is σ-semistable if $\forall L \subseteq E$: sub line bundle, we have $\deg L \leq \begin{cases} \frac{1}{2}\deg E - \sigma, & \text{if } \phi \in H^0(L) \\ \frac{1}{2}\deg E + \sigma, & \text{if } \phi \notin H^0(L) \end{cases}$. It's σ-stable if both "$\leq$" are strict.

Main Theorem $\exists$ a projective moduli $M(\sigma, \Lambda)$ of σ-semistable pairs (E, ϕ) s.t. $\Lambda^2 E = \Lambda$

$M(\sigma, \Lambda) \neq \emptyset \Leftrightarrow \sigma \leq \frac{d}{2}$.

We begin with several basic facts of σ-semistable pairs.

Proposition $\forall \sigma \in \mathbb{Q}^+$, $\exists$ a σ-semistable pair of determinant $\Lambda \Leftrightarrow \sigma \leq \frac{d}{2}$

Pf "$\Rightarrow$": If $\sigma > \frac{d}{2}$, then if $\phi \in H^0(L)$, σ-semistability $\Rightarrow \deg L < 0$ $\rightarrow\!\leftarrow$.

"$\Leftarrow$": If $\sigma \leq \frac{d}{2}$, let $L \to X$ be a line bundle of degree $= \lfloor \frac{d}{2} - \sigma \rfloor$ having a nonzero section ϕ.

Define E as a nonsplit extension $0 \to L \to E \to \underset{\Lambda L^{-1}}{\Lambda \otimes L^{-1}} \to 0$.

- existence : We need $Ext^1(\Lambda L^{-1}, L) \neq 0$

$Ext^1(\Lambda L^{-1}, L) \simeq Hom(L, \Lambda L^{-1} \omega_X)^\vee = H^0(\Lambda L^{-2} \omega_X)^\vee$

R-R : $h^0(\Lambda L^{-2}\omega_X) - \underbrace{h^1(\Lambda L^{-2}\omega_X)}_{0 \text{ by Kodaira vanishing}} = (d - 2\lfloor \frac{d}{2} - \sigma \rfloor + 2g - 2) + 1 - g > 0$

- Now if $\exists M \subseteq E$: a sub line bundle with $\deg M > \frac{d}{2} + \sigma$, then $\exists M \xrightarrow{\neq 0} \Lambda L^{-1}$

Since $\deg \Lambda L^{-1} < \frac{d}{2} + \sigma + 1$, it's an isom., and hence the extension split $\rightarrow\!\leftarrow$.

- For $M \subseteq E$ with $\phi \in H^0(M)$, $\deg M \leq \frac{d}{2} - \sigma$ is obvious $\Rightarrow (E, \phi)$: σ-semistable. $\square$

Proposition (E, ϕ) : a pair. $\exists$ at most one σ-destabilizing $L \subseteq E$ s.t. $\phi \in H^0(L)$ and at most one σ-destabilizing $M \subseteq E$ s.t. $\phi \notin H^0(M)$. If both L and M exist, then $E = L \oplus M$.

Pf. The first statement is clear : $\exists$ at most one line bundle $L \subseteq E$ s.t. $\phi \in H^0(L)$.

Since $L \hookrightarrow E$, $\phi \in H^0(E) = Hom(\mathcal{O}_X, E)$. Then $\phi \in H^0(L)$ means L is the image of ϕ.

- If $\exists M \subseteq E$: σ-destabilizing with $\phi \notin H^0(M)$ i.e. $\deg M \geq \frac{d}{2} + \sigma$, then M will be an ordinary destabilizing bundle (w.r.t. slope stability) since $\deg M > \frac{d}{2}$

- If both L and M exists, then $M \to E \to \Lambda L^{-1}$ is nonzero, since $\phi \in H^0(L)$ but

$\phi \notin H^0(M)$. However, $\deg M \geq \frac{d}{2}+\sigma \geq \deg \Lambda L^{-1} \Rightarrow M = \Lambda L^{-1} \Rightarrow E$: split $\rightarrow\leftarrow$. $\square$

Proposition $(E_1, \phi_1), (E_2, \phi_2)$: σ-stable pairs of degree d, $\psi : E_1 \longrightarrow E_2$: a map s.t. $\psi\phi_1 = \phi_2 \Rightarrow \psi$: isom.

Pf. Consider $\ker\psi$: a subsheaf of a loc. free sheaf on a smooth curve, i.e. $\ker\psi$ is locally free.

- If $rk(\ker\psi) = 2 \Rightarrow \psi = 0$ and $\psi\phi_1 \neq \phi_2$ $\rightarrow\leftarrow$.
- If $rk(\ker\psi) = 1 \Rightarrow \ker\psi$ is a sub line bundle of E_1.
  $\Rightarrow \psi$ descends to $\Lambda L^{-1} \longrightarrow E_2$ s.t. $\phi_2 \in H^0(\Lambda L^{-1})$. (E_2, ϕ_2) : σ-stable $\Rightarrow \deg \Lambda L^{-1} < \frac{d}{2} - \sigma \Rightarrow \deg L > \frac{d}{2} + \sigma$, $\rightarrow\leftarrow$ since (E_1, ϕ_1) is σ-stable
- So $rk(\ker\psi) = 0 \Rightarrow \psi$: mono. Also, $\mathrm{cok}\,\psi$: coherent sheaf on a curve, with $rk. = \deg. = 0 \Rightarrow \mathrm{cok}\,\psi = 0$ $\square$

Proposition (E, ϕ) : σ-stable pair. $\nexists f \in End(E)$ s.t. $f(\phi) = 0$ and $f \neq 0$. Equivalently, $\nexists g \in End(E)$ s.t. $g \neq 1_E$ and $g\phi = \phi$.

Pf. If $\exists f \in End(E)$ s.t. $f(\phi) = 0$, then f will annihilate $L := \langle\phi\rangle$, so f will descend to $E/L \longrightarrow E$. By σ-stability, E/L is a line bundle of $\deg. \geq \frac{d}{2} + \sigma$ $\Rightarrow$ the image is either 0 or would generate a line bundle of $\deg. \geq \frac{d}{2} + \sigma$ which would be a destabilizing $\Rightarrow f = 0$. $\square$

Lemma If (E, ϕ) is σ-(semi)stable, then so is $(E(D), \phi(D))$ for any effective divisor D. Similarly, if ϕ vanishes on an effective divisor D and (E, ϕ) : σ-(semi) stable, then so is $(E(-D), \phi(-D))$

Pf. Let $L \subseteq E$ be any line bundle. "$\phi(D) \in H^0(L(D))$" $\Leftrightarrow$ "$\phi \in H^0(L)$"

Also, $\deg L(D) = \deg L + \deg D$ and $\frac{1}{2}\deg E(D) = \frac{1}{2}\deg E + \deg D$. $\square$

Hence, if the moduli space $M(\sigma, \Lambda)$ exists for sufficiently large d, then the moduli space for smaller d will be contained inside them.

Proposition For fixed g and σ, $d \gg 0$, (E, ϕ) : σ-semistable $\Rightarrow H^1(E) = 0$ and E is globally generated.

$\exists L, M$ s.t. $0 \to L \to E \to M \to 0$

Lemma in Vakil: if $h^0(KE^*) \geq 1$, can take $\mathcal{O}(D)$

Pf. Suppose that $H^1(E) \neq 0$. Serre dual $\Rightarrow H^0(KE^*) \neq 0 \Rightarrow \exists\, 0 \to \mathcal{O}(D) \to KE^*$

$\Rightarrow \exists K^{-1}(D) \hookrightarrow E^*$ for some effective D $\Rightarrow \exists K^{-1}\Lambda(D) \hookrightarrow \Lambda E^* \simeq E$

(zero of a section in $H^0(E)$

Since $\deg K^{-1}\Lambda(D) \geq 2-2g+d$, σ-semistability $\Rightarrow 2-2g+d \leq \frac{d}{2}+\sigma$

$\Rightarrow d \leq 4g-4+2\sigma$. So for $d \gg 0$, $H^1(E)=0$.

• Similarly, if $d > 4g-2+2\sigma$, $H^1(E(-x))=0 \quad \forall x \in X \Rightarrow E$: globally generated $\square$

Mumford $\Rightarrow$ Generally, $\chi(E) = \deg E + rk(E)\cdot(1-g)$

This proposition implies that for (E,ϕ): σ-stable, $\dim H^0(E) = \chi(E) = d+2-2g =: \chi$

filter as $0 \to L \to E \to M \to 0$, use R-R. $(d \gg 0)$

If we fix an isom. $s: \mathbb{C}^\chi \longrightarrow H^0(E) \rightsquigarrow \Lambda^2\mathbb{C}^\chi \xrightarrow{s} \Lambda^2 H^0(E) \to H^0(\Lambda) = H^0(\Lambda^2 E)$

Since E is globally generated, the composition is nonzero, so we can associate a point $T(E,s) \in \mathbb{P}\mathrm{Hom}(\Lambda^2\mathbb{C}^\chi, H^0(\Lambda)) =: \mathbb{P}\mathrm{Hom}$

Consider the pair $(T(E,s), s^{-1}\phi) \in \mathbb{P}\mathrm{Hom} \times \mathbb{P}\mathbb{C}^\chi$. $M(\sigma,\Lambda)$ will be a GIT quotient of the set of such pairs.

Two such isom.s are related by an element of $SL(\chi,\mathbb{C})$, the group action will be the diagonal action $SL(\chi) \curvearrowright \mathbb{P}\mathrm{Hom} \times \mathbb{P}\mathbb{C}^\chi$. Choose the ample line bundle $\mathcal{O}(\chi+2\sigma, 4\sigma)^N$ and we can define (semi) stable points in the sense of GIT w.r.t. this linearization. We now prove that this stability is compatible with σ-stability.

<u>Theorem</u>. If (E,ϕ) is σ-(semi)stable, then $(T(E,s), s^{-1}\phi)$ is a (semi) stable point w.r.t. the linearization above.

Pf. Suppose that $T := (T(E,s), s^{-1}\phi)$ is unstable.

Hilbert-Mumford criterion $\Rightarrow \exists$ a nontrivial 1-parameter subgroup $\lambda: \mathbb{C}^\times \longrightarrow SL(\chi)$ s.t. $\forall\, \tilde{T}$ in the fiber of the dual of $\mathcal{O}(\chi+2\sigma,4\sigma)^N$ over T, we have $\lim_{t\to 0}\lambda(t)\cdot\tilde{T} = 0$

Explicitly, since any 1-parameter subgroup of $SL(\chi)$ can be diagonalized, so $\exists$ a basis e_i of $\mathbb{C}^\chi$ s.t. $\lambda(t)\cdot e_i = t^{r_i}e_i$ ($r_i \in \mathbb{Z}$, not all zero, $\sum_i r_i = 0$, $r_i \leq r_j$ for $i \leq j$). $\lim_{t\to 0}\lambda(t)\cdot\tilde{T} = 0$ means any basis element $(e_i^* \wedge e_j^* \otimes v, e_k) \in \mathrm{Hom}(\Lambda^2\mathbb{C}^\chi, H^0(\Lambda)) \oplus \mathbb{C}^\chi$ which is acted on with <u>weight</u> ≤ 0 has coeff. zero in the basis expansion.

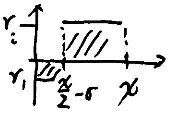
$-(r_i+r_j)\cdot(-(\chi+2\sigma)) + r_k\cdot(-4\sigma) \leq 0$

Namely, $T(E,s)(e_i,e_j) = 0$ whenever $r_i + r_j \leq \frac{2\sigma}{\frac{\chi}{2}+\sigma}\cdot r_\ell$ where $\ell := \max\{i \mid \text{coeff. of } e_i \text{ in } s^{-1}\phi \neq 0\}$

Let $L \subseteq E$ be the line bundle generated by $s(e_1)$.

Case 1. $\phi \in H^0(L)$.

Note that for $i \le \frac{\chi}{2} - \sigma + 1$, $(\frac{\chi}{2} - \sigma) r_1 + (\frac{\chi}{2} + \sigma) r_i \le \sum_j r_j = 0$.

(LHS = 

. Its value on $[j-1, j) \le r_j$)

$\Rightarrow -2\sigma r_1 + (\frac{\chi}{2} + \sigma)(r_1 + r_i) \le 0 \Rightarrow r_1 + r_i \le \frac{2\sigma}{\frac{\chi}{2}+\sigma} r_1 \le \frac{2\sigma}{\frac{\chi}{2}+\sigma} r_\ell$

$\Rightarrow T(E,s)(e_1, e_i) = s(e_1) \wedge s(e_i) = 0$ i.e. $s(e_i)$ is a section of L

$\Rightarrow \dim H^0(L) > \frac{\chi}{2} - \sigma = \frac{d}{2} + 1 - g - \sigma$

R–R $\Rightarrow$ $\deg L + 1 - g = h^0(L) - h^1(L)$, so if "$H^1(L) = 0$", $\deg L > \frac{d}{2} - \sigma$

· L is globally generated. Consider $0 \to L(-x) \to L \to L \otimes \mathcal{O}_x \to 0$

$\rightsquigarrow$ $0 \to H^0(L(-x)) \to H^0(L) \twoheadrightarrow H^0(L|_x)$ $\Rightarrow H^1(L(-x)) = 0 \Rightarrow H^1(L) = 0$

$\to H^1(L(-x)) \to H^1(L) \to H^1(L|_x) = 0$ $\forall x \in X$.

Case 2. $\phi \notin H^0(L)$

As above, for $i \le \frac{\chi}{2} + \sigma + 1$, $(\frac{\chi}{2} + \sigma) r_1 + (\frac{\chi}{2} - \sigma) r_i \le 0$

$\Rightarrow r_1 + r_i \le \frac{2\sigma}{\frac{\chi}{2}+\sigma} r_i$

Claim $\ell > \frac{\chi}{2} + \sigma + 1$.

If not, then $\forall i \le \ell$, $r_1 + r_i \le \frac{2\sigma}{\frac{\chi}{2}+\sigma} r_\ell$ $\Rightarrow$ $s(e_i)$ would be in the same line bundle as $s(e_1)$. ϕ is a linear combination of e_i for $i \le \ell$ (by defn. of ℓ) $\Rightarrow \phi \in H^0(L) \nrightarrow$.

$\Rightarrow r_1 + r_i \le \frac{2\sigma}{\frac{\chi}{2}+\sigma} r_\ell$ for $i \le \frac{\chi}{2} + \sigma + 1$; hence $s(e_i) \in H^0(L)$ as in Case 1 $\Rightarrow h^0(L) > \frac{\chi}{2} + \sigma$ and again (E, ϕ) is not σ-semistable.

The proof for stability is similar; we only need to replace "$\le$" by $<$. □

Theorem. (E, ϕ): a pair, $s: \mathbb{C}^\chi \longrightarrow H^0(E)$: a linear map, $v \in \mathbb{C}^\chi$ and $s(v) = \phi$.

Write T_s for the composition $\Lambda^2 \mathbb{C}^\chi \xrightarrow{s} \Lambda^2 H^0(E) \xrightarrow{\wedge} H^0(\Lambda)$

If (T_s, v) is semistable, then s: isom. and (E, ϕ): σ-semistable.

Pf. Claim. s is injective.

Assume the contrary, say if $s(w) = 0$ for some w, then put $e_1 := w$, $e_2 := v$ and extend it to a basis $\{e_i\}$ of $\mathbb{C}^\chi$. $\Rightarrow \ell = 2$

Take the 1-parameter subgroup defined by $r_1 := 2 - \chi$, $r_2 = 0$, $r_3 = \cdots = r_\chi = 1$. $\Rightarrow$ "$r_i + r_j \le \frac{2\sigma}{\frac{\chi}{2}+\sigma} r_\ell$" means $r_i + r_j \le 0$.

$\Rightarrow$ either $i=1$, or $j=1$, or $i=j=2$. In these cases, $T_s(e_i, e_j) = 0$ i.e.

(T_s, v) is not semistable. ↯ This proves the <u>Claim</u>.

Now if (E, ϕ) is σ-unstable, we distinguish two cases: (say $L \subseteq E$: σ-des.)

<u>Case 1</u>. $d - \deg L > 2g-2$.

Serre duality: $H^1(\Lambda L^{-1}) = H^0(K \otimes (\Lambda L)^{-1*}) = 0$ in this case.

Also, $H^1(L) = H^0(KL^{-1}) = 0$ since L is σ-destabilizing $\Rightarrow \deg L > \frac{d}{2} - \sigma$ which is large relative to g.

Consider $0 \to L \to E \to \Lambda L^{-1} \to 0$: exact and its cohomology sequence

$\Rightarrow H^1(E) = 0 \Rightarrow h^0(E) = \chi$ and s : isom.

Choose a basis $e_1, \dots, e_p$ for $s^{-1}(H^0(L))$ and extend it to a basis $e_1, \dots, e_\chi$ of $\mathbb{C}^\chi$. Take the 1-parameter subgroup defined by $r_i := \begin{cases} p - \chi, & i \le p \\ p, & i > p \end{cases}$

$$\Rightarrow r_\ell = \begin{cases} p - \chi, & \text{if } \phi \in H^0(L) \\ p, & \text{if } \phi \notin H^0(L) \end{cases}$$

L is σ-destabilizing. R–R $\Rightarrow \begin{cases} p > \frac{\chi}{2} - \sigma & (\text{note } H^1(L) = 0\,!) \text{ if } \phi \in H^0(L) \\ p > \frac{\chi}{2} + \sigma & \text{if } \phi \notin H^0(L) \end{cases}$

Either way, "$r_i + r_j \le \frac{2\sigma}{\frac{\chi}{2}+\sigma} r_\ell$" $\Rightarrow$ "$i, j \le p$":

- If $\phi \in H^0(L)$, and say $i > p$, then:

$$r_i + r_j - \frac{2\sigma}{\frac{\chi}{2}+\sigma} r_\ell \ge p + (p-\chi)\left(1 - \frac{2\sigma}{\frac{\chi}{2}+\sigma}\right) = \frac{p\chi}{\frac{\chi}{2}+\sigma} - \frac{\chi(\frac{\chi}{2}-\sigma)}{\frac{\chi}{2}+\sigma}$$

$$> \left(\frac{\chi}{2} - \sigma\right) \frac{\chi}{\frac{\chi}{2}+\sigma} - \chi \cdot \frac{(\frac{\chi}{2}-\sigma)}{(\frac{\chi}{2}+\sigma)} = 0$$

- If $\phi \notin H^0(L)$, and say $j > p$, then $r_i + r_j - \frac{2\sigma}{\frac{\chi}{2}+\sigma} r_\ell \ge p - \chi + p - \frac{2\sigma p}{\frac{\chi}{2}+\sigma}$

$$= p \cdot \frac{\chi}{\frac{\chi}{2}+\sigma} - \chi > \chi - \chi = 0$$

Now if $i, j \le p$, then $s(e_i), s(e_j) \in H^0(L) \Rightarrow T_s(e_i, e_j) = 0 \Rightarrow (T_s, v)$: unstable

<u>Case 2</u>. $d - \deg L \le 2g-2$. (Serre dual $\Rightarrow \phi \notin H^0(L) \Rightarrow L$: ordinary des. bundle $\Rightarrow 0 \to L \to E \to M \to 0$ is just H–N) $\rightsquigarrow$ M : torsion free and $M = \Lambda L^{-1}$

Consider M s.t. $\deg M < 2g-1$ and $h^0(M) > g$. Choose D eff. $\deg D = 2g-1-\deg M$ $\Rightarrow M(D)$ has deg $2g-1$ and $\underset{\underset{g}{\shortparallel}}{h^0(M(D))} > g$ ↯

In this case, $h^0(\Lambda L^{-1}) \le g$.

From the exact sequence as above, we deduce that codim. of $H^0(L)$ in $H^0(E) \le g \Rightarrow \operatorname{codim}_{\mathbb{C}^\chi} s^{-1}(H^0(L)) \le g$.

Choose a basis $e_1, \dots, e_p$ for $s^{-1}(H^0(L))$ and extend it to a basis $e_1, \dots, e_\chi$ for $\mathbb{C}^\chi$.

Take the 1-parameter subgroup define by $r_i := \begin{cases} p-\chi, & i \le p \\ p, & i > p \end{cases}$

Since $p \ge \chi - g$ and $\chi := d+2-2g \gg \sigma, g$, $p > \frac{\chi}{2} + \sigma$

$\Rightarrow$ repeat the proof of Case 1.

Now (E, ϕ) is σ-semistable $\Rightarrow h^0(E) = \chi$. s : injective $\Rightarrow s$: isom. $\square$

Consider the Grothendieck Quot scheme parametrizing flat quotients of $\mathcal{O}_X^{\chi}$ with degree d. Let Quot $(\Lambda) \subseteq$ Quot be the locally closed subset consisting of loc. free quotients E with $\wedge^2 E = \Lambda$, and let $U \subseteq$ Quot (Λ) be the open set where the quotients induces $s : \underset{H^0(\mathcal{O}^{\chi})}{\overset{\shortparallel}{\mathbb{C}^{\chi}}} \xrightarrow{\sim} \underset{\phi}{\overset{}{H^0(E)}}$.

$SL(\chi) \curvearrowright U$ and $T \times 1 : U \times \mathbb{P}\mathbb{C}^{\chi} \longrightarrow \mathbb{P}\mathrm{Hom} \times \mathbb{P}\mathbb{C}^{\chi}$ intertwines actions on two sets.

$$(E, [\phi]) \longmapsto (T(E,s), [\phi])$$

Now let $V'(\sigma) \subseteq \mathbb{P}\mathrm{Hom} \times \mathbb{P}\mathbb{C}^{\chi}$ be the semistable set w.r.t. the linearization $\mathcal{O}(\chi+2\sigma, 4\sigma)^N$

Then two theorems above implies that $(T\times 1)^{-1}(V'(\sigma))$ is the set of σ-semistable pairs, denoted by $V(\sigma)$ $\rightsquigarrow$ $T \times 1 : V(\sigma) \longrightarrow V'(\sigma)$

We'll use some facts in the geometric invariant theory.

<u>Fact</u>. X : irr. quasi-proj., $\mathcal{L}$: G-linearized bundle on X. Then $\exists$ a good quotient $\pi : X^{ss}(\mathcal{L}) \longrightarrow \underline{X^{ss}(\mathcal{L}) /\!/ G}$

Furthermore, $\exists\, U \underset{\text{open}}{\subseteq} X^{ss}(\mathcal{L}) /\!/ G$ s.t. $\pi^{-1}(U) = X^{s}(\mathcal{L})$ and $\pi : \pi^{-1}(U) \longrightarrow U$ is a geometric quotient. (cf. Dolgachev, Lectures on Invariant Theory, or MFK, GIT)

<u>Fact</u> (Gieseker) G : reductive group, M_1, M_2 : G-spaces. $f : M_1 \longrightarrow M_2$: finite G-morphism.

If $\exists$ a good quotient $M_2 /\!/ G$, then $\exists$ a good quotient $M_1 /\!/ G$ and the induced map $M_1 /\!/ G \longrightarrow M_2 /\!/ G$ is finite.

So we need to check that $(T \times 1)$ is finite (on $V(\sigma)$).

<u>Proposition</u>. $(E_1, \phi_1), (E_2, \phi_2)$: σ-semistable. If $\exists\, s_1, s_2$ s.t. $(T(E_1, s_1), s_1^{-1}\phi_1) = (T(E_2, s_2), s_2^{-1}\phi_2)$ then $\exists$ an isom. $(E_1, \phi_1) \simeq (E_2, \phi_2)$. ($\Rightarrow T\times 1$ is injective)

証 That is, we need to recover (E, ϕ) and s from a given $(T(E,s), s^{-1}\phi)$

Since E_i are globally generated, the components $s_i(e_j) \wedge s_i(e_k)$ of $T(E_i, s_i)$ will define a map $X \longrightarrow Gr(\chi-2, \mathbb{C}^{\chi})$ such that E_i = the pull-back of the rank 2 tautological bundle on Grassmannian.

$\Rightarrow \phi_i$ = the pullback of the section defined by $s_i^{-1}(\phi_i)$ (projection). $\square$

Fact. F : vector bundle of rk r on X, generated by sections of $V \subseteq H^0(F)$.

Then $\exists$ a regular map $\varphi_V : X \longrightarrow Gr(r-1, \mathbb{P}(V^*))$ s.t. $\varphi_V^*(\mathcal{U}^*) = F$ and

$H^0(Gr(r-1, \mathbb{P}(V^*)), \mathcal{U}^*) = V$.

Now we use valuative criterion to check that $T\times 1$ is proper ($\Rightarrow T\times 1$: finite)

<u>Proposition</u> C : smooth affine curve, $P \in C$, $(\mathbb{E}, \Phi)$: a loc. free family of pairs on $X \times (C \setminus P)$ and suppose $\mathbb{E}$ is generated by finitely many sections s_i.

$\Rightarrow$ (after possibly rescaling Φ by a function on $C\setminus P$,) $(\mathbb{E}, \Phi)$ and s_i extend over P s.t. $\mathbb{E}$ is still loc. free, $\Phi_P \neq 0$ and s_i generate $\mathbb{E}_P$ at the generic point.

Pf. Choose $\mathcal{L}$: ample line bundle on $X \times (C \setminus P)$ s.t. $\mathbb{E}^* \otimes \mathcal{L}$: g.b.g.s.

$\Rightarrow \bigoplus_j \mathcal{O} \twoheadrightarrow \mathbb{E}^* \otimes \mathcal{L} \Rightarrow \mathbb{E} \hookrightarrow \bigoplus_j \mathcal{L}$.

$\bigoplus_j \mathcal{L}$ can be extended over P in such a way that s_i extend too.

Consider $\overset{\mathcal{F}}{\overset{\shortparallel}{\langle s_i \rangle}} \subseteq \bigoplus_j \mathcal{L}$: a subsheaf of a loc. free sheaf $\Rightarrow$ torsion-free

$\Rightarrow Sing(\mathcal{F})$: of codim. ≥ 2. Furthermore, $\mathcal{F} \hookrightarrow \mathcal{F}^{\vee\vee}$ and $Sing(\mathcal{F}^{\vee\vee})$: of codim. ≥ 3

$\Rightarrow Sing(\mathcal{F}^{\vee\vee}) = \emptyset$ i.e. $\mathcal{F}^{\vee\vee}$: loc. free extension of $\mathbb{E}$ over P.

As for Φ, it extends with a possible pole at P, so it's just need to multiply it by a func. on C vanishing to some order at P. $\square$

Now let C be a smooth curve, $P \in C$, $\Psi : C \setminus P \longrightarrow V(\sigma)$: a map s.t. $(T\times 1)\Psi$ extends to a map $C \longrightarrow V(\sigma)$. On $C\setminus P$, we then have a family $(\mathbb{E}, \Phi)$ of pairs s.t. $\mathbb{E}$ is generated by $s(e_1), \dots, s(e_\chi)$ $\Rightarrow$ on an open affine of C containing P, $(\mathbb{E}, \Phi)$ extends over P in such a way that $\Phi_P \neq 0$ and $s(e_i)$ generically generated $\mathbb{E}_P$.

$\Rightarrow T(\mathbb{E}_P, s)$ is defined and so by continuity $(T(\mathbb{E}_P, s), s^{-1}\Phi_P) = ((T\times 1)\Psi)(p)$.

$\Rightarrow s : \mathbb{C}^\chi \longrightarrow H^0(\mathbb{E}_P)$ is an isom. and $(\mathbb{E}_P, \Phi_P)$: σ-s.s. $\Rightarrow (\mathbb{E}_P, s^{-1}\Phi_P) \in V(\sigma)$ and Ψ extends to a map $C \to V(\sigma)$. $\Rightarrow T\times 1$: proper

We conclude that $V(\sigma)$ has a good quotient. Moreover, since X : proj., $\mathcal{O}(\chi+2\sigma, 4\sigma)$: ample

$\Rightarrow$ the quotient is projective.

Fact. The stable subset of the moduli space is fine. Namely, $\exists$ a univ. pair over $M_s(\sigma, \Lambda) := \{(E,\phi) \in M(\sigma, \Lambda) \mid (E,\phi) : \sigma\text{-stable}\}$.

<u>Theorem</u> Let $T_{(E,\phi)}$ be the tangent space of $M(\sigma,\Lambda)$ at a σ-stable point (E,ϕ).

(i) $T_{(E,\phi)}$ is canonically isom. to H^1 of the complex:

$$\mathcal{A}^\cdot : 0 \longrightarrow C^0(End_0 E) \oplus \mathbb{C} \xrightarrow{p} C^1(End_0 E) \oplus C^0(E) \xrightarrow{q} C^1(E) \longrightarrow 0$$

with $p(g,c) := (dg, (g+c)\phi)$, $q(f,\psi) := f\phi - d\psi$.

(ii) $H^0(\mathcal{A}^\cdot) = 0$, $H^2(\mathcal{A}^\cdot) = 0$.

(iii) $\exists$ a natural exact seq. $0 \to H^0(End\, E) \xrightarrow{\phi} H^0(E) \to T_{(E,\phi)} \to H^1(End_0 E) \xrightarrow{\phi} H^1(E) \to 0$

(iv.) $\dim T_{(E,\phi)} = d + g - 2$.

Pf. (i) Let $R := \mathbb{C}[\varepsilon]/(\varepsilon^2)$. Hartshorne <u>Ex</u>(II.2.8) $\Rightarrow$ $T_{(E,\phi)} = \{$ isom. classes of maps $f: Spec\, R \longrightarrow M(\sigma,\Lambda) \mid f((\varepsilon)) = (E,\phi)\}$

Let $\pi: Spec\, R \times X \longrightarrow X$ be the projection. Note that $T_{(E,\phi)}$ is equal to the set $\{$ isom. classes of families $(\mathbb{E},\Phi)$ on $Spec R \times X \mid (\mathbb{E},\Phi)_{(\varepsilon)} = (E,\phi)$ and $\Lambda^2\mathbb{E} = \pi^*\Lambda\}$: each such class comes from a pull-back by the existence of the universal pair.

Any bundle $\mathbb{E}$ over $Spec R \times X$ can be trivialized on $\{Spec R \times U_\alpha\}$ for some open cover $\{U_\alpha\}$ of X.

Recall that a transition function can determine a bundle $\mathbb{E}$ and if two transition functions are conjugate, they determine the same bundle.

- If $(\mathbb{E},\Phi)_{(\varepsilon)} = (E,\phi)$, the transition functions give a Čech cochain of the form $1+\varepsilon f_{\alpha\beta}$ for some $f \in C^1(End\, E)$. Also in order for $\Lambda^2\mathbb{E} = \pi^*\Lambda$, their transition functions must be conjugate, i.e. $(1+\varepsilon h_\alpha)(\det(1+\varepsilon f_{\alpha\beta}))(1-\varepsilon h_\alpha) = 1$ for some $h \in C^0(\mathcal{O})$ $\Rightarrow$ $dh + trf = 0$.

  But if such h exists, then $\tilde{f} := f + \frac{dg}{2}$ is trace-free, and $1+\varepsilon\tilde{f}$ is conjugate to $1+\varepsilon f$. Hence we only need to consider trace-free $f \in C^1(End_0 E)$.

- If $\exists\, \Phi \in H^0(\mathbb{E})$ s.t. $\Phi_{(\varepsilon)} = \phi$, then w.r.t. the local trivialization above, $\Phi = \phi + \varepsilon\psi_\alpha$ for some Čech cochain $\psi \in C^0(E)$.

  It must be compatible with the transition func.s: $(1+\varepsilon f_{\alpha\beta})(\phi+\varepsilon\psi_\beta) = (\phi+\varepsilon\psi_\alpha)$

  $\Rightarrow f\phi - d\psi = 0$ i.e. $(f,\psi) \in Ker\, q$.

- Two choices of $(f,\psi) \in C^1(End_0 E) \oplus C^0(E)$ are related by a change of trivialization on $Spec\, R \times U_\alpha$, but since (E,ϕ) has no automorphism (except 1), we may assume it's of the form $1+\varepsilon g_\alpha$.

The action of g is given by : $1+\varepsilon f_{\alpha\beta} \longmapsto (1+\varepsilon g_\alpha)(1+\varepsilon f_{\alpha\beta})(1-\varepsilon g_\beta)$
$= 1+\varepsilon(f_{\alpha\beta}+g_\alpha-g_\beta)$

i.e. $f \mapsto f+dg$

Note that dg is trace-free if and only if $g \in C^0(\mathrm{End}\,E)$ is the sum of a trace-free cocycle and a constant. Similarly, the action of g on ψ is $\psi \mapsto \psi + g\phi$.

Hence, (f,ψ) and $(\tilde{f},\tilde{\psi})$ determine the same pair $(\mathbb{E},\Phi)$ if and only if they are the same in $H^1(\mathcal{A}^\cdot)$. This proves (i).

(ii), (iii): Let $\mathcal{B}^\cdot$ be the complex : $0 \to C^0(\mathrm{End}_0 E)\oplus \mathbb{C} \to C^1(\mathrm{End}_0 E) \to 0 \to 0 \cdots$

Consider $0 \to \mathcal{C}^\cdot(E)[-1] \to \mathcal{A}^\cdot \to \mathcal{B}^\cdot \to 0$: exact

$$\Rightarrow \quad 0 \to H^0(\mathcal{A}^\cdot) \to H^0(\mathrm{End}_0 E)\oplus\mathbb{C} \simeq H^0(\mathrm{End}\,E) \to H^0(E)$$
$$\to H^1(\mathcal{A}^\cdot) \to H^1(\mathrm{End}_0 E) \to H^1(E)$$
$$\to H^2(\mathcal{A}^\cdot) \to 0 \quad : \text{exact}$$

- (E,ϕ) : σ-stable $\Rightarrow \nexists$ endo. annihilating ϕ, i.e. $H^0(\mathrm{End}\,E) \xrightarrow{\phi} H^0(E)$ is injective $\Rightarrow H^0(\mathcal{A}^\cdot)=0$

- We'll show that $H^0(KE^*) \xrightarrow{\phi} H^0(K\,\mathrm{End}_0 E^*)$ is injective.

  Note that $E^* \hookrightarrow \mathrm{End}\,E^*$ (locally, if we write $E_x = \langle v_1, v_2\rangle$ and $\phi_x = a_1v_1+a_2v_2$ then $E^*_x \xrightarrow{\phi} (\mathrm{End}\,E^*)_x$ is given by the matrix $\begin{pmatrix} a_1 & a_2 \\ a_1 & a_2\end{pmatrix}$), and its image consists of endomorphisms of rank ≤ 1.

  The projection $\mathrm{End}\,E^* \twoheadrightarrow \mathrm{End}_0 E^*$ has kernel consisting of multiples of 1

  $\Rightarrow E^* \hookrightarrow \mathrm{End}_0 E^*$ and hence $KE^* \hookrightarrow K\,\mathrm{End}_0 E^*$.

(iv.) By above exact sequence, $\dim T_{(E,\phi)} = \chi(E) - \chi(\mathrm{End}_0 E) - 1$

$= (d+2-2g) - (3-3g) - 1 = d+g-2$ (use G-R-R). $\square$

- $\forall\, \sigma \in (\max\{0, \frac{d}{2}-i-1\}, \frac{d}{2}-i)$ where $i \in \mathbb{Z}$ and $0 \leq i \leq \frac{d-1}{2}$, the σ-semistability remains the same and implies σ-stability.

Hence, for σ in that interval, we get a fixed moduli space, denoted by $M_i(\Lambda)$.

Example When $i=0$, $M_0(\Lambda) = \mathbb{P}(H^1(\Lambda^{-1}))$:

The first inequality of σ-semistability says that "$\phi \in H^0(L) \Rightarrow \deg L \leq 0$"

$\Rightarrow L = \mathcal{O}$ i.e. E is an extension of $\mathcal{O}$ by $\Lambda = \Lambda L^{-1}$.

The second inequality says that E has no subline bundles of deg. $\geq d$. This is equivalent to not being split, since "$M \to E \to \Lambda$ is nonzero and $\deg M \geq d = \deg \Lambda \Leftrightarrow M = \Lambda$". Hence, $M_0(\Lambda) = \mathbb{P}(\mathrm{Ext}^1(\Lambda, \mathcal{O})) = \mathbb{P}(H^1(\Lambda^{-1}))$.

<u>Theorem</u>. M_i are all smooth rational integral projective varieties of dim. $= d+g-2$. If $i>0$, then $\exists$ a birational map $M_i \dashleftarrow\dashrightarrow M_1$, which is an isom. except on sets of codimension ≥ 2.

Pf. • The first statement clearly holds for $i=0$ by above <u>Example</u> and R-R.

Now assume $0 < i \leq \frac{d-1}{2}$. Let $\pi : X_i \times X \longrightarrow \underbrace{X_i}_{i\text{-th sym. product}}$ be the projection

Let $\Delta \subseteq X_i \times X$ be the universal divisor.

Then define $W_i^- := (R^0\pi)\, \mathcal{O}_\Delta \Lambda(-\Delta)$, $W_i^+ := (R^1\pi)\Lambda^{-1}(2\Delta)$.

• $\Delta \xrightarrow{\pi|_\Delta} X$ has degree $i \Rightarrow W_i^-$ is a loc. free sheaf of rank i on X_i

• W_i^+ is also loc. free, and of rank $= \dim H^1(X, \Lambda^{-1}(2\Delta))$

$$\text{R-R} \Rightarrow \underbrace{h^0(\Lambda^{-1}(2\Delta))}_{=0} - h^1(\Lambda^{-1}(2\Delta)) = (2i-d)+1-g \Rightarrow W_i^+ \text{ is of } rk = d+g-2i-1$$

<u>Claim</u>. $\exists$ a family over $\mathbb{P}W_i^+$ (resp. $\mathbb{P}W_i^-$), parametrizing exactly those pairs represented in M_i but not M_{i-1} (resp. M_{i-1} but not M_i)

Were this proven, then by the universal property of the moduli space, $\exists$ inclusions $\mathbb{P}W_i^+ \hookrightarrow M_i$ and $\mathbb{P}W_i^- \hookrightarrow M_{i-1}$, and hence there is an isomorphism

$$M_{i-1} \setminus \mathbb{P}W_i^- \simeq M_i \setminus \mathbb{P}W_i^+$$

Note that $\begin{cases} \dim \mathbb{P}W_i^- = 2i-1 < d-1 < d+g-2 \\ \dim \mathbb{P}W_i^+ = d+g-2-i < d+g-2 \end{cases}$, so $\dim M_i = \dim M_{i-1} = \cdots = \dim M_0 = d+g-2$

• By previous theorem, the Zariski tangent space to M_i has const. dim. $d+g-2$

$\Rightarrow M_i$: smooth.

• Also, for $i>1$, $\begin{cases} \operatorname{codim}_{M_{i-1}} \mathbb{P}W_i^- = d+g-2i-1 \geq 2 \\ \operatorname{codim}_{M_i} \mathbb{P}W_i^+ = i \geq 2 \end{cases}$. This proves the second statement.

It remains to prove the <u>Claim</u>. We'll prove the case for $\mathbb{P}W_i^+$.

First note that when passing from i to $i-1$, σ increases and hence the first inequality get stronger and the second gets weaker.

So those pairs in M_i but not in M_{i-1} is an extension :

$0 \to \mathcal{O}(D) \to E \to \Lambda(-D) \to 0$ where $\deg D = i$ and ϕ is the section of $\mathcal{O}(D)$ vanishing on D.

Conversely, any such pair is stable unless it splits, i.e. $E = \mathcal{O}(D) \oplus \Lambda(-D)$. Indeed, if $L \subseteq E$ and $\phi \notin H^0(L)$, then $L \hookrightarrow E \to \Lambda(-D)$ is nonzero, so $\deg L \le \deg \Lambda(-D) = d - i$ with equality only if $L = \Lambda(-D)$.

(note that $\sigma \in (\max\{0, \frac{d}{2} - i\}, \frac{d}{2} - i + 1))$

But $\mathbb{P}W_i^+$ is the base of a family parametrizing all such nonsplit pairs : let $\mathbb{E}$ be the tautological extension $0 \to \mathcal{O}(\Delta) \to \mathbb{E} \to \Lambda(-\Delta)(-1) \to 0$ and $\Phi :=$ the section of $\mathcal{O}(\Delta)$ vanishing on Δ. $\square$

Example. The case $i = 1$: W_1^- is a line bundle on $X_1 = X \Rightarrow \mathbb{P}W_1^- \simeq X$

$\Rightarrow X \hookrightarrow \mathbb{P}H^1(\Lambda^{-1})$. It can be proved that this inclusion is given by $|K_X \Lambda|$.

Now let $\widetilde{M}_i^+$ (resp. $\widetilde{M}_{i-1}^-$) be the blow-up of M_i along $\mathbb{P}W_i^+$ (resp. M_{i-1} along $\mathbb{P}W_i^-$). Then the exceptional divisors will be $E_i^\pm = \mathbb{P}W_i^- \oplus \mathbb{P}W_i^+$ and $\widetilde{M}_i^+ \simeq \widetilde{M}_{i-1}^-$, and thus we denote it simply by $\widetilde{M}_i$. That is, M_i is obtained from M_{i-1} by blowing up $\mathbb{P}W_i^-$ and then blowing down the same exceptional divisor in another direction.

In particular, the flip degenerates to an ordinary blow-up in the case of M_0 (Since W_1^- is a line bundle, there's nothing to blow down) $\Rightarrow M_i$ is the blow-up of $M_0 = \mathbb{P}(H^1(\Lambda^{-1}))$ along X (embedded via $|K_X \Lambda|$).

We are also interested in the other extreme case, i.e. M_w where $w := [\frac{d-1}{2}]$. We already have the following diagram :

(all maps are birational)

$$\begin{tikzcd}[column sep=small]
 & \widetilde{M}_2 \arrow[dl] \arrow[dr] & & \widetilde{M}_3 \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \widetilde{M}_w \arrow[dl] \arrow[dr] & \\
M_1 \arrow[d] & & M_2 & & M_3 & \cdots & M_{w-1} & & M_w \\
M_0 & & & & & & & &
\end{tikzcd}$$

Proposition $\exists$ a natural "Abel-Jacobi" map $M_w \to \mathcal{N}$, where $\mathcal{N} :=$ the moduli space of rank 2 μ-semistable bundles of determinant Λ, with fiber $\mathbb{P}H^0(E)$ over a stable bundle E.

Pf. $i = w \Rightarrow \sigma \in (0, [\frac{d}{2}] + 1 - \frac{d}{2})$. Hence "$(E, \phi)$: σ-stable $\Rightarrow E$: μ-semistable"

This gives a map $M_w \to \mathcal{N}$.

Moreover "E : μ-stable $\Rightarrow (E, \phi)$: σ-stable $\forall \phi \overset{\neq 0}{\in} H^0(E)$", so the fiber over a stable E is just $\mathbb{P}H^0(E)$. $\square$

Theorem (Koseki – Toda) Let $M^T(r,d)$ (resp. $M(r,d)$) be the moduli space (resp. coarse moduli space) of Thaddeus pairs (resp. S-equivalent classes of μ-semistable bundles) on X with $(\mathrm{rank}, \chi) = (r, d)$. Then:

$$\begin{tikzcd} M^T(r,d) \arrow[dr, "\pi^+"'] & & M^T(r,-d) \arrow[dl, "\pi^-"] \\ & M(r,d) & \end{tikzcd}$$

(Serre dual)

is a d-critical flip for $d > 0$ and $\exists$ a
(flop) ($d = 0$)

fully faithful functor $\Phi_M : D^b(M^T(r,-d)) \hookrightarrow D^b(M^T(r,d))$.