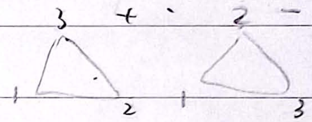
II.

Topology of $\mathcal{MS}_{1,1}(\theta)$. with conical sing. of angle $2\pi\theta$

$\mathcal{MS}^{(2)}_{1,1}(\theta)$: The moduli space of spherical tori with labelled 2-torsion

$\mathcal{MS}^{\pm}_{bal}(\theta) = \mathcal{MS}^{+}_{bal}(\theta) \sqcup \mathcal{MS}^{-}_{bal}(\theta) / \sim$

$\mathcal{MS}_{bal}(\theta)$ with reversing label

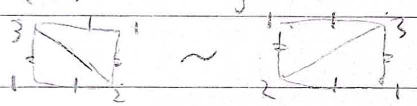


$\sim$: identify semi-balanced triangle with its anticonformal image

Prop. There's homeom. $\mathcal{MT}^{\pm}_{bal}(\theta) \underset{\Delta^{(2)}}{\overset{T^{(2)}}{\rightleftarrows}} \mathcal{MS}^{(2)}_{1,1}(\theta)$ for $\theta \notin 2\mathbb{Z}+1$

$T^{(2)}(\Delta)$ = gluing two pieces of Δ along three sides through orientation-reversing isometry.

$\Delta^{(2)}(T)$ = Triangulation of T invariant under σ



for semi-balanced triangle

$\leftrightarrow$ Tori with extra anticonformal involution

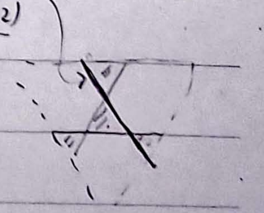
Prop. As a topological space, $\mathcal{MS}^{(2)}_{1,1}(\theta)$ $(\theta \notin 2\mathbb{Z}+1)$ is a connected surface of genus $(m-1)(m-2)/2$ with $3m$ punctures

pf. There are $3m$ boundary edges for $\mathcal{MS}_{bal}(\theta)$, $\left(m = \left\lfloor \frac{\theta+1}{2} \right\rfloor\right)$ corresponding to semi-balanced Δ (Seen from $\mathcal{C}rp_{bal}(\theta)$)

Thus $\chi(\mathcal{MS}^{\pm}_{bal}(\theta)) = 2(E-N) - 3m = \begin{cases} 2(m^2 - 3m(m-1)/2) - 3m, & \theta \le 2m \\ 2(m^2 + 3m - 3m(m+1)/2) - 3m, & \theta > 2m \end{cases}$

$= -m^2$

The $3m$ punctures correspond to the $3m$ "ideal edges" on the carpet and so $\chi(\mathcal{MS}^{\pm}_{bal}(\theta)) = 2 - 2g - 3m \Rightarrow g = \frac{(m-1)(m-2)}{2}$



Prop. As an orbifold, $\mathcal{M}S_{1,1}(\theta)$ $(\theta \notin 2\mathbb{Z}+1)$ is isom. to the quotient of its underlying top. space by trivial $\mathbb{Z}_2$-action (σ) and hence $\chi(\mathcal{M}S^{(2)}_{1,1}(\theta)) = \chi(\mathcal{M}T^{\pm}_{bal}(\theta))/2 = -m^2/2$

Prop. The forgetful map $\mathcal{M}S^{(2)}_{1,1}(\theta) \to \mathcal{M}S_{1,1}(\theta)$ is an unramified S_3-cover of orbifold, so $\chi(\mathcal{M}S_{1,1}(\theta)) = \chi(\mathcal{M}S^{(2)}_{1,1}(\theta))/6 = -m^2/12$

There are some spherical torus with $\mathbb{Z}_4$, $\mathbb{Z}_6$ autom., correspond to square / regular hexagonal flat torus $\leftrightarrow$ triangle with angles $(\pi\theta/2, \pi\theta/4, \pi\theta/4)$ or $(\pi\theta/3, \pi\theta/3, \pi\theta/3)$

$d_1(\theta, 4\mathbb{Z}) > 1$ $\qquad$ $d_1(\theta, 6\mathbb{Z}) > 1$

Odd case. $\mathcal{M}S_{1,1}(\theta)^{\sigma}$: Those tori whose σ is an isometry $(\theta \in 2\mathbb{Z}+1)$

Prop. $\mathcal{M}T_{bal}(\underset{2m+1}{\theta})$ is a disjoint union of $m(m+1)/2$ open triangles $\mathring{\Delta}^2$

$= \# \{(m_1, m_2, m_3) \in \mathbb{Z}^3 \mid m_1+m_2+m_3 = 2m+1 \wedge |m_1 - m_2| < m_3 < m_1+m_2\}$

There's a bijection
$$\mathcal{M}S_{1,1}(\theta)^{\sigma} \leftrightarrow \mathcal{M}T_{bal}(\theta)/A_3$$
and so there's $\lceil \frac{m(m+1)}{6} \rceil$ components in $\mathcal{M}S_{1,1}(\theta)^{\sigma}$

If $m \equiv 1 \pmod 3$, there's one component on which A_3 act as permutation on coordinates of δ^2.

Prop. As an orbifold $\mathcal{M}S^{(2)}_{1,1}(\theta)$ is isom. to the quotient of $\mathcal{M}T^{\pm}_{bal}(2m+1) \times \mathbb{R}$ by flipping the sign of $\mathbb{R}$, and the forgetful map $\mathcal{M}S^{(2)}_{1,1}(\theta) \to \mathcal{M}S_{1,1}(\theta)$ is an unramified cover of orbifold, thus
$$\mathcal{M}S_{1,1}(\theta) \simeq (\mathcal{M}T_{bal}(\theta) \times \mathbb{R})/(A_3 \times \mathbb{Z}_2)$$

$\mathcal{MS}^{(2)}_{1,1}(2m)$ as a Belyi curve

cpt riemann surface

Def. A <u>Belyi function</u> is a holom. map $\psi: S \to \mathbb{CP}^1$ s.t. it only ramifies over $0, 1, \infty$. The <u>dessin</u> of ψ is the 3-partite graph on S obtained as the preimage of $\mathbb{RP}^1 \subset \mathbb{CP}^1$

Thm. If $\mathcal{MS}^{(2)}_{1,1}(2m)$ is given the complex structure via the forgetful map $\mathcal{MS}^{(2)}_{1,1}(2m) \to \mathcal{M}^{(2)}_{1,1}$, then there's a Belyi function $\psi_{Bel}: \mathcal{MS}^{(2)}_{1,1}(2m) \to \mathbb{CP}^1$ of degree m^2 s.t. the dessin of ψ_{Bel} is composed of tori with one integral angle in $\Delta(\hat{T})$.

The cycle type of ramification at $\{0, 1, \infty\}$ would be $(1, 3, \cdots, 2m-1)$

Prop. The monodromy of $(S, x) \in \mathcal{MS}^{(2)}_{1,1}(2m)$ is given by the Klein-four: $K_4 = \mathbb{Z}_2 \times \mathbb{Z}_2 \subset SO(3)$ (So the developing map f can be chosen s.t. $f(z+w_1) = -f(z)$, $f(z+w_2) = \frac{1}{f(z)}$) (nontrivial)

pf. Since $\rho(\gamma) = [\rho(\alpha), \rho(\beta)] = I \in SO(3)$ and the only noncoaxial abelian subgroup in $SO(3)$ is the Klein-four. ...

Prop. $\forall (S, x) \in \mathcal{MS}^{(2)}_{1,1}(2m)$, there is a (unique) branched cover map $\varphi_S: S \to S^2/K_4$ ($\cong \mathbb{CP}^1$) of degree $4m-2$ s.t. $f(p_1) = 0$, $f(p_2) = 1$, $f(p_3) = i$, which is locally isometric outside branch pts. (p_i 2-torsion pt of T)

moreover $f(x) \neq 0, 1, i$

pf. Consider the developing map $f: \tilde{S} \to S^2$, by taking quotient of $\pi_1(S)$ we obtain $\varphi_S: S \to S^2/K_4$

$\deg(\varphi_{Kl}) = \text{Area}(T) / \text{Area}(S^2/K_4) = 2\pi(2m-1)/\pi = 4m-2$

If $f(x) =$ some branch pt, then the degree at x would be $(2m \cdot 2\pi)/\pi = 4m > 4m-2$ $\times$

Prop. $MS_{1,1}^{(2)}(2m)$ is isom. to the Hurwitz space of connected

deg $4m-2$ cover ramified over $0, 1, \infty, \lambda \in \mathbb{CP}^1$

with cycle type $(2, \cdots, 2)$ for $0, 1, \infty$

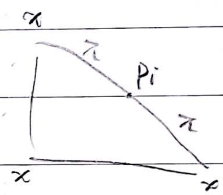
$(1, \cdots, 1, 2m)$ for λ, $\lambda \neq 0, 1, \infty$

via an isom. $\mathbb{CP}^1 / K_4 \cong \mathbb{CP}^1$

sending $0, 1, i$ to $0, 1, \infty$ (i.e, by $g(z) = \dfrac{4z^2}{(z^2+1)^2}$)

Now we can define $\varphi_{Bel}((S, x)) = g(\varphi_S(x))$

φ_{Bel} is unramified covering of $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$



Dessin : If $\Delta(\tau)$ has an integral angle, then

x is of distance π with some p_i

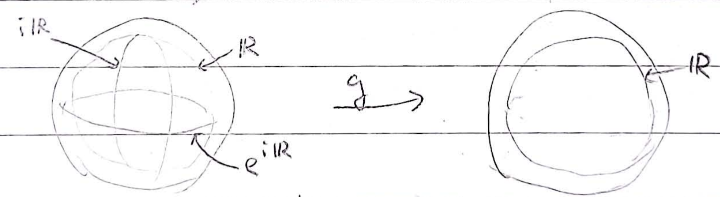
$\Rightarrow \varphi_S(x) \in \mathbb{R}$ or $i\mathbb{R}$ or $e^{i\mathbb{R}}$ in $\mathbb{CP}^1 / K_4$

$\Rightarrow g(\varphi_S(x)) \in \mathbb{R}$. Conversely,

If $\lambda = g(\varphi(x)) \in \mathbb{R}$, then wlog $\varphi(x) \in \mathbb{R}$,

Then the geodesic loop of x through p_3

has length π $\Rightarrow$ angle of $\Delta(\tau)$ opposite to p_3 is integral



$\mathbb{CP}^1 / K_4$ $\qquad$ $\mathbb{CP}^1$

There are totally $3m^2$ edge, hence $\deg \varphi_{Bel} = m^2$