II

Monodromy constraint. Given $(S, \vec{x}) \in MS_{g,n}(\theta)$, $\dot{S} = S \setminus \vec{x}$

Let $\rho : \pi_1(\dot{S}) \to SO(3)$ be the monodromy associated with the developing map f

$(= \langle \alpha, \beta, \gamma \mid [\alpha, \beta] = \gamma \rangle$ loop at x if $(g,n) = (1,1))$

Prop. The monodromy can be lifted to $\hat{\rho} : \pi_1(\dot{S}) \to SU(2)$, and two such lifting differ by a homom. $\pi_1(S) \to \{\pm 1\}$

The developing map can be extended to the completion of universal cover of $\dot{S}$

$\left(SU(2) \curvearrowright \mathbb{CP}^1 \text{ by } \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} z = \frac{az+b}{-\bar{b}z+\bar{a}} \right)$

Def. A subgp G of $SU(2)$ $(SO(3))$ is called coaxial if G lies in a one-parameter subgp of $SU(2)$ $(SO(3))$

Lemma. (i) $G \subset SU(2)$ is abelian iff G is coaxial

(ii) If $G \subset SU(2)$ is non coaxial, and $\tau \in PSL(2, \mathbb{C})$ commute with the image of G in $SO(3)$, then $\tau \in SO(3)$

proof. (i) Abelian subgp of $SU(2)$ is simultaneously diagonalizable.

(ii) Let $\hat{\tau} = \begin{pmatrix} \lambda & z \\ 0 & \lambda^{-1} \end{pmatrix} \in SL(2, \mathbb{C})$, $\hat{\tau}\gamma = \pm \gamma \hat{\tau}$ $\forall \gamma \in G$,

Consider $h = \hat{\tau}^* \hat{\tau} = \begin{pmatrix} |\lambda|^2 + |z|^2 & z\bar{\lambda}^{-1} \\ \bar{z}\lambda^{-1} & |\lambda|^{-2} \end{pmatrix}$,

Let μ^2, μ^{-2} be eigenvalues of h, $\mu \geq 1$

then $\|\hat{\tau}(v)\| \leq \mu v$ $\forall v \in \mathbb{C}^2$, and eq. holds iff v is a μ^2-eigenvalue of h, but $\|\hat{\tau}\gamma(v)\| = \|\gamma\hat{\tau}(v)\| = \|\hat{\tau}(v)\|$

$\leq \mu v = \mu(\gamma(v))$ $\forall \gamma \in G \Rightarrow G$ preserves μ^2-eigenspace of h

Since G is not coaxial, $\mu^2 = 1$, $|\lambda| = 1$ and $z = 0 \Rightarrow \tau \in SO(3)$

Given $(S,x) \in \mathcal{MS}_{1,1}(\theta)$, consider the conformal involution σ

Prop. (i) If $\theta \notin 2\mathbb{Z}+1$, then σ is an isometry

(ii) If $\theta \in 2\mathbb{Z}+1$, then the projective equivalence class of (S,x) is parametrized by $\mathbb{R}$, (developing map differ by a Mobius transform)

and σ act on the class by $t \to -t$

proof. Consider the schwarz derivative $S(f) = (\frac{f''}{f'})' - \frac{1}{2}(\frac{f''}{f'})^2$ of the developing map f, it is a meromorphic function on S, with the only pole at x of order at most 2.

The space of such function is spanned by $\wp$ and 1, which implies that $S(f) = S(f\circ\sigma)$ since $\wp$ and 1 are even

$\Rightarrow f\circ\sigma = \tilde{\sigma}\circ f$ for some mobius transform $\tilde{\sigma}$ $(\in PSL(2,\mathbb{C}))$

Note that $\hat{\rho}(\gamma)$ has eigenvalue $e^{\pm i\pi(\theta-1)}$

If $\theta \notin 2\mathbb{Z}+1$, then $\hat{\rho}(\gamma) = [\hat{\rho}(\alpha), \hat{\rho}(\beta)] \neq 1$

$\Rightarrow$ The monodromy of (S,x) is not coaxial and $\tilde{\sigma}$ commute with the monodromy group (in $PSL(2,\mathbb{C})$)

so $\tilde{\sigma} \in SO(3)$ by Lemma. $\Rightarrow \sigma$ is an isometry.

If $\theta = 2\mathbb{Z}+1$, then $\hat{\rho}(\gamma) = 1 \Rightarrow [\hat{\rho}(\alpha), \hat{\rho}(\beta)] = 1$

$\Rightarrow$ The monodromy of (S,x) is coaxial (and is nontrivial since o.w. S would cover $\mathbb{CP}^1$ with unique ramification) and $\tilde{\sigma}$ commute with them

If $\hat{\rho}(\alpha) = e^X \neq I$ $(X \in su_2)$, then the proj. equiv. class of (S,x) have developing map $e^{itX}\circ f$ $(t \in \mathbb{R})$

Denote such a metric by h_t

Suppose $\sigma^*(h_0) = h_a$, then $\sigma^*(h_t) = h_{a-t}$ since $\hat{\rho}(\sigma(\alpha)) = \hat{\rho}(\alpha^{-1})$

This suggest that replace h_0 by $h_{\frac{a}{2}}$, then we have $\sigma^*(h_t) = h_{-t}$

In particular σ is an isometry on (S, x) with metric h_0

Construction of the lifting to $SU(2)$:

Choose a real vector field V on S, having vanishing order $2-2g$ at some $q \in S \setminus \vec{x}$, and nonvanish on $D = S \setminus \{q\}$

$\forall [\gamma] \in \pi_1(\dot{S})$ (base point p in D) we may assume γ lies in D,

The $\rho(\gamma)$ is the unique element in $SO(3)$ mapping $df(V_{\gamma_0})$ $(\in T_{f(\gamma_0)} S^2)$ to $df(V_{\gamma_1})$ ($\gamma : [0,1] \to \dot{S}$, can be lifted to $\hat{S}$)

Indeed, if we identify $T^1 S^2$ and $SO(3)$,

then we may construct a map $D' \to SO(3)$ s.t. when restricted to the preimage of p gives the monodromy mapping.

(by pushing forward $\tilde{V}$ and normalize) $D' :=$ preimage of D in $\hat{S}$ (universal cover of $\dot{S}$)

The map can be uniquely lift to $SU(2)$ since the winding number of $\tilde{V}$ at any preimage of q is even.

Thus the monodromy can be lifted to $SU(2)$