$\mathcal{M}_{g,n}$: The moduli space of conformal structures on surface of genus g with n puncture

$\mathcal{M}_{g,n}(\vec{\theta})$: The moduli space of spherical metrics on surface of genus g with n conical singularities of angle $2\pi\theta_1, \cdots, 2\pi\theta_n$

There is a forgetful map $F: \mathcal{M}_{g,n}(\vec{\theta}) \to \mathcal{M}_{g,n}$

Goal of this talk: Describe $\mathcal{M}_{1,1}(\theta)$

(As an application, the area element $e^u dz d\bar{z}$ (on $\mathbb{C}/\Lambda$) of a spherical metric satisfies the Liouville equation

$$\Delta u + 2e^u = 2\pi(\theta-1)\delta_\Lambda$$

On the other hand, the developing map $f: \mathbb{C}/\Lambda \to S^2 \cong \mathbb{C}P^1$ is also the ratio of two sol. of the Lame equation

$$w'' - \Big(\frac{\theta^2-1}{4}\wp(z) + \underbrace{B}_{\text{some const.}}\Big)w = 0 \qquad (f = w_1/w_2)$$

u and f are related by the Fubini-Study metric $\frac{4\,dz\,d\bar{z}}{(1+|z|^2)^2}$:

$$u = \log \frac{4|f'|^2}{(1+|f|^2)^2} \quad)$$

Voronoi Diagram

Def. Given a spherical surface S with conical pts. $x_1, \cdots, x_n$,

The <u>Voronoi function</u> $V_S : (S, \vec{x}) \to \mathbb{R}$ is defined by

$$V_S(p) := d(p, \vec{x}) \quad (= \min_i d(p, x_i))$$

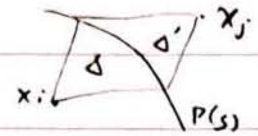
The <u>Voronoi graph</u> $\Gamma(S)$ is the locus of pts p with ≥ 2 minimal geodesics of length $V_S(p)$ joining p to $\vec{x}$

The <u>Voronoi domain</u> D_i is the connected comp. of $S \setminus \Gamma(S)$ containing x_i.

Observations. (i) $V_S < \pi$ (ii) $\Gamma(S)$ consists of geodesics

(iii) Vertices of $\Gamma(S)$ has valency ≥ 3, thus at most $6g-6+3n$ edges

(iv) $\bar{D}_i$ are convex and star-shaped.

(v) 

Δ and Δ' are anticonformally isometric

Cor. For a spherical torus (S,x) with one conical pt x,
$\Gamma(S)$ is either a trefoil $\ominus$ or an eight ∞
(not $\bigcirc\!-\!\bigcirc$ since faces has different # of vertices)

Isometric involution

Given $(S,x) \in \mathcal{MS}_{1,1}(\theta)$, consider the flat torus $(T = \mathbb{C}/\Lambda, x)$
conformal to (S,x) outside x

Let σ be the conformal involution $z \mapsto -z$ on T

We consider S s.t. σ is an isometry on S

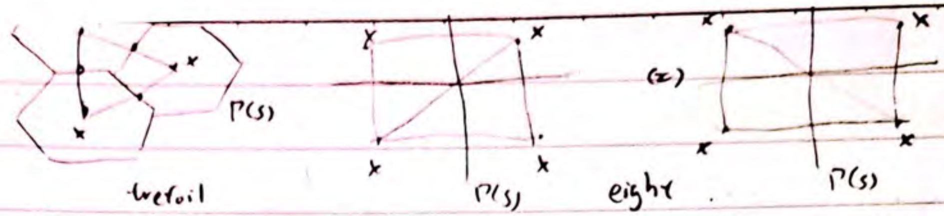
Def. A balanced (spherical) triangle is a triangular surface with spherical metric and geodesic edges s.t. the three angles $2\pi\theta_1$, $2\pi\theta_2$, $2\pi\theta_3$ satisfies the triangle ineq. $|\theta_1 - \theta_2| \leq \theta_3 \leq \theta_1 + \theta_2$
If some of the equal holds, then it is called semi-balanced.

Prop. If (S,x) has such isometry σ, then there are two cases:

(i) $\Gamma(S) = \ominus$, then (S,x) is 1-1 corresponding to a strictly balanced triangle (with labelled 2-torsion points)

(ii) $\Gamma(S) = \infty$, then such (S,x) is corresponding to a semi-balanced triangle and its anticonformal image (triangle with labelled vertices)

Diagram.



(The "circumcenter" of a balanced triangle can be found by Voronoi diagram of its "doubling")

Moduli of balanced triangle.

Prop. Let $\pi\theta_1, \pi\theta_2, \pi\theta_3, l_1, l_2, l_3$ be the angles and edges of a spherical triangle, then (θ_i, l_i) completely determines this triangle.

$\mathcal{MT}_{(bal)}(\theta)$: Moduli of (balanced) spherical triangle with total interior angle $\pi\theta$ $(= \pi + \text{Area})$

$\mathcal{MT}_{(bal)}$: Moduli of (balanced) spherical triangle.

Prop. Consider $\Psi : \mathcal{MT} \to \mathbb{R}^6_+$ given by (θ_i, l_i), then $\Psi(\mathcal{MT})$ is a connected smooth manifold of 3-dim.

The three equations $\cos l_i \sin(\pi\theta_j)\sin(\pi\theta_k) = \cos(\pi\theta_i) + \cos(\pi\theta_j)\cos(\pi\theta_k)$ gives the constraint.

$Crp_{(bal)}(\theta)$: Image of $\mathcal{MT}_{(bal)}$ in $\theta_1, \theta_2, \theta_3$

Prop. Suppose $(\theta_1, \theta_2, \theta_3)$ satisfies Δ ineq. and $\theta_1 + \theta_2 + \theta_3 = \theta$, then

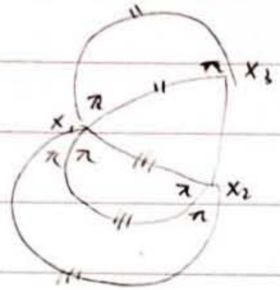
(i) If θ_i has no integer, then $(\theta_1, \theta_2, \theta_3) \in Crp_{bal}(\theta)$ iff $d((\theta_1, \theta_2, \theta_3), \mathbb{Z}^3_e) > 1$ $(\mathbb{Z}^3_e = (a,b,c) \in \mathbb{Z}^3,\ 2 \mid a+b+c)$

(ii) If $\theta_1 \in \mathbb{Z}$ and $\theta_2, \theta_3 \notin \mathbb{Z}$, then $(\theta_1, \theta_2, \theta_3) \in Crp_{bal}(\theta)$ iff $\theta_1 + \theta_2 - \theta_3 \in 2\mathbb{Z}+1$. Fiber over (θ_i) is parametrized by l_2

(iii) If $\theta_1, \theta_2 \in \mathbb{Z}$, then $(\theta_1, \theta_2, \theta_3) \in Crp_{bal}(\theta)$ iff $\theta_1 + \theta_2 + \theta_3 \in 2\mathbb{Z}+1$. Fiber over (θ_i) is parametrized by l_1, l_2

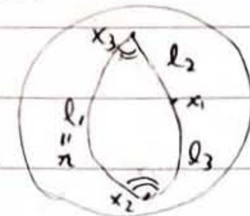
(They can be constructed from adding digons to simple triangles)

Triangles



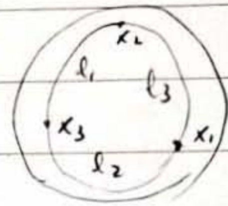
adding digons

(ii)



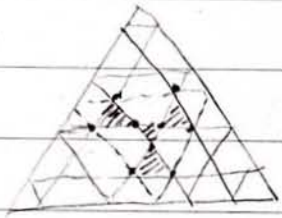
$l_2 + l_3 = \pi$

(iii)



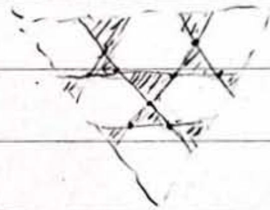
$l_1 + l_2 + l_3 = 2\pi$

Carpets



$\theta = 4$

$\theta = 3.5$



$\theta = 4.5$

Prop. Suppose Crp_{bal} intersects E open triangles, and has N inner nodes, then $E = \begin{cases} m^2 & \text{if } \theta \le 2m \\ m^2+3m & \theta > 2m, \end{cases}$ $N = \begin{cases} 3m(m-1)/2 & \text{if } \theta \le 2m \\ 3m(m+1)/2 & \text{if } \theta > 2m \end{cases}$ $(m = \lfloor \frac{\theta+1}{2} \rfloor)$

The map $\Theta : MT_{bal}(\theta) \to Crp_{bal}(\theta)$ is a real blowup at a node.