Invariance of Plurigenera

Let M be a smooth projective variety over $\mathbb{C}$. The plurigenera

$$P_m(M) := \dim H^0(M, \mathcal{O}(mK_M)) \quad (m = 1, 2, \ldots)$$

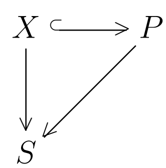
are fundamental birational invariants of M. It was conjectured that the pluregenera are invariant under smooth projective deformation (although they are not invariant under nonprojective deformation [Nakamura 1975]). This has been proved by Yum-Tong Siu in a series of remarkable papers [Siu 1998, 2000, 2005].

In this note, we are going to derive Yum-Tong Siu's theorem on the deformation invariance of plurigenera. In the original proof [Siu 2000], the theorem is proved using two key tools. The first is the global generation of multiplier ideal sheaves [Skoda 1972], and the second is the L^2 extension theorem [Ohsawa-Takegoshi 1987]. We present here the simplified proof [Păun 2007], which can be made do with only the L^2 extension theorem.

1. Introduction and Preliminaries

A family of compact complex manifolds is by definition a holomorphic map $X \xrightarrow{\pi} S$ of complex manifolds that is a proper surjective submersion. Assume that S is connected. The compact manifolds $X_t := \pi^{-1}(t)$ for $t \in S$, are called deformations of each other.

A family $X \xrightarrow{\pi} S$ is said to be projective if there is an embedding $X \hookrightarrow P$ into some holomorphic projective bundle P over S such that the following diagram commutes:



In this case, the pull-back of $\mathcal{O}_P(1)$ gives a π-ample line bundle $\mathcal{O}_X(1)$ over X, and that each manifold X_t is projective algebraic by Chow's theorem.

Theorem. For a projective family $X \xrightarrow{\pi} S$ over a connected base, the m-th plurigenus $P_m(X_t)$ is independent of $t \quad (m = 1, 2, \ldots)$.

Since the statement is local, by connecting any two points of S by finitely many coordinate disks, we may assume $S = \Delta$ is the unit disk in $\mathbb{C}$; moreover, we may shrink Δ if necessary when proving the theorem. It is a known fact (see, e.g., Hartshorne III. 12) that $P_m(X_t)$ is upper semicontinuous. Thus to prove the constancy of $P_m(X_t)$, it suffices to prove that every element of $H^0(X_t, \mathcal{O}(mK_{X_t}))$ extends to an element of $H^0(X, \mathcal{O}(mK_X))$.

Let $X \xrightarrow{\pi} \Delta$ be a projective family, and let L be a holomorphic line bundle over X with a singular hermitian metric h such that:

(1) the curvature current $i\Theta_h(L) \geq 0$;

(2) $h\,|_{X_0}$ defines a singular hermitian metric on X_0.

Then locally we can write $\|\cdot\|_h^2 \overset{a.e.}{=} |\cdot|^2 e^{-\varphi}$ for some psh function $\varphi \in L^1_{loc}$ whose restriction to the central fiber X_0 is L^1_{loc} (in the Lebesgue measure of local coordinates of X_0), or equivalently, a global expression $\|\cdot\|_h^2 \overset{a.e.}{=} \|\cdot\|_{h_{sm}}^2 e^{-\varphi}$, where h_{sm} is a smooth hermitian metric of L and $\varphi \in L^1_{loc}(X)$ is a psh function such that $\varphi\,|_{X_0} \in L^1_{loc}(X_0)$.†

The multiplier ideal sheaf $\mathcal{J}(h) \subset \mathcal{O}_X$ is dy definition the ideal sheaf of germs of holomorphic functions f such that $|f|^2 e^{-\varphi} \in L^1_{loc}$. Let ω be a smooth hermitian metric on X. The global sections of $\mathcal{O}(mK_{X_0} + L\,|_{X_0}) \otimes \mathcal{J}(h\,|_{X_0})$ are thus the global sections u of $\mathcal{O}(mK_{X_0} + L\,|_{X_0})$ such that

$$\int_{X_0} \|\, u \,\|_{\omega\otimes h}^2 \, dV_\omega < \infty.$$

We now come to the main technical tool of the proof, namely the Ohsawa-Takegoshi extension theorem; the next version was established by Y.-T. Siu in [Siu 2000].

L^2 extension theorem.
Let $X \xrightarrow{\pi} \Delta$ be a projective family, and let L be a holomorphic line bundle over X with a singular hermitian metric $h = e^{-\varphi}$ such that (1) and (2) holds.$^{(\star)}$ Then every section u of $\mathcal{O}(K_{X_0} + L\,|_{X_0}) \otimes \mathcal{J}(h\,|_{X_0})$ over X_0 extends to a section $\widetilde{u}$ of $\mathcal{O}(K_X + L) \otimes \mathcal{J}(h)$ over X that satisfies the following L^2 estimate:

$$\int_X \|\, \widetilde{u} \,\|_{\omega\otimes h}^2 \, dV_\omega \leq C \int_{X_0} \|\, u \,\|_{\omega\otimes h}^2 \, dV_\omega$$

where $C = 8\pi e\sqrt{2 + \frac{1}{e}}$.

An important point is that the constant C is universal (independent of everything). This is a key factor in establishing the following statement.

Extension of twisted pluricanonical sections.
Under the same hypothesis$^{(\star)}$, every section of $\mathcal{O}(mK_{X_0} + L\,|_{X_0}) \otimes \mathcal{J}(h\,|_{X_0})$ over X_0 extends to a section of $\mathcal{O}(mK_X + L) \otimes \mathcal{J}(h)$ over X $\;(m = 1, 2, \ldots)$.

Remark that the invariance of plurigenera is obtained by taking (L, h) to be trivial.

† In what follows, we will always fix a smooth hermitian metric for each bundle we consider. For every tensor product bundle, we then obtain an induced smooth hermitian metric as tensor product. We will denote these smooth metrics by $|\cdot|^2$, in order to avoid too many suffixes for norms.

2. Proof

Strategy of the proof.

Let σ be the section to be extended. It suffices to construct a metric $H = e^{-\chi}$ for mK_X+L such that $e^{-\chi}|_{X_0} = |\sigma|^{-2}$, for then

$$\int_{X_0} |\sigma|^2 e^{-\frac{(m-1)\chi+\varphi}{m}}\, dV_\omega = \int_{X_0} |\sigma|^{2/m} e^{-\frac{\varphi}{m}}\, dV_\omega$$
$$\leq \operatorname{Vol}_\omega(X_0)^{\frac{m-1}{m}} \Big(\int_{X_0} |\sigma|^2 e^{-\varphi}\, dV_\omega \Big)^{\frac{1}{m}} < \infty.$$

and the L^2 extension theorem completes the proof. Thus we have substituted the problem of extending the section σ for the problem of extending the metric $|\sigma|^{-2}$. The improvement is that we can take roots of metrics. Thus we can introduce an auxiliary line bundle A, and try to extend the metric

$$|\sigma^{\,q}|^{-2}\, e^{-a_r}$$

to a metric of

$$q(mK_X + L) + rK_X + A$$

where e^{-a_r} is a prescribed metric of $rK_X + A$ and $0 \leq r \leq m-1$ (for then we may take root of this extension, and pass to the limit $q \to \infty$ to obtain H). Here A is chosen to be sufficiently ample so that each bundle $F_r := rK_X + A$ is generated by finitely many global sections $\widetilde{u}_1^{(r)}, ..., \widetilde{u}_{N_r}^{(r)}$. This allows us to prescribe a nice enough metric on F_r: $e^{-a_r} := 1/\sum_j |\widetilde{u}_j^{(r)}|^2$ (the metric induced by $\widetilde{u}_1^{(r)}, ..., \widetilde{u}_{N_r}^{(r)}$) for which we can apply the L^2 extension theorem.

The extension process is by induction.[‡] We first put $F_p = pK_X + qL + A$ for $p = qm + r$, where $0 \leq r \leq m-1$ and $q \geq 0$, so that the $\mathbb{Q}$-line bundles

$$\frac{1}{p}\, F_p \to K_X + \frac{1}{m}\, L \quad \text{as} \quad p \to \infty.$$

For each $p \geq m$, we use the sections $\{\widetilde{u}_j^{(p-1)}\}$ of F_{p-1} to define a metric on F_p, for which we apply the L^2 extension theorem to obtain global sections $\{\widetilde{u}_j^{(p)}\}$ of F_p such that

$$e^{-a_p} := 1/\textstyle\sum_j |\widetilde{u}_j^{(p)}|^2 \quad \text{extends} \quad |\sigma^{\,q}|^{-2}\, e^{-a_r}.$$

We then use $\{\widetilde{u}_j^{(p)}\}$ to define a metric on F_{p+1} and iterate this procedure. An important point is that the estimates provided by the L^2 extension theorem guarantees that the family $\{\frac{1}{p}\, a_p\}$ (of weight functions of the metrics $e^{-\frac{1}{p}a_p}$ on $\frac{1}{p}\, F_p$) is locally bounded above, so we are able to pass to the limit and obtain the desired metric.

[‡] The induction is a so-called "tower" in Siu's and Păun's proofs; this extension process corresponds to the second tower in Siu's original proof.

Proof. Let σ be a section of $\mathcal{O}(mK_{X_0} + L\,|_{X_0}) \otimes \mathcal{J}(h\,|_{X_0})$ over X_0. Assume that $\sigma \not\equiv 0$. Let A be a holomorphic line bundle over X such that for every $0 \le r \le m-1$, $F_r := rK_X + A$ is generated by its global sections $\widetilde{u}_1^{(r)}, ..., \widetilde{u}_{N_r}^{(r)}$. Such a line bundle exists by Lemma 0.1. Define F_p inductively by

$$\begin{cases} F_p = F_{p-1} + K_X & p \not\equiv 0 \text{ mod m} \\ F_p = F_{p-1} + K_X + L & p \equiv 0 \text{ mod m} \end{cases}$$

We will construct for each $p \ge m$, a set of global sections $\{\widetilde{u}_1^{(p)}, ..., \widetilde{u}_{N_p}^{(p)}\}$ of F_p such that whenever $p = qm + r$ with $0 \le r \le m-1$, we have $N_p = N_r$, and

$$\widetilde{u}_j^{(p)} \ \text{ extends } \ \sigma^{\otimes q} \otimes \widetilde{u}_j^{(r)}\,\big|_{X_0} \ \text{ for all } \ j.$$

Denote

$$u_j^{(r)} = \widetilde{u}_j^{(r)}\,\big|_{X_0} \ \text{ and } \ u_j^{(p)} = \sigma^{\otimes q} \otimes u_j^{(r)}.$$

By induction, we consider the hermitian bundles

$$\Big(F_{p-1},\ \frac{|\cdot|^2}{\sum_j |\widetilde{u}_j^{(p-1)}|^2}\ \Big),\ (K_X,\, \omega) \ \text{ and } \ (L,\, h).$$

These bundles induce a tensor product bundle $(F_p,\, \|\cdot\|^2)$. To apply the L^2 extension theorem, we write

$$\sum_j \| u_j^{(p)} \|^2 = \begin{cases} \dfrac{\sum_j |u_j^{(p)}|^2}{\sum_k |u_k^{(p-1)}|^2} = \dfrac{\sum_j |u_j^{(r)}|^2}{\sum_k |u_k^{(r-1)}|^2} & \text{if } \ p = qm + r \ \text{ with } \ 0 < r \le m-1\,; \\[2ex] \dfrac{\sum_j |u_j^{(p)}|^2}{\sum_k |u_k^{(p-1)}|^2}\, e^{-\varphi} = \dfrac{\sum_j |u_j^{(0)}|^2}{\sum_k |u_k^{(m-1)}|^2}\, |\sigma|^2\, e^{-\varphi} & \text{if } \ m \,|\, p\,. \end{cases}$$

The ratios on the right-hand sides are bounded on X_0, and since $|\sigma|^2\, e^{-\varphi} \in L^1_{loc}(X_0)$, there exists a positive constant C_1 such that

$$\int_{X_0} \textstyle\sum_j \| u_j^{(p)} \|^2\, dV_\omega \le C_1 \ \text{ for all } \ p \ge m.$$

By the L^2 extension theorem, each $u_j^{(p)}$ extends to a section $\widetilde{u}_j^{(p)}$ over X; moreover, we have the following estimates:

$$\int_X \textstyle\sum_j \| \widetilde{u}_j^{(p)} \|^2\, dV_\omega \le 200\, C_1 \ \text{ for all } \ p \ge m.$$

The Hölder inequality (Lemma 0.2) then implies that the family $\{\, \big(\sum_j |\widetilde{u}_j^{(p)}|^2\big)^{1/p}\, \}$ is bounded in L^1 norm on every compact set. By the mean value inequality, the family $\{\, w_p := \dfrac{1}{p} \log \sum_j |\widetilde{u}_j^{(p)}|^2\, \}$ of psh functions on X is locally uniformly bounded from above.

On X_0, we have $w_p \to \frac{1}{m}\log|\sigma|^2 \not\equiv -\infty$. Hence there exists a subsequence $\{w_{p_\nu}\}$ which converges to some psh function w in $L^1_{loc}(X)$ (Lemma 0.3, 0.4). The metric $H := e^{-mw}$ defined on $mK_X + L$ satisfies

$$i\Theta_H \geq 0 \quad \text{and} \quad H\Big|_{X_0} = |\sigma|^{-2}.$$

Thus the metric $\widetilde{H} := H^{\frac{m-1}{m}} h^{\frac{1}{m}}$ defined on $(m-1)K_X + L$ satisfies $i\Theta_{\widetilde{H}} \geq 0$ and

$$\int_{X_0} \|\,\sigma\,\|^2_{\omega\otimes\widetilde{H}}\, dV_\omega = \int_{X_0} \|\,\sigma\,\|^{2/m}_{\omega\otimes h}\, dV_\omega < \infty$$

by the Hölder inequality. The L^2 extension theorem now completes the proof. □

List of lemmas.

Lemma 0.1 (Hartshorne III. 12)**.** *Let $X \xrightarrow{\pi} \Delta$ be a projective family, and let F be a holomorphic vector bundle over X. Then for sufficiently large N, $F \otimes \mathcal{O}_X(N)\,|_U$ is generated by finitely many global sections, where U is a neighborhood of X_0.*

Lemma 0.2. *Let $r \in (0, +\infty)$ and $r_1, ..., r_p \in (0, +\infty]$ such that*

$$\frac{1}{r_1} + ... + \frac{1}{r_p} = \frac{1}{r}\,.$$

Then for any measurable functions $f_1, ..., f_p$,

$$\int \prod_{j=1}^{p} |f_j|\; d\mu \leq \prod_{j=1}^{p} \Big(\int |f_j|^{r_j}\, d\mu \Big)^{1/r_j}$$

unless $\int |f_j|^{r_j}\, d\mu = 0$ for some j, in which case the right-hand side does not make sense.

Lemma 0.3. *Let $\{u_j\}$ be a sequence of psh functions which is locally uniformly bounded from above on a domain $\Omega \subset \mathbb{C}^n$. Then either $u_j \to -\infty$ uniformly on compact sets, or else there is a subsequence $\{u_{j_\nu}\}$ which converges in $L^1_{loc}(\Omega)$.*

Lemma 0.4. *Let $\{u_j\}$ be a sequence of locally integrable psh functions on a domain $\Omega \subset \mathbb{C}^n$. If $u_j \to T$ in $\mathcal{D}'(\Omega)$, then T is defined by some $u \in L^1_{loc}(\Omega)$, and $u_j \to u$ in $L^1_{loc}(\Omega)$; moreover,*

$$u \overset{a.e.}{=} Reg(u) := \lim_{\epsilon\to 0} \sup_{B(z,\epsilon)} u \in Psh(\Omega).$$

Appendix: Psh functions − definition and basic properties.

An upper semicontinuous function $u : \Omega \to [-\infty, +\infty)$ defined on an open set $\Omega \subset \mathbb{C}^n$ is said to be plurisubharmonic (psh for short) if for every complex affine line $L \subset \mathbb{C}^n$, $u\,|_{\Omega \cap L}$ is subharmonic on $\Omega \cap L$. This condition is equivalent to the statement that u satisfies the mean value inequality over circles in Ω:

$$(1) \qquad u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + e^{i\theta}\xi)\, d\theta$$

for every $a \in \Omega$ and $\xi \in \mathbb{C}^n$ with $|\xi| \leq dist(a, \partial\Omega)$. If $u \in C^2(\Omega)$, we write $L : z = a + \omega\,\xi$ as a function of $\omega \in \mathbb{C}$, so that the subharmonicity of $u\,|_{\Omega \cap L}$ is equivalent to

$$0 \leq \frac{1}{4}\,\Delta u\,|_{\Omega \cap L} = \frac{\partial^2 u}{\partial\omega\partial\bar{\omega}} = \sum_{1 \leq j,\, k \leq n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \xi_j \bar{\xi_k}.$$

Therefore, u is psh if and only if $i\partial\overline{\partial}u \geq 0$. A similar statement also holds for $v \in L^1_{loc}(\Omega)$, i.e., v is equal to a psh function on Ω a.e. if and only if $i\partial\overline{\partial}v \geq 0$ in the sense of currents. We denote the sets of subharmonic functions and psh functions on Ω by $Sh(\Omega)$ and $Psh(\Omega)$, respectively. An integration of (1) over $\xi \in S(0, r)$ shows that $Psh(\Omega) \subset Sh(\Omega)$.

The following is a short list of basic properties of psh functions.

Proposition 0.5. *If* $\{u_j\}_{j \in \mathbb{N}} \subset Psh(\Omega)$ *and* $u_j \searrow u$ *as* $j \to \infty$*, then* $u \in Psh(\Omega)$*.*

Proposition 0.6. *Let* $u_1, ..., u_p \in Psh(\Omega)$ *and let* $\chi : \mathbb{R}^p \to \mathbb{R}$ *be a convex function such that* $\chi(t_1, ..., t_p)$ *is nondecreasing in each* t_j*. If* χ *is extended by continuity into a function* $[-\infty, +\infty)^p \to [-\infty, +\infty)$*, then*

$$\chi(u_1, ..., u_p) \in Psh(\Omega).$$

In particular, $u_1 + ... + u_p$*,* $\max\{u_1, ..., u_p\}$*,* $\exp(u_1 + ... + u_p) \in Psh(\Omega)$*.*

Proposition 0.7. *If* Ω *is connected and* $u \in Psh(\Omega)$*, then either* $u \equiv -\infty$ *or* $u \in L^1_{loc}(\Omega)$*.*

Let $\rho \in C^\infty(\mathbb{R}^m)$ be a radially symmetric function (i.e., $\rho(x)$ depends only on $|\,x\,|$) such that

$$\rho \geq 0, \qquad \text{Supp}\,\rho \in B(0, 1) \quad \text{and} \quad \int_{\mathbb{R}^m} \rho(x)\, dV = 1.$$

Then

$$\rho_\epsilon(x) := \frac{1}{\epsilon^m}\,\rho(\,\frac{x}{\epsilon}\,) \ \text{ satisfies } \ \text{Supp}\,\rho_\epsilon \in B(0, \epsilon) \quad \text{and} \quad \int_{\mathbb{R}^m} \rho_\epsilon(x)\, dV = 1.$$

We call $\{\rho_\epsilon\}$ a family of smoothing kernels, due to the next proposition:

Proposition 0.8. *Let $\{\rho_\epsilon\}$ be a family of smoothing kernels. Suppose $u \in Psh(\Omega)$ is such that $u \not\equiv -\infty$ on each connected component of Ω. Then the convolution $u * \rho_\epsilon$ is smooth and psh on $\Omega_\epsilon := \{ z \in \Omega \,|\, dist(z, \partial\Omega) > \epsilon \}$ and $u * \rho_\epsilon \searrow u$ as $\epsilon \to 0$.*

Proposition 0.9. *The convex cone $Psh(\Omega) \cap L^1_{loc}(\Omega)$ is closed in the topological vector space $L^1_{loc}(\Omega)$ (Fréchet topology of convergence in L^1 norm on every compact subset of Ω), and it has the property that every bounded subset is relatively compact.*

Proposition 0.10. *Let $\{u_\alpha\}_{\alpha\in I} \subset Psh(\Omega)$ be locally uniformly bounded from above, and let $u := sup\, u_\alpha$ be the upper envelope (pointwise supremum). Then the regularized upper envelope $Reg(u) := \lim_{\epsilon\to 0} \sup_{B(z,\epsilon)} u$ is psh and is equal to u almost everywhere.*

References

[Skoda 1972] H. Skoda — *Application des techniques L^2 à la théorie des ideaux d'un algèbre de fonctions holomorphes avec poids,* Ann. Sci. Ec. Norm. Sup. **5** (1972), 548-580.

[Nakamura 1975] I. Nakamura — *Complex parallelisable manifolds and their small deformations,* J. Differential Geom. **10** (1975), 85-112.

[Ohsawa-Takegoshi 1987] T. Ohsawa and K. Takegoshi, — *On the extension of L^2 holomorphic funcion,* Math. Z. **195** (1987), 197-204.

[Siu 1998] Y.-T. Siu — *Invariance of plurigenera,* Inv. Math. **134** (1998), 661-673.

[Siu 2000] Y.-T. Siu — *Extension of twisted pluricanonical sections with plurisubharmonic weight and invariance of semi-positively twisted plurigenera for manifolds not necessarily of general type,* Complex Geometry (Göttingen, 2000), Springer, Berlin, 2002, 223-277.

[Siu 2005] Y.-T. Siu — *Multiplier ideal sheaves in complex and algebraic geometry,* Sci. China Math. **48** (2005), 1-31.

[Păun 2007] M. Păun — *Siu's invariance of plurigenera: a one-tower proof,* J. Differential Geom. **76** (2007), 485-493.

[Demailly 2007] J.P. Demailly — *Complex analytic and differential geometry,* online book: `https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf`

[McNeal-Mustaţă 2010] J. McNeal and M. Mustaţă — *Analytic and algebraic geometry: common problems, different methods,* IAS/Park City Math. Ser. **17**, AMS/IAS (2010).

[Demailly 2012] J.P. Demailly — *Analytic methods in algebraic geometry,* Surveys of Modern Mathematics **1**, International Press of Boston, Inc. (2012).