Multiplier Ideal Sheaves and Nadel's Vanishing Theorem

For a singular hermitian metric $h = e^{-\varphi}$ of a holomorphic line bundle L over a complex manifold X, we define its *multiplier ideal sheaf* $\mathcal{J}(h) \subset \mathcal{O}_X$ to be the ideal sheaf of germs of holomorphic functions f such that $|f|^2 e^{-\varphi} \in L^1_{loc}$. Suppose that the curvature current $i\Theta_h(L) = i\partial\overline{\partial}\varphi \geq 0$ (or equivalently, the local weigh function $\varphi \overset{a.e.}{=} \phi \in Psh$). In this case, $\mathcal{J}(h)$ has many nice properties, such as coherence and Nadel vanishing.

Hörmander's L^2 estimates for the $\overline{\partial}$-operator. *Let (X, ω) be a weakly pseudoconvex Kähler manifold, L a holomorphic line bundle over X endowed with a smooth hermitian metric $h = e^{-\varphi}$, and $q \in \{1, 2, ..., n = dimX\}$. Suppose $i\Theta_h(L) = i\partial\overline{\partial}\varphi \geq c\,\omega$ for some positive constant c. Then for any $g \in L^2(X, \Lambda^{n,q}T^\star X \otimes L)$ such that $\overline{\partial}\, g = 0$, there exists $f \in L^2(X, \Lambda^{n,q-1}T^\star X \otimes L)$ such that*

$$\overline{\partial}\, f = g \quad \text{and} \quad \int_X |f|^2_\omega\, e^{-\varphi}\, dV_\omega \;\leq\; \frac{1}{qc} \int_X |g|^2_\omega\, e^{-\varphi}\, dV_\omega.$$

Proposition 0.1. *$\mathcal{J}(h)$ is coherent.*

Lemma 0.2 (Strong noetherian property)**.** *Let $\mathcal{F}$ be a coherent analytic sheaf over X and let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset ...$ be an increasing sequence of coherent subsheaves of $\mathcal{F}$. Then the sequence $\{\mathcal{F}_k\}$ is stationary on every compact subset of X.*

Lemma 0.3 (Krull's intersection theorem)**.** *Let R be a noetherian local ring and let $\mathfrak{m}$ be the maximal ideal of R. Then for every finitely generated R-module F and every submodule E of F,*

$$\bigcap_{k=1}^{\infty} E + \mathfrak{m}^k F = E.$$

Lemma 0.4. *If the Lelong number*

$$\nu(\varphi, x) := \liminf_{z \to x} \frac{\varphi(z)}{\log|z - x|} \;\leq\; 2(n + k)$$

for some $x \in X$ and $k \in \mathbb{N}$, then $\mathcal{J}(h)_x \subset \mathfrak{m}_x^k$, where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$.

Theorem 0.5 (Nadel's vanishing theorem)**.** *Let X be a compact complex projective algebraic manifold and let L be a holomorphic line bundle over X endowed with a singular hermitian metric $h = e^{-\varphi}$. Suppose $i\Theta_h(L) = i\partial\overline{\partial}\varphi \geq \omega$ for some Kähler form ω on X. Then*

$$H^p\big(X, \mathcal{O}(K_X + L) \otimes \mathcal{J}(h)\big) = 0 \quad \text{for} \quad p \geq 1.$$

Lemma 0.6 (Chow's theorem)**.** *An analytic subspace of a complex projective space that is closed in the strong topology is closed in the Zariski topology.*

Lemma 0.7. *Let V be an analytic variety of an open set U in $\mathbb{C}^n$. Then every f in $L^2(U) \cap \mathcal{O}(U \setminus V)$ is equal to a holomorphic function on U almost everywhere.*

Theorem 0.8 (Hörmander, Andreotti-Vesentini, Skoda)**.** *Let (X, ω) be a Kähler manifold and let E be a holomorphic vector bundle over X endowed with a hermitian metric h. Fix $q \in \{1, 2, ..., n = dimX\}$. Let $\Omega \subset\subset X$ be a smoothly bounded domain whose boundary is q-positive with respect to ω. Assume that there is a smooth strictly positive $(1,1)$-form γ on X such that $i\Theta_h(E) + Ricci(\omega) - \gamma$ is q-positive with respect to ω. Then for any E-valued $(0,q)$-form g on Ω such that*

$$\overline{\partial}\, g = 0 \quad \textit{and} \quad \int_\Omega |g|^2_{h,\,\omega;\,\gamma}\, dV_\omega < +\infty$$

there exists an E-valued $(0, q-1)$-form f such that

$$\overline{\partial}\, f = g \quad \textit{and} \quad \int_\Omega |f|^2_{h,\,\omega}\, dV_\omega \;\le\; \int_\Omega |g|^2_{h,\,\omega;\,\gamma}\, dV_\omega.$$