Ohsawa-Takegoshi L^2 extension theorem

In this note, we are going to derive a version of the Ohsawa-Takegoshi extension theorem [1] that was historically used to prove the invariance of plurigenera [5]. We will closely follow the arguments in [5] and provide details for the proof.

Our setting is as follows. Let Y be an n-dimensional Kähler manifold. Assume there exists a holomorphic function w on Y such that $\sup_Y |w| \leq 1$ and dw is nonzero at any point of $Z := w^{-1}(0)$. We also assume there exists an analytic hypersurface V in Y such that $Z \nsubseteq V$ and $Y - V$ is Stein.

The main example we have in mind is a projective family $Y \xrightarrow{\pi} \Delta$, in which case $Y \hookrightarrow \Delta \times \mathbb{P}^k$ by shrinking the base, and V could be taken to be a hyperplane section so that $Y - V$ is embedded as a closed submanifold of $\Delta \times (\mathbb{P}^k - \mathbb{P}^{k-1}) \cong \Delta \times \mathbb{C}^k$, hence Stein.

Theorem. Suppose given a holomorphic line bundle L over Y with a singular hermitian metric $h = e^{-\varphi}$ such that

(i) the curvature current $i\Theta_h(L) \geq 0$ and

(ii) $h|_Z$ defines a singular hermitian metric on $L|_Z$.

Then for every section f of $\mathcal{O}(K_Z + L|_Z) \otimes \mathcal{J}(h|_Z)$ over Z, $f \wedge dw$ extends to a section F of $\mathcal{O}(K_Y + L) \otimes \mathcal{J}(h)$ over Y that satisfies the following L^2 estimate:

$$\int_Y |F|^2 e^{-\varphi} \leq C \int_Z |f|^2 e^{-\varphi}$$

where $C = 8\pi e\sqrt{2 + \frac{1}{e}}$.

An important point is that the constant C is universal (i.e., independent of everything). This is a key factor in proving the invariance of plurigenera [5]. Also, the numerical value of C given above is not optimal (i.e., minimal), as we will see later.

1. Standard Approximation

Lemma 1.1. *There exist*

(i) *an increasing sequence of Stein domains* $\Omega_1 \Subset \ldots \Subset \Omega_\nu \Subset \Omega_{\nu+1} \Subset \ldots$ *with smooth strongly pseudoconvex boundaries such that* $\bigcup_\nu \Omega_\nu = Y - V$.

(ii) *a decreasing sequence of psh functions* $\varphi_\nu \in C^\infty(\Omega_\nu)$ *with* $\sqrt{-1}\partial\bar{\partial}\varphi_\nu > 0$ *such that* $\varphi_\nu \searrow \varphi$ *as* $\nu \to \infty$.

Proof. Recall that the following statements are equivalent for a complex manifold X:

(I) X is Stein (i.e., holomorphically convex and holomorphically separable).

(II) X is biholomorphic to a closed complex submanifold of $\mathbb{C}^m$ for some m.

(III) X is strongly pseudoconvex (i.e., there exists $\rho \in C^\infty(X,\mathbb{R})$ such that $\sqrt{-1}\partial\overline{\partial}\rho > 0$ and $\Omega_r := \{x \in X \,|\, \rho(x) < r\} \Subset X$ for all $r \in \mathbb{R}$).

(IV) $H^p(X,\mathcal{F}) = 0$ for every coherent analytic sheaf $\mathcal{F}$ on X and every $p \geq 1$.

Choose a function ρ on $X = Y - V$ as in (III). Exponentiating it, we may assume $\rho \geq 0$. Observe that Ω_r is itself strongly pseudoconvex (with $1/(r-\rho)$ being a smooth strictly psh exhaustion function on Ω_r). Moreover, by Sard's theorem, $\partial\Omega_r = \rho^{-1}(r)$ is smooth for almost every $r \in \mathbb{R}$. Thus we obtain (i) after reindexing the domains Ω_r.

Smooth Regularization of Singular Metrics The singular hermitian metric h on L is given by $\|\cdot\|_h^2 = |\cdot|^2 e^{-\varphi}$, where $\varphi \in Psh(X)$ and $|\cdot|^2$ is a fixed smooth hermitian metric on L. In order to regularize the L^1_{loc} function φ, we first embed X into $\mathbb{C}^m$ for some m. Invoking a well-known theorem [2]:

Every Stein submanifold X of a complex analytic space V admits a Stein neighborhood U and a holohorphic retract $r : U \to X$.

we see there exists a Stein open set $U \subset \mathbb{C}^m$ such that $X \subset U$, and a holomorphic retract $r : U \to X$. We then use the map r to pull back φ and apply the standard convolution techniques in $\mathbb{C}^m$; let $\tilde{\varphi} := \varphi \circ r \in Psh(U)$ and $\{\varrho_\varepsilon\}$ a family of smoothing kernels, so

$$\varrho_\varepsilon(x) := \frac{1}{\varepsilon^{2m}}\,\varrho_1\Big(\frac{x}{\varepsilon}\Big)$$

where $\varrho_1 \in C^\infty(\mathbb{C}^m,\mathbb{R})$ is a radially symmetric function (i.e., $\varrho_1(x)$ depends only on $|x|$) such that

$$\varrho_1 \geq 0, \qquad \mathrm{Supp}\,\varrho_1 \Subset B(0,1) \quad \text{and} \quad \int_{\mathbb{C}^m} \varrho_1(x)\,dV = 1.$$

Then $\tilde{\varphi} * \varrho_\varepsilon$ is smooth and psh on $U^\varepsilon := \{\, x \in U \,|\, dist(x,\partial U) > \varepsilon \,\}$, and

$$\tilde{\varphi} * \varrho_\varepsilon \searrow \tilde{\varphi} \ \text{ as } \ \varepsilon \to 0.$$

Moreover,

$$\sqrt{-1}\partial\overline{\partial}(\tilde{\varphi} * \varrho_\varepsilon) = (\sqrt{-1}\partial\overline{\partial}\tilde{\varphi}) * \varrho_\varepsilon = (r^*\sqrt{-1}\partial\overline{\partial}\varphi) * \varrho_\varepsilon > 0.$$

By the Stein property, there is an exhaustion $\{U_\nu\}$ of U by bounded Stein domains U_ν. Hence $\{X_\nu := X \cap U_\nu\}$ is an exhaustion of X by relatively compact open subsets. Finally, consider an exhaustion $\{\Omega_\nu\}$ of X as in (i). For each ν, we choose j_ν and ε_ν such that $\Omega_\nu \subset X_{j_\nu} \subset U_{j_\nu} \subset U^{\varepsilon_\nu}$. Then $\varphi_\nu := \tilde{\varphi} * \varrho_{\varepsilon_\nu}$ verifies (ii). □

The proof of the theorem consists of two steps. The first is showing that the result holds for each pair $(\Omega_\nu, \varphi_\nu)$, and the second is passing to a subsequential limit $\nu_j \to \infty$.

Lemma 1.2. *Suppose for each* ν,

$$f \wedge dw\,|_{Z_\nu := Z \cap \Omega_\nu}$$

extends to a section F_ν *of* $\mathcal{O}(K_{\Omega_\nu} + L|_{\Omega_\nu})$ *over* Ω_ν *that satisfies the estimate:*

$$\int_{\Omega_\nu} |F_\nu|^2 e^{-\varphi_\nu} \le C \int_{Z_\nu} |f|^2 e^{-\varphi_\nu}.$$

Then the theorem holds true.

Proof. Since

(i) $$\int_{\Omega_\nu} |F_\nu|^2 e^{-\varphi_\nu} \le C \int_{Z_\nu} |f|^2 e^{-\varphi_\nu} \le C \int_Z |f|^2 e^{-\varphi} < \infty,$$

(ii) $|F_\nu|^2 \in Psh(\Omega_\nu)$ (for $\sqrt{-1}\partial\overline{\partial}|F_\nu|^2 = \sqrt{-1}\partial F_\nu \wedge \overline{\partial F_\nu} \ge 0$) and

(iii) $\{e^{-\varphi_\nu}\}$ is locally uniformly bounded below (φ_1 being upper semicontinuous),

the mean value inequality for psh functions implies that the family $\{F_\nu\}$ is locally uniformly bounded. By Montel's theorem, there is a subsequence $\{F_{\nu_j}\}$ converging to some $F \in H^0(X, K_X + L|_X)$, where $X = \bigcup_\nu \Omega_\nu = Y - V$. Moreover, we have the estimate

$$\int_X |F|^2 e^{-\varphi} \le \liminf_{\nu\to\infty} \int_X \chi_{\Omega_\nu} |F_\nu|^2 e^{-\varphi_\nu} \le C \int_Z |f|^2 e^{-\varphi}$$

by Fatou's lemma. Now since pluripolar sets are removable for L^2 holomorphic functions, we conclude the proof of the theorem. □

2. Preliminaries and L^2 Estimates for $\overline{\partial}$

We have therefore reduced the original problem to the case of a smoothly bounded Stein domain Ω in X and a smooth hermitian metric $e^{-\kappa}$ on $L|_\Omega$.[†] Since Ω is Stein, we can extend $f \wedge dw$ to an L-valued holomorphic n-form $\tilde{F}$ on Ω. Moreover, by extending to a Stein neighborhood of Ω, we may also assume that

$$\int_\Omega |\tilde{F}|^2 e^{-\kappa} < \infty. \tag{1}$$

Of course, we have no better estimate of this extension, for it is obtained merely by the Stein property. In particular, the estimate could degenerate as $\nu \to \infty$.

[†] In view of Lemma 1.2, we will henceforth discard the original singular metric, with the notation $h = e^{-\varphi}$ being saved for later use. We take here $\Omega = \Omega_\nu$ and $\kappa = \varphi_\nu$ for a fixed ν.

In order to tame the growth of $\tilde{F}$, we first multiply it by a cutoff function $\chi(w) = \varrho(|w|^2)$ which is $\equiv 1$ on $Z \cap \Omega$. However the resulting extension $\chi\tilde{F}$ of $f \wedge dw$ may not be holomorphic. So to get a holomorphic correction, we turn to solve the $\overline{\partial}$-equation

$$\overline{\partial}\alpha = w^{-1}\,\overline{\partial}(\chi\tilde{F}).$$

Then $F := \chi\tilde{F} - w\alpha$ is holomorphic and $F|_{\{w=0\}} = f \wedge dw$.

Also, to obtain an estimate for F with respect to $F|_{\{w=0\}}$, we introduce a variable ε and use the cutoff function $\chi_\varepsilon(w) := \varrho(|w|^2/\varepsilon^2)$ instead of $\chi(w)$. Then F is obtained by solving the $\overline{\partial}$-equations

$$\overline{\partial}\alpha_\varepsilon = w^{-1}\,\overline{\partial}(\chi_\varepsilon\tilde{F}) = \varepsilon^{-2}\,\chi'\,\tilde{F}\,\overline{\partial w},$$

with L^2 estimates, setting $F_\varepsilon := \chi_\varepsilon\,\tilde{F} - w\alpha_\varepsilon$ and passing to the limit as $\varepsilon \to 0$.

The practical situation is, however, fairly complicated. The main difficult part is to keep track of the estimates. In fact, to apply the L^2 method, we consider

$$\int_\Omega |\varepsilon^{-2}\,\chi'\,\tilde{F}|^2\,e^{-\kappa},$$

which is of order ε^{-2} (because of (1) and that $\operatorname{Supp}\varrho$ is compact). To offset this order, we introduce the weight function $\log(|w|^2 + \varepsilon^2)$ so that the additional curvature is

$$\sqrt{-1}\,\partial\overline{\partial}\log(|w|^2 + \varepsilon^2) = \frac{\varepsilon^2}{(|w|^2 + \varepsilon^2)^2}\,\sqrt{-1}\,\partial w \wedge \overline{\partial w}.$$

The Bochner-Kodaira formula (3) (on page 9) then implies an a priori inequality

$$\|\overline{\partial}^* u\|^2 + \|\overline{\partial} u\|^2 \geq \Big\|\,\frac{\varepsilon}{|w|^2 + \varepsilon^2}\,\langle u, dw\rangle\,\Big\|^2$$

where $\langle u, dw\rangle = dw(u)$ is the $(n,0)$-form obtained by contracting u and $dw = \partial w$ with respect to the Kähler metric tensor (whose definition can be found on page 8). Applying the Schwarz inequality, we obtain

$$|\langle\!\langle u, \varepsilon^{-2}\,\chi'\,\tilde{F}\,\overline{dw}\,\rangle\!\rangle|^2 = |\langle\!\langle dw(u), \varepsilon^{-2}\,\chi'\,\tilde{F}\,\rangle\!\rangle|^2$$

$$\leq \left\|\,\frac{|w|^2 + \varepsilon^2}{\varepsilon}\,\varepsilon^{-2}\,\chi'\,\tilde{F}\,\right\|^2 \cdot \left\|\,\frac{\varepsilon}{|w|^2 + \varepsilon^2}\,\langle u, dw\rangle\,\right\|^2$$

$$\leq \left\|\,\frac{|w|^2 + \varepsilon^2}{\varepsilon}\,\varepsilon^{-2}\,\chi'\,\tilde{F}\,\right\|^2 \cdot \left(\|\overline{\partial}^* u\|^2 + \|\overline{\partial} u\|^2\right)$$

where $\frac{(|w|^2+\varepsilon^2)^2}{\varepsilon^2}$ is of order ε^2 exactly canceling the unwanted factor ε^{-2}. However, the introduced weight $\frac{1}{|w|^2+\varepsilon^2}$ for the L^2 norm $\|\cdot\|$ also contributes back a factor ε^{-2}, and we are back to the beginning. Nevertheless, we see that the ideal situation would be to get the contribution of curvature without returning any factor. This is impossible for the Bochner-Kodaira formula, because all norms in the formula are equally weighted.

Following T. Ohsawa and K. Takegoshi, we can use the twisted Bochner-Kodaira formula to obtain the a priori inequality

$$\int_\Omega (\eta+\gamma)\,|\,\overline{\partial}_\psi^* u\,|^2\, e^{-\psi} + \int_\Omega \eta\,|\,\overline{\partial} u\,|^2\, e^{-\psi} \geq \int_\Omega -\sqrt{-1}\partial\overline{\partial}\eta\,(u,u)\, e^{-\psi}, \tag{2}$$

where $\psi = \kappa + |\,w\,|^2$, η is the weight we introduce, which produces the right-hand sided term similar to the curvature term before, and γ is to insure that this inequality holds so that considering the operators

$$T = \overline{\partial} \circ \sqrt{\eta+\gamma} \quad \text{and} \quad S = \sqrt{\eta}\;\overline{\partial},$$

the left-hand side of (2) reads $\|T^* u\|^2 + \|Su\|^2$ and we can use the theory of Hilbert spaces to solve the equation:

$$T\alpha = \varepsilon^{-2}\,\chi'\,\tilde{F}\,\overline{\partial w}.$$

2.2. **Unbounded operators on Hilbert spaces.** We introduce here some background in functional analysis. Let H_i be complex Hilbert spaces ($i = 1, 2, 3$). By a densely defined operator T, we shall mean a linear map T whose domain $D(T)$ is a dense subspace of H_1 and whose range $R(T)$ lies in H_2, often denoted by abuse of notation with $T : H_1 \to H_2$.

We are mainly interested in closed and densely defined operators, that is, we assume in addition that the graph $G(T) = \{(x, Tx) \,|\, x \in D(T)\}$ of T is closed in $H_1 \times H_2$. Such an operator T has the nice property that its adjoint $T^* : H_2 \to H_1$ is also closed and densely defined, and T comes naturally with the orthogonal decompositions of Hilbert spaces: $H_2 = N(T) \oplus \overline{R(T^*)}$ and its dual $H_1 = N(T^*) \oplus \overline{R(T)}$.

The following lemma is of fundamental importance in solving $\overline{\partial}$-equations.

Lemma 2.1. *Let $T : H_1 \to H_2$ and $S : H_2 \to H_3$ be closed and densely defined operators such that $ST = S \circ T = 0$. Then for every $C \geq 0$ and $\beta \in H_2$ such that $S\beta = 0$, the following two statements are equivalent:*

(i) *There exists $\alpha \in H_1$ such that $T\alpha = \beta$ and $\|\alpha\|_1 \leq C$.*

(ii) $|\langle u, \beta\rangle_2|^2 \leq C^2(\|T^*u\|_1^2 + \|Su\|_3^2)$ *for all $u \in D(S) \cap D(T^*)$.*

Proof. If (i) holds, then $|\langle u, \beta\rangle_2| = |\langle T^*u, \alpha\rangle_1| \leq \|T^*u\|_1 \|\alpha\|_1$ for every $u \in D(T^*)$, hence (ii) is true. Conversely, suppose (ii) holds. For each $u \in D(T^*)$, we write $u = u_1 + u_2$ with $u_1 \in N(S)$ and $u_2 \in \overline{R(S^*)}$. Then since $T^*S^* = 0$ and $\beta \in N(S)$, we have $T^*u = T^*u_1$ and $\langle u, \beta\rangle_2 = \langle u_1, \beta\rangle_2$. Therefore,

$$|\langle u, \beta\rangle_2| = |\langle u_1, \beta\rangle_2| \leq C\|T^*u_1\|_1 = C\|T^*u\|_1$$

for every $u \in D(T^*)$, where the inequality follows from (ii). Thus we have a well defined continuous linear form $T^*u \mapsto \langle u, \beta\rangle_2$ on $D(T^*)$ whose norm is $\leq C$. By the Hahn-Banach theorem, it extends to a continuous linear form on H_1 with norm $\leq C$, which we represent it as $h \mapsto \langle h, \alpha\rangle_1$ where $\alpha \in H_1$ and $\|\alpha\|_1 \leq C$. In particular, $\langle T^*u, \alpha\rangle_1 = \langle u, \beta\rangle_2$ for all $u \in D(T^*)$. Hence $\alpha \in D(T^{**}) = D(T)$ and $T\alpha = \beta$. □

In a nutshell, the inequality $|\langle u, \beta\rangle|^2 \leq C^2(\|T^*u\|^2 + \|Su\|^2)$ for all $u \in D(S) \cap D(T^*)$ is equivalent to the inequality $|\langle u, \beta\rangle| \leq C\|T^*u\|$ for all $u \in D(T^*)$, which by the Hahn-Banach theorem and Riesz representation, means that $T^*u \mapsto \langle u, \beta\rangle$ can be extended to $h \mapsto \langle h, \alpha\rangle$, hence $\langle T^*u, \alpha\rangle = \langle u, \beta\rangle$ for all $u \in D(T^*)$, and we have $T\alpha = \beta$.

The abstract existence theorem formulates suitable inequalities that will guarantee the solvability of an equation together with an estimate for the norm of solutions. In our case, this inequality is carried out by the basic estimate (Lemma 2.2). Before stating it, we first introduce some preliminaries.

2.3. **Basics in the L^2 theory.** We briefly review here some background in the L^2 theory. Let (E,h) be a hermitian holomorphic vector bundle over a complex manifold X. The underlying space where the L^2 theory develops is the Hilbert space $L^2(X,E)$ of square integrable global sections of E. For $u,v \in L^2(X,E)$, we denote the corresponding norm, resp. inner product, by

$$\|u\|^2 = \int_X |u|^2 \, dV, \text{ resp. } \langle\!\langle u,v\rangle\!\rangle = \int_X \langle u,v\rangle \, dV.$$

We will consider the spaces

$$\mathcal{D}(X,E) \subset L^2(X,E) \subset \mathcal{D}'(X,E)$$

where $\mathcal{D}(X,E)$ denotes the space of testing sections of E (compactly supported smooth sections) and $\mathcal{D}'(X,E)$ denotes the space of distributional (or generalized) sections of E.

Given a linear differential operator $P : C^\infty(X,E) \to C^\infty(X,F)$, it extends to a closed and densely defined operator $P_{\mathcal{H}} : L^2(X,E) \to L^2(X,F)$ as follows. First, by computing as distributions, P extends to an operator $\widetilde{P} : \mathcal{D}'(X,E) \to \mathcal{D}'(X,F)$. Then we set [‡]

$$Dom(P_{\mathcal{H}}) := \{u \in L^2(X,E) \,|\, \widetilde{P}u \in L^2(X,E)\} \quad \text{and} \quad P_{\mathcal{H}}u := \widetilde{P}u.$$

Since $\mathcal{D}(X,E)$ is dense in $L^2(X,E)$, $P_{\mathcal{H}}$ is densely defined. Suppose (u_j,v_j) is a sequence in the graph of $P_{\mathcal{H}}$ converging to some $(u,v) \in L^2(X,E) \times L^2(X,F)$. Then $u_j \to u$ weakly in $\mathcal{D}'(X,E)$ and thus $v_j = P_{\mathcal{H}}u_j \to \widetilde{P}u$ weakly in $\mathcal{D}'(X,F)$. Hence $\widetilde{P}u = v$ and so $P_{\mathcal{H}}$ is closed and densely defined.

The adjoint $(P_{\mathcal{H}})^*$ is constructed as follows. First, $Dom((P_{\mathcal{H}})^*)$ is defined as

$$\{v \in L^2(X,F) \mid \exists\, C \geq 0 \text{ s.t. } \forall\, u \in D(P_{\mathcal{H}}) \quad \Big|\int_X \langle P_{\mathcal{H}}u, v\rangle \, dV\Big|^2 \leq C \int_X |\, u\,|^2 \, dV\},$$

i.e., the domain of $(P_{\mathcal{H}})^*$ consists of $v \in L^2(X,F)$ such that the linear form

$$Dom(P_{\mathcal{H}}) \ni u \mapsto \langle\!\langle P_{\mathcal{H}}u, v\rangle\!\rangle$$

is bounded. Applying the Hahn-Banach theorem and the Riesz representation theorem, we see that there is $w \in L^2(X,E)$ such that $\langle\!\langle u,w\rangle\!\rangle = \langle\!\langle P_{\mathcal{H}}u,v\rangle\!\rangle$ for every $u \in Dom(P_{\mathcal{H}})$. Now since $Dom(P_{\mathcal{H}})$ is dense, such an element w is unique, and we set $(P_{\mathcal{H}})^*\, v := w$.

The above construction goes in the exactly same way for every closed and densely defined operator $T : H_1 \to H_2$. To show that T^* is closed and densely defined, we have $H_1 \times H_2 = G(-T) \oplus G(T^*)$ by a direct verification. In particular, $G(T^*)$ is closed, and each $v \in H_2$ can be written as $v = y + TT^*y$ for some $y \in Dom(T^*)$. Thus

$$\langle v, y\rangle_2 = \|y\|_2^2 + \|T^*y\|_1^2 \quad \text{implies} \quad Dom(T^*)^\perp = \{0\}.$$

Hence T^* is closed and densely.

[‡] For greater notational clarity, the domain of an operator T will henceforward be denoted by $Dom(T)$.

2.4. **Contraction and pairing.** For a hermitian vector bundle (E, h) over a complex manifold X, we associate a (fiberwise) bilinear map

$$B_E(\cdot\,,\cdot) : \Lambda^{p,q} T^*X \otimes E \times \Lambda^{r,s} T^*X \otimes \bar{E} \to \Lambda^{p+r,q+s} T^*X$$

by combining the wedge product $(\alpha, \beta) \mapsto \alpha \wedge \beta$ with $h(\cdot\,,\bar{\cdot}) \in C^\infty(X, (E \otimes \bar{E})^*)$ as tensor product of bilinear maps. Next we combine B_E with $E_1 \times E_2 \xrightarrow{\otimes} E_1 \otimes E_2$, and take $E = \Lambda^{1,0} T^*X$, $E_1 = \mathbb{C} \times X$ and $E_2 = L_2$ being a line bundle, to obtain a map

$$\Lambda^{1,0} T^*X \times \Lambda^{n,1} T^*X \otimes L_2 \to \Lambda^{n,0} T^*X \otimes L_2$$

which we will denote it by $(f, g) \mapsto f(g)$, or sometimes by $(f, g) \mapsto \langle g, f\rangle$ following the notation of [5].

Under local coordinates $(z_1, ..., z_n)$, suppose $g_{\alpha\overline{\beta}}\, dz^\alpha \otimes d\overline{z}^\beta$ is a hermitian metric on X, $\xi = \xi_\alpha\, dz^\alpha$ and $u = u_{\overline{\beta}}\, e_{L_2} \otimes dz \wedge d\overline{z}^\beta$, where e_{L_2} is a local frame of L_2 and $dz = dz^1 \wedge ... \wedge dz^n$. Then

$$\xi(u) = \xi_\alpha\, g^{\alpha\overline{\beta}} u_{\overline{\beta}}\, e_{L_2} \otimes dz.$$

Suppose now that L_2 is endowed with a smooth hermitian metric $h_{L_2} = e^{-\varphi_{L_2}}$. Then every $\omega \in C^\infty(X, \Lambda^{1,1} T^*X)$ induces a hermitian form $\omega(\cdot\,,\cdot)$ on $\Lambda^{n,1} T^*X \otimes L_2$:

$$\omega(u, v) := \left\langle \sqrt{-1} \cdot B_{\Lambda^{n,0} T^*X \otimes L_2}(u, \overline{v}),\ \overline{\omega} \right\rangle.$$

Locally if $\omega = \sqrt{-1}\, \omega_{\alpha\overline{\beta}}\, dz^\alpha \wedge d\overline{z}^\beta$, $v = v_{\overline{\gamma}}\, e_{L_2} \otimes dz \wedge d\overline{z}^\gamma$ and $|e_{L_2}| = 1$, then

$$\omega(u, v) = \omega_{\alpha\overline{\beta}} \left(g^{\alpha\overline{\gamma}} u_{\overline{\gamma}}\right) \det(g^{\mu\overline{\nu}}) e^{-\varphi_{L_2}}\, \overline{\left(g^{\beta\overline{\delta}} v_{\overline{\delta}}\right)}.$$

2.5. **The basic estimate.**

Lemma 2.2. *Let Ω and L be as before. Let η and γ be positive smooth functions on Ω. Fix a smooth hermitian metric $\widetilde{h} = e^{-\psi}$ on $L|_\Omega$, and denote $\overline{\partial}^*_\psi$ the Hilbert space adjoint of $\overline{\partial}$ with respect to $\widetilde{h}$. Then*

$$\int_\Omega (\eta + \gamma)\, |\overline{\partial}^*_\psi u|^2\, e^{-\psi} + \int_\Omega \eta\, |\overline{\partial} u|^2\, e^{-\psi}$$

$$\geq \int_\Omega (\eta\sqrt{-1}\partial\overline{\partial}\psi - \sqrt{-1}\partial\overline{\partial}\eta)\, (u, u)\, e^{-\psi} + 2\, Re \int_\Omega \langle \partial\eta(u), \overline{\partial}^*_\psi u\rangle\, e^{-\psi} + \int_\Omega \gamma\, |\overline{\partial}^*_\psi u|^2\, e^{-\psi}$$

for every $(n, 1)$-form u in $Dom(\overline{\partial}^*_\psi) \cap Dom(\overline{\partial})$.

Proof of Lemma 2.2. As before, ρ is a smooth strictly psh exhaustion function on X that defines the smooothly bounded Stein domain Ω. Let $h = e^{-\varphi}$ denote a *smooth* hermitian metric on $L|_{\Omega}$. We begin with the usual Bochner-Kodaira formula for L-valued $(n,1)$-forms. Suppose u is such a form that is also smooth in a neighborhood of $\overline{\Omega}$, and u lies in the domain of $\overline{\partial}_{\varphi}^{*}$ on Ω. Then it is a standard result (see e.g. [3]) that:

$$\int_{\Omega} |\overline{\partial}_{\varphi}^{*} u|^2 e^{-\varphi} + \int_{\Omega} |\overline{\partial} u|^2 e^{-\varphi} = \int_{\Omega} \sqrt{-1}\partial\overline{\partial}\varphi(u,u)\, e^{-\varphi} + \int_{\Omega} |\nabla^{0,1} u|^2\, e^{-\varphi} + \int_{\partial\Omega} \sqrt{-1}\partial\overline{\partial}\rho(u,u)\, e^{-\varphi}. \tag{3}$$

Next we pass to the twisted formula. The main idea is to consider a twist of the original metric $e^{-\varphi}$, and use it to define the adjoint of $\overline{\partial}$. That is to say, we consider another metric $e^{-\psi}$ on $L|_{\Omega}$. For any such metric, there is a positive smooth function η such that $e^{-\varphi} = \eta\, e^{-\psi}$, and conversely every η defines a twist of the original metric. Then we have a relation between the formal adjoints with respect to $e^{-\varphi}$ and $e^{-\psi}$:

$$\overline{\partial}_{\varphi}^{*} u = -\eta^{-1}\partial\eta\,(u) + \overline{\partial}_{\psi}^{*} u,$$

since

$$\overline{\partial}_{\varphi}^{*} u = -\sum_{j} e^{\varphi} \frac{\partial}{\partial z_j} \left(e^{-\varphi} u_{\bar{j}}\right)\, e_L \otimes dz$$

if locally $u = u_{\overline{\beta}}\, e_L \otimes dz \wedge d\overline{z}^{\beta}$, where e_L is a local frame for $L|_{\Omega}$ and $dz = dz^1 \wedge \ldots \wedge dz^n$. Also, we have

$$\partial\overline{\partial}\varphi = \partial\overline{\partial}\psi - \eta^{-1}\partial\overline{\partial}\eta + \eta^{-2}\partial\eta \wedge \overline{\partial}\eta.$$

Substitution of these identities into (3) gives:

$$\begin{aligned}
&\int_{\Omega} |\overline{\partial}_{\psi}^{*} u|^2\, e^{-\varphi} + \int_{\Omega} |\overline{\partial} u|^2\, e^{-\varphi} \\
&= \int_{\Omega} \sqrt{-1}\,\partial\overline{\partial}\varphi\,(u,u)\, e^{-\varphi} + 2\,\mathrm{Re} \int_{\Omega} \langle\, \eta^{-1}\partial\eta(u), \overline{\partial}_{\psi}^{*} u \rangle\, e^{-\varphi} - \int_{\Omega} \|\eta^{-1}\partial\eta(u)\|^2\, e^{-\varphi} \\
&\qquad + \int_{\Omega} |\nabla^{0,1} u|^2\, e^{-\varphi} + \int_{\partial\Omega} \sqrt{-1}\partial\overline{\partial}\rho(u,u)\, e^{-\varphi} \\
&= \int_{\Omega} (\sqrt{-1}\,\partial\overline{\partial}\psi - \sqrt{-1}\,\eta^{-1}\partial\overline{\partial}\eta)\,(u,u)\, e^{-\varphi} + 2\,\mathrm{Re} \int_{\Omega} \langle\, \eta^{-1}\partial\eta(u), \overline{\partial}_{\psi}^{*} u \rangle\, e^{-\varphi} \\
&\qquad + \int_{\Omega} |\nabla^{0,1} u|^2\, e^{-\varphi} + \int_{\partial\Omega} \sqrt{-1}\partial\overline{\partial}\rho(u,u)\, e^{-\varphi},
\end{aligned}$$

the so-called twisted Bochner-Kodaira formula. Observe that the last two terms are ≥ 0, hence adding $\int_{\Omega} \gamma\, |\overline{\partial}_{\psi}^{*} u|^2\, e^{-\psi}$ to both sides of the equation yields

$$\int_\Omega (\eta+\gamma)\,|\overline{\partial}^*_\psi u|^2\, e^{-\psi} + \int_\Omega \eta\,|\overline{\partial} u|^2\, e^{-\psi}$$

$$\geq \int_\Omega (\eta\sqrt{-1}\partial\overline{\partial}\psi - \sqrt{-1}\partial\overline{\partial}\eta)\,(u,u)\, e^{-\psi} + 2\,\mathrm{Re}\int_\Omega \langle \partial\eta(u), \overline{\partial}^*_\psi u\rangle\, e^{-\psi} + \int_\Omega \gamma\,|\overline{\partial}^*_\psi u|^2\, e^{-\psi}$$

for every L-valued $(n,1)$-form u in the domain of $\overline{\partial}^*_\psi$ on Ω that is also smooth in neighborhood of $\overline{\Omega}$. Now it is well-known that such forms are dense in $Dom(\overline{\partial}^*_\psi) \cap Dom(\overline{\partial})$ with respect to the graph norm

$$u \mapsto \|u\| + \|\overline{\partial} u\| + \|\overline{\partial}^*_\psi u\|,$$

that is, for each $(n,1)$-form $u \in Dom(\overline{\partial}^*_\psi) \cap Dom(\overline{\partial})$, we can find a sequence

$$u_j \in Dom(\overline{\partial}^*_\psi) \cap Dom(\overline{\partial})$$

of smooth $(n,1)$-forms defined in a neighborhood of $\overline{\Omega}$, such that

$$u_j \to u, \quad \overline{\partial} u_j \to \overline{\partial} u \quad \text{and} \quad \overline{\partial}^*_\psi u_j \to \overline{\partial}^*_\psi u \quad \text{in} \ \ L^2(X,E).$$

We thus conclude the proof of Lemma 2.2. □

3. Proof of the Theorem

Let

$$\eta = \log \frac{A}{|\,w\,|^2 + \varepsilon^2} \quad \text{and} \quad \gamma = \frac{1}{|\,w\,|^2 + \varepsilon^2}$$

(the motivation has been explained in the previous section), where $A > e$ is a constant. Whenever $\varepsilon < \sqrt{\frac{A}{e} - 1}$, we have $\eta > 1$,

$$\partial\eta = -\frac{\overline{\omega}\,\partial\omega}{|\,w\,|^2 + \varepsilon^2}, \quad \overline{\partial}\eta = -\frac{\omega\,\overline{\partial\omega}}{|\,w\,|^2 + \varepsilon^2} \quad \text{and} \quad -\sqrt{-1}\,\partial\overline{\partial}\,\eta = \frac{\varepsilon^2}{(|\,w\,|^2 + \varepsilon^2)^2}\sqrt{-1}\,\partial w \wedge \overline{\partial w}.$$

We will obtain the a priori inequality (2) from the basic estimate Lemma 2.2. In fact, taking $\psi = |\,w\,|^2 + \kappa$ in Lemma 2.2, the sum of the last two terms and the term involving $\eta\sqrt{-1}\partial\overline{\partial}\psi$ is ≥ 0 :

From $\eta\sqrt{-1}\partial\overline{\partial}\psi(u,u) \geq \sqrt{-1}\partial\overline{\partial}\,|\,w\,|^2(u,u) = |\langle u, dw\rangle|^2$, we deduce that

$$\int_\Omega \eta\sqrt{-1}\partial\overline{\partial}\psi\,(u,u)\,e^{-\psi} + \int_\Omega \gamma\,|\,\overline{\partial}_\psi^* u\,|^2\,e^{-\psi} \geq \int_\Omega \frac{|\,w\,|^2}{|\,w\,|^2 + \varepsilon^2}\,|\langle u, dw\rangle|^2\,e^{-\psi} + \int_\Omega \gamma\,|\,\overline{\partial}_\psi^* u\,|^2\,e^{-\psi}$$

$$\geq 2\int_\Omega \frac{|\,w\,|}{|\,w\,|^2 + \varepsilon^2}\,|\langle u, dw\rangle|\,|\,\overline{\partial}_\psi^* u\,|\,e^{-\psi} \geq \left|\,2\,\mathrm{Re}\int_\Omega \langle \partial\eta(u), \overline{\partial}_\psi^* u\rangle\,e^{-\psi}\,\right|,$$

where the middle inequality follows from $a^2 + b^2 \geq 2ab$ and the Hölder inequality. Hence Lemma 2.2 reduces to

$$\int_\Omega (\eta + \gamma)\,|\,\overline{\partial}_\psi^* u\,|^2\,e^{-\psi} + \int_\Omega \eta\,|\,\overline{\partial} u\,|^2\,e^{-\psi}$$

$$\geq \int_\Omega (\eta\sqrt{-1}\partial\overline{\partial}\psi - \sqrt{-1}\partial\overline{\partial}\eta)\,(u,u)\,e^{-\psi} + 2\,\mathrm{Re}\int_\Omega \langle \partial\eta(u), \overline{\partial}_\psi^* u\rangle\,e^{-\psi} + \int_\Omega \gamma\,|\,\overline{\partial}_\psi^* u\,|^2\,e^{-\psi}$$

$$\geq \int_\Omega -\sqrt{-1}\partial\overline{\partial}\eta\,(u,u)\,e^{-\psi} = \int_\Omega \frac{\varepsilon^2}{(|\,w\,|^2 + \varepsilon^2)^2}\,|\langle u, dw\rangle|^2\,e^{-\psi}.$$

As before, we consider

$$T = \overline{\partial} \circ \sqrt{\eta + \gamma} \quad \text{and} \quad S = \sqrt{\eta}\, \overline{\partial},$$

so that $ST = 0$, and the a priori inequality reads

$$\|T^* u\|^2 + \|Su\|^2 \geq \Big\| \frac{\varepsilon}{|w|^2 + \varepsilon^2} \langle u, dw \rangle \Big\|^2, \tag{4}$$

where $\|\cdot\|$ is the L^2 norm over Ω with respect to the weight $e^{-\psi}$.

3.2. **Choice of the cutoff function.** Fix $0 < \delta < 1$, and let $0 \leq \varrho(x) \leq 1$ be a smooth function on $[0, +\infty)$ such that

$$\varrho \equiv 1 \text{ on } [0, \frac{\delta}{2}], \quad \varrho \equiv 0 \text{ on } [1, +\infty) \text{ and } \sup |\varrho'| \leq 1 + \delta.$$

Such a ϱ exists by smoothing out the piecewise linear function $P : [0, +\infty) \to [0, 1]$ which is $\equiv 1$ on $[0, \frac{7\delta}{12}]$, linear on $[\frac{7\delta}{12}, \frac{11\delta}{12}]$ and $\equiv 0$ on $[\frac{11\delta}{12}, +\infty)$, that is, we set $\varrho := P * \varrho_{\frac{\delta}{12}}$, where $\{\varrho_\varepsilon\}$ a family of smoothing kernels on $\mathbb{R}$ defined as on page 2 (by replacing $\mathbb{C}$ with $\mathbb{R}$ and $2m$ with m).

Let $\chi_\varepsilon(w) := \varrho(\frac{|w|^2}{\varepsilon^2})$ and

$$\beta_\varepsilon := w^{-1}\, \overline{\partial}(\chi_\varepsilon \tilde{F}) = \frac{\tilde{F}}{\varepsilon^2}\, \varrho'(\frac{|w|^2}{\varepsilon^2})\, \overline{dw}.$$

Our goal is to solve $T\alpha_\varepsilon = \beta_\varepsilon$ for $(n, 0)$-forms α_ε using Lemma 2.1. Toward that end, we apply the Schwarz inequality: for every $u \in Dom(T^*) \cap Dom(S)$,

$$\begin{aligned} |\langle\!\langle u, \beta_\varepsilon \rangle\!\rangle|^2 &= \Big| \int_\Omega \Big\langle dw(u), \frac{\tilde{F}}{\varepsilon^2}\, \varrho'(\frac{|w|^2}{\varepsilon^2}) \Big\rangle e^{-\psi} \Big|^2 \\ &\leq \Big(\int_\Omega \Big| \frac{\tilde{F}}{\varepsilon^2}\, \varrho'(\frac{|w|^2}{\varepsilon^2}) \Big|^2 \frac{(|w|)^2 + \varepsilon^2}{\varepsilon^2}\, e^{-\psi} \Big) \\ &\quad \cdot \Big(\int_\Omega |\langle u, dw \rangle|^2 \frac{\varepsilon^2}{(|w|^2 + \varepsilon^2)^2}\, e^{-\psi} \Big) \\ &\leq C_\varepsilon (\|T^* u\|^2 + \|Su\|^2), \end{aligned}$$

where

$$C_\varepsilon := \int_\Omega \Big| \frac{\tilde{F}}{\varepsilon^2}\, \varrho'(\frac{|w|^2}{\varepsilon^2}) \Big|^2 \frac{(|w|)^2 + \varepsilon^2}{\varepsilon^2}\, e^{-\psi}$$

and the last inequality follows from (4). Thus Lemma 2.1 guarantees that the equation

$$T\alpha_\varepsilon = \overline{\partial}(\sqrt{\eta+\gamma}\,\alpha_\varepsilon) = \frac{\tilde{F}}{\varepsilon^2}\,\varrho'(\frac{|w|^2}{\varepsilon^2})\,\overline{dw} = \beta_\varepsilon$$

has a solution α_ε with $\|\alpha_\varepsilon\| \le C_\varepsilon$.

3.3. **Estimate of the constant** C_ε**.** Fix $p \in \overline{\Omega}$. Let $(z^j = x^j + \sqrt{-1}\,y^j)$ be local coordinates on an open set $U \subset X$ centered at p, and e_L a local frame of L on U, such that

(i) $z^n = w$;

(ii) $U = D \times \{r < \varepsilon\}$ where $r := |z^n|$ and D is an $(n-1)$-dimensional polydisc;

(iii) $\tilde{F} = \tilde{F}(z^1, ..., z^n)\, e_L \otimes dz^1 \wedge ... \wedge dz^n$ and $f = f(z^1, ..., z^{n-1})\, e_L \otimes dz^1 \wedge ... \wedge dz^{n-1}$ on U.

From $\tilde{F}\big|_{Z\cap\Omega} = f \wedge dw$, it follows that

$$\tilde{F}(z^1, ..., z^{n-1}, 0) = f(z^1, ..., z^{n-1}).$$

Let $\mathfrak{U} = \{U_1, ..., U_k\}$ be a finite open cover of $\overline{\Omega}$ with each U_j being chosen as U above. Let $\{\rho_1, ..., \rho_k\}$ be a partition of unity subordinate to $\mathfrak{U}$. Then

$$C_\varepsilon \le (1+\delta^2)\sum_{j=1}^{k}\int_{U_j\cap\Omega}\rho_j\,|\tilde{F}(z^1, ..., z^n)|^2\,\frac{(|z^n|^2+\varepsilon^2)^2}{\varepsilon^6}\,e^{-\kappa-|z^n|^2}dx^1\wedge dy^1\wedge ...\wedge dx^n\wedge dy^n.$$

Now $f_j(z^1, ..., z^n) := \rho_j\,|\tilde{F}(z^1, ..., z^n)|^2\,e^{-\kappa-|z^n|^2}$ is smooth on U_j, so since $U_j \cap \overline{\Omega}$ compact in U_j, there is a constant $M > 0$ such that

$$\Bigg|\int_{U_j\cap\Omega}\rho_j\,|\tilde{F}(z^1, ..., z^n)|^2\,\frac{(|z^n|^2+\varepsilon^2)^2}{\varepsilon^6}\,e^{-\kappa+|z^n|^2}dx^1\wedge dy^1\wedge ...\wedge dx^n\wedge dy^n$$

$$-\int_{U_j\cap\Omega} f_j(z^1, ..., z^{n-1}, 0)\,\frac{(|z^n|^2+\varepsilon^2)^2}{\varepsilon^6}\,dx^1\wedge dy^1\wedge ...\wedge dx^n\wedge dy^n\Bigg|$$

$$\le \int_{U_j\cap\Omega} M|\,z^n\,|\,\frac{(|z^n|^2+\varepsilon^2)^2}{\varepsilon^6}\,dx^1\wedge dy^1\wedge ...\wedge dx^n\wedge dy^n = O(\varepsilon).$$

Hence

$$C_\varepsilon \le (1+\delta^2)\sum_{j=1}^{k}\int_{U_j\cap\Omega} f_j(z^1, ..., z^{n-1}, 0)\,\frac{(|z^n|^2+\varepsilon^2)^2}{\varepsilon^6}\,dx^1\wedge dy^1\wedge ...\wedge dx^n\wedge dy^n + O(\varepsilon).$$

Now

$$\int_{U_j\cap\Omega} f_j(z^1,...,z^{n-1},0)\,\frac{(|\,z^n\,|^2+\varepsilon^2)^2}{\varepsilon^6}\,dx^1\wedge dy^1\wedge...\wedge dx^n\wedge dy^n$$

$$=\Big(\int_0^{2\pi}\int_0^{\varepsilon}\frac{(r^2+\varepsilon^2)^2}{\varepsilon^6}\,r\,dr\,d\theta\Big)\cdot\int_{U_j\cap\Omega\cap Z} f_j(z^1,...,z^{n-1},0)\,dx^1\wedge dy^1\wedge...\wedge dx^{n-1}\wedge dy^{n-1}$$

$$=\frac{7\pi}{3}\int_{U_j\cap\Omega\cap Z}\rho_j\,|\,f\,|^2\,e^{-\kappa}.$$

Hence

$$\limsup_{\varepsilon\to0} C_\varepsilon\;\le\;\frac{7\pi}{3}\,(1+\delta^2)\int_{Z\cap\Omega}|\,f\,|^2\,e^{-\kappa}.$$

3.4. **The Final Step.** It has been solved that

$$\overline{\partial}(\sqrt{\eta+\gamma}\,\alpha_\varepsilon)=w^{-1}\overline{\partial}(\chi_\varepsilon\,\tilde{F})\;\text{ where }\int_\Omega|\,\alpha_\varepsilon\,|^2\,e^{-\psi}\le C_\varepsilon.$$

The solution α_ε is smooth by the ellipticity of $\overline{\partial}$, thus

$$F_\varepsilon:=\chi_\varepsilon\,\tilde{F}-w\sqrt{\eta+\gamma}\,\alpha_\varepsilon$$

is holomorphic and $F_\varepsilon|_{Z\cap\Omega}=f\wedge dw$. Now

$$\lim_{\varepsilon\to0}\int_\Omega|\,\chi_\varepsilon\,\tilde{F}\,|^2\,e^{-\kappa}=0$$

by the dominated convergence theorem (because of (1) and that $Z\cap\Omega$ is of measure zero). Furthermore, since $\max_{y>0} y\log\frac{1}{y}=\frac{1}{e}$ and by assumption $\sup_\Omega|\,w\,|\le1$, we have

$$\sup_\Omega|\,\omega\sqrt{\eta+\gamma}\,|\;\le\;\sup_{x\in[0,1]}\sqrt{x^2\Big(\log A+\log\frac{1}{x^2+\varepsilon^2}+\frac{1}{x^2+\varepsilon^2}\Big)}\;\le\;\sqrt{\log A+\frac{1}{e}+1}.$$

Finally, the uniform boundedness of $\{F_\varepsilon\}$ in the L^2 norm $\|\cdot\|$ implies the local uniform boundedness of $\{F_\varepsilon\}$ in the smooth hermitian metric $|\cdot|^2$ (as on page 3). Hence by Montel's theorem, we can extract a convergent subsequence $\{F_{\varepsilon_j}\}$ whose limit $F_{A,\delta}$ is a holomorphic extension of $f \wedge dw|_{Z\cap\Omega}$ over Ω such that

$$\int_\Omega |F_{A,\delta}|^2 e^{-\kappa} \le \sqrt{\log A + \frac{1}{e} + 1} \cdot \limsup_{j\to\infty} \int_\Omega |\alpha_{\varepsilon_j}|^2 e^{-\kappa} \le \sqrt{\log A + \frac{1}{e} + 1} \cdot e \cdot \limsup_{\varepsilon\to\infty} C_\varepsilon.$$

A diagonal process with $A \to e$ and $\delta \to 0$ then yields a holomorphic extension F of $f \wedge dw|_{Z\cap\Omega}$ such that

$$\int_\Omega |F|^2 e^{-\kappa} \le \frac{7\pi}{3} e \sqrt{2 + \frac{1}{e}} \int_{Z\cap\Omega} |f|^2 e^{-\kappa}.$$

The proof is complete.

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