Formal Deformation Theory
and
Examples of Deformation Functors

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0 Notations and basic definitions

Notation 0.1. *We work over $k = \bar{k}$; all schemes are over k. Let*
$\mathcal{A}$ = the category of local artinian k-algebras with residue field k;
$\hat{\mathcal{A}}$ = the category of complete local noetherian k-algebras with residue field k;
$\mathcal{A}^*$ = the category of local noetherian k-algebras with residue field k. Note $\mathcal{A} \supset \hat{\mathcal{A}} \supset \mathcal{A}^*$. An algebraic schemes means a scheme over k of finite type.

Definition 0.2. *Let X be an algebraic scheme. A deformation of X parametrized by S is a fiber product*

$$\eta : \begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi:\textit{flat surj}} \\ k & \xrightarrow{\ s\ } & S \end{array} .$$

Morphism between deformations is a ϕ making a commutative diagram

$$\begin{array}{ccccc} & & X & & \\ & \swarrow & & \searrow & \\ \mathcal{X} & & \xrightarrow{\quad\phi\quad} & & \mathcal{Y} \\ & \searrow & & \swarrow & \\ & & S & & \end{array} .$$

We denote a deformation by (S, η). A deformation is called infinitesimal if $S = \text{Spec } A$ for some $A \in \mathcal{A}$; it's called first-order if $S = \text{Spec } k[\epsilon]$.

Remark 0.3. *Let (S, η) be a deformation. A morphism of algebraic schemes $(S', s') \to (S, s)$ induces a fiber product digram:*

$$\begin{array}{ccccc} X & \longrightarrow & \mathcal{X} \times_S S' & \longrightarrow & \mathcal{X} \\ \big\downarrow & & \big\downarrow{\scriptstyle \eta'} & & \big\downarrow{\scriptstyle \eta} \\ k & \xrightarrow{\ s'\ } & S' & \longrightarrow & S \end{array} ;$$

in particular (S', η') is a deformation of X over S'. When $S' = k[\epsilon]$, this construction yields a map $\rho : T_{S,s} \to Def_X(k[\epsilon])$. If X is a smooth variety, $Def_X(k[\epsilon])$ coincides with $H^1(X, T_X)$, and in this case we call $\rho : T_{S,s} \to H^1(X, T_X)$ the Kodaira-Spencer map.

Remark 0.4. *(cf. Lemma 1.2.3 in [1]) Suppose $Z_0 \subseteq_{C.S.} Z$ and $\mathcal{I}_{Z_0}$ is nilpotent (where "C.S." stands for "closed subscheme"). Then the affineness of Z_0 implies that of Z. Consequently, every infinitesimal deformation of an affine scheme is affine.*

Remark 0.5. *Suppose $X = Spec\ B_0$ is affine and we have a morphism between infinitesimal deformation:*

$$\begin{array}{ccccc} & \nearrow & B_0 & \nwarrow & \\ B & & \xrightarrow{\quad \phi \quad} & & B' \\ & \nwarrow & A & \nearrow & \end{array} .$$

Then, $B \otimes_A k \cong B_0 \cong B' \otimes_A k$ and so $\tilde{\phi} : B/\mathfrak{m}_A B \to B'/\mathfrak{m}_A B'$ is an isomorphism. Using flatness and Nakayama lemma, one can show that ϕ is an isomorphism as well.

1 Functor of Artin rings

Notation 1.1. *For $\Lambda \in \mathcal{A}^*$, consider the following categories:*
$\mathcal{A}_\Lambda$ = the category of local artinian Λ-algebras with residue field k;
$\hat{\mathcal{A}}_\Lambda$ = the category of complete local noetherian Λ-algebras with residue field k;
$\mathcal{A}^*_\Lambda$ = the category of local noetherian Λ-algebras with residue field k.

Definition 1.2. *A functor of Artin rings is a covariant functor*

$$F : \mathcal{A}_\Lambda \to (Sets), \text{ where } \Lambda \in \mathcal{A}^*.$$

We say F is prorepresentable if $F \cong h_{R/\Lambda} : A \mapsto Hom_{\hat{\mathcal{A}}_\Lambda}(R, A)$ for some $R \in \hat{\mathcal{A}}_\Lambda$.
(Of course, a representable functor is prorepresentable.)

Remark 1.3. *A fiber product*

$$\begin{array}{ccc} A' \times_A A'' & \longrightarrow & A'' \\ \downarrow & & \downarrow \\ A' & \longrightarrow & A \end{array}$$

in $\mathcal{A}_\Lambda$ induces $\alpha : F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$. If $F(k) = *$ and α is bijective when $A = k$ and $A'' = k[\epsilon]$, then $F(k[\epsilon])$ inherits a structure of k-vector space, as follows: The addition $F(k[\epsilon]) \times F(k[\epsilon]) \to F(k[\epsilon])$ is induced by

$$k[\epsilon] \times_k k[\epsilon] \to k[\epsilon]$$
$$(a + b\epsilon, a + b'\epsilon) \mapsto a + (b + b')\epsilon,$$

the multiplication by $c \in k$ is

$$k[\epsilon] \to k[\epsilon]$$
$$a + b\epsilon \mapsto a + (cb)\epsilon,$$

and the zero element 0 *is* $im(F(k) \to F(k[\epsilon]))$. $F(k[\epsilon]) =: t_F$ *is called the tangent space of* F. *For a natural transformation* $f : F \to G$ *between such functors,* $df : t_F \to t_G$ *is called the differential of* f. *It is* k*-linear.*

Definition 1.4. *(formal element) A functor of Artin rings* $F : \mathcal{A}_\Lambda \to (Sets)$ *can be extended to* $\hat{F} : \hat{\mathcal{A}}_\Lambda \to (Sets)$ *by setting*

$$\hat{F}(R) = \lim_{\leftarrow} F(R/m_R^{n+1}) \ (n \geq 0)$$

and the morphism $\hat{F}(\varphi)$ *is given by* $\{F(R/m_R^{n+1}) \to F(R/m_S^{n+1})\}_{n \geq 0}$.
An element $\hat{u} = \{u_n \in F(R/m_R^{n+1})\} \in \hat{F}(R)$ *is called a formal element.*

Proposition 1.5. *Let* $R \in \mathcal{A}_\Lambda$. *Then there exists a bijection*

$$\{\hat{u} \in \hat{F}(R)\} \longleftrightarrow \{\text{natural transformations } h_{R/\Lambda} \to F\}.$$

Definition 1.6.
(1) $f : F \to G$ *is smooth if* $\forall$ *surjection* $B \to A$, $F(B) \to F(A) \times_{G(A)} G(B)$ *is also surjective.*
(2) F *is smooth if* $\forall$ *surjection* $B \to A$, $F(B) \to F(A)$ *is surjective (ie. the natural transformation from* F *to the trivial functor is smooth).*

Definition 1.7. *Fix* $\hat{u} \in \hat{F}(R)$, *where* $R \in \hat{\mathcal{A}}_\Lambda$. *Recall that* $\hat{u}$ *induces* $\hat{u} : h_{R/\Lambda} \to F$.
(1) We call $\hat{u}$ *semi-universal if* $\hat{u} : h_{R/\Lambda} \to F$ *is smooth, and* $t_{R/\Lambda} \to t_F$ *is bijective (where* $t_{R/\Lambda}$ *is the tangent space for* $h_{R/\Lambda}$*).*
(2) We call $\hat{u}$ *universal if* $\hat{u} : h_{R/\Lambda} \to F$ *is an isomorphism.*

Definition 1.8. *Let* R *be a local* k*-algebra with residue field* k. *A small extension of* R *is a* k*-extension of* R *by* k:

$$0 \longrightarrow k \longrightarrow R' \longrightarrow R \longrightarrow 0$$

such that $k^2 = 0$ *in* R'. *In this case* $k \trianglelefteq R'$ *can be proved to be principle.*

Theorem 1.9. *(Schelessinger) Suppose* $F(k) = *$. *Recall the notation*

$$\alpha : F(A'' \times_A A') \to F(A') \times_{F(A)} F(A'')$$

.

(1) F *has a semi-universal element if and only if the following three conditions hold:*
- *(i)* $(\bar{H})$ α *is surjective for every small extension* $A'' \to A$.
- *(ii)* (H_ϵ) α *is bijective when* $A = k$ *and* $A'' = k[\epsilon]$.
- *(iii)* (H_f) $\dim_k(t_F) < \infty$.

(2) F *has a universal element if and only if, moreover, the following condition* (H) *holds:*

$$F(A' \times_A A') \to F(A') \times_{F(A)} F(A')$$

is surjective for every small extension $A' \to A$.

Example 1.10. *For an algebraic scheme X, define*

$$\begin{aligned}\mathrm{Def}_X : \mathcal{A} &\to (Sets) \\ A &\mapsto \{deformation\ of\ X\ over\ A\}/isomorphism\end{aligned}$$

Then $\mathrm{Def}_X(k) = *$, *and* Def_X *satisfies* $\bar{H}$ *and* H_ϵ.

2 Formal/Algebraic deformations

In this section, we discuss when a series of infinitesimal deformation comes from a "true" deformation, that is, a deformation over some $A \in \hat{\mathcal{A}}$.

Definition 2.1. *Let X be an algebraic scheme and $A \in \hat{\mathcal{A}}$. A formal deformation of X over A is $\hat{\eta} \in \widehat{\mathrm{Def}}_X(A)$, ie.,*

$$\hat{\eta} = \left\{ \eta_n : \begin{array}{ccc} X & \xrightarrow{f_n} & \mathcal{X}_n \\ \downarrow & & \downarrow{\scriptstyle \pi_n} \\ k & \longrightarrow & \mathrm{Spec}\ A_n \end{array} \right\}_{n \geq 0},$$

where $A_n = A/m_A^{n+1}$, each η_n is a fiber product, and the pullback $\mathcal{X}_n \otimes_{A_n} A_{n-1}$ is isomorphic to $\mathcal{X}_{n-1}$.

A natural question is, is it true that all these fiber products are pullbacks of some deformation over A? That is, we ask for the existence of a deformation $\pi : \mathcal{X} \to A$, making the fiber product diagram:

$$\begin{array}{ccccccccccc} X & \xrightarrow{f_0} & \cdots & \longrightarrow & \mathcal{X}_{n-1} & \longrightarrow & \mathcal{X}_n & \longrightarrow & \cdots & \longrightarrow & \mathcal{X} \\ \downarrow & & & & \downarrow{\scriptstyle \pi_{n-1}} & & \downarrow{\scriptstyle \pi_n} & & & & \downarrow{\scriptstyle \pi} \\ k & \longrightarrow & \cdots & \longrightarrow & \mathrm{Spec}\ A_{n-1} & \longrightarrow & \mathrm{Spec}\ A_n & \longrightarrow & \cdots & \longrightarrow & \mathrm{Spec}\ A \end{array}$$

Definition 2.2. *If such a deformation*

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ k & \longrightarrow & \mathrm{Spec}\ A \end{array}$$

exists, we call (A, η_n) effective.

Remark 2.3. *(Grothendieck) If X is a proper algebraic scheme, then two such "liftings" must be isomorphic.*

Remark 2.4. *A formal deformation is the same as giving a morphism of formal schemes*

$$\bar{\pi} : \bar{\mathcal{X}} \to \mathrm{Specf}(A)$$

where $\bar{\mathcal{X}} = (X, \lim\limits_{\leftarrow} \mathcal{O}_{\mathcal{X}_n})$ *and* $\mathrm{Sepcf}(A)$ *is a formal spectrum;* $\mathrm{Sepcf}(A) = \widehat{\mathcal{O}}_{\mathrm{Spec}\ A} = (*, \lim\limits_{\leftarrow} \mathrm{Spec}(A/m_A^n))$.

Remark 2.5. *A formal deformation is effective is and only if there exists a deformation*

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ k & \longrightarrow & \mathrm{Spec}\ A \end{array}$$

such that $\bar{\mathcal{X}}$ is the completion of $\mathcal{X}$ along X, ie., $\bar{\mathcal{X}} = \hat{\mathcal{X}} = (X, \varprojlim \mathcal{O}_{\mathcal{X}}/\mathcal{I}_X^n)$. For example, for $X = \mathbb{P}^r$, the formal deformation $\left\{\eta_n : \begin{array}{ccc} \mathbb{P}^r & \longrightarrow & \mathbb{P}^r_{A_n} \\ \downarrow & & \downarrow \\ k & \longrightarrow & \text{Spec } A_n \end{array}\right\}$ *is effective; the associated formal scheme is the completion of $\mathbb{P}^r_A$ along $\mathbb{P}^r$, denoted by $\mathcal{P}^r_A := (\mathbb{P}^r, \varprojlim \mathcal{O}_{\mathbb{P}^r_{A_n}})$.*

Theorem 2.6. *(Grothendieck) Let X be a projective scheme.*
(1) Let $A \in \hat{\mathcal{A}}$ and $\bar{\pi} : \bar{\mathcal{X}} \to \text{Specf}(A)$ be a formal deformation of X over A. Assume $\exists j$ such that the diagram

$$\begin{array}{ccc} \bar{\mathcal{X}} & \overset{j}{\underset{C.S.}{\hookrightarrow}} & \mathcal{P}^r_A \\ & \searrow^{\bar{\pi}} & \downarrow p \\ & & \text{Specf}(A) \end{array}$$

is commutative, where p is the projection. Then $\bar{\pi}$ is effective.
(2) When $H^2(X, \mathcal{O}_X) = 0$, every formal deformation of X is effective.

3 Obstruction space

We first define the space of extension classes, which inherits a module structure.

Definition 3.1. *Given an A-algebra R and an A-module I, the space of isomorphism classes of A-extensions of R by I is denoted by $\text{Ex}_A(R, I)$; we denote an extension*

$$0 \longrightarrow I \longrightarrow R' \longrightarrow R \longrightarrow 0$$

by (R', φ) and its class by $[R', \varphi]$. Here, we always demand $I^2 = 0$.

Definition 3.2. *Let F be a functor of Artin rings. An obstruction space for F is a vector space over k, denoted by $v(F)$, such that $\forall A \in \mathcal{A}_\Lambda$, $\forall \zeta \in F(A)$, there is a linear transformation $\zeta_v : \text{Ex}_\Lambda(A, k) \to v(F)$, with*

$$\ker \zeta_v = \left\{[\ 0 \longrightarrow I \longrightarrow \tilde{A} \longrightarrow A \longrightarrow 0\ ] \mid \zeta \in \text{im}(F(\tilde{A}) \to F(A))\right\}.$$

Remark 3.3. *Take F to be a deformation functor and consider an deformation $\zeta \in F(A)$. Intuitively, the kernel of ζ_v collects all extension classes that lift ζ.*

3.1 Module structure on $\text{Ex}_A(R, I)$

An extension is called trivial if it has a section, that is, it splits. A trivial extension can be constructed by considering the A-algebra $R\tilde{\oplus}I$, whose module structure is $R \oplus I$, and multiplication is given by $(r, i) \cdot (s, j) := (rs, rj + si)$. With the projection $p : R\tilde{\oplus}I \to R$, $(R\tilde{\oplus}I, p)$ is clearly a trivial extension.

Remark 3.4. *In fact, every trivial extension (R', φ) is isomorphic to $(R\tilde{\oplus}I, p)$.*

The module structure on $\mathrm{Ex}_A(R, I)$ is based on two operations.

Definition 3.5. *(pullback) Given*

$$(R', \varphi): \ 0 \longrightarrow I \longrightarrow R' \xrightarrow{\varphi} R \longrightarrow 0 \ ,$$

and $f: S \to R$ an A-algebra homomorphism, the pullback of (R', φ) by f is the A-extension $f^*(R', \varphi)$:

$$f^*(R', \varphi): \ 0 \longrightarrow I \longrightarrow R' \times_R S \longrightarrow S \longrightarrow 0 \ \in \mathrm{Ex}_A(S, I)$$

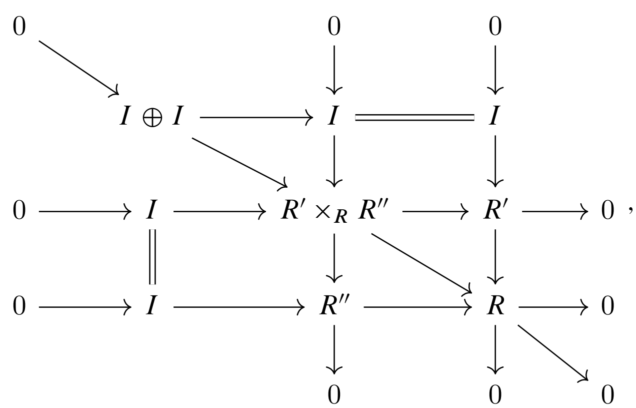
Definition 3.6. *(pushout) Given (R', φ) and $\lambda: I \to J$ an R-module homomorphism, the pushout of (R', φ) by λ is the A-extension $\lambda_*(R', \varphi)$:*

$$\lambda_*(R', \varphi): \ 0 \longrightarrow J \longrightarrow R' \textstyle\coprod_I J \longrightarrow R \longrightarrow 0 \ \in \mathrm{Ex}(R, J),$$

where

$$R' \coprod_I J := \frac{R' \tilde{\oplus} J}{\{(-\alpha(i), \lambda(i)) \mid i \in I\}}.$$

Definition 3.7. *Given $[R', \varphi]$ and $[R'', \psi] \in \mathrm{Ex}_A(R, I)$, we have the following diagram:*



which defines $(R' \times_R R'', \zeta): \ 0 \longrightarrow I \longrightarrow I \longrightarrow R' \times_R R'' \xrightarrow{\zeta} R \longrightarrow 0$. Let $\delta: I \oplus I \to I$ be defined by $(i, j) \mapsto i + j$. Then the addition is

$$[R', \varphi] + [R'', \varphi] := [\delta_* p(R' \times_R R'', \delta)].$$

On the other hand, for $[R', varphi] \in \mathrm{Ex}_A(R, I)$, $r \in R$, let $r: I \to I$ be the multiplication by r. Define $r \cdot [R', \varphi] := [r_*(R', \varphi)]$. The identity element in $\mathrm{Ex}_A(R, I)$ is the trivial extension $[R \tilde{\oplus} I, p]$.

4 More on Def_{B_0}

Here we give more concrete examples of the deformation functor Def_{B_0}.

Definition 4.1. $\mathrm{Ex}_A(R, R) =: T^1_{R/A}$ *is the first cotangent module of R over A.*

Definition 4.2. *(extension of schemes) Let X be an S-scheme. An extension of X over S is*

$$\mathcal{E} : 0 \to I \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0,$$

where I is an $\mathcal{O}_X$-module with $I^2 = 0$ and φ is an $\mathcal{O}_S$-algebra homomorphism.
Define $\mathrm{Ex}(X/S, I)$ *to be the isomorphism classes of extensions. The trivial extension $\mathcal{O}_X \tilde{\oplus} I$ can be defined similarly as in the module case.*

Theorem 4.3. *Let X be a finite type S-scheme, and both are algebraic scheme. Suppose I is locally free with finite rank on S, and X is reduced and S-smooth on a dense open set. Then*

$$\mathrm{Ex}(X/S, I) \cong \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_{X/S}, I)$$

proof. The maps are given by

$$\begin{aligned} \mathrm{Ex}(X/S, I) \to & \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_{X/S}, I) \\ \Big(0 \to I \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0\Big) \mapsto & \Big(0 \to I \to \Omega_{X'/S}|_X \to \Omega_{X/S} \to 0\Big) \end{aligned}$$

and

$$\begin{aligned} \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_{X/S}, I) \to & \mathrm{Ex}(X/S, I) \\ \Big(0 \to I \to \mathcal{A} \to \Omega_{X/S} \to 0\Big) \mapsto & \Big(0 \to I \to \mathcal{A} \times_{\Omega_{X/S}} \mathcal{O}_X \to \mathcal{O}_X \to 0\Big), \end{aligned}$$

where, the algebra structure on $\mathcal{A} \times_{\Omega_{X/S}} \mathcal{O}_X$ is $(a, f) \cdot (a', f') := (fa' + f'a, ff')$. □

Corollary 4.4. $T^1_X \cong \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$.

Proposition 4.5. *For a k-algebra B_0,* $\mathrm{Def}_{B_0}(k[\epsilon]) \cong T^1_{B_0}$.

Remark 4.6. *Suppose $B_0 = k[x_1, ..., x_d]/J$, with J prime. Then there is an exact sequence:*

$$0 \to \mathrm{Hom}(\Omega_{B_0/k}, \Omega_{B_0}) \to \mathrm{Hom}(\Omega_{k[x_1,...,x_d]/k} \otimes B_0, B_0) \to \mathrm{Hom}(J/J^2, B_0) \to T^1_{B_0} \to 0,$$

and thus $T^1_{B_0}$ can be computed. The result is: If J is generated by a regular sequence, then

$$T^1_{B_0} \cong \frac{k[x_1, ..., x_d]^n}{\left(\begin{pmatrix} \frac{\partial f_1}{\partial x_1} \\ \vdots \\ \frac{\partial f_n}{\partial x_1} \end{pmatrix}, \cdots, \begin{pmatrix} \frac{\partial f_1}{\partial x_d} \\ \vdots \\ \frac{\partial f_n}{\partial x_d} \end{pmatrix}\right)} \otimes_{k[x_1,...,x_d]} B_0.$$

For example,

(1) For hypersurface $B_0 = V(f)$, $T^1_{B_0} \cong \dfrac{k[x_1, ..., x_d]}{(f, \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_d})}$.

(2) For $f = x^2 - y^2$, $T^1_{B_0} \cong \dfrac{k[x, y]}{(x^2 - y^2, x, y)} \cong k$;

For $f = y^3 - x^2$, $T^1_{B_0} \cong \dfrac{k[x, y]}{(y^3 - x^2, y^2, x)} \cong k^2$.

5 Obstruction space for a nonsingular algebraic variety

Proposition 5.1.
(1) For a nonsingular algebraic variety X, $H^2(X, T_X)$ is an obstruction space for Def_X.
(2) Let $e : 0 \to (t) \to \tilde{A} \to A \to 0$ be a small extension. Then, if $o_\xi(e) = 0$, there exists a transitive action of $H^1(X, T_X)$ on {liftings to $\tilde{A}$}.

Example 5.2.

(1) Nonsingular curves are unobstructed.

(2) Let X be nonsingular. Then $H^1(X, T_X) = 0 \iff X$ is rigid. Indeed, for the "if" part, note that the first order deformation is trivial and use statement (2) in the proposition.
In particular, $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is rigid.

6 Second cotangent module and obstructions

In this section we introduce another obstruction space for Def_{B_0}, other than $H^2(X, T_X)$.
Let $B_0 = P/J$, where P is a smooth k-algebra. Take

$$\eta : 0 \longrightarrow R \xrightarrow{\iota} F \xrightarrow{j} J \longrightarrow 0,$$

where F is free over P. Define $\lambda : \bigwedge^2 F \to F$ by $x \wedge y \mapsto (jx)y - (jy)x$, and let $R^{tr} := \mathrm{im}(\lambda)$ be *the module of trivial relations.* The *second cotangent module of* B_0 is the B_0-module defined by the exact sequence:

$$\mathrm{Hom}_{B_0}(J/J^2, B_0) \to \mathrm{Hom}_{B_0}(F \otimes_P B_0, B_0) \to \mathrm{Hom}_{B_0}(R/R^{tr}, B_0) \to T^2_{B_0} \to 0.$$

Example 6.1. *If $J = (f_1, ..., f_n)$, and $F = P^n$, then $R/R^{tr} = H_1(K_\bullet(f_1, ..., f_n))$, ie., the first cohomology group of the Koszul complex. Therefore, if J is generated by a regular sequence, then $T^2_{B_0} = 0$, since $R/R^{tr} = 0$.*

Definition 6.2. *For an algebraic scheme X, let T^2_X be the sheaf glued by $T^2_{B_0}$. It's the "second cotangent sheaf".*

Proposition 6.3. *$T^2_{B_0}$ is an obstruction space for* Def_{B_0}.

Remark 6.4. *Let $X = \mathrm{Spec}\ B_0$ and $\dim_k(T^1_{B_0}) < \infty$. By Schlessinger's theorem, there exists a semiuniversal element $(R, \{\eta_n\})$, and in fact we have the inequalities*

$$\dim_k(T^1_{B_0}) \geq \dim(R) \geq \dim_k(T^1_{B_0}) - \dim_k(T^2_{B_0}).$$

Moreover, the first is an equality if and only if X_0 is unobstructed; in particular if B_0 is l.c.i., then X is unobstructed.

7 Local Hilbert functor

Let $X \subseteq_{C.S.} Y$ be algebraic schemes. A family of deformations of X in Y, parametrized by S, is a fiber product

$$\begin{array}{ccccc} X & \longrightarrow & \mathcal{X} & \hookrightarrow & Y \times S \\ \downarrow & & \downarrow{\scriptstyle \pi: flat} & \swarrow{\scriptstyle \pi} & \\ k & \longrightarrow & S & & \end{array} .$$

This defines a functor

$$\begin{aligned} H_X^Y : \mathcal{A} &\to (Sets) \\ A &\mapsto \{\text{deformations of } X \text{ in Y over A}\}/\text{isomorphisms}, \end{aligned}$$

called the local Hilbert functor of X in Y.

Proposition 7.1.
(1) H_X^Y satisfies H_0, H_ϵ, $\bar{H}$ and H in Schlessinger's theorem.
(2) $H_X^Y(k[\epsilon]) = H^0(X, N_{X/Y})$.

Proposition 7.2. *If $X \subseteq Y$ is a regular closed embedding, then $H^1(N_{X/Y})$ is an obstruction space for H_X^Y.*

8 Deformation of morphism leaving domain and target fixed

In this section we consider the following fiber product:

$$\begin{array}{ccc} X & \longrightarrow & X \times S \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle F} \\ Y & \longrightarrow & Y \times S \\ \downarrow & & \downarrow{\scriptstyle \pi: projection} \\ k & \longrightarrow & S \end{array}$$

with $\pi \circ F$ flat. This is a deformation of f wfdat (with fixed domain and target). The associated deformation functor is $\mathrm{Def}_{X/f/Y} : \mathcal{A} \to (Sets)$.

Proposition 8.1. *Let $f : X \to Y$ be a morphism between algebraic schemes, with X projective, reduced, and Y nonsingular. Then:*
(1) $\mathrm{Def}_{X/f/Y} \cong H_{\Gamma_f}^{X \times Y}$
(2) $\mathrm{Def}_{X/f/Y}(k[\epsilon]) = H^0(X, f^*T_Y)$
(3) $H^1(X, f^*T_Y)$ is an obstruction space for $Def_{X/f/Y}$.

Remark 8.2. *The reducibility and nonsingularity are used only in (2) and (3).*

Example 8.3. *Suppose $\mathbb{P}^1 \cong R \subseteq S$, S is nonsingular, and $E^2 = -n \leq -1$, ie. $\deg \mathcal{O}_E(E) = -n$. Consider the SES*

$$0 \longrightarrow T_E \longrightarrow T_S|_E \longrightarrow N_{E/S} \longrightarrow 0 \ .$$

Note $\mathrm{Ext}^1_{\mathcal{O}_E}(N_{E/S}, T_E) \cong H^1(E, \mathcal{O}_E(2) \oplus \mathcal{O}_E(n)) = 0$, *and thus* $T_S|_E \cong \mathcal{O}_E(2) \oplus \mathcal{O}_E(-n)$ *has* $h^0 = 3$. *This means, even though* E *is rigid in* S, *the inclusion* $i : R \to S$ *has* 3 *dimensional family of deformations.*

9 Deformation of morphism with target fixed

In this section, the fiber product diagram becomes more general:

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle F} \\ Y & \longrightarrow & Y \times S \\ \downarrow & & \downarrow{\scriptstyle \pi:projection} \\ k & \longrightarrow & S \end{array}$$

again with $\pi \circ F$ flat. This is a deformation of f with target Y. An isomorphism between such two deformations is commutative diagrams

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ \mathcal{X} & \xrightarrow{\phi} & \mathcal{X}' \\ \searrow & & \swarrow \\ & S & \end{array}, \text{ and } \begin{array}{ccc} \mathcal{X} & & \\ \downarrow{\scriptstyle \phi} & \searrow & \\ & & Y \times S \\ & \nearrow & \\ \mathcal{X}' & & \end{array}.$$

The associated deformation functor is $\mathrm{Def}_{f/Y}$. We say a deformation with target fixed is locally trivial, if the very deformation of X

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ k & \longrightarrow & S \end{array}$$

is locally trivial, ie. there exists an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ such that for all i, $\mathcal{X}|_{U_i}$ is a trivial deformation of U_i. The associated functor is denoted by $\mathrm{Def}'_{f/Y}$.

Definition 9.1. *Let* $f : X \to Y$ *be a morphism between algebraic schemes, with* X *projective, and let* $\mathcal{U} = \{U_i\}_{i \in I}$ *be an open affine cover of* X. *Define*

$$D_{X/Y} := \frac{\{(v, t) \in \mathcal{C}^0(\mathcal{U}, f^*T_Y) \times \mathcal{Z}^1(\mathcal{U}, T_X) \mid \delta v = df(t)\}}{(df(w), \delta w) \mid w \in \mathcal{C}^0(\mathcal{U}, T_X)} \text{ and}$$

$$D^1_{X/Y} := \frac{\{(\zeta, s) \in \mathcal{C}^1(\mathcal{U}, f^*T_Y) \times \mathcal{Z}^2(\mathcal{U}, T_X) \mid \delta \zeta = df(s)\}}{(df(u), \delta u) \mid u \in \mathcal{C}^1(\mathcal{U}, T_X)},$$

where δ *is the coboundary map in Cech complex and* $d : T_X \to f^*T_Y$.

Theorem 9.2. *With the above settings,*
(1) $\mathrm{Def}'_{f/Y}$ *has a semiuniversal element.*
(2) $Def'_{f/Y}(k[\epsilon]) = D_{X/Y}$; *an obstruction space is* $D^1_{X/Y}$.

10 Reference

[1] Edoardo Sernesi, Deformations of Algebraic Schemes.