Note on the Riemann-Hilbert Problem and the Birkhoff Problem on the Complex Projective Line

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This note is based on Chapters IV and V of [**Sabbah**], C. Sabbah's book "Isomonodromic Deformations and Frobenius Manifolds: An Introduction."

1 The Riemann-Hilbert problem on $\mathbb{P}^1$

1.1 Statement of the problem

The Riemann-Hilbert problem. Consider $\Sigma = \{m_0 = \infty, m_1, \cdots, m_p\} \subset \mathbb{P}^1 = \mathbb{CP}^1$, and give a monodromy representation

$$\rho : \pi_1(\mathbb{P}^1 \setminus \Sigma, o) = \pi_1(\mathbb{C} \setminus \{m_1, \cdots, m_p\}, o) \to \mathrm{GL}(d, \mathbb{C}).$$

Recall ρ can be determined by monodromies $T_1, \cdots, T_p \in \mathrm{GL}(d, \mathbb{C})$ around $m_1, \cdots, m_p$. The Riemann-Hilbert correspondence gives a meromorphic bundle $(\mathscr{M}^d, \nabla)$ on $\mathbb{P}^1$ which has regular singularities at Σ and which associates to ρ. The problem here is: Can we find a global frame for $\mathscr{M}$ so that the 1-form of ∇ takes the form

$$\sum_{k=1}^{p} \frac{A_k}{t - m_k}$$

where t is the coordinate of $\mathbb{C} \subset \mathbb{P}^1$ and $A_1, \cdots, A_p \in M(d, \mathbb{C})$?

Remark. If this problem has a solution, in general we do not have $T_k = \exp(2\pi i A_k)$.

1.2 Observations

Observation 1.1. The Riemann-Hilbert problem on $\mathbb{P}^1$ has a solution if and only if there is a trivial lattice $(\mathscr{E} \simeq \mathscr{O}^d_{\mathbb{P}^1}, \nabla)$ of $(\mathscr{M}, \nabla)$.

Sketch of proof: Indeed, $(\mathscr{E}, \nabla)$ has a global connection matrix $\Omega \in \Gamma(T^*\mathscr{O}_{\mathbb{P}^1}(\Sigma))$ with logarithmic poles along Σ. We may reduce to the case $\Omega =$ scalar 1-form, which can be verified with the aid of $\Gamma(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(\infty)) \simeq \Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-1)) = 0$. ◇

Observation 1.2. There is a trivial lattice of $(\mathscr{M}, \nabla)$ if and only if there is a quasi-trivial lattice $(\mathscr{E}, \nabla)$ (namely $\mathscr{E} \simeq \mathscr{O}_{\mathbb{P}^1}(l)^{\oplus d}$ for some $l \in \mathbb{Z}$) of $(\mathscr{M}, \nabla)$.

Proof: If we have a quasi-trivial lattice $\mathscr{E} \simeq \mathscr{O}_{\mathbb{P}^1}(l)^{\oplus d}$, $\mathscr{O}_{\mathbb{P}^1}(-l \cdot \infty) \otimes_{\mathscr{O}_{\mathbb{P}^1}} \mathscr{E} \simeq \mathscr{O}^{\oplus d}_{\mathbb{P}^1}$ is a trivial lattice of $\mathscr{M}$. □

Therefore, the Riemann-Hilbert problem on $\mathbb{P}^1$ is now equivalent to finding a quasi-trivial lattice of $(\mathscr{M}, \nabla)$.

1.3 Solutions to the Riemann-Hilbert problem

The following theorem gives solutions to the Riemann-Hilbert problem on some suitable conditions:

Theorem 1.3 (**[Sabbah, Thm.IV.2.2]**)**.** Consider the following setting:

- $(\mathscr{M}, \nabla)$ is a meromorphic bundle over $\mathbb{P}^1$ having $\Sigma = \{m_0, \cdots, m_p\} \subset \mathbb{P}^1$ as its only singularities;
- $(\mathscr{E}, \nabla) \subset (\mathscr{M}, \nabla)$ is a lattice which is of pole order $r_1, \cdots, r_p \geq 0$ at $m_1, \cdots, m_p$ respectively, and which is of logarithmic pole at m_0 with local connection 1-form $A\dfrac{\mathrm{d}t}{t}$ $(A \in M(d, \mathbb{C}))$ where t is a local coordinate centered at m_0.

Assume moreover that one of the following two conditions is fulfilled:

(a) (Plemelj) A is semi-simple;

(b) (Bolibrukh-Kostov) $(\mathscr{M}, \nabla)$ is irreducible, say there does not exist $0 \neq \mathscr{N} \subsetneq \mathscr{M}$ such that $\nabla(\mathscr{N}) \subset \Omega^1_{\mathbb{P}^1} \otimes_{\mathscr{O}_{\mathbb{P}^1}} \mathscr{N}$.

Then there is a lattice $\mathscr{E}' \subset \mathscr{E}$ which is quasi-trivial, logarithmic at m_0 and which is the same as $\mathscr{E}$ when both are restricted to $\mathbb{P}^1 \setminus \{m_0\}$.

Proof: We only prove (a) here. First we take the following lemmas for granted:

Lemma 1.4 (**Birkhoff-Grothendieck, [Sabbah, Thm.I.4.4]**)**.** For any bundle $\mathscr{E}$ on $\mathbb{P}^1$ with rank d, $\exists!$ $\{a_1 \geq \cdots \geq a_d\} \subset \mathbb{Z}$ such that $\mathscr{E} \simeq \mathscr{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^1}(a_d)$.

Lemma 1.5 (**[Sabbah, Cor.I.4.14]**)**.** In the decomposition of Lemma 1.4, let us define the defect of E to be $\delta(E) := \sum_{k=1}^{d}(a_1 - a_k) \geq 0$. If $L^0 \subset E_{m^0}$ $(m^0 \in \mathbb{P}^1)$ is NOT contained in $\mathscr{O}_{\mathbb{P}^1}(a_1)_{m^0}$ (via $\mathscr{E} \simeq \oplus_{k=1}^{d}\mathscr{O}_{\mathbb{P}^1}(a_k)$), then $\delta(E(L^0m^0)) = \delta(E) - 1$. (Here $\mathscr{E}(L^0m^0)$ is a subsheaf of $\mathscr{E}(m^0)$ whose germ at $m \neq m^0$ is the same of that of $\mathscr{E}(m^0)$, and whose germ at m_0 is modified as the set of meromorphic sections of $\mathscr{E}_{m^0}$ having at most a simple pole at m^0 with the residue at m^0 lying in L^0.)

Now we come back to prove (a):

- $\nabla = A\dfrac{\mathrm{d}t}{t}$ $(A = \mathrm{diag}\,(\alpha_1, \cdots, \alpha_d))$ locally $\Rightarrow$ there is a local basis $e_1, \cdots, e_d$ such that $\nabla e_k = \dfrac{\alpha_k}{t} e_k$.
- There is a line $L^0 \subset E_{m_0}$ passing through $e_k(m_0)$ for some k such that $L^0 \not\subset \mathscr{O}_{\mathbb{P}^1}(\alpha_1)_{m^0}$. Now may assume $L^0 = \mathbb{C} \cdot e_d(m_0)$. Then $\{e_1, \cdots, e_{d-1}, \dfrac{e_d}{t}\}$ is a local basis for $(\mathscr{E}(L^0m_0), \nabla = \mathrm{diag}\,(\alpha_1, \cdots, \alpha_{d-1}, \alpha_d - 1))$; also, $\nabla(e_d/t) = (\alpha_d - 1/t)(e_d/t)$. So $\delta(E(L^0m_0)) < \delta(E)$.

- Inductively, we get a zero defect lattice $\mathscr{E}'' \subset \mathscr{E}(\delta(E))$. Then the desired $\mathscr{E}'$ may be taken as $\mathscr{E}' = \mathscr{E}''(-\delta(E))$.

□

Corollary 1.5.1 (**[Sabbah, Cor.IV.2.6]**)**.** In Theorem 1.3, the two conditions (a) and (b) can be replaced by the following conditions (a') and (b') respectively, such that Theorem 1.3 is still valid:

(a') (Plemelj) The monodromy at m_0 is semi-simple;

(b') (Bolibrukh-Kostov) $\rho : \pi_1(\mathbb{P}^1 \setminus \Sigma) \to \mathrm{GL}(d, \mathbb{C})$ associated to $(\mathscr{M}, \nabla)$ is irreducible.

2 The Birkhoff problem on $\mathbb{P}^1$

2.1 Statement of the problem

Three equivalent Birkhoff problems.

(a) Local analytic version: Assume $\tau \in D \subset \mathbb{C}$ local coordinate, (D: the unit disc), $A(\tau)$: $d \times d$ meromorphic square matrix of pole order (Poincaré rank) r at $\tau = 0$. Does there exist a change of variables $P(\tau) \in \mathrm{GL}(d, \mathscr{O}_0)$ such that

$$B(\tau) := P^{-1}AP + P^{-1}P' = \frac{B_{-(r+1)}}{\tau^{r+1}} + \cdots + \frac{B_{-1}}{\tau}$$

for some constant matrices $B_{-(r+1)}, \cdots, B_{-1}$?

(b) Sheaf version: Assume $(\mathscr{M}^d, \nabla)$ is a meromorphic bundle on $\mathbb{P}^1$ which has regular singularity at ∞ and which has a pole of order $r \geq 1$ at 0 with a specified local lattice $\mathscr{E}^0$. Does there exist a lattice $(\mathscr{E}, \nabla)$ of $(\mathscr{M}, \nabla)$ such that $(\mathscr{E}, \nabla)$ is logarithmic at ∞, equal to $\mathscr{E}^0$ near 0, and trivial?

(c) Algebraic version: Assume $(\mathbb{M}, \nabla) \simeq \mathbb{C}[\tau, \tau^{-1}]^d$ has poles at 0 and ∞ only where ∞ is regular. Also assume $\mathbb{E}_0 \subset \mathbb{M}$ is a lattice of order $r \geq 1$ and a free $\mathbb{C}[\tau]$-module. Does there exist a lattice $\mathbb{E}_\infty \subset \mathbb{M}$ on U_∞ (a chart around ∞) which is logarithmic at ∞ and which satisfies $\mathbb{E}_0 = (\mathbb{E}_0 \cap \mathbb{E}_\infty) \oplus \tau\mathbb{E}_0$?

2.2 Solutions to the Birkhoff problems

Theorem 2.1 (**[Sabbah, Cor.IV.5.7]**)**.** If $(\mathscr{M}, \nabla)$ is irreducible or if the monodromy of its restriction to $\mathbb{C}^\times$ is semi-simple, then the Birkhoff problems are solvable.

Proof: Utilize the results of Theorem 1.3 and Corollary 1.5.1. □

Theorem 2.2 (**M. Saito's criterion, [Sabbah, Thm.IV.5.9 & Lem.IV.5.10]**)**.** Consider the following setting:

- τ, τ' are coordinates around $0, \infty \in \mathbb{P}^1$ respectively such that $\tau\tau' = 1$;
- $M'(= \mathscr{M}_\infty) = \mathbb{M}_\infty$;
- V' is a Deligne (logarithmic) lattice of (M', ∇) such that $\mathrm{Re}(\mathrm{Spec}(\mathrm{Res}_{V'}\nabla)) \subset (-1, 0] \subset \mathbb{R}$;
- $\mathbb{V}'$ is the $\mathbb{C}[\tau']$-submodule of $\mathbb{M}$ whose analytic germ at ∞, defined as $\mathbb{C}\{\tau'\} \otimes_{\mathbb{C}[\tau']} \mathbb{V}'$, is V';
- $H' := V'/\tau'V' = \mathbb{V}'/\tau'\mathbb{V}'$ is equipped with the monodromy $T' = \exp(2\pi i \mathrm{Res}_{V'}\nabla)$;
- $G'_\bullet := \dfrac{\mathbb{V}' \cap \tau'^{\bullet}\mathbb{E}_0}{(\tau'\mathbb{V}') \cap \tau'^{\bullet}\mathbb{E}_0}$ is an increasing exhaustive filtration on H'.

Then Birkhoff problem (c) has a solution if and only if one of the following equivalent conditions holds:

- $H' = \oplus_{p\in\mathbb{Z}} H'_p$ such that $T'(H'_p) \subset H'_p \oplus H'_{p+1} \oplus \cdots$ for any p and $G'_k = \oplus H'_p$ for any k;
- There is an exhaustive decreasing filtration $H'^{\bullet}$ of H' which is stable by T' (i.e. $T'(H'^{\bullet}) = H'^{\bullet}$) and which is opposite to $G'_\bullet$ (i.e. for all $k \neq l$, $H'^{l} \cap G'_k = (H'^{l+1} \cap G'_k) + (H'^{l} \cap G'_{k-1})$).

3 The Fourier-Laplace transform

3.1 The 1-variable Weyl algebra and its modules

The Weyl algebra in a variable t is defined as

$$\mathbb{C}\langle t, \partial_t\rangle := \{\text{The free } \mathbb{C}\text{-algebra generated by two variables } t \text{ and } \partial_t\}/([\partial_t, t] = 1).$$

One can regard ∂_t is the Weyl algebra as "differentiation with respect to t." The Weyl algebra $\mathbb{C}\langle t, \partial_t\rangle$ contains $\mathbb{C}[t]$ and $\mathbb{C}[\partial_t]$ as sub-algebras, and is itself a left and right Noetherian ring (as can be verified with reference to the proof of the Hilbert basis theorem for polynomial rings).

We would like to mention the following properties and definitions for modules over the Weyl algebra $\mathbb{C}\langle t, \partial_t\rangle$:

- Any right $\mathbb{C}\langle t, \partial_t\rangle$-module $\mathbb{M}$ is endowed with a canonical left $\mathbb{C}\langle t, \partial_t\rangle$-module structure via $P \cdot m = m \cdot P^T$ for $m \in \mathbb{M}$ and $P \in \mathbb{C}\langle t, \partial_t\rangle$, and vice versa.
- We define the **holonomic $\mathbb{C}\langle t, \partial_t\rangle$-modules** to be those $\mathbb{C}\langle t, \partial_t\rangle$-module $\mathbb{M}$ of rank 1, i.e. those $\mathbb{M}$ isomorphic to
$$\left(\frac{\mathbb{C}\langle t, \partial_t\rangle}{(P)}\right)\Big/(\text{torsion submodule})$$
for some $P \in \mathbb{C}\langle t, \partial_t\rangle \setminus \mathbb{C}$.

- A lattice $\mathbb{E}$ of a $\mathbb{C}\langle t, \partial_t\rangle$-module $\mathbb{M}$ is defined as a $\mathbb{C}[t]$-submodule of $\mathbb{M}$ for which $\mathbb{M} = \mathbb{C}\langle t, \partial_t\rangle \otimes_{\mathbb{C}[t]} \mathbb{E}$.
- The singularities of a holonomic $\mathbb{C}\langle t, \partial_t\rangle$-module $\mathbb{M}$ at $t = 0$ (for example) is defined to be the same as that of the analytic germ $M := \mathbb{C}\{t\}[t^{-1}] \otimes_{\mathbb{C}[t]} \mathbb{M}$ of $\mathbb{M}$ at $t = 0$.
- $\mathbb{M} = \dfrac{\mathbb{C}\langle t, \partial_t\rangle}{(P)}$ has a regular singularity at $t = 0$ if and only if $P \in \mathbb{C}\langle t, \partial_t\rangle$ is.

3.2 The Fourier-Laplace transform

Consider the following "Fourier-Laplace transform" between the t-plane and the τ'-plane:

$$\begin{array}{ccc} \mathbb{C}\langle t, \partial_t\rangle & \overset{\sim}{\longleftrightarrow} & \mathbb{C}\langle \tau', \partial_{\tau'}\rangle \\ \hline t & \longleftrightarrow & -\partial_{\tau'} \\ \partial_t & \longleftrightarrow & \tau' \\ \hline P \in \mathbb{C}\langle t, \partial_t\rangle & \longleftrightarrow & \widehat{P} \in \mathbb{C}\langle \tau', \partial_{\tau'}\rangle \\ \mathbb{M} : \mathbb{C}\langle t, \partial_t\rangle\text{-module} & \longleftrightarrow & \widehat{\mathbb{M}} : \mathbb{C}\langle \tau', \partial_{\tau'}\rangle\text{-module.} \end{array}$$

Proposition 3.1 ([**Sabbah, Prop.V.2.2**])**.** Assume the $\mathbb{C}\langle t, \partial_t\rangle$-holonomic module $\mathbb{M} = \dfrac{\mathbb{C}\langle t, \partial_t\rangle}{(P)}$ has a singularity at $t = \infty$ (it might have other singularities). Then:

(a) $\widehat{\mathbb{M}}$ has singularities ONLY at $\tau' = 0$ (regular singularity) and $\tau' = \infty$ (Poincaré rank ≤ 1);

(b) The left action $\partial_t : \mathbb{M} \to \mathbb{M}$ has the following properties: $\dim\ker \partial_t < \infty$, $\dim \operatorname{coker} \partial_t < \infty$, and $\dim \operatorname{coker} \partial_t - \dim\ker \partial_t = \operatorname{rank} \widehat{\mathbb{M}} = \operatorname{rank} \mathbb{M}$.
In particular, if $\mathbb{M} = \mathbb{M}[\partial_t^{-1}] := \mathbb{C}[\partial_t, \partial_t^{-1}] \otimes_{\mathbb{C}[\partial_t]} \mathbb{M}$, then $\operatorname{rank} \widehat{\mathbb{M}} = \operatorname{rank} \mathbb{M}$.

Proof: We only prove (a) here. Using the relation $[\partial_t, t] = 1$, we may write $P = \sum_{k=0}^{d} \partial_t^k a_k(t)$ where $a_k(t) \in \mathbb{C}[t]$ with $\widehat{k} := \deg a_k(t)$ and $a_d(t) \not\equiv 0$. Then, set $a_k(t) = \sum_{j=0}^{\widehat{k}} a_{kj} t^j$, and then we get

$$P = a_{d\widehat{d}} \partial_t^d t^{\widehat{d}} + \sum_{j<\widehat{d},\, k\leq d} a_{kj} \partial_t^k t^j \quad (a_{d\widehat{d}} \neq 0)$$

and therefore, via $\tau := 1/\tau'$ and $\partial_{\tau'} = -\tau^2 \partial_\tau$,

$$\widehat{P} = a_{d\widehat{d}} \tau'^d (-\partial_{\tau'})^{\widehat{d}} + \sum_{j<\widehat{d},\, k\leq d} a_{kj} \tau'^k (-\partial_{\tau'})^j = a_{d\widehat{d}} \tau^{-d} (\tau^2 \partial_\tau)^{\widehat{d}} + \sum_{j<\widehat{d},\, k\leq d} a_{kj} \tau^{-k} (\tau^2 \partial_\tau)^j. \quad (1)$$

Now, from equality (1), we derive the following:

- Since $a_{d\widehat{d}} \neq 0$, the leading term $a_{d\widehat{d}}\tau'^d(-\partial_{\tau'})^{\widehat{d}}$ in (1) shows that $\widehat{\mathbb{M}}$ does not have any singularity away from $\tau' = 0, \infty$;
- We can use Fuchs condition on the equality in τ' in (1) to show $\widehat{\mathbb{M}}$ has a regular singularity at $\tau' = 0$;
- The equality in τ in (1) implies furthermore

$$\tau^d \widehat{P} == a_{d\widehat{d}}\tau^{-d}(\tau^2\partial_\tau)^{\widehat{d}} + \sum_{j<\widehat{d},\, k\leq d} a_{kj}\tau^{d-k}(\tau^2\partial_\tau)^j, \tag{2}$$

so that $\widehat{\mathbb{M}}[\tau'^{-1}] = \dfrac{\mathbb{C}\langle \tau, \tau^{-1}, \partial_\tau\rangle}{(\widehat{P})} \simeq \mathbb{C}[\tau,\tau^{-1}]^{\oplus \widehat{d}}$ with the free basis $\beta = \{(\tau^2\partial_\tau)^j : 0 \leq j < \widehat{d}\}$; also, from (2) and the basis β we see that

$$[\tau^2\partial_\tau]_\beta = A(\tau) \in M(\widehat{d}, \mathbb{C})[\tau] \Rightarrow [\partial_\tau]_\beta = \frac{1}{\tau^2}A(\tau);$$

that is, $\widehat{\mathbb{M}}$ has Poincaré rank ≤ 1 at $\tau' = \infty$.

□

3.3 Relations of the Riemann-Hilbert problem and the Birkhoff problem via the Fourier-Laplace transform

The following proposition provides a way to comprehend the Riemann-Hilbert problem and the Birkhoff problem as dualities through the Fourier-Laplace transform:

Proposition 3.2 ([**Sabbah, Prop.V.2.10**]). Let $\widehat{\mathbb{E}} \simeq \mathbb{C}[\tau]^{\oplus \widehat{d}}$ be equipped with a meromorphic connection $\widehat{\nabla}$ having singularities ONLY at $\tau = 0$ (Poincaré rank ≤ 1) and $\tau = \infty$ (regular singularity). Suppose furthermore $\widehat{\mathbb{E}}$ has a basis $\mathbf{e}$ so that in this basis the matrix of $\widehat{\nabla}$ is

$$[\widehat{\nabla}]_{\mathbf{e}} = \left(\frac{B_0}{\tau} + B_\infty\right)\frac{\mathrm{d}\tau}{\tau}$$

where B_∞ has no eigenvalue in $\mathbb{Z}_{\geq 0}$. We may extend $\widehat{\nabla}$ to $\tau = \infty$ on the trivial bundle $\widehat{\mathbb{E}}$ on $\widehat{\mathbb{P}}^1$, and consider the inverse Fourier transform $\mathbb{E}$ of $\widehat{\mathbb{E}}$ which is defined as follows:

- As sets $\mathbb{E} := \widehat{\mathbb{E}}$;
- Actions on $\mathbb{E}$: $t := \tau^2\widehat{\nabla}_{\partial_\tau}$, $\nabla_{\partial_t} = \partial_t := \tau^{-1}$.

Then the module $(\mathbb{E}, \nabla)$ has the following properties:

(a) $\mathbb{E} \simeq \mathbb{C}[t]^{\oplus d}$ with $d = \widehat{d}$;

(b) $[\nabla]_{\mathbf{e}} = (B_\infty - I)(tI - B_0)^{-1}\mathrm{d}t$; therefore, $(\mathbb{E}, \nabla)$ is logarithmic if and only if B_0 has distinct eigenvalues.

Proof:

(b) Since $t = \tau^2\partial_\tau$, $\tau^{-1}t = t\tau^{-1} + 1$, so $\partial_t = \tau^{-1}$ is indeed a (meromorphic) connection on $\mathbb{E}$. From $\partial_\tau \mathbf{e} = \left(\frac{B_0^T}{\tau} + B_\infty^T\right)\frac{1}{\tau}\mathbf{e}$ we get

$$\partial_t \mathbf{e} = \tau^{-1}\mathbf{e} = (t - B_0^T)^{-1}(B_\infty^T - 1)\mathbf{e} \tag{3}$$

and therefore $[\partial_t]_{\mathbf{e}} = (B_\infty - 1)(t - B_0)^{-1}\mathrm{d}t$.

(a) First we show rank $\mathbb{E} = \widehat{d}$. Utilize $\tau^{-1}t = t\tau^{-1} + 1$ and induction, we get $\tau^n t = t\tau^n - n\tau^{n+1}$ for all $n \in \mathbb{Z}$. Therefore by applying $\tau^{n-1}\cdot$ on the assumption $t\mathbf{e} = B_0^T\mathbf{e} + B_\infty^T\tau\mathbf{e}$ and substituting (3), we derive

$$((B_\infty)^T + (n-1)I)\tau^n\mathbf{e} = (tI - B_0^T)\tau^{n-1}\mathbf{e}. \tag{4}$$

Use (4) and the initial conditions $\tau\mathbf{e} = (B_\infty^T)^{-1}(tI - B_0^T)\mathbf{e}$ and $\tau^{-1}\mathbf{e} = (t - B_0^T)^{-1}(B_\infty^T - 1)\mathbf{e}$, we get:

$$\tau^k\mathbf{e} = \prod_{l=0}^{k-1}[(B_\infty^T + l)^{-1}(t - B_0^T)]\mathbf{e} \ \ (k \geq 1); \tag{5}$$

$$\tau^{-k}\mathbf{e} = \prod_{l=-k}^{-1}[(t - B_0^T)^{-1}(B_\infty + l)]\mathbf{e} \ \ (k \geq 1). \tag{6}$$

From (5) we see $\mathbb{E}$ is generated by $\mathbf{e}$ over $\mathbb{C}[t]$; from (6) we see that for $\widehat{\mathbb{M}} := \widehat{\mathbb{E}}[\tau^{-1}]$, $\mathbb{M} \subset \mathbb{C}(t) \otimes_{\mathbb{C}[t]} \mathbb{E}$. Therefore $\mathbb{C}(t) \otimes_{\mathbb{C}[t]} \mathbb{M} = \mathbb{C}(t) \otimes_{\mathbb{C}[t]} \mathbb{E}$ and hence with the aid of Proposition 3.1 (noting that $\mathbb{M} = \mathbb{M}[\partial_t^{-1}]$),

rank $\mathbb{E} = \dim_{\mathbb{C}(t)}(\mathbb{C}(t) \otimes_{\mathbb{C}[t]} \mathbb{E} = \mathbb{C}(t) \otimes_{\mathbb{C}[t]} \mathbb{M}) =$ rank $\mathbb{M} =$ (Prop. 3.1) rank $\widehat{\mathbb{M}} = \widehat{d}$.

Next, we derive $\mathbb{E} \simeq \mathbb{C}[t]^{\oplus\widehat{d}}$. Since rank $\mathbb{E} = \widehat{d}$ as shown above, we can use $\mathbf{e}$ to get a surjective map $\varphi : \mathbb{C}[t]^{\oplus\widehat{d}} \twoheadrightarrow \mathbb{E}$, which induces an isomorphism

$$\mathbb{C}(t)^{\oplus\widehat{d}} \simeq \mathbb{C}(t) \otimes_{\mathbb{C}[t]} \mathbb{C}[t]^{\oplus\widehat{d}} \xrightarrow[\sim]{1\otimes\varphi} \mathbb{C}(t) \otimes_{\mathbb{C}[t]} \mathbb{E}.$$

For $x \in \ker\varphi$, $(1 \otimes \varphi)(1 \otimes x) = 0$, so $1 \otimes x = 0 \in \mathbb{C}(t) \otimes_{\mathbb{C}[t]} \mathbb{C}[t]^{\oplus\widehat{d}}$; via $\mathbb{C}(t)^{\oplus\widehat{d}} \simeq \mathbb{C}(t) \otimes_{\mathbb{C}[t]} \mathbb{C}[t]^{\oplus\widehat{d}}$, the counterpart of $0 = 1 \otimes x \in \mathbb{C}(t) \otimes_{\mathbb{C}[t]} \mathbb{C}[t]^{\oplus\widehat{d}}$ in $\mathbb{C}(t)^{\oplus\widehat{d}}$ is x, so that $x = 0$. Consequently $\ker\varphi = 0$ and then φ is an isomorphism.

$\square$

4 Reference

[Sabbah] C. Sabbah, Isomonodromic Deformations and Frobenius Manifolds: An Introduction, Springer, 2008.