Note on The Riemann-Hilbert Correspondence and Deformation of Lattices

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This note is based on Chapters II and III of [**Sabbah**], C. Sabbah's book "Isomonodromic Deformations and Frobenius Manifolds: An Introduction."

1 Introduction: Sheaves and their non-abelian cohomologies

Consider the system of linear differential equations on $t \in \mathbb{C}$:

$$\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t} + A(t)\mathbf{s} = 0, \tag{1}$$

where $\mathbf{s} = [s_1(t), \cdots, s_d(t)]^T$ and $A(t)$ is a $d \times d$ (holomorphic or meromorphic) matrix function on $\mathbb{C}$.

We can use the language of connections and sheaves to rewrite the equation (1):

- Linear differential equation (1) $\Rightarrow$ Bundles with connections:
  - $M := \mathbb{C}$;
  - $E := M \times \mathbb{C}^d$ trivial bundle over M;
  - Connection ∇ on E with connection 1-form $A(t)\mathrm{d}t$ with respect to the canonical frame $e_1, \cdots, e_d$ of this trivial bundle.
- Sheafify E (in fact a general process for any vactor bundle E over M):
  - $E \rightsquigarrow \mathscr{E}$, $\mathscr{E}(U) := \Gamma(U, E)$ for open subsets $U \subset M$; thus get a sheaf $\mathscr{E}$ over M, the sheafification of E;
  - The induced connection ∇ on $\mathscr{E}$: a sheaf morphism
    $\nabla : \mathscr{E} \to \Omega^1_M \otimes_{\mathscr{O}_M} \mathscr{E}$ (holomorphic)
    or $\nabla : \mathscr{E} \to \Omega^1_M(*Z) \otimes_{\mathscr{O}_M} \mathscr{E}$ (meromorphic with poles along $Z \subset M$).
- Solutions to equation (1) $\Leftrightarrow$ Horizontal sections of ∇:
  - $E^\nabla := \ker[\nabla : \mathscr{E} \to \Omega^1_M \otimes_{\mathscr{O}_M} \mathscr{E}]$ (a sheaf over M); sections of E^∇ are called horizontal sections with respect to ∇;
  - $\mathbf{s} = \sum_{i=1}^d s_i e_i \in E^\nabla(M) \Leftrightarrow \nabla\mathbf{s} = \sum_{i,j}(\mathrm{d}s_i + A_{ij}s_j\mathrm{d}t) \otimes e_i = 0 \Leftrightarrow \frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t} + A(t)\mathbf{s} = 0$.
- The gauge transformation $\mathbf{s} = P\tilde{\mathbf{s}}$ in (1) corresponds to the base change $[\tilde{e_1} \cdots \tilde{e_d}] = [e_1 \cdots e_d]P$ of the connection 1-form of ∇.

1.1 The 1st nonabelian cohomology

Let M be a complex analytic manifold, $\mathscr{G}$ a sheaf of groups (not necessarily abelian) over M, and $\mathcal{U}$ an open cover of M. We define the following two objects:

- 1-cocycles: those $(\psi_{UV} \in \Gamma(U \cap V, \mathscr{G}))$ satisfying the cocycle condition $\psi_{UV}\psi_{VW} = \psi_{UW}$ for all $U, V, W \in \mathcal{U}$;
- 1-coboundaries: those $(\eta_U \eta_V^{-1} \in \Gamma(U \cap V, \mathscr{G}))$ where $\eta_U \in \Gamma(U, \mathscr{G})$ for each $U \in \mathcal{U}$.

Then we define the 1st nonabelian cohomology of $\mathscr{G}$ relative to $\mathcal{U}$ as

$$H^1(\mathcal{U}, \mathscr{G}) := \{\text{1-cocycles}\}/\sim$$

where $\psi \sim \psi' \Leftrightarrow$ there is a 1-coboundary η such that $\psi'_{UV} = \eta_U \psi_{UV} \eta_V^{-1}$ on $U \cap V$.

Finally, we define the 1st nonabelian cohomology of $\mathscr{G}$ as

$$H^1(M, \mathscr{G}) := \varinjlim_{\mathcal{U}} H^1(\mathcal{U}, \mathscr{G}) \text{ (in fact } = \bigcup_{\mathcal{U}} H^1(\mathcal{U}, \mathscr{G}));$$

note that each $H^1(\mathcal{U}, \mathscr{G}) \hookrightarrow H^1(M, \mathscr{G})$.

2 The Riemann-Hilbert correspondence for regular singularities on a Riemann surface

2.1 The monodromy representation

Consider the following setting:

$$\begin{array}{ccc} \gamma^{-1}\mathscr{F} & \longrightarrow & \mathscr{F} \\ \downarrow & & \downarrow \text{locally constant sheaf} \\ ([0,1], 0) & \xrightarrow{\gamma} & (X, o). \end{array}$$

Define the monodromy $T_\gamma^{\mathscr{F}}$ as:

$$\begin{array}{rcccl} T_\gamma^{\mathscr{F}}: & \mathscr{F}_o = (\gamma^{-1}\mathscr{F})_0 & \longrightarrow & (\gamma^{-1}\mathscr{F})_1 = \mathscr{F}_0 & \\ & s_0 & \longmapsto & s(1) & (\exists!\ s \in \Gamma([0,1], \gamma^{-1}\mathscr{F}) \text{ such that } s(0) = s_0). \end{array}$$

We find that $T_\gamma^{\mathscr{F}}$ remains invariant when γ varies in the same homotopy class in $\pi_1(X, o)$. Thus we get a monodromy representation

$$T^{\mathscr{F}} : \pi_1(X, o) \to \mathrm{GL}(\mathscr{F}_o).$$

2.2 The main theorem

Theorem 2.1 (**The Riemann-Hilbert correspondence for regular singularities, [Sabbah, Sec.II.3.a]**)**.** Let M be a (connected) Riemann surface and let Σ be a discrete subset of points of M (to be the regular singularities). Consider the following three categories:

- $\mathfrak{B}$: the category of meromorphic bundles with connections having regular singularities along Σ;
- $\mathfrak{S}$: the category of locally constant sheaves of $\mathbb{C}$-vector spaces on $M \setminus \Sigma$;
- $\mathfrak{R}$: the category of finite-dimensional representations of $\pi_1(M \setminus \Sigma)$.

They are equivalent categories via the following identification:

$$\begin{array}{ccccc} \mathfrak{B} & \xrightarrow{\sim} & \mathfrak{S} & \xrightarrow{\sim} & \mathfrak{R} \\ (\mathscr{M}, \nabla) & \longmapsto & (\mathscr{M}^{\nabla})|_{M\setminus\Sigma} & & \\ & & \mathscr{F} & \longmapsto & T^{\mathscr{F}}. \end{array}$$

Sketch of proof: We take $\mathfrak{S} \simeq \mathfrak{R}$ for granted. Let us show $\mathfrak{B} \simeq \mathfrak{S}$. First we consider this problem locally, and then may assume $\Sigma = \{p\}$ a point, and assume M is a disc centered at p, so that $\pi_1(M \setminus \{p\}) = \mathbb{Z}$. Then given any finite-dimensional representation of $\pi_1(M \setminus \{p\}) = \mathbb{Z}$ is equivalent to give any $T \in \mathrm{GL}(d, \mathbb{C})$ ($d := \dim M$). If we are given a $T \in \mathrm{GL}(d, \mathbb{C})$, we may use the technique in §1 and consult the theory of linear differential equations with regular singularities to produce a differential equation so that the associated horizontal bundle has monodromy T (one may need the fact $\exp : M(d, \mathbb{C}) \to \mathrm{GL}(d, \mathbb{C})$ is surjective). The global problem follows from gluing the local horizontal bundles and then tensoring the global bundle with $\mathscr{O}_M(*\Sigma)$. ♢

3 The Riemann-Hilbert correspondence for irregular singularities via the Stokes sheaf

In this section X is a complex analytic manifold, and D is the unit disc in $\mathbb{C}$.

3.1 The sheaf $\mathscr{A}$

- Polar coordinate $\begin{array}{rrcl} \pi : & \tilde{D} := [0, r^0) \times S^1 & \longrightarrow & D \\ & (r, e^{i\theta}) & \longmapsto & t = re^{i\theta}. \end{array}$

  (We call $\tilde{D}$ the real blow-up of D at the origin.)
- $\mathscr{C}^\infty_{\tilde{D}\times X} := i^{-1}\mathscr{C}^\infty_{(-\varepsilon, r^0)\times S^1\times X}$ via the inclusion $\tilde{D} \times X \hookrightarrow (-\varepsilon, r^0) \times S^1 \times X$.
- $\mathscr{A}_{\tilde{D}\times X}$ is defined to be the intersection

$$\ker\left(\bar{t}\frac{\partial}{\partial \bar{t}} = \frac{1}{2}\left(r\frac{\partial}{\partial r} + i\frac{\partial}{\partial \theta}\right) \in \mathrm{End}(\mathscr{C}^\infty_{\tilde{D}\times X})\right) \cap \left(\bigcap_j \ker\left(\frac{\partial}{\partial \overline{x^j}} \in \mathrm{End}(\mathscr{C}^\infty_{\tilde{D}\times X})\right)\right).$$

3.2 The good model

A "good model" is a meromorphic bundle $\mathscr{M}^{\text{good}}$ over $D \times X$ equipped with a *flat* connection ∇^{good} with poles along $\{0\} \times X$, such that the following property holds: For all $x^0 \in X$, there are $\varphi_1, \cdots, \varphi_p \in t^{-1}\mathscr{O}_{X,x^0}[t^{-1}]$ and nonzero bundles $\mathscr{R}_{\varphi_1}, \cdots, \mathscr{R}_{\varphi_p}$ with regular singularities along $\{0\} \times X$, such that:

- $\mathscr{M}^{\text{good}} \simeq \oplus_k(\mathscr{E}^{\varphi_k} \otimes \mathscr{R}_{\varphi_k})$ in a neighborhood of x^0 (here $\mathscr{E}^{\varphi} = \mathbb{C}\{t, x\}[t^{-1}]$ with connection 1-form $-\mathrm{d}\varphi$);
- $\mathrm{ord}_{t=0}(\varphi_k - \varphi_l)(t, x)$ does not depend on x in a neighborhood of x^0 for all $k \neq l$.

This good model also induces the following sheaves of bundles that will be used later:

- $\widetilde{\mathscr{M}}^{\text{good}} := \mathscr{A}_{\tilde{D}\times X} \otimes_{\mathscr{O}_{D\times X}} \mathscr{M}^{\text{good}}$;
- $\widehat{\mathscr{M}}^{\text{good}} := \widehat{\mathscr{O}}_{D\times X} \otimes_{\mathscr{O}_{D\times X}} \mathscr{M}^{\text{good}}$;
- $\mathrm{Aut}^{<X}(\widetilde{\mathscr{M}}^{\text{good}}) := \ker\left(\mathrm{Aut}(\widetilde{\mathscr{M}}^{\text{good}}|_{S^1\times X}) \to \mathrm{Aut}(\widehat{\mathscr{M}}^{\text{good}}|_{S^1\times X})\right)$ where the automorphisms on the right hand side are required to be compatible with connections. Its local sections are "Stokes matrices."

3.3 The Stokes sheaf and the sheaf $\mathscr{H}_X$

From now on, till the end of this section, we will FIX a good model $(\mathscr{M}^{\text{good}}, \nabla^{\text{good}})$.

Now we introduce two additional sheaves and a morphism between them, which will be the main objects in this section:

- The Stokes sheaf $\mathbf{St}_X$ over X is defined by

  $$\mathbf{St}_X(U) := H^1(S^1 \times U, \mathrm{Aut}^{<X}(\widetilde{\mathscr{M}}^{\text{good}})) \text{ for open subsets } U \subset X.$$

  Here we use the 1st nonabelian cohomology introduced in §1.1.

- The sheaf $\mathscr{H}_X$ over X is defined by

  $$\mathscr{H}_X(U) := \frac{\{(\mathscr{M}, \nabla, \hat{f}) \text{ on } D \times U | \hat{f} : (\widehat{M}, \widehat{\nabla})|_{\{0\}\times U} \xrightarrow{\sim} (\widehat{\mathscr{M}}^{\text{good}}, \widehat{\nabla}^{\text{good}})|_{\{0\}\times U}\}}{(\mathscr{M}, \nabla, \hat{f}) \sim (\mathscr{M}', \nabla', \hat{f}') \Leftrightarrow \exists\, g : (\mathscr{M}, \nabla) \xrightarrow{\sim} (\mathscr{M}', \nabla') \text{ with } \hat{f} = \hat{f}' \circ \hat{g}}.$$

  for any open subset $U \subset X$.

- The morphism $\Phi : \mathscr{H}_X \to \mathbf{St}_X$ is defined as follows: for any open subset $U \subset X$, define

  $$\begin{array}{rrcl} \Phi(U): & \mathscr{H}_X(U) & \longrightarrow & \mathbf{St}_X(U) \\ & [(\mathscr{M}, \nabla, \hat{f})] & \longmapsto & [(f_j f_i^{-1}|_{W_{ij}})_{i,j}] \end{array}$$

  where (W_i) is an open cover of $S^1 \times U$ and $f_i : (\widetilde{\mathscr{M}}, \widetilde{\nabla})|_{W_i} \xrightarrow{\sim} (\widetilde{\mathscr{M}}^{\text{good}}, \widetilde{\nabla}^{\text{good}})|_{W_i}$ such that $\hat{f}_i = \hat{f}$ (the existence of W_i and f_i is guaranteed by [**Sabbah, Thm. II.5.12**]).

  One may use the definition of the 1st nonabelian cohomology to check that $\Phi(U)$ is well-defined.

3.4 The main theorem

Theorem 3.1 (**Classification of meromorphic connections with fixed formal type; the Riemann-Hilbert correspondence for irregular singularities, [Sabbah, Thm.II.6.3]**)**.** The morphism $\Phi : \mathscr{H}_X \to \mathbf{St}_X$ is an isomorphism of sheaves of sets over X.

Proof: It suffices to show that this is a germ-wise isomorphism, namely to show that for every $x^0 \in X$, if $i : S^1 \times \{x^0\} \hookrightarrow S^1 \times X$ is the inclusion, then $\Phi_{x^0} : \mathscr{H}_{X,x^0} \to H^1(S^1, i^{-1}\mathrm{Aut}^{<X}(\widetilde{\mathscr{M}}^{\mathrm{good}}))$ is an isomorphism.

- **Injectivity of Φ_{x^0}:** Suppose $(M, \nabla, \hat{f}), (M', \nabla', \hat{f}') \in \mathscr{H}_{X,x^0}$ such that there images under Φ_{x^0} are both $\lambda \in H^1(S^1, i^{-1}\mathrm{Aut}^{<X}(\widetilde{\mathscr{M}}^{\mathrm{good}}))$. To make this more precise, may assume there is a finite cover (I_i) of S^1 and an open neighborhood $x^0 \in V \subset X$ such that $f_i : \widetilde{\mathscr{M}} \to \widetilde{\mathscr{M}}^{\mathrm{good}}, f_i' : \widetilde{\mathscr{M}'} \to \widetilde{\mathscr{M}}^{\mathrm{good}}$ associated to $\hat{f}, \hat{f}'$ are defined over $I_i \times V$ and make $\lambda = [(f_j f_i^{-1})] = [(f_j' f_i'^{-1})]$.
  Then, by definition of the 1st nonabelian cohomology, there exist $g_i \in \Gamma(I_i \times V, \mathrm{Aut}^{<X}(\widetilde{\mathscr{M}}^{\mathrm{good}}))$ such that on $I_{ij} \times V = (I_i \cap I_j) \times V$, $f_j' f_i'^{-1} = g_j f_j f_i^{-1} g_i^{-1}$ or
  $$f_i^{-1} g_i^{-1} f_i' = f_j^{-1} g_j^{-1} f_j'. \tag{2}$$
  So we may set $\sigma_i := f_i^{-1} g_i^{-1} f_i'$ which are compatible with connections, and which glue to $\sigma \in \Gamma(S^1 \times V, \underline{\mathrm{Hom}}(\widetilde{\mathscr{M}'}, \widetilde{\mathscr{M}}))$ by (2). Now $\hat{f}' = \hat{f} \circ \hat{\sigma}$ (since $\hat{g}_i = 1$), which implies $(M, \nabla, \hat{f}) \sim (M', \nabla', \hat{f}')$.

- **Surjectivity of Φ_{x^0}:** Suppose $\lambda \in H^1(S^1, i^{-1}\mathrm{Aut}^{<X}(\widetilde{\mathscr{M}}^{\mathrm{good}}))$. Choose an open cover (I_i) of S^1 such that $\lambda = (\lambda_{ij}) \in \Gamma(I_{ij}, i^{-1}\mathrm{Aut}^{<X}(\widetilde{\mathscr{M}}^{\mathrm{good}})) = \mathrm{GL}_d^{<X}(i^{-1}\mathscr{A}_{\tilde{D}\times X})$. Then
  $$\lambda_{ij}\lambda_{jk} = \lambda_{ik}; \quad \hat{\lambda}_{ij} = \mathrm{id}_{\mathbb{C}^d};$$
  so that by [**Sibuya, Thm. 6.4.1**], λ is a coboundary with value in $\mathrm{Aut}^{<X}(\widetilde{\mathscr{M}}^{\mathrm{good}})$, i.e. $\lambda = f_j f_i^{-1}$ for some $f_i \in \Gamma(I_i, i^{-1}\mathrm{Aut}(\widetilde{\mathscr{M}}^{\mathrm{good}}))$.
  Now we construct the desired object which maps to λ via Φ_{x^0}. Since $\lambda_{ij} = f_j f_i^{-1}$ is compatible with ∇^{good},$\nabla^{\mathrm{good}}(\lambda_{ij}s) = \lambda_{ij}(\nabla^{\mathrm{good}}s)$ for all s, so that
  $$f_j^{-1}\nabla^{\mathrm{good}} f_j = f_i^{-1}\nabla^{\mathrm{good}} f_i \text{ over } I_{ij}. \tag{3}$$
  On I_i consider $(\widetilde{\mathscr{M}}^{\mathrm{good}}, f_i^{-1}\nabla^{\mathrm{good}} f_i)$; by (3) they glue to $(\widetilde{\mathscr{M}}^{\mathrm{good}}, \nabla)$ with $\nabla|_{I_i} = f_i^{-1}\nabla^{\mathrm{good}} f_i$. Similarly $(\hat{f}_i)$ glue to $\hat{f} : (\widehat{\mathscr{M}}^{\mathrm{good}}, \nabla) \to (\widehat{\mathscr{M}}^{\mathrm{good}}, \nabla^{\mathrm{good}})$. Thus the image of $(\widehat{\mathscr{M}}^{\mathrm{good}}, \nabla, \hat{f})$ under Φ is λ.

The proof is now complete. □

3.5 Remarks on the Stokes sheaf

- $\mathbf{St}_X = \mathbf{St}_X(\mathscr{M}^{\text{good}})$ is a locally constant sheaf of (pointed) sets.
- Therefore, via the isomorphism $\Phi : \mathscr{H}_X \xrightarrow{\sim} \mathbf{St}_X$, we find $\mathscr{H}_X$ is also a locally constant sheaf.

4 The rigidity of logarithmic lattices

Theorem 4.1 (Rigidity of deformation of logarithmic lattices, [Sabbah, Prop. III.1.19]). Suppose we are given the following objects:

- A vector bundle E^0 over the unit disc $D \subset \mathbb{C}$ equipped with a flat connection ∇ having only a logarithmic pole at $0 \in D$;
- A simply connected complex analytic manifold X and a specified point $x^0 \in X$.

Then there exists, unique up to isomorphism, a vector bundle E over $D \times X$ equipped with a flat connection ∇ having only logarithmic poles along $\{0\} \times X \subset D \times X$, such that the restriction of (E, ∇) to $D \times \{x^0\} \subset D \times X$ is (E^0, ∇^0).

Sketch of proof: Consider the canonical projection $p : D \times X \to D$ and put $(E, \nabla) = p^*(E^0, \nabla^0)$. ♢

5 Deformation of lattices with connections of pole order 1

In this section, D is the unit disc on $\mathbb{C}$, and X is a connected complex analytic manifold of dimension n with a base point $x^0 \in X$. We say a connection is of pole order 1 along some singularities if it has Poincaré rank 1 there.

5.1 The rank 1 case: local classification

Theorem 5.1 ([Sabbah, Prop.III.2.11]). Suppose $\mathscr{E}$ is a holomorphic *line bundle* over $D \times X$ equipped with a flat connection ∇ having only poles of order 1 along $\{0\} \times X$. Define $(E, \nabla) := (\mathscr{E}, \nabla)_{(0,x^0)}$. Then:

(a) There exist unique $\lambda(x) \in \mathscr{O}(X)$ and $\mu \in \mathbb{C}$ such that in any local basis of E, the polar part of ∇ is $\omega_{\text{pol}} = -\mathrm{d}\left(\frac{\lambda(x)}{t}\right) + \mu\frac{\mathrm{d}t}{t}$.

(b) (E, ∇) admits a non-identically zero holomorphic horizontal section if and only if $\lambda \equiv 0, \mu \in \mathbb{Z}_{\leq 0}$.

5.2 The formal decomposition

Consider the following assumption which will be used later:

Assumption 5.2. $\lambda_1, \cdots, \lambda_d : X \to \mathbb{C}$ are holomorphic functions, and $\lambda_1^0, \cdots, \lambda_d^0 \in \mathbb{C}$, such that

- For all $i \neq j$, $\lambda_i - \lambda_j$ is non-vanishing on X;
- For all i, $\lambda_i(x^0) = \lambda_i^0$.

Theorem 5.3 (The formal decomposition of a lattice of order 1, [Sabbah, Thm.III.2.15]). Suppose that

- E is a holomorphic vector bundle over $D \times X$ (we use (t, x) to denote coordinates on $D \times X$) endowed with a flat connection ∇ having only poles of order 1 along $\{0\} \times X$;
- The "second residue" $R_0(x)$ of ∇, namely the residue of $t\nabla$, has eigenvalues $\lambda_i(x)$ that satisfy Assumption 5.2.

Then there exists a unique formal decomposition

$$(\widehat{\mathscr{E}}, \widehat{\nabla}) \simeq \bigoplus_{i=1}^{d} (\widehat{\mathscr{E}}_i, \widehat{\nabla})$$

subjected to the condition that the second residue of $(\widehat{\mathscr{E}}_i, \widehat{\nabla})$ is $\lambda_i(x)$ for all i.

5.3 The main theorem

Theorem 5.4 (Rigidity of deformation of lattices having pole-order 1, [Sabbah, Thm.III.2.10]). Suppose that

- E^0 is a vector bundle over the unit disc $D \subset \mathbb{C}$ equipped with a flat connection ∇ having only a order-1 pole at $0 \in D$;
- The second residue R_0^0 of ∇^0 has eigenvalues $\lambda_1^0, \cdots, \lambda_d^0$;
- The parameter space X is simply-connected.

Then there exists, unique up to isomorphism, a holomorphic bundle (E, ∇) over $D \times X$ such that

- The second residue $R_0(x)$ of ∇ has eigenvalues $\lambda_i(x)$ satisfying Assumption 5.2;
- The restriction of (E, ∇) to $D \times \{x^0\}$ is isomorphic to (E^0, ∇^0).

Sketch of proof:

- Construction of the good formal model: Let $\mathscr{M}^{\text{good}}$ be the trivial bundle on $D \times X$ equipped with a connection ∇^{good} having connection 1-form $\text{diag}(\omega_1, \cdots, \omega_d)$ with $(\omega_i)_{\text{pol}} = -\mathrm{d}\left(\frac{\lambda_i(x)}{t}\right) + \mu_i \frac{\mathrm{d}t}{t}$ where μ_i are determined by (E^0, ∇^0). Then we get the good formal model $(\widehat{\mathscr{M}}^{\text{good}}, \widehat{\nabla}^{\text{good}})$.

- Construction of the meromorphic bundle: Since X is simply-connected, by §3.5, $\mathbf{St}_X$ is a constant sheaf and so is $\mathscr{H}_X$. So there is a unique section σ of $\mathscr{H}_X$ such that $\sigma_{x^0} = (M^0, \nabla^0, \hat{f}^0) \in \mathscr{H}_{X,x^0}$ (M^0 is induced by E^0). This σ corresponds to $(\mathscr{M}, \nabla, \hat{f})$ whose restriction to $D \times \{x^0\}$ is isomorphic to $(M^0, \nabla^0, \hat{f}^0)$.

- Construction of (E, ∇): Take $(\mathscr{E}, \nabla) = (\widehat{\mathscr{E}} \cap \mathscr{M}, \nabla)$ and then we get the corresponding (E, ∇).

The uniqueness of (E, ∇) can be deduced by appropriately using the local constancy of $\text{Aut}(\widehat{E}^{\text{good}}, \widehat{\nabla}^{\text{good}})$ and the constancy of $\mathscr{H}_X$. $\diamondsuit$

6 Reference

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