THE MODULI SPACE OF STABLE CURVES

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We introduce the proof of the groupoid of stable, \(n\)-pointed, genus \(g\) curves as a Deligne-Mumford stack in three stages. First, as a quotient groupoid, we show that it is isomorphic to the Hilbert scheme of \(\nu\)-log-canonically embedded, stable, \(n\)-pointed genus \(g\) curves quotient the projective general linear group. Then we briefly recall the Grothendiecks descent theory for quasi-coherent sheaves and use it to show that the moduli groupoid is a stack and, finally, Deligne-Mumford stack.

1. HIllBERT SCHEME

Let \((C; p_1, \ldots, p_n)\) be a stable, \(n\)-pointed, genus \(g\) curves and let \(D = \sum_{i=1}^{n} p_i\). We have the following fact:

**Fact 1.1.** \((w_C(D))^\nu\) is very ample if \(\nu \geq 3\), where \(w_C\) denotes the dualizing sheaf of \(C\).

We embeds \(C\) in \(\mathbb{P}^r\) via \((w_C(D))^\nu\), where \(r = (2\nu t - 1)(g - 1) + vn - 1\). Its Hilbert polynomial is

\[ p_\nu(t) = (2\nu t - 1)(g - 1) + vnt. \]

The embedding given by \((w_C(D))^\nu\) is called the \(\nu\)-log canonical embedding.

**Definition 1.2.** We define \(H_{\nu,g,n}\) as the Hilbert scheme of stable, \(n\)-pointed, genus \(g\) curve given by the \(\nu\)-log canonical embedding.

**Remark 1.3.** \(H_{\nu,g,n}\) is a smooth locally closed subscheme of the product \(\text{Hilb}_{\mathbb{P}^r}^{p_\nu(t)} \times (\mathbb{P}^r)^n\) of dimension \(3g - 3 + n + (r + 1)^2 - 1\). The natural action of \(\text{PGL}(r + 1)\) on this product restricts to the action on \(H_{\nu,g,n}\).

2. GROUPOID

Let \(S\) be a scheme and consider the category \(\text{Sch}/S\) of schemes over \(S\). From now on, the scheme will implicitly assumed to be of finite type over \(C\).

**Definition 2.1.** A category fibered in groupoids over \(\text{Sch}/S\) or, more simply, a groupoid over \(S\), is a pair \((C_M, p_M)\), where \(C_M\) is a category, and

\[ p_M : C_M \to \text{Sch}/S \]

is a functor satisfying the following two conditions:
(A) Let $f : T \to T'$ be a morphism in $\text{Sch}/S$, and let $\eta \in \text{Ob}(C_M)$ such that $p_M(\eta) = T'$. Then there exists (not necessarily unique) $\xi \in C_M$ and a morphism $\phi : \xi \to \eta$ in $C_M$ with $p_M(\phi) = f$.

(B) Every morphism $\phi : \xi \to \eta$ is cartesian in the following sense. Given other arrow $\phi' : \xi' \to \eta$ and a morphism $h : p_M(\xi) \to p_M(\xi')$ such that $p_M(\phi')h = p_M(\phi)$, there exists a unique morphism $\psi : \xi \to \xi'$ such that $p_M(\psi) = h$ and $\phi' \psi = \phi$.

A morphism $\alpha : M \to M'$ of groupoids over $\text{Sch}/S$ is a functor $\alpha : C_M \to C_{M'}$ such that $p_{M'} \alpha = p_M$. When $\alpha$ is an equivalence of categories, we say that it is an isomorphism of groupoids.

Example 2.2.

(1) Let $X$ be a scheme. We will consider $X$ as a groupoid $X = (C_X, p_X)$, where the objects of $C_X$ are pairs $(T, f)$ with $f : T \to X$ a morphism of schemes. The morphism $\phi : (T, f) \to (T', f')$ are the morphisms $h : T \to T'$ such that $f'h = f$. Finally, the functor $p_X$ is defined by $p_X(T, f) = T$.

(2) Let $C$ be the category in which the objects are the families

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\xi} & T \\
\downarrow & & \\
T & \end{array}
$$

of smooth (resp. stable, $n$-pointed) curves of genus $g$ and in which a morphism

$$
\phi : \xi' \to \xi
$$

between two families $\xi' : \mathcal{X}' \to T'$ and $\xi : \mathcal{X} \to T$ is a cartesian product

$$
\begin{array}{ccc}
\mathcal{X}' & \longrightarrow & \mathcal{X} \\
\xi' & \downarrow & \xi \\
T' & \xrightarrow{f} & T
\end{array}
$$

The functor $p$ assigns to a family $\xi : \mathcal{X} \to T$ its parameter space $T : p(\xi) = T$. For the morphism, we set $p(\phi) = f$. It is not hard to check that the pair $(C, p)$ is a groupoid.

We denote $\mathcal{M}_g,n$ (resp. $\overline{\mathcal{M}}_{g,n}$) the groupoid of smooth (resp. stable, $n$-pointed), genus $g$ curves.

Definition 2.3 (The category $\mathcal{M}(T)$). Given a groupoid $M = (C, p)$, denote by $\mathcal{M}(T)$ the category whose objects are objects $\xi \in C$ with $p(\xi) = T$ and whose morphisms are morphisms $\phi$ in $C$ with $p(\phi) = \text{id}_T$. The condition (B) tells us that a morphism $\phi$ in $C$ is an isomorphism if and only if $p(\phi)$ is. Hence $\mathcal{M}(T)$ is a groupoid in the sense that all morphisms are isomorphisms. The category $\mathcal{M}(T)$ is called the category of sections of $M$ over $T$. 
It is important to check whether two groupoids \((C_M, p_M)\) and \((C_{M'}, p_{M'})\) are isomorphic. As we already mentioned before, there must exist an equivalence of categories \(F : C_M \to C_{M'}\) such that \(p_{M'}F = p_M\). It is well-known that the functor \(F\) is equivalence if and only if the following conditions hold:

(i) \(F\) is fully faithful in the following sense, for any \(\xi, \xi' \in \text{Ob}(C_M)\), the induced map
\[
\text{Hom}_{C_M}(\xi, \xi') \to \text{Hom}_{C_{M'}}(F(\xi), F(\xi'))
\]
is bijective.

(ii) \(F\) is essentially surjective, meaning that every \(\eta \in \text{Ob}(M')\) is isomorphic to \(F(\xi)\) for some \(\xi \in M\).

For groupoids, we have the following lemma.

**Lemma 2.4.** A morphism \(F : M \to M'\) of groupoids over \(\text{Sch}/S\) is an isomorphism if and only if for every \(T\) in \(\text{Sch}/S\), the induced functor on fibers \(F_T : M(T) \to M'(T)\) is an equivalence of categories.

Suppose that a group scheme \(G\) acts on a scheme \(X\). Then one can form the quotient groupoid
\[
[X/G] = (P_{G,X}, P),
\]
where \(P_{G,X}\) is the category whose objects are pairs \((\pi, \sigma_\pi)\), where \(\pi : E \to T\) is a principal \(G\)-bundle, and \(\sigma(\pi) : E \to X\) is a \(G\)-equivariant map. A morphism between \((\pi, \sigma_\pi)\) and \((\pi', \sigma'_\pi)\) is a pair of commutative diagrams
\[
\begin{array}{ccc}
E' & \xrightarrow{\phi} & E \\
\downarrow{\pi'} & & \downarrow{\pi} \\
T' & \xrightarrow{f} & T
\end{array} \quad \begin{array}{ccc}
E' & \xrightarrow{\phi} & E \\
\downarrow{\sigma'_{\pi}} & & \downarrow{\sigma_\pi} \\
X
\end{array}
\]
where the first one is cartesian. At last, the functor \(p : P_{G,X} \to \text{Sch}\) is given by \((\pi, \sigma_\pi) \to T\). Hence \(P_{G,X}(T)\) is the category of principal \(G\)-bundle over \(T\), equipped with a \(G\)-equivariant map from their total space to \(X\).

**Example 2.5.** For \(X = \{pt\}\), we have \([\{pt\}/G] = BG\).

**Theorem 2.6.** The moduli groupoid \(\mathcal{M}_{g,n}\) is isomorphic to the quotient groupoid \([H_{v,g,n}/\text{PGL}(N)]\).

**Proof.** We first define a morphism
\[
\Phi : \mathcal{M}_{g,n} \to [H_{v,g,n}/\text{PGL}(N)].
\]
Let \(\xi : \mathcal{X} \to T\) be a family of stable \(n\)-pointed genus \(g\) curves, which is an object of \(\mathcal{M}_{g,n}\). \(\Phi(\xi)\) must consist of a \(G\)-bundle \(\pi : E \to T\) and a \(G\)-equivariant map \(\sigma_\pi : E \to H_{v,g,n}\).
First let $\pi : E \to T$ be the principal $G$-bundle associated to the projective bundle $P_{\xi} := \mathbb{P}(\xi(w^\nu_\xi(vD))) \to T$. Consider the canonically trivialized $G$-bundle

$$\pi^*P_{\xi} \to E$$

and the pull-back family

$$\eta : Z = \mathcal{X} \times_\pi E \to E$$

There is a canonical isomorphism

$$P_{\eta} \cong \pi^*P_{\xi}.$$  

We can therefore view $Z \to E$ as a family of $\nu$-log canonically embedded curves via the canonical trivialization of $P_{\eta}$. This gives a $G$-equivariant morphism $\sigma_\pi : E \to H_{v,g,n}$. This completes the definition of $\Phi$.

To show $\Phi$ is an isomorphism of groupoid, we use Lemma (2.4). We must show that $\Phi_T$ is fully faithful and essentially surjective for every scheme $T$.

For the first part, since any two objects of $\mathcal{M}_{g,n}(T)$ are isomorphism, we only need to prove that $\Phi_T$ induces a bijection

$$\text{Hom}_{\mathcal{M}_{g,n}(T)}(\tilde{\xi}, \tilde{\eta}) \xrightarrow{\sim} \text{Hom}_{P(T)}(\Phi(\tilde{\xi}), \Phi(\tilde{\eta})).$$

It is equivalent to the statement that the automorphism of a family of stable, $n$-pointed, genus $g$ curves $\tilde{\xi} : \mathcal{X} \to T$ and the automorphisms of the projective bundle $P_{\xi}^* \to T$ determine each other. Let $\gamma$ be the automorphism of the family $\tilde{\xi} : \mathcal{X} \to T$. Then $\gamma$ induces an automorphism of $\mathbb{P}(H^0(\xi(w_\xi^\nu(vD))))$ and therefore induces the automorphism of $P_{\xi}^* \to T$. For the other direction, we observe that any non-trivial element $\phi \in \text{PGL}(r+1)$ which leaves a $\nu$-log canonically embedded curve $C \hookrightarrow \mathbb{P}^r$ invariant must act non-trivial on $C$, since $C$ does not lies in the proper linear subspace, the fixed locus of $\phi$. Hence any non-trivial automorphism of $P_{\xi}^* \to T$ will induce an non-trivial automorphism of $\tilde{\xi} : \mathcal{X} \to T$.

For the essential surjectivity part, let $(\pi, \sigma_\pi) \in \text{Ob}(P)$, so that $\pi : E \to T$ is a principal $G$-bundle, and $\sigma_\pi : E \to H_{v,g,n}$ is a $G$-equivariant map. Now we consider the universal family $\mathcal{Y} \to H_{v,g,n}$ and the following cartesian diagram

$$\begin{diagram}
  \node{Z} \arrow{e} \node{\mathcal{Y}} \\
  \node{E} \arrow{s, \eta} \node{H_{v,g,n}} \arrow{s, \sigma_\pi}
\end{diagram}$$

The group $G$ acts equivariantly and freely on $E$ and $Z$. We can then induce the quotient family

$$\tilde{\xi} : Z/G = \mathcal{X} \to T = E/G.$$  

The last part is to check that $\Phi_T(\tilde{\xi})$ is isomorphic to $(\pi, \sigma_\pi)$. We left it to the reader.  \qed
3. The Theory of Descent for Quasi-Coherent Sheaves

Consider a morphism of schemes \( X \to Y \), and a quasicoherent \( \mathcal{O}_X \)-module \( \mathcal{F} \). Let \( p_1 \) and \( p_2 \) be the first and second projection of \( X \times_Y X \to X \), \( p_{12}, p_{13}, \) and \( p_{23} \) to indicate the projections of \( X \times_Y X \times_Y X \to X \times_Y X \) by omitting the third, second, and first components, respectively, and \( q_1, q_2, q_3 \) to indicate the three projections of \( X \times_Y X \times_Y X \to X \). Notice that \( p_1 p_{12} = q_1 = p_1 p_{13}, p_2 p_{12} = q_2 = p_1 p_{23}, \) and \( p_2 p_{13} = q_3 = p_2 p_{23} \). The descent data for \( \mathcal{F} \) relative to \( X \to Y \) is an isomorphism \( \phi : \mathcal{F}_1 \to \mathcal{F}_2 \) such that the following diagram commutes:

![Diagram](image)

, where \( \mathcal{F}_1 = p_1^* \mathcal{F} \) and \( \mathcal{F}_2 = p_2^* \mathcal{F} \). We will call this the cocycle condition.

When \( \mathcal{F} \) is the pullback of a quasicoherent \( \mathcal{O}_Y \)-module, there is a canonical isomorphism between \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) with descent data. We are interested in the covered part, that is, whether a quasicoherent \( \mathcal{O}_X \)-module with descent data comes from an \( \mathcal{O}_Y \)-module. Similar problem can be asked on the morphism of quasicoherent \( \mathcal{O}_X \)-modules. Here we introduce Grothendieck's descent theory:

**Theorem 3.1.** Let \( \pi : X \to Y \) be a faithfully flat and quasi-compact morphism of schemes. Then the pullback functor

\[
\{ \text{quasicoherent } \mathcal{O}_Y \text{-modules} \} \to \{ \text{quasicoherent } \mathcal{O}_X \text{-modules with descent data relative to } \pi \}
\]

is an equivalence of categories.

4. Moduli Space of Curve as a Stack

Let \( \mathcal{M} = (C, \rho) \) be a groupoid, \( T \) be a scheme, \( \xi \) be an object of \( \mathcal{M}(U) \), and \( f : U \to T \) be an étale surjective morphism. As before, we denote \( p_{12}, p_{13}, \) and \( p_{23} \) to be the projections of \( U \times_T U \times_T U \to U \times_T U \) by omitting the third, second, and the first component, respectively and \( q_1, q_2, q_3 \) to indicate the three projections of \( U \times_T U \times_T U \to U \) so that \( p_1 p_{12} = q_1 = p_1 p_{13}, p_2 p_{12} = q_2 = p_1 p_{23}, \) and \( p_2 p_{13} = q_3 = p_2 p_{23} \). A descent datum for \( \xi \), relative to \( f : U \to T \), is an isomorphism \( \phi : p_{12}^* \xi \to p_{23}^* \xi \) such that the
following diagram commutes:

\[
\begin{array}{ccc}
p_{12}^* & \cong & p_{23}^* p_1^* \\
\downarrow p_{12} \phi & & \downarrow p_{23} \phi \\
p_{13}^* p_1^* & \cong & p_{13}^* p_2^* \\
\end{array}
\]

A descent datum for $\zeta$ relative to $f$, is said to be effective if there exist an object $\eta \in T$ and an isomorphism $\psi : f^*(\eta) \to \zeta$ such that

\[\psi = (p_2^* \psi) \circ (p_1^* \psi)^{-1}\]

Now we are ready to define a stack in groupoids for the étale topology or, more simply, a stack. A stack is a groupoid $M = (C, p)$ having the following two properties.

1. Every descent datum is effective.
2. Given a scheme $S$ and objects $\xi$ and $\eta$ in $M(S)$, the functor

   \[\text{Isom}_S(\xi, \eta) : \text{Sch}/S \to \text{Sets}\]

   which associates to a morphism $f : T \to S$ the set of isomorphisms in $M(T)$ between $f^*(\xi)$ and $f^*(\eta)$ is a sheaf in the étale topology.

A contravariant functor $F : \text{Sch}/S \to \text{Sets}$ is a sheaf in the étale topology if for every étale surjective morphism $f : U \to T$ of $S$-schemes, the diagram

\[
\begin{array}{ccc}
F(T) & \xrightarrow{F(f)} & F(U) \\
\downarrow F(p_1) & & \downarrow F(p_2) \\
F(U \times_T U) & &
\end{array}
\]

is exact. We recall the following theorem due to Grothendieck.

**Theorem 4.1.** Let $S$ be a scheme. Let $F : \text{Sch}/S \to \text{Sets}$ be a contravariant, representable functor. Then $F$ is a sheaf for the étale topology.

**Theorem 4.2.** The groupoids $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ are stacks.

**Proof.** We consider $\overline{\mathcal{M}}_{g,n}$ case only. The case of $\mathcal{M}_{g,n}$ is similar. We deal with the condition (2) in the definition of stack first. Given two families of stable curves $\tilde{\zeta} : X \to S$ and $\eta : Y \to S$, objects in $\overline{\mathcal{M}}_{g,n}(S)$, the functor $\text{Isom}_S(\tilde{\zeta}, \eta)$ is represented by the scheme $\text{Isom}_S(X, Y)$, where $\text{Isom}_S(X, Y)$ represents the functor which associates to each scheme $T$ over $S$ the set of all isomorphisms, as schemes over $T$, from $X \times_S T$ to $Y \times_S T$.

The turn to condition (1). Let $T \to T'$ be a surjective étale morphism, and let

\[\tilde{\zeta} : X \to T\]
be a family of stable curves with descent data \( \phi : p^*_1(\xi) \to p^*_2(\xi) \),

\[
p^*_1(X) = T \times_{T'} X \xrightarrow{\phi} X \times_{T'} T = p^*_2(X)
\]

To check the effectiveness, we need to produce a family of stable curves \( \eta : Y \to T' \) such that \( \xi = \pi^*(\eta) \). This construction is a typical descent construction and will be reduced to the theory of descent of quasicoherent sheaves. This reduction has two steps

We first give some notation. Consider the family \( \xi : X \to T \). We consider the dual direct image bundle \( E_\xi = \xi^*(w_3(3D))^* \). The total space \( X \) of the family \( \xi \) can be viewed as embedded in \( \mathbb{P}(E_\xi) \):

\[
X \subset \mathbb{P}(E_\xi)
\]

\[
\downarrow \xi
\]

\[
\downarrow T
\]

**Step 1.** From the descent data for \( \xi \) we deduce descent data for the vector bundle (or better, locally free sheaf) \( E_\xi \). From the theory of descent on \( \text{QCoh} \), we get a vector bundle \( E' \) over \( T' \) with \( \pi^*(E') = E \).

**Step 2.** We consider the following diagram

\[
\mathbb{P} = \mathbb{P}(E_\xi) \xrightarrow{q} \mathbb{P}(E') = \mathbb{P}'
\]

\[
\downarrow T \xrightarrow{\pi} T'
\]

The descent data for \( \xi : X \to T \) relative to the étale cover \( \pi : T \to T' \) determine descent data for the ideal sheaf \( \mathcal{I}_X \subset \mathcal{O}_P \) with respect to the étale cover \( q : \mathbb{P} \to \mathbb{P}' \). Using the theory of descent for \( \text{QCoh} \), we get the subscheme \( Y \subset \mathbb{P}' \) and the family \( \eta : Y \to T' \).

Before proving the two steps, we have some preparation. Given a morphism \( S \to S' \) and a cartesian diagram of families of stable curves

\[
Z \xrightarrow{h} Z' \\
\alpha \downarrow \quad \downarrow \beta
\]

\[
S \xrightarrow{f} S'
\]

We have the canonical isomorphisms:

\[
\sigma_{h,f} : h^*(L_\beta) \sim L_\alpha,
\]

\[
\tau_{h,f} : f^*(E_\beta) \sim E_\alpha .
\]
Given two composable cartesian squares of families of stable curves

\[
\begin{array}{ccc}
Z & \xrightarrow{h} & Z' \\
\downarrow & & \downarrow \\
S & \xrightarrow{g} & S'
\end{array}
\quad
\begin{array}{ccc}
Z'' & \xrightarrow{k} & Z'' \\
\downarrow & & \downarrow \\
S' & \xrightarrow{f} & S''
\end{array}
\]

one can check the following equalities:

\[
\sigma_{kh,gf} = \sigma_{h,f} h^* (\sigma_{kg}) : (kh)^* (L_\gamma) \xrightarrow{\sim} L_a, \\
\tau_{kh,gf} = \tau_{h,f} f^* (\tau_{kg}) : (gf)^* (E_\gamma) \xrightarrow{\sim} E_a.
\]

(4.1)

We return to the étale cover \( \pi : T \rightarrow T' \) and to the descent datum \( \phi : p_1^*(X) \rightarrow p_2^*(X) \) for the family \( \xi : X \rightarrow T \). We consider the diagram:

\[
\begin{array}{ccc}
X & \xleftarrow{p_1} & p_1^*(X) \\
\downarrow & & \downarrow \\
T & \xleftarrow{p_2} & T \times_{T'} T
\end{array}
\quad
\begin{array}{ccc}
p_2^*(X) & \xrightarrow{\phi} & p_2^*(X) \\
\downarrow & & \downarrow \\
T & \xrightarrow{p_1} & T \times_{T'} T
\end{array}
\]

Using (4.1), we can see that the isomorphism

\[
\phi_\xi = \tau^{-1}_{p_2,p_2} \tau^{-1}_{\phi,\id} \tau_{p_1,p_1} : p_1^* E_\xi \rightarrow p_2^* E_\xi
\]

satisfies the cocycle condition for the étale cover \( \pi : T \rightarrow T' \), therefore defining descent data for the coherent \( \mathcal{O}_T \)-module \( E_\xi \). From the theory of descent on \( \text{QCoh} \), we get a quasicoherent \( \mathcal{O}_{T'} \)-module \( E' \) such that \( E_\xi = \pi^* (E') \). The remaining part is to show that \( E' \) is locally free. We can easily see that by recalling a well-known lemma:

**Lemma 4.3.** Let \( A \rightarrow A' \) be a faithfully flat ring homomorphism. Then an \( A \)-module \( M \) is finitely generated (resp. free) if and only if \( M \otimes_A A' \) is finitely generated (resp. free) \( A' \)-module.

For the second step, as we already mentioned, the descent data for \( \xi : X \rightarrow T \), relative to the étale cover \( \pi : T \rightarrow T' \), determine descent data for the ideal sheaf \( I_X \subset \mathcal{O}_T \) with respect to the étale cover \( q : \mathbb{P} \rightarrow \mathbb{P}' \). By descent theorem in \( \text{QCoh} \), we get an \( \mathcal{O}_{T'} \)-module \( G \) such that \( q^* (G) = I_X \). As \( q \) is étale, and hence faithfully flat, the previous lemma follows that \( G \) is a sheaf of ideals in \( \mathcal{O}_{T'} \). This sheaf of ideals defines a subscheme \( Y \subset \mathbb{P}' \) such that \( X \cong q^* (Y) = Y \times_{\mathbb{P}'} \mathbb{P} \). We also have

\[
X \cong Y \times_{\mathbb{P}'} \mathbb{P} = Y \times_{\mathbb{P}'} T \cong Y \times_{T'} T.
\]

We then get a cartesian square

\[
\begin{array}{ccc}
X & \xrightarrow{\xi} & Y \\
\downarrow & \downarrow & \downarrow \\
T & \xrightarrow{\eta} & T'
\end{array}
\]
Since $\pi$ is étale, $\eta : Y \to T'$ is a family of stable curves, as wanted. \hfill $\square$

We define the fiber products of stacks. Let $\alpha : M \to P$ and $\beta : N \to P$ are morphisms of stacks. Then $M \times_P N$ is the groupoid whose objects are defined by

$$(M \times_P N)(T) = \{(\xi, \eta, \phi) : (\xi, \eta) \in M(T) \times N(T), \phi \in \text{Isom}_T(\alpha(\xi), \beta(\eta))\},$$

for every scheme $T$. A morphism between two object $(\xi, \eta, \phi)$ and $(\xi', \eta', \phi')$ is a pair $(\psi_1, \psi_2)$, where $\psi_1 : \xi \to \xi'$ is a morphism in $M$ and $\psi_2 : \eta \to \eta'$ is a morphism in $N$, with $p_M(\psi_1) = p_N(\psi_2)$ and $\phi' \alpha(\psi_1) = \beta(\psi_1) \phi$. It can be checked that such a groupoid is actually a stack.

5. Moduli space of curve as a Deligne-Mumford stack

As we already seen in example (2.2), we may view scheme $S$ as a stack, by considering the stack associated to the functor of points of $S$. We will talk about morphisms between schemes and stacks. A morphism $f$ from a scheme $S$ to a stack $M$ is equivalent to given an object $\xi \in M(S)$; indeed, $\xi = f(\text{id}_S)$. We say a stack is represented by a scheme if it is isomorphic to a scheme.

Example 5.1. Given a groupoid $M$, a scheme $S$, and a morphism $S \to M$, the groupoid $M \times_M S$ is represented by $S$.

A morphism of stacks $f : M \to N$ is said to be representable if for every scheme $S$ and every morphism $S \to N$, the fibre product $M \times_N S$ is a scheme. The following lemma explains what the representability of diagonal morphism of stack means.

Lemma 5.2. $\bigtriangleup : M \to M \times M$ is representable if and only if every morphism from a scheme to $M$ is.

Let $P$ be a property of morphisms of schemes which is stable under base change. For example, flat, étale, unramified, separated, or of finite type. Then a representable morphism $f : M \to N$ satisfies $P$ if for every morphism $S \to M$, where $S$ is a scheme, the morphism of schemes $M \times_N S \to S$ satisfies $P$.

A Deligne-Mumford stack is a stack $M$ having the following two properties.

(1) The diagonal $\Delta : M \to M \times M$ is representable, quasi-compact, and separated.

(2) There exist a scheme $X$ and an étale surjective morphism $\alpha : X \to M$.

The morphism $\alpha$ is also called an atlas for $M$.

Theorem 5.3. $M_{g,n}$ and $\overline{M}_{g,n}$ are Deligne-Mumford stack.
Proof. Again, we prove the theorem for $\overline{M}_{g,n}$, the proof for $\mathcal{M}_{g,n}$ is similar. Set $\mathcal{M} = \overline{M}_{g,n}$. We first prove the representability of $\Delta : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is straightforward. Let $h : S \to \mathcal{M} \times \mathcal{M}$ be a morphism. It is equivalent to giving two families of stable pointed curves $\xi : X \to S$ and $\eta : Y \to S$ in $\mathcal{M}(S)$. We observe that

$$
(M \times M_{\mathcal{M}} S)(T) = \{(f, a) | f : T \to S, a \in \text{Isom}_T(f^*\xi, f^*\eta)\}
$$

(5.2)

$$
= \{(f, \beta) | f : T \to S, \beta \in \text{Hom}_S(T, \text{Isom}_S(\xi, \eta))\}
$$

$$
= \text{Hom}(T, \text{Isom}_S(\xi, \eta)).
$$

Therefore, $\mathcal{M} \times M_{\mathcal{M}} S$ is represented by $\text{Isom}_S(\xi, \eta)$, which is separated and quasi-compact. Hence we prove condition (1). For the second condition, we observe that, given two morphisms $f : S \to \mathcal{M}$ and $g : T \to \mathcal{M}$, where $S$ and $T$ are schemes, or equivalently given two families of stable curves, $\xi : X \to S$ in $\mathcal{M}(S)$ and $\eta : Y \to T$ in $\mathcal{M}(T)$ in $\mathcal{M}(T)$, we get

$$
S \times M T = \text{Isom}_{S \times T}(p_1^*\xi, p_2^*\eta),
$$

where $p_1 : S \times T \to S$ and $p_2 : S \times T \to T$ are two projections. We recall that there is a smooth variety $X$ which is the disjoint union of a finite number of “slices”, $X_1, \ldots, X_N$, in the Hilbert scheme $H_{g,n}$. Each slice is a smooth affine $(3g - 3 + n)$-dimensional subvariety of $H_{g,n}$ which is transversal to the orbits of $G = \text{PGL}(N)$. We sketch of the construction of $X$. First we introduce the following theorem of Kuranishi family:

**Definition 5.4.** Let $(C; p_1, \ldots, p_n)$ be a $n$-pointed nodal curve. A deformation

$$
\phi \downarrow \sigma_i, i = 1, \ldots, n
$$

$$(B, b_0) \quad \chi : (C; p_1, \ldots, p_n) \to (\phi^{-1}(b_0); \sigma_1(b_0), \ldots, \sigma_n(b_0))
$$

of $(C; p_1, \ldots, p_n)$ is said to be a Kuranishi family for $(C; p_1, \ldots, p_n)$ if it satisfies the following condition:

- For any deformation $\psi : D \to (E, e_0)$ of $(C; p_1, \ldots, p_n)$ and for any sufficiently small connected neighborhood $U$ of $e_0$, there is a unique morphism of deformations of $n$-pointed curves

$$
\begin{array}{ccc}
D_U & \xrightarrow{F} & C \\
\downarrow \phi & & \downarrow \\
(U, e_0) & \xrightarrow{f} & (B, b_0)
\end{array}
$$

**Theorem 5.5.** Let $\nu \geq 3$ be an integer. Let $(C; p_1, \ldots, p_n) \subset \mathbb{P}^r$ be a stable $n$-pointed genus $g$ curve, embedded in $\mathbb{P}^r$, $r = (2\nu - 1)(g - 1) + \nu n - 1$, via the $\nu$-fold log-canonical system. Let $x_0 \in H_{g,n}$ be the corresponding Hilbert point, and let $Aut(C; p_1, \ldots, p_n) = G_{x_0} \subset G = \text{PGL}(r + 1)$ be the stabilizer of $x_0$. Then there is a locally closed $(3g - 3 + n)$-dimensional smooth subscheme $X_0$ of
passing through $x_0$ such that the restriction of $X$ to the universal family over $H_{v,g,n}$ is a Kuranishi family for all of its fibers and hence, in particular, a Kuranishi family for $(C, p_1, \ldots, p_n)$. In addition, one can choose an $X_0$ with the following properties:

(a) $X_0$ is affine;
(b) the family is Kuranishi at every point of $X_0$;
(c) the action of the group $G_{x_0}$ on the central fiber extends to compatible actions on $C$ and $X_0$;
(d) for every $y \in X$, the automorphism group $G_y$ is equal to the stabilizer of $y$ in $G_{x_0}$. In particular, $G_y$ is a subgroup of $G_{x_0}$;
(e) for every $y \in X$, there is a $G_y$-invariant neighborhood $U$ of $y$ in $X_0$, for the analytic topology, such that any isomorphism (of $n$-pointed curves) between fibers over $U$ is induced by an element of $G_y$.

Such an $X_0$ can be obtained as follows. Consider the orbit $O(x_0) \subset H$ of $x_0$ under $G$; this is a smooth subvariety of $H$ of dimension $(r + 1)^2 - 1$ passing through $x_0$. Since the linear subspace $T$ of $\mathbb{P}^M$ tangent to $O(x_0)$ at $x_0$ is obviously $G_{x_0}$-invariant, there is a $G_{x_0}$-invariant linear subspace $L$ of $\mathbb{P}^M$ of the complementary dimension such that $L \cap T = \{x_0\}$. Now $X_0$ is obtained by a Zariski-open neighborhood of $x_0$ in $H \cap L$.

We go back to the construction of $X$. By compactness we can cover $H_{v,g,n}$ with finitely many sets of the type $G \times X_i$, $i = 1, \ldots, N$. We set $X = \bigsqcup_{i=1}^N X_i$.

The restriction to $X$ of the universal family over $H_{v,g,n}$ gives a family of stable curves $\xi : C \to X$ and hence a morphism

$$\alpha \to \mathcal{M}$$

The remaining part is to show that $\alpha$ is étale and surjective. By definition, we must prove that, for every morphism $f$ from a scheme $S$ to $\mathcal{M}$, the induced morphism

$$X \times_{\mathcal{M}} S = \text{Isom}_{X \times S}(p_1^* \xi, p_2^* \eta) \to S$$

is étale and surjective. Let $\eta : \chi \to S$ be the family corresponding to the morphism $f : S \to \mathcal{M}$. Since being étale is a local property and since, locally on $S$, the family $\eta$ is the pullback of the family $\tilde{\xi} : \tilde{C} \to X$, we are reduced to showing that the natural projections

$$X \times_{\mathcal{M}} X = \text{Isom}_{X \times X}(p_1^* \tilde{\xi}, p_2^* \tilde{\xi}) \to X$$

are étale and surjective.

Recall, from property (e) of Theorem (5.5), that every point $y$ in $X$ possesses a $G_y$-invariant neighborhood $U$ such that $\{\gamma \in G | \gamma U \cap U \neq 0\} \subset G_y = \text{Aut}(\tilde{C}_y)$. Let $\alpha : C_U \to U$ be the restriction to $U$ of the family $\tilde{\xi}$ over $X$. The following lemma gives a local description of the two maps $q$ and $q_1$. 
Lemma 5.6. Consider the Kuranishi family $\alpha : C_U \to U$. Let $p_1$ and $p_2$ be the two projections from $U \times U$ to $U$. Consider the natural diagram

$$
\begin{aligned}
\text{Isom}_{U \times U}(p_1^*\alpha, p_2^*\alpha) & \xrightarrow{q_1} U \\
\downarrow q & \\
U \times U & 
\end{aligned}
$$

Let $C = C_{u_0}$ be the central fiber of $\alpha$. Let $H = \text{Aut}(C)$. Then there is an isomorphism $\chi : H \times U \to \text{Isom}_{U \times U}(p_1^*\alpha, p_2^*\alpha)$ such that $q_1\chi(g, u) = u$ and $q_\chi(g, u) = (gu, u)$. In particular, $q_1$ is étale and surjective.

Set $I = \text{Isom}_{U \times U}(p_1^*\alpha, p_2^*\alpha)$. Define $\chi : H \times U \to I$ by setting

$$
\chi(g, u) = \{g^{-1} : C_{gu} \to C_u\}.
$$

Since every isomorphism between two fibers of $\alpha$ is uniquely induced by an element of $H$, the morphism $\chi$ is set-theoretically, a bijection. Set $k = |H|$. We then have a decomposition of $I = I_1 \cup I_2 \cdots \cup I_k$. We also have induced bijective morphisms $\chi : U \to I_i$ having the property that $q_1\chi = \text{id}_U$. But then $\chi$ is unramified. Thus $I_i$ must be smooth and $\chi$ is an isomorphism. This proves the lemma.

The proof of the theorem now follows directly from the lemma.

\[ \square \]

REFERENCES