THE MODULI SPACE OF STABLE CURVES

YOU-CHENG CHOU

We introduce the proof of the groupoid of stable, n-pointed, genus g curves as a Deligne-Mumford stack in three stages. First, as a quotient groupoid, we show that it is isomorphic to the Hilbert scheme of ν-log-canonically embedded, stable, n-pointed genus g curves quotient the projective general linear group. Then we briefly recall the Grothendiecks descent theory for quasi-coherent sheaves and use it to show that the moduli groupoid is a stack and, finally, Deligne-Mumford stack.

1. HILBERT SCHEME

Let $(C; p_1, \ldots, p_n)$ be a stable, n-pointed, genus g curves and let $D = \sum_{i=1}^{n} p_i$. We have the following fact:

Fact 1.1. *$(w_C(D))^\nu$ is very ample if $\nu \geq 3$, where w_C denotes the dualizing sheaf of C.*

We embeds C in $\mathbb{P}^r$ via $(w_C(D))^\nu$, where $r = (2\nu t - 1)(g-1) + \nu n - 1$. Its Hilbert polynomial is

$$p_\nu(t) = (2\nu t - 1)(g-1) + \nu n t.$$

The embedding given by $(w_C(D))^\nu$ is called the ν-log canonical embedding.

Definition 1.2. *We define $H_{\nu,g,n}$ as the Hilbert scheme of stable, n-pointed, genus g curve given by the ν-log canonical embedding.*

Remark 1.3. $H_{\nu,g,n}$ is a smooth locally closed subsheme of the product $\mathrm{Hilb}_{\mathbb{P}^r}^{p_\nu(t)} \times (\mathbb{P}^r)^n$ of dimension $3g - 3 + n + (r+1)^2 - 1$. The natural action of $\mathrm{PGL}(r+1)$ on this product restricts to the action on $H_{\nu,g,n}$.

2. GROUPOID

Let S be a scheme and consider the category Sch/S of schemes over S. From now on, the scheme will implicitly assumed to be of finite type over $\mathbb{C}$.

Definition 2.1. *A category fibered in groupoids over Sch/S or, more simply, a groupoid over S, is a pair $(C_{\mathcal{M}}, p_{\mathcal{M}})$, where $C_{\mathcal{M}}$ is a category, and*

$$p_{\mathcal{M}} : C_{\mathcal{M}} \to Sch/S$$

is a functor safistying the following two conditions:

(A) *Let $f : T \to T'$ be a morphism in Sch/S, and let $\eta \in Ob(C_{\mathcal{M}})$ such that $p_{\mathcal{M}}(\eta) = T'$. Then there exists (not necessary unique) $\xi \in C_{\mathcal{M}}$ and a morphism $\phi : \xi \to \eta$ in $C_{\mathcal{M}}$ with $p_{\mathcal{M}}(\phi) = f$.*

(B) *Every morphism $\phi : \xi \to \eta$ is cartesian in the following sense. Given other arrow $\phi' : \xi' \to \eta$ and a morphism $h : p_{\mathcal{M}}(\xi) \to p_{\mathcal{M}}(\xi')$ such that $p_{\mathcal{M}}(\phi')h = p_{\mathcal{M}}(\phi)$, there exists a unique morphism $\psi : \xi \to \xi'$ such that $p_{\mathcal{M}}(\psi) = h$ and $\phi'\psi = \phi$.*

A morphism $\alpha : \mathcal{M} \to \mathcal{M}'$ of groupoids over Sch/S is a functor $\alpha : C_{\mathcal{M}} \to C_{\mathcal{M}'}$ such that $p_{\mathcal{M}'}\alpha = p_{\mathcal{M}}$. When α is an equivalence of categories, we say that it is an isomorphism of groupoids.

Example 2.2.

(1) Let X be a scheme. We will consider X as a groupoid $X = (C_X, p_X)$, where the objects of C_X are pairs (T, f) with $f : T \to X$ a morphism of schemes. The morphism $\phi : (T, f) \to (T', f')$ are the morphisms $h : T \to T'$ such that $f'h = f$. Finally, the functor p_X is defined by $p_X(T, f) = T$.

(2) Let C be the category in which the objects are the families

$$\begin{array}{c} \mathcal{X} \\ \big\downarrow {\scriptstyle \xi} \\ T \end{array}$$

of smooth (resp. stable, n-pointed) curves of genus g and in which a morphism

$$\phi : \xi' \to \xi$$

bwtween two families $\xi' : \mathcal{X}' \to T'$ and $\xi : \mathcal{X} \to T$ is a cartesian product

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ {\scriptstyle \xi'}\big\downarrow & & \big\downarrow{\scriptstyle \xi} \\ T' & \xrightarrow{f} & T \end{array}$$

The functor p assigns to a family $\xi : \mathcal{X} \to T$ its parameter space T: $p(\xi) = T$. For the morphism, we set $p(\phi) = f$. It is not hard to check that the pair (C, p) is a groupoid.

We denote $\mathcal{M}_{g,n}$ (resp. $\overline{\mathcal{M}}_{g,n}$) the groupoid of smooth (resp. stable), n-pointed, genus g curves.

Definition 2.3 (The category $\mathcal{M}(T)$)**.** *Given a groupoid $\mathcal{M} = (C, p)$, denote by $\mathcal{M}(T)$ the category whose objects are objects $\xi \in C$ with $p(\xi) = T$ and whose morphisms are morphisms ϕ in C with $p(\phi) = id_T$. The condition (B) tells us that a morphism ϕ in C is an isomorphism if and only if $p(\phi)$ is. Hence $\mathcal{M}(T)$ is a groupoid in the sense that all morphisms are isomorphisms. The category $\mathcal{M}(T)$ is called the category of sections of M over T.*

It is important to check whether two groupoids $(C_{\mathcal{M}}, p_{\mathcal{M}})$ and $(C_{\mathcal{M}'}, p_{\mathcal{M}'})$ are isomorphic. As we already mentioned before, there must exist an equivalence of categories $F : C_{\mathcal{M}} \to C_{\mathcal{M}'}$ such that $p_{\mathcal{M}'}F = p_{\mathcal{M}}$. It is well-known that the functor F is equivalence if and only if the following conditions hold:

(i) F is fully faithful in the following sense, for any $\xi, \xi' \in Ob(C_{\mathcal{M}})$, the induced map

$$\mathrm{Hom}_{C_{\mathcal{M}}}(\xi, \xi') \to \mathrm{Hom}_{C_{\mathcal{M}'}}(F(\xi), F(\xi'))$$

is bijective.

(ii) F is essentially surjective, meaning that every $\eta \in Ob(\mathcal{M}')$ is isomorphic to $F(\xi)$ for some $\xi \in \mathcal{M}$.

For groupoids, we have the following lemma.

Lemma 2.4. *A morphism $F : \mathcal{M} \to \mathcal{M}'$ of groupoids over Sch/S is an isomorphism if and only if for every T in Sch/S, the induced functor on fibers $F_T : \mathcal{M}(T) \to \mathcal{M}'(T)$ is an equivalence of categories.*

Suppose that a group scheme G acts on a scheme X. Then one can form the quotient groupoid

$$[X/G] = (P_{G,X}, P),$$

where $P_{G,X}$ is the category whose objects are pairs (π, σ_π), where $\pi : E \to T$ is a principal G-bundle, and $\sigma(\pi) : E \to X$ is a G-equivariant map. A morphism between (π, σ_π) and (π', σ'_π) is a pair of commutative diagrams

$$\begin{array}{ccc} E' & \xrightarrow{\phi} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ T' & \xrightarrow{f} & T \end{array} \qquad \begin{array}{ccc} E' & \xrightarrow{\phi} & E \\ & {\scriptstyle \sigma_{\pi'}}\searrow & \downarrow{\scriptstyle \sigma_\pi} \\ & & X \end{array}$$

where the first one is cartesian. At last, the functor $p : P_{G,X} \to Sch$ is given by $(\pi, \sigma_\pi) \to T$. Hence $P_{G,X}(T)$ is the category of principal G-bundle over T, equipped with a G-equivariant map from their total space to X.

Example 2.5. For $X = \{pt\}$, we have $[\{pt\}/G] = BG$.

Theorem 2.6. *The moduli groupoid $\overline{\mathcal{M}}_{g,n}$ is isomorphic to the quotient groupoid $[H_{\nu,g,n}/PGL(N)]$.*

Proof. We first define a morphism

$$\Phi : \overline{\mathcal{M}}_{g,n} \to [H_{\nu,g,n}/PGL(N)].$$

Let $\xi : \mathcal{X} \to T$ be a family of stable n-pointed genus g curves, which is an object of $\overline{\mathcal{M}}_{g,n}$. $\Phi(\xi)$ must consist of a G-bundle $\pi : E \to T$ and a G-equivariant map $\sigma_\pi : E \to H_{\nu,g,n}$.

First let $\pi : E \to T$ be the principal G-bundle associated to the projective bundle $\mathbb{P}_\xi := \mathbb{P}(\xi_*(w_\xi^\nu(\nu D))) \to T$. Consider the cononically trivialized G-bundle

$$\pi^*\mathbb{P}_\xi \to E$$

and the pull-back family

$$\eta : \mathcal{Z} = \mathcal{X} \times_\pi E \to E$$

There is a canonical isomorphism

$$\mathbb{P}_\eta \cong \pi^*\mathbb{P}_\xi.$$

We can therefore view $\mathcal{Z} \to E$ as a family of ν-log canonically embedded curves via the canonical trivialization of $\mathbb{P}_\eta$. This gives a G-equivariant morphism $\sigma_\pi : E \to H_{\nu,g,n}$. This completes the definition of Φ.

To show Φ is an isomorphism of groupoid, we use Lemma (2.4). We must show that Φ_T is fully faithful and essentially surjective for every scheme T.

For the first part, since any two objects of $\overline{\mathcal{M}}_{g,n}(T)$ are isomorphism, we only need to prove that Φ_T incuces a bijection

$$\mathrm{Hom}_{\overline{\mathcal{M}}_{g,n}(T)}(\xi, \xi) \xrightarrow{\sim} \mathrm{Hom}_{P(T)}(\Phi(\xi), \Phi(\xi)).$$

It is equivalent to the statement that the automorphism of a family of stable, n-pointed, genus g curves $\xi : \mathcal{X} \to T$ and the automorphisms of the projective bundle $\mathbb{P}_\xi^* \to T$ determine each other. Let γ be the automorphism of the family $\xi : \mathcal{X} \to T$. Then γ induces an automorphism of $\mathbb{P}(H^0(w_\xi^\nu(\nu D)))$ and therefore induces the automorphism of $\mathbb{P}_\xi^* \to T$. For the other direction, we observe that any non-trivial element $\phi \in PGL(r+1)$ which leaves a ν-log canonically embedded curve $C \hookrightarrow \mathbb{P}^r$ invariant must act non-trivial on C, since C does not lies in the proper linear subspace, the fixed locus of ϕ. Hence any non-trivial automorphism of $\mathbb{P}_\xi^* \to T$ will induce an non-trivial automorphism of $\xi : \mathcal{X} \to T$.

For the essential surjectivity part, let $(\pi, \sigma_\pi) \in Ob(P)$, so that $\pi : E \to T$ is a principal G-bundle, and $\sigma_\pi : E \to H_{\nu,g,n}$ is a G-equivariant map. Now we consider the universal family $\mathcal{Y} \to H_{\nu,g,n}$ and the following cartesian diagram

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & \mathcal{Y} \\ {\scriptstyle\eta}\downarrow & & \downarrow \\ E & \xrightarrow{\sigma_\pi} & H_{\nu,g,n} \end{array}$$

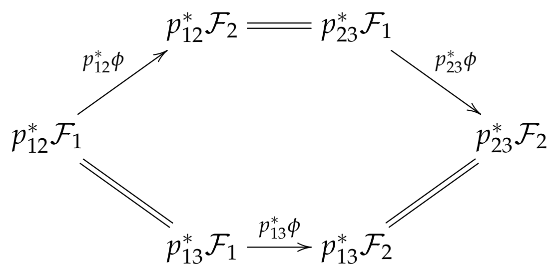
The group G acts equivariantly and freely on E and $\mathcal{Z}$. We can then induce the quotient family

$$\xi : \mathcal{Z}/G = \mathcal{X} \to T = E/G.$$

The last part is to check that $\Phi_T(\xi)$ is isomorphic to (π, σ_π). We left it to the reader. $\square$

3. The theory of descent for quasi-coherent sheaves

Consider a morphism of schemes $X \to Y$, and a quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$. Let p_1 and p_2 be the fist and second projection of $X \times_Y X \to X$, p_{12}, p_{13}, and p_{23} to indicate the projections of $X \times_Y X \times_Y X \to X \times_Y X$ by omitting the third, second, and first components, respectively, and q_1, q_2, q_3 to indicate the three projections of $X \times_Y X \times_Y X \to X$. Notice that $p_1 p_{12} = q_1 = p_1 p_{13}$, $p_2 p_{12} = q_2 = p_1 p_{23}$, and $p_2 p_{13} = q_3 = p_2 p_{23}$. The descent data for $\mathcal{F}$ relative to $X \to Y$ is an isomorphism $\phi : \mathcal{F}_1 \to \mathcal{F}_2$ such that the following diagram commutes:



, where $\mathcal{F}_1 = p_1^*\mathcal{F}$ and $\mathcal{F}_2 = p_2^*\mathcal{F}$ We will call this the cocycle condition.

When $\mathcal{F}$ is the pullback of a quasicoherent $\mathcal{O}_Y$-module, there is a canonical isomorphism between $\mathcal{F}_1$ and $\mathcal{F}_2$ with descent data. We are interested in the convered part, that is, whether a quasicoherent $\mathcal{O}_X$-module with descent data comes from an $\mathcal{O}_Y$-module. Similar problem can be asked on the morphism of quasicoherent $\mathcal{O}_X$-modules. Here we introduce Grothendieck's descent theory:

Theorem 3.1. *Let $\pi : X \to Y$ be a faithfully flat and quasi-compact morphism of schemes. Then the pullback functor*

$$\{ \textit{ quasicoherent } \mathcal{O}_Y\textit{-modules } \} \to \left\{ \begin{matrix} \textit{quasicoherent } \mathcal{O}_X\textit{-modules with} \\ \textit{descent data relative to } \pi \end{matrix} \right\}$$

is an equivalence of categories.

4. Moduli space of curve as a stack

Let $\mathcal{M} = (C, p)$ be a groupoid, T be a scheme, ξ be an object of $\mathcal{M}(U)$, and $f : U \to T$ be an étale surjective morphism. As before, we denote p_{12}, p_{13}, and p_{23} to be the projections of $U \times_T U \times_T U$ to $U \times_T U$ by omitting the third, second, and the first component, respectively and q_1, q_2, q_3 to indicate the three projections of $U \times_T U \times_T U$ to U so that $p_1 p_{12} = q_1 = p_1 p_{13}$, $p_2 p_{12} = q_2 = p_1 p_{23}$, and $p_2 p_{13} = q_3 = p_2 p_{23}$. A descent datum for ξ, relative to $f : U \to T$, is an isomorphism $\phi : p_1^*\xi \to p_2^*\xi$ such that the following diagram commutes:

$$\begin{array}{ccccc} & p_{12}^*p_2^*\xi & = & p_{23}^*p_1^*\xi & \\ {\scriptstyle p_{12}^*\phi}\nearrow & & & & \searrow{\scriptstyle p_{23}^*\phi} \\ p_{12}^*p_1^*\xi & & & & p_{23}^*p_2^*\xi \\ \| & & & & \| \\ & p_{13}^*p_1^*\xi & \xrightarrow{p_{13}^*\phi} & p_{13}^*p_2^*\xi & \end{array}$$

A descent datum for ξ relative to f, is said to be effective if there exist an object $\eta \in \mathcal{T}$ and an isomorphism $\psi : f^*(\eta) \to \xi$ such that

$$\psi = (p_2^*\psi) \circ (p_1^*\psi)^{-1}$$

Now we are ready to define a stack in groupoids for the étale topology or, more simply, a stack. A stack is a groupoid $\mathcal{M} = (C, p)$ having the following two properties.

(1) Every descent datum is effective.
(2) Given a scheme S and objects ξ and η in $\mathcal{M}(S)$, the functor

$$Isom_S(\xi, \eta) : Sch/S \to Sets$$

which associates to a morphism $f : T \to S$ the set of isomorphisms in $\mathcal{M}(T)$ between $f^*(\xi)$ and $f^*(\eta)$ is a sheaf in the étale topology.

A contravariant functor $F : Sch/S \to Sets$ is a sheaf in the étale topology if for every étale surjective morphism $f : U \to T$ of S-schemes, the diagram

$$F(T) \xrightarrow{F(f)} F(U) \underset{F(p_2)}{\overset{F(p_1)}{\rightrightarrows}} F(U \times_T U)$$

is exact. We recall the following theorem due to Grothendieck.

Theorem 4.1. *Let S be a scheme. Let $F : Sch/S \to Sets$ be a contravariant, representable functor. Then F is a sheaf for the étale topology.*

Theorem 4.2. *The groupois $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ are stacks.*

Proof. We consider $\overline{\mathcal{M}}_{g,n}$ case only. The case of $\mathcal{M}_{g,n}$ is similar. We deal with the condition (2) in the definition of stack first. Given two families of stable curves $\xi : X \to S$ and $\eta : Y \to S$, objects in $\overline{\mathcal{M}}_{g,n}(S)$, the functor $Isom_S(\xi, \eta)$ is represented by the scheme $\mathbf{Isom}_S(X, Y)$, where $\mathbf{Isom}_S(X, Y)$ represents the functor which associates to each scheme T over S the set of all isomorphisms, as schemes over T, from $X \times_S T$ to $Y \times_S T$.

The turn to condition (1). Let $T \to T'$ be a surjective étale morphism, and let

$$\xi : X \to T$$

be a family of stable curves with descent data $\phi : p_1^*(\xi) \to p_2^*(\xi)$,

$$\begin{array}{ccc} p_1^*(X) = T \times_{T'} X & \xrightarrow{\quad\phi\quad} & X \times_{T'} T = p_2^*(X) \\ & {\scriptstyle p_1^*(\xi)}\searrow \quad \swarrow{\scriptstyle p_2^*(\xi)} & \\ & T \times_{T'} T & \end{array}$$

To check the effectiveness, we need to produce a family of stable curves $\eta : Y \to T'$ such that $\xi = \pi^*(\eta)$. This construction is a typical descent construction and will be reduced to the theory of descent of quasicoherent sheaves. This reduction has two steps

We first give some notation. Consider the family $\xi : X \to T$. We consider the dual direct image bundle $E_\xi = \xi_*(w_\xi^3(3D))^*$. The total space X of the family ξ can be viewed as embedded in $\mathbb{P}(E_\xi)$:

$$\begin{array}{c} X \subset \mathbb{P}(E_\xi) \\ \big\downarrow{\scriptstyle \xi} \\ T \end{array}$$

Step 1. From the descent data for ξ we deduce descent data for the vector bundle (or better, locally free sheaf) E_ξ. From the theory of descent on $QCoh$, we get a vector bundle E' over T' with $\pi^*(E') = E$.

Step 2. We consider the following diagram

$$\begin{array}{ccc} \mathbb{P} = \mathbb{P}(E_\xi) & \xrightarrow{q} & \mathbb{P}(E') = \mathbb{P}' \\ \big\downarrow & & \big\downarrow \\ T & \xrightarrow{\pi} & T' \end{array}$$

The descent data for $\xi : X \to T$ relative to the étale cover $\pi : T \to T'$ determine descent data for the ideal sheaf $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}}$ with respect to the étale cover $q : \mathbb{P} \to \mathbb{P}'$. Using the theory of descent for $QCoh$, we get the subscheme $Y \subset \mathbb{P}'$ and the family $\eta : Y \to T'$.

Before proving the two steps, we have some preparation. Given a morphism $S \to S'$ and a cartasian diagram of families of stable curves

$$\begin{array}{ccc} Z & \xrightarrow{h} & Z' \\ {\scriptstyle \alpha}\big\downarrow & & \big\downarrow{\scriptstyle \beta} \\ S & \xrightarrow{f} & S' \end{array}$$

We have the canonical isomorphisms:

$$\sigma_{h,f} : h^*(L_\beta) \xrightarrow{\sim} L_\alpha,$$

$$\tau_{h,f} : f^*(E_\beta) \xrightarrow{\sim} E_\alpha \qquad .$$

Given two composable cartesian squares of families of stable curves

$$\begin{array}{ccccc} Z & \xrightarrow{h} & Z' & \xrightarrow{k} & Z'' \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow \\ S & \xrightarrow{f} & S' & \xrightarrow{g} & S'' \end{array}$$

one can check the follwoing equalities:

$$\begin{aligned} \sigma_{kh,gf} &= \sigma_{h,f} h^*(\sigma_{k,g}) : (kh)^* \ (L_\gamma) \xrightarrow{\sim} L_\alpha, \\ \tau_{kh,gf} &= \tau_{h,f} f^*(\tau_{k,g}) : (gf)^* \ (E_\gamma) \xrightarrow{\sim} E_\alpha. \end{aligned} \tag{4.1}$$

We trturn to the étale cover $\pi : T \to T'$ and to the descent datum $\phi : p_1^*(X) \to p_2^*(X)$ for the family $\xi : X \to T$. We consider the diagram:

$$\begin{array}{ccccccc} X & \xleftarrow{p_1} & p_1^*(X) & \xrightarrow{\phi} & p_2^*(X) & \xrightarrow{p_2} & X \\ {\scriptstyle\xi}\downarrow & & {\scriptstyle p_1^*(\xi)}\downarrow & & {\scriptstyle p_2^*(\xi)}\downarrow & & {\scriptstyle\xi}\downarrow \\ T & \xleftarrow{p_1} & T \times_{T'} T & = & T \times_{T'} T & \xrightarrow{p_2} & T \end{array}$$

Using (4.1), we can see that the isomorphism

$$\phi_\xi = \tau^{-1}_{p_2,p_2} \tau^{-1}_{\phi,id} \tau_{p_1,p_1} : p_1^* E_\xi \to p_2^* E_\xi$$

satisfies the cocycle condition for the étale cover $\pi : T \to T'$, therefore defining descent data for the coherent $\mathcal{O}_T$-module E_ξ. From the theory of descent on *QCoh*, we get a quasicoherent $\mathcal{O}_{T'}$-module E' such that $E_\xi = \pi^*(E')$. The remaining part is to show that E' is locally free. We can easily see that by recalling a well-known lemma :

Lemma 4.3. *Let $A \to A'$ be a faithfully flat ring homomorphism. Then an A-module M is finitely generated (resp. free) if and onlly if $M \otimes_A A'$ is finitely generated (resp. free) A'-module.*

For the second step, as we already mentioned, the descent data for $\xi : X \to T$, relative to the étale cover $\pi : T \to T'$, determine descent data for the ideal sheaf $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}}$ with respect to the étale cover $q : \mathbb{P} \to \mathbb{P}'$. By descent theorem in *QCoh*, we get an $\mathcal{O}_{\mathbb{P}'}$-module $\mathcal{G}$ such that $q^*(\mathcal{G}) = \mathcal{I}_X$. As q is étale, and hence faithfully flat, the previous lemma follows that $\mathcal{G}$ is a sheaf of ideals in $\mathcal{O}_{\mathbb{P}'}$. This sheaf of ideals defines a subscheme $Y \subset \mathbb{P}'$ such that $X \cong q^*(Y) = Y \times_{\mathbb{P}'} \mathbb{P}$. We also have

$$X \cong Y \times_{\mathbb{P}'} \mathbb{P} = Y \times_{\mathbb{P}'} \times_{T'} T \cong Y \times_{T'} T.$$

We then get a cartesian square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle\xi}\downarrow & & {\scriptstyle\eta}\downarrow \\ T & \xrightarrow{\pi} & T' \end{array}$$

Since π is étale, $\eta : Y \to T'$ is a family of stable curves, as wanted. □

We define the fiber products of stacks. Let $\alpha : \mathcal{M} \to \mathcal{P}$ and $\beta : \mathcal{N} \to \mathcal{P}$ are morphisms of stacks. Then $\mathcal{M} \times_{\mathcal{P}} \mathcal{N}$ is the groupoid whose objects are defined by

$$(\mathcal{M} \times_{\mathcal{P}} \mathcal{N})(T) = \{(\xi, \eta, \phi) : (\xi, \eta) \in \mathcal{M}(T) \times \mathcal{N}(T), \phi \in \mathrm{Isom}_T(\alpha(\xi), \beta(\eta))\},$$

for every scheme T. A morphism between two object (ξ, η, ϕ) and (ξ', η', ϕ') is a pair (ψ_1, ψ_2), where $\psi_1 : \xi \to \xi'$ is a morphism in $\mathcal{M}$ and $\psi_2 : \eta \to \eta'$ is a morphism in $\mathcal{N}$, with $p_{\mathcal{M}}(\psi_1) = p_{\mathcal{N}}(\psi_2)$ and $\phi'\alpha(\psi_1) = \beta(\psi_1)\phi$. It can be checked that such a groupoid is actually a stack.

5. Moduli space of curve as a Deligne-Mumford stack

As we already seen in example (2.2), we may view scheme S as a stack, by considering the stack associated to the functor of points of S. We will talk about morphisms bwtween schemes and stacks. A morphism f from a scheme S to a stack $\mathcal{M}$ is equivalent to given an object $\xi \in \mathcal{M}(S)$; indeed, $\xi = f(\mathrm{id}_S)$. We say a stack is represented by a scheme if it is isomorphic to a scheme.

Example 5.1. Given a groupoid $\mathcal{M}$, a scheme S, and a morphism $S \to \mathcal{M}$, the groupoid $\mathcal{M} \times_{\mathcal{M}} S$ is represented by S.

A morphism of stacks $f : \mathcal{M} \to \mathcal{N}$ is said to be representable if for every scheme S and every morphism $S \to \mathcal{N}$, the fibre product $\mathcal{M} \times_{\mathcal{N}} S$ is a scheme. The following lemma explains what the representability of diagonal morphism of stack means.

Lemma 5.2.
bigtriangleup $: \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ *is representable if and only if every morphism from a scheme to* $\mathcal{M}$ *is.*

Let **P** be a property of morphisms of schemes which is stable under base change. For example, flat, étale, unramified, separated, or of finite gype. Then a representable morphism $f : \mathcal{M} \to \mathcal{N}$ satisfies **P** if for every morphism $S \to \mathcal{M}$, where S is a scheme, the morphism of schemes $\mathcal{M} \times_{\mathcal{N}} S \to S$ satisfies **P**.

A Deligne-Mumford stack is a stack $\mathcal{M}$ having the following two proberties.

(1) The diagonal $\triangle : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is representable, quasi-compact, and separated.
(2) There exist a scheme X and an étale surjective morphism $\alpha : X \to \mathcal{M}$.

The morphism α is also called an atlas for $\mathcal{M}$.

Theorem 5.3. $\mathcal{M}_{g,n}$ *and* $\overline{\mathcal{M}}_{g,n}$ *ar Deligne-Mumford stack.*

Proof. Again, we prove the theorem for $\overline{\mathcal{M}}_{g,n}$, the proof for $\mathcal{M}_{g,n}$ is similar. Set $\mathcal{M} = \overline{\mathcal{M}}_{g,n}$. We first prove the representability of $\triangle : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is straightforward. Let $h : S \to \mathcal{M} \times \mathcal{M}$ be a morphism. It is equivalent to giving two families of stable pointed curves $\xi : X \to S$ and $\eta : Y \to S$ in $\mathcal{M}(S)$. We observe that

$$
\begin{aligned}
(\mathcal{M} \times_{\mathcal{M}\times\mathcal{M}} S)(T) &= \{(f,\alpha) | f : T \to S, \alpha \in \mathrm{Isom}_T(f^*\xi, f^*\eta)\} \\
&= \{(f,\beta) | f : T \to S, \beta \in \mathrm{Hom}_S(T, \mathbf{Isom}_S(\xi,\eta))\} \qquad (5.2) \\
&= \mathrm{Hom}(T, \mathbf{Isom}_S(\xi,\eta)).
\end{aligned}
$$

Therefore, $\mathcal{M} \times_{\mathcal{M}\times\mathcal{M}} S$ is represented by $\mathbf{Isom}_S(\xi,\eta)$, which is separated and quasi-compact. Hence we prove condition (1). For the second condition, we observe that, given two morphisms $f : S \to \mathcal{M}$ and $g : T \to \mathcal{M}$, where S and T are schemes, or equivalently given two families of stable curves, $\xi : X \to S$ in $\mathcal{M}(S)$ and $\eta : Y \to T$ in $\mathcal{M}(T)$ in $\mathcal{M}(T)$, we get

$$S \times_{\mathcal{M}} T = \mathbf{Isom}_{S\times T}(p_1^*\xi, p_2^*\eta),$$

where $p_1 : S \times T \to S$ and $p_2 : S \times T \to T$ are two projections. We recall that there is a smooth variety X which is the disjoint union of a finite number of "slices", $X_1, \ldots, X_N$, in the Hilbert scheme $H_{\nu,g,n}$. Each slice is a smooth affine $(3g-3+n)$-dimensional subvariety of $H_{\nu,g,n}$ which is transversal to the orbits of $G = PGL(N)$. We sketch of the construction of X. First we introduce the following theorem of Kuranishi family:

Definition 5.4. *Let $(C; p_1, \ldots, p_n)$ be a n-pointed nodal curve. A deformation*

$$
\begin{array}{c}
\mathcal{C} \\
\phi \downarrow \uparrow \sigma_i,\ i=1,\ldots,n \\
(B, b_0)
\end{array}
\qquad \chi : (C; p_1, \ldots, p_n) \tilde{\to} (\phi^{-1}(b_0); \sigma_1(b_0), \ldots, \sigma_n(b_0))
$$

of $(C; p_1, \ldots, p_n)$ is said to be a Kuranishi family for $(C; p_1, \ldots, p_n)$ if it satisfies the following condition:

- *For any deformation $\psi : \mathcal{D} \to (E, e_0)$ of $(C; p_1, \ldots, p_n)$ and for any sufficiently small connected neighborhood U of e_0, there is a unique morphism of deformations of n-pointed curves*

$$
\begin{array}{ccc}
\mathcal{D}_U & \xrightarrow{F} & \mathcal{C} \\
\downarrow & & \downarrow \phi \\
(U, e_0) & \xrightarrow{f} & (B, b_0)
\end{array}
$$

Theorem 5.5. *Let $\nu \geq 3$ be an integer. Let $(C; p_1, \ldots, p_n) \subset \mathbb{P}^r$ be a stable n-pointed genus g curve, embedded in $\mathbb{P}^r$, $r = (2\nu-1)(g-1) + \nu n - 1$, via the ν-fold log-canonical system. Let $x_0 \in H_{\nu,g,n}$ be the corresponding Hilbert point, and let $\mathrm{Aut}(C; p_1, \ldots, p_n) = \mathbb{G}_{x_0} \subset \mathbb{G} = PGL(r+1)$ be the stabilizer of x_0. Then there is a locally closed $(3g-3+n)$-dimensional smooth subscheme X_0 of*

$H_{\nu,g,n}$ passing through x_0 such that the restriction of X to the universal family over $H_{\nu,g,n}$ is a Kuranishi family for all of its fibers and hence, in particular, a Kuranishi family for $(C; p_1, \dots, p_n)$. In addition, one can choose an X_0 with the following properties:

(a) X_0 is affine;
(b) the family is Kuranishi at every point of X_0;
(c) the action of the group $\mathbb{G}_{x_0}$ on the central fiber extends to compatible actions on $\mathcal{C}$ and X_0;
(d) for every $y \in X$, the automorphism group $\mathbb{G}_y$ is equal to the stabilizer of y in $\mathbb{G}_{x_0}$. In particular, $\mathbb{G}_y$ is a subgroup of $\mathbb{G}_{x_0}$;
(e) for every $y \in X$, there is a $\mathbb{G}_y$-invariant neighborhood U of y in X_0, for the analytic topology, such that any isomorphism (of $n-$ pointed curves) between fibers over U is induced by an element of $\mathbb{G}_y$.

Such an X_0 can be obtained as follows. Consider the orbit $O(x_0) \subset H$ of x_0 under $\mathbb{G}$; this is a smooth subvariety of H of dimension $(r+1)^2 - 1$ passing through x_0. Since the linear subspace T of $\mathbb{P}^M$ tangent to $O(x_0)$ at x_0 is obviously $\mathbb{G}_{x_0}$ invariant, there is a $\mathbb{G}_{x_0}$-invariant linear subspace L of $\mathbb{P}^M$ of the complementary dimension such that $L \cap T = \{x_0\}$. Now X_0 is obtained by a Zariski-open neighborhood of x_0 in $H \cap L$.

We go back to the construction of X. By compactness we can cover $H_{\nu,g,n}$ with finitely many sets of the type $G \times X_i$, $i = 1, \dots, N$. We set $X = \coprod_{i=1}^{N} X_i$.

The restriction to X of the universal family over $H_{\nu,g,n}$ gives a family of stable curves $\xi : \mathcal{C} \to X$ and hence a morphism

$$\alpha \to \mathcal{M}$$

The remaining part is to show that α is étale and surjective. By definition, we must prove that, for every morphism f from a scheme S to $\mathcal{M}$, the induced morphism

$$X \times_{\mathcal{M}} S = \mathbf{Isom}_{X \times S}(p_1^* \xi, p_2^* \eta) \to S$$

is étale and surjective. Let $\eta : \chi \to S$ be the family corresponding to the morphism $f : S \to \mathcal{M}$. Since being étale is a local property and since, locally on S, the family η is the pullback of the family $\xi : \mathcal{C} \to X$, we are reduced to showing that the natural projections

$$X \times_{\mathcal{M}} X = \mathbf{Isom}_{X \times X}(p_1^* \xi, p_2^* \xi) \to X$$

are étale and surjective.

Recall, from property (e) of Theorem (5.5), that every point y in X possesses a $\mathbb{G}_y$-invariant neighborhood U such that $\{\gamma \in G | \gamma U \cap U \neq 0\} \subset \mathbb{G}_y = \mathrm{Aut}(\mathcal{C}_y)$. Let $\alpha : \mathcal{C}_U \to U$ be the restriction to U of the family ξ over X. The following lemma gives a local description of the two maps q and q_1.

Lemma 5.6. *Consider the Kuranishi family $\alpha : \mathcal{C}_U \to U$. Let p_1 and p_2 be the two projections from $U \times U$ to U. Consider the natural diagram*

$$\begin{array}{ccc} \boldsymbol{Isom}_{U\times U}(p_1^*\alpha, p_2^*\alpha) & \xrightarrow{q_1} & U \\ \downarrow{\scriptstyle q} & & \\ U \times U & & \end{array}$$

Let $C = C_{u_0}$ be the central fiber of α. Let $H = \mathrm{Aut}(C)$. Then there is an isomorphism $\chi : H \times U \to \boldsymbol{Isom}_{U\times U}(p_1^*\alpha, p_2^*\alpha)$ such that $q_1\chi(g,u) = u$ and $q\chi(g,u) = (gu, u)$. In particular, q_1 is étale and surjective.

Set $\mathbf{I} = \mathbf{Isom}_{U\times U}(p_1^*\alpha, p_2^*\alpha)$. Define $\chi : H \times U \to \mathbf{I}$ by setting

$$\chi(g,u) = \{g^{-1} : C_{gu} \to C_u\}.$$

Since every isomorphism between two fibers of α is uniquely induced by an element of H, the morphism χ is set-theoretically, a bijection. Set $k = |H|$. We then have a decomposition of $\mathbf{I}= I_1 \cup I_2 \cdots \cup I_k$. We also have induced bijective morphisms $\chi : U \to I_i$ having the property that $q_1\chi = \mathrm{id}_U$. But then χ is unramified. Thus I_i must be smooth and χ is an isomorphism. This proves the lemma.

The proof of the theorem now follows directly from the lemma. □

References

[1] Enrico Arbarello, maurizio Cornalba, Pillip A. Griffiths: Geometry of Algebraic Curves, vol. 2, chap. 9-12.