KONTSEVICH'S PROOF OF WITTEN'S CONJECTURE

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0.1. **Combinatorial Model.** Let X be a compact Riemann surface and ρ be a meromorphic quadratic differential on X. Locally, $\rho = \phi(z)(dz)^2$, where $\rho(z)$ is a meromorphic function. In our case, we assume ρ has only simple or double poles.

We define the horizontal line field as

$$\{\nu \in TX \mid \phi(z)(dz(\nu))^2 > 0\}.$$

For a generic quadratic differential, a generic trajectory of the horizontal line field is non closed. However, there exist special quadratic differential such that all horizontal trajectories except for a finite set are closed and we defined it below.

Definition 0.1. *A Jenkins-Strebel quadratic differential is a quadratic differential with all horizontal trajectory except for a finite set are closed.*

By a local analysis, we can see that if z_0 is a d-tuple zero of ρ, then there are $d+2$ horizontal trajectories issuing from z_0. If z_0 is a simple pole, then there is a unique horizontal trajectory issuing from z_0. Finally, if z_0 is a double pole with negative residue, then z_0 is surrounded by closed horizontal trajectories. We further list some properties of Jenkins-Strebel quadratic differential that we will need later.

Proposition 0.2. *1. The connected component of $X\setminus$ $\{$graph of nonclosed horizontal trajectory$\}$ is either open annulus or open disk.*
2. All closed horizontal trajectory in the same connected component have the same length. (We use the metric $dl^2 = |\phi(z)||dz|^2$.)
3. If the length of closed trajectory associated to a double pole is p_i, then ρ can be written as $-(\frac{p_i dz}{2\pi z})^2$ at the neighborhood of the pole.

Theorem 0.3. *(Strebel) Let $2g-2+n > 0$. Then for any $2n+1$-tuples $(X; x_1, \dots, x_n, p_1, \dots, \_n)$, where X is a Riemann surface of finite type, x_i are distinct point of X, and $p_i > 0$. Then there exists a unique Jenkins-Strebel quadratic differential with double pole at x_i and no other poles such that the connected component of $X\setminus$ graph of nonclosed horizontal trajectory are open disks, and the length of closed trajectory associated to the i-th pole is p_i.*

We call the unique Jenkins-Strebel quadratic differential defined above the canonical Jenkins-Strebel quadratic differential.

Conversely, given an embedded graph with each valencies of vertex ≥ 3, face marked by $\{x_1, \dots, x_n\}$, and fixed lengths of its edges. There exists unique complex structure such that its corresponding canonical Jenkins-Strebel differential determines the given embedded graph.

Now, we define $M_{g,n}^{comb} :=$ {the space of genus g connected embedded graphs with n-marked points with all vertices of valencies ≥ 3 and endowed with a metric.}

Theorem 0.4. *$M_{g,n} \times \mathbb{R}_+^n \cong M_{g,n}^{comb}$ as real orbifolds.*

We can further generalize the above discussion to stable curve.

Definition 0.5. *A Jenkins-Strebel differential on the stable curve is a quadratic differential such that*
1. It has double poles at the marked points and at worst simple poles at the nodal points, and no other poles.
2. $\phi \equiv 0$ on the unmarked components.
3. ϕ is the Jenkins-Strebel quadratic differential on the puntured marked components.

$\bar{M}_{g,n}^{comb} :=$ { the space of stable genus g embedded graphs with n-marked points, with vertices of valencies ≥ 3 on smooth points, with at most one valency on nodal points and endowed with metric.}

To determine the relation between $\bar{M}_{g,n}$ and $\bar{M}_{g,n}^{comb}$, we introduce the equivalent relation as follows: Let C be a stable curve with genus g and n marked points. We can canonically decompose C as the union of two curves $C = C^+ \cup C^0$, where C^+ is the union of all the components of C containing marked points, and C^0 is the union of those containing no marked points. Let $\xi_1, \dots, \xi_u$ be the points that C^+ has in common with C^0. We say that $[(C; x_1, \dots, x_n)]$ is equivalent to $[(C', x_1', \dots, x_n')]$ if there is a family of nodal curves $\{C_s^0\}_{s \in S}$ over a connected base S, together with sections of smooth points $\tau_1, \dots, \tau_u$, with the property that $(C; x_1, \dots, x_n)$ (resp.,$(C', x_1', \dots, x_n')$) can be obtained from C^+ and C_s^0 (resp.,$C_{s'}^0$) by identifying ξ_i with $\tau_i(s)$ (resp.,$\tau_i(s')$) for $i = 1, \dots, u$.

It is easy to check that what we just defined is an equivalence relation. We let

$$Q : \bar{M}_{g,n} \to \bar{M}_{g,n}'$$

denote the projection via the equivalence relation. Now we can state the similar identifications for stable curves:

Theorem 0.6. *$H : \bar{M}_{g,n}' \times \mathbb{R}_+^n \to \bar{M}_{g,n}^{comb}$ is an homeomorphism.*

0.2. **Matrix Integral Model.** Let $\Lambda = (\Lambda_i)_{1 \leq i \leq N}$ be a diagonal matrix with positive entries and $H = (h_{ij}) = (x_{ij} + iy_{ij})$ be a Hermitian matrix. We consider the following measure on the space of Hermitian matrices

$$d\mu_\Lambda(H) = C_{\Lambda,N} e^{e^{\frac{1}{2}H^2\Lambda}} \prod_{i=1}^{N} dx_{ii} \prod_{i<j} dx_{ij} dy_{ij},$$

where $C_{\Lambda,N}$ is chosen such that

$$\int_{\mathcal{H}_N} d\mu_\Lambda(H) = 1.$$

By direct computation, we have $C_{\Lambda,N} = (2\pi)^{-\frac{N^2}{2}} \prod_{i=1}^N \Lambda_i^{\frac{1}{2}} \prod_{i<j}(\Lambda_i + \Lambda_j)$.

Now we introduce the Kontsevich Model:

$$\log \int_{\mathcal{H}_N} e^{\frac{i}{6} tr(H^3)} d\mu_\Lambda(H).$$

Before explaining the meaning of this model, we recall some facts about matrix integral.

Let B be a $n \times n$ positive definite symmetric matrix. We consider the integral

$$c \int_{\mathbb{R}^n} e^{-\frac{1}{2}(Bx,x)} \prod_{i=1}^n dx_i,$$

where c is chosen such that the integral equals 1. With this normalization we have

$$< x_i x_j >:= c \int_{\mathbb{R}^n} x_i x_j e^{-\frac{1}{2}}(Bx,x) \prod_{i=1}^n dx_i = (B^{-1})_{ij}.$$

We can further generalized this computation.

Theorem 0.7 (Wick's formula)**.** *Let $f_1, \dots, f_{2k}$ be linear functions of $x_1, \dots, x_n$. Then*

$$< f_1 f_2 \cdots f_{2k} >= \sum_{\substack{p_1 < \cdots < p_k \\ q_1 < \cdots < q_k}} < f_{p_1} f_{q_1} > \cdots < f_{p_k} f_{q_k} > .$$

Go back to our case, in coordinated x_{ii}, x_{ij}, y_{ij}, we can write $tr(H^2\Lambda) = (Bx, x)$, where

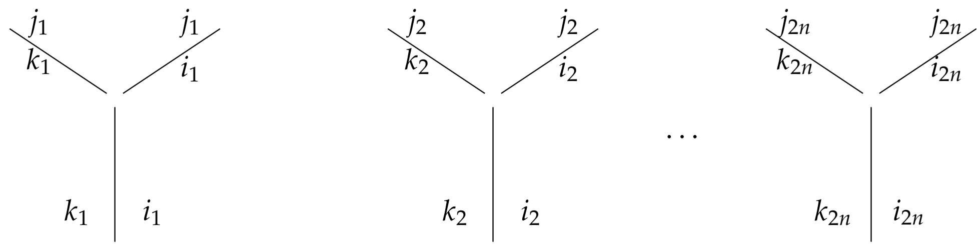
$$B = \begin{pmatrix} \Lambda_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \Lambda_N & & & & & & \\ & & & \Lambda_1 + \Lambda_2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & \Lambda_{N-1} + \Lambda_N & & & \\ & & & & & & \Lambda_1 + \Lambda_2 & & \\ & & & & & & & \ddots & \\ & & & & & & & & \Lambda_{N-1} + \Lambda_N \end{pmatrix}.$$

We compute $< x_{ii}^2 >= \frac{1}{\Lambda_1}$, $< x_{ij}^2 >=< y_{ij}^2 >= \frac{1}{\Lambda_i + \Lambda_j}$. Also, we have $< h_{ij} h_{ji} >= \frac{2}{\Lambda_i + \Lambda_j}$ and $< h_{ij} h_{kl} >= 0$ if $(i,j) \neq (l,k)$.

Now we compute

$$\int_{\mathcal{H}_N} e^{\frac{i}{6} tr(H^3)} d\mu_\Lambda(H) = \int_{\mathcal{H}_N} \Big(1 - \frac{1}{2!} \frac{1}{6^2} (tr(H^3))^2 + \frac{1}{4!} \frac{1}{6^4} (tr(H^3))^4) - \dots \Big) d\mu_\Lambda(H).$$

By Wick's formula, the right hand side can be presented as the monomial of $< h_{i_n j_n} h_{i_m k_m} >$. Notice that each term can correspond to the gluing of 3-stars.



In this case an edge of the gluing corresponds to a pair $< h_{i_n j_n} h_{i_m k_m} >$. We define the weight of an gluing is the product

$$\prod \frac{2}{\Lambda_i + \Lambda_j}$$

taken over all edges of the gluing.

Now the meaning of Kontsevich model,$\int_{\mathcal{H}_N} e^{\frac{i}{6}tr(H^3)} d\mu_\Lambda(H)$, can be express as the sum of the weight of all the gluing of 3-stars. Taking log means the enumeration of connected gluings.

0.3. **Witten's Conjecture.**
We fix some notation first. Let L_i be the i-th point bundle on $\bar{M}_{g,n}$ and $\psi_i := c_1(L_i)$. The intersection number is defined by

$$< \tau_{d_1} \cdots \tau_{d_n} >:= \int_{\bar{M}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n},$$

where $< \tau_{d_1} \cdots \tau_{d_n} >= 0$ if $d_1 + \cdots + d_n \neq 3g - 3 + n$.Finally, the generating series for intersection numbers is the formal power series

$$F(t_0, t_1, \dots) = \sum_{d_1 \geq 0, \dots, d_n \geq 0} \frac{1}{n!} < \tau_{d_1} \cdots \tau_{d_n} > t_{d_1} \cdots t_{d_n}.$$

We also recall the KdV-hierarchy. Let $U(t_0, t_1, \dots)$ be a formal power series. We say U satisfies KdV if and only if

$$\frac{\partial U}{\partial t_i} = \frac{\partial}{\partial t_0} R_i[U],$$

where $R_0 = U$ and $\frac{\partial R_{n+1}}{\partial t_0} = \frac{1}{2n+1}(\frac{\partial U}{\partial t_0} + 2U\frac{\partial}{\partial t_0} + \frac{1}{4}\frac{\partial^3}{\partial t_0^3})R_n$. The formal power series $\tau(t_0, t_1, \dots)$ is a τ-function for KdV-hierarchy if

$$2\frac{\partial^2}{\partial t_0^2} \log \tau(t_0, \dots) = U(t_0, \dots)$$

Theorem 0.8 (Witten's Conjecture). *e^F is the τ-function for KdV with respect to variables $T_{2i+1} = \frac{t_i}{(2i+1)!!}$*

The first step of the proof is to give a combinatorial formula for ψ_i. Let $\bar{\pi} S^1(L_i) \to \bar{M}_{g,n}$. We want to find a closed 2-form w_i on $\bar{M}_{g,n}$ such that $\bar{\pi}^*(w_i) = d\phi$ and $\int_{S^1} \phi|_{\text{fiber}} = 1$.

Fix $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_+$, we have the commutative diagram:

$$\begin{array}{ccc} & \bar{M}_{g,n} & \\ {\scriptstyle Q}\swarrow & & \searrow{\scriptstyle f} \\ \bar{M}'_{g,n} & \xrightarrow[\sim]{h} & \bar{M}^{comb}_{g,n}(r) \end{array}$$

By the way Q was constructed, the line bundle L_i restricts to a trivial line bundle on the fibers of Q and therefore drops to a well-defined line bundle L'_i on $\bar{M}'_{g,n}$ with $Q^*(L'_i) = L_i$. Let L^{comb}_i be the pullback of L'_i via h^{-1}, so that

$$h^*(L^{comb}_i) = L'_i,\ f^*(l^{comb}_i) = L_i.$$

Now our goal becomes giving the combinatorial expression for its first Chern class.

We recall a few facts about piecewise linear forms. Let K be a simplicial complex.

$$\sigma : \triangle_n \to |K|$$

be the n-simplex. A PL-form ϕ of degree ν on $|K|$ is a collection

$$\phi = \{\phi_\sigma\}_{\sigma \in K},$$

where

$$\phi_\sigma = \sum \phi_{i_1 \ldots i_\nu}, 0 \le i_k \le \dim\sigma$$

is a ν-form on the hyperplane $\sum t_i = 1$ in $\mathbb{R}^{\dim\sigma+1}$, having as polynomials in the t_i with rational coefficients, and such that

$$\phi_\sigma|_\tau = \phi_\tau$$

whenever τ is a face of σ. We can then define the complex of PL-forms on K. Its cohomology is denoted by $H^*_{PL}(K)$. We have an important fact that

$$H^*_{PL}(K) \cong H^*(K, \mathbb{Q}).$$

Now we use PL-forms on $\bar{M}^{comb}_{g,n}$ to compute the first Chern class of L^{comb}_i.

Let $|a|/\Gamma_a$ be an orbicell of $\bar{M}^{comb}_{g,n}(r)$, where a corresponds to an embedded graph $(G_a; x_1, \ldots, x_n)$ whose i-th half-perimeter is equal to r_i and $\Gamma_a = Aut((G_a; x_1, \ldots, x_n))$. The coordinates relative to the cell $|a|$ are the lengths

$$\{l_e\}_{e \in E(G_a)}$$

of the edges of G_a. At each point x_i, we consider a cyclically ordered set of oriented edges of G_a

$$(\overrightarrow{e_1}, \ldots, \overrightarrow{e_\nu})$$

with possible repetitions. A repetition happens when the edge in question bounds, on both sides, the same boundary component of G_a. We set

$$(w_i)_a = \sum_{1<s<t\le v-1} d\Big(\frac{l_{e_s}}{r_i}\Big) \wedge \Big(\frac{l_{e_t}}{r_i}\Big)$$

One can check that

$$w_i = \{(w_i)_a\}_{|a|\subset \bar{M}^{comb}_{g,n}(r)}$$

is a PL-form on $\bar{M}^{comb}_{g,n}(r)$.

Lemma 0.9. *For each x_i and $r \in \mathbb{R}^n_+$,*

$$[w_i] = c_1(L_i^{comb}) \in H^2_{PL}(\bar{M}^{comb}_{g,n}(r)).$$

In particular,

$$[f^*(w_i)] = c_i(L_i) \in H^2(\bar{M}_{g,n}, \mathbb{Q}).$$

Now we can rewrite the intersection number

$$< \tau_{d_1}, \ldots, \tau_{d_n} >= \int_{\bar{M}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} = \int_{\bar{M}^{comb}_{g,n}(r)} w_1^{d_1} \cdots w_n^{d_n}$$
$$= \int_{M^{comb}_{g,n}(r)} w_1^{d_1} \cdots w_n^{d_n}$$

The last equality is true since the boundary is measure zero.

Let $\Omega = \sum_{i=1}^n r_i^2 w_i$.

$$\int_{\mathbb{R}^n_{\ge 0}} e^{-\sum \lambda_i r_i} \Big(\int_{M^{comb}_{g,n}(r)} \frac{\Omega^d}{d!} \Big) dr_1 \cdots dr_n$$
$$= \sum_{d_1+\cdots d_n = d} < \tau_{d_1} \cdots \tau_{d_n} > \prod_{i=1}^n \frac{2d_i!}{d_i!} \lambda_i^{-2(d_i+1)} = (1),$$

where Re(λ_i)>0 and $d = 3g-3+n$.

We use the combinatorial theorem due to Kontsevich.

Theorem 0.10. $\frac{\Omega^d}{d!} dr_1 \wedge \cdots \wedge dr_n = 2^{2n+5g-5} dl_{e_1} \wedge \cdots \wedge dl_{e_{6g-6+3n}}$.

We have

$$(1) = \int_{\mathbb{R}^n_{\ge 0}} e^{-\sum \lambda_i r_i} \Big(\int_{M^{comb}_{g,n}(r)} 2^{2n+5g-5} dl_{e_1} \wedge \cdots \wedge dl_{e_{6g-6+3n}} \Big)$$
$$= \sum_{G \in \mathcal{G}^{3,c}_{g,n}} \frac{2^{n+5g-5}}{|AutG|} \int_{|a(G)|} e^{-\sum \lambda_i r_i} dl_{e_1} \wedge \cdots \wedge dl_{e_{6g-6+3n}},$$

where $\mathcal{G}^{3,c}_{g,n}$ is the isomorphism class of connected 3-valent embedded graph with genus g and $n-$marked points. We further do some change of variables.

$$\sum_{i=1}^n \lambda_i r_i = \sum_{e \in E(G)} (\lambda_e + \lambda'_e) l_e,$$

where λ_e and λ'_e are the perimeter of the two faces adjacent to the edge e.

Now we have the relation $(*)$

$$\sum_{\mathcal{G}^{3,c}_{g,n}} \frac{2^{-|V(G)|}}{|Aut(G)|} \prod_{e\in E(G)} \frac{2}{\lambda_e+\lambda'_e} = \sum_{d_1+\cdots+d_n=d} < \tau_{d_1}\cdots\tau_{d_n} > \prod_{i=1}^{n} \frac{(2d_i-1)!!}{\lambda_i^{2d_i+1}}.$$

Theorem 0.11. *Let* $F(t_0,t_1,\cdots) = \sum_{n\geq 0}\sum_{d_1,\ldots,d_n\geq 0} \frac{1}{n!} < \tau_{d_1}\cdots\tau_{d_n} > t_{d_1}\cdots t_{d_n}$. *Set* $\Lambda = diag(\Lambda_1,\ldots\Lambda_N)$ *with* $Re(\Lambda_i)>0$ *and* $t_i(\Lambda) = -(2i-1)!!Tr\Lambda^{-2i+1}$. *Then*

$$F(t_0(\Lambda),t_1(\Lambda),\ldots) = \sum_{G\in\bar{\mathcal{G}}^{3,c,N}} \frac{(\frac{\sqrt{-1}}{2})^{|V(G)|}}{|Aut(G)|} \prod_{e\in E(G)} \frac{2}{\Lambda_e+\Lambda'_e},$$

where $\bar{\mathcal{G}}^{3,c,N}$ *is the isomorphism class of connected* 3*-valent embedded graph with genus* g *and* N*-colored boundary component.*

Notice that this theorem and the relation $(*)$ are similar except for the constant $\frac{1}{n!}$. This is because we sum over the graph with n marked points in relation $(*)$, but we are not in the theorem.

On the other hand, we consider the Kontsevich model:

$$\begin{aligned}\int_{\mathcal{H}_N} e^{\frac{\sqrt{-1}}{6}tr(H^3)} d\mu_\Lambda(H) &= \sum_{i=1}^{\infty} (\frac{\sqrt{-1}}{2})^i \frac{1}{i!3!} \int_{\mathcal{H}_N} (tr(H^3))^i d\mu_\Lambda(H) \\ &= \sum_{G\in\bar{\mathcal{G}}^{3,N}} \frac{(\frac{\sqrt{-1}}{2})^{|V(G)|}}{|AutG|} \prod_{e\in E(G)} \frac{2}{\Lambda_e+\Lambda'_e}\end{aligned}$$

The last equality is the special case of the following proposition.

Proposition 0.12.

$$\begin{aligned}&\int_{\mathcal{H}_N} (trH)^{\alpha_1}\cdots(tr(H^k))^{\alpha_k} d\mu_\Lambda(H) \\ &\qquad = \alpha_1!\cdots\alpha_k! 2^{\alpha_2}\cdots k^{\alpha_k} \sum_{G\in\bar{\mathcal{G}}^{3,N}} \frac{1}{|AutG|} \prod_{e\in E(G)} \frac{2}{\Lambda_e+\Lambda'_e}.\end{aligned}$$

Now we get the important relation between intersection number and Kontsevich model.

$$e^{F(t_0(\Lambda),\cdots)} = < e^{\frac{\sqrt{-1}}{6}tr(H^3)} >_{\Lambda,N}$$

Here we recall some properties of Airy function. We first study its asymptotic behaviour by the stationary phase method. The asymptotic expansion for $A(y)$ as $y\to\infty$ is the sum of terms corresponding to the critical points of the function $\frac{x^3}{3} - xy$.

$$\begin{aligned}A(y) &= \int_{\infty}^{\infty} e^{\sqrt{-1}(\frac{x^3}{3}-xy)}dx \\ &\sim \int_{U(\sqrt{y})} e^{\sqrt{-1}(\frac{x^3}{3}-xy)}dx + \int_{U(\sqrt{-y})} e^{\sqrt{-1}(\frac{x^3}{3}-xy)}dx \\ &\sim \sum_{\pm\sqrt{y}} \text{const}\cdot y^{\frac{-3}{4}} e^{-\frac{2\sqrt{-1}}{3}y^{\frac{3}{2}}} f_1(y^{-\frac{1}{2}}).\end{aligned}$$

Also we have

$$A^{j-1}(y) \sim \sum_{\pm\sqrt{y}} \text{const}\cdot y^{\frac{-3}{4}} e^{-\frac{2\sqrt{-1}}{3}y^{\frac{3}{2}}} f_j(y^{-\frac{1}{2}}),$$

where $f_j(y) = y^{-j}(1 + \circ(1))$.

We can obtain the relation between the Kontsevich's model and matrix Airy function by substitute Λ by $2\sqrt{-1}Y^{1/2}$

$$\begin{aligned}A(Y) &= \int_{\mathcal{H}_N} e^{\sqrt{-1}(\frac{X^3}{3}-XY)}dX \\ &\sim \text{const}\cdot Y_i^{-\frac{1}{4}} \prod_{i<j}(Y_i^{\frac{1}{2}} + Y_j^{\frac{1}{2}})^{-\frac{1}{2}} \sum_{Y^{\frac{1}{2}}} e^{-\frac{2\sqrt{-1}}{3}trY^{3/2}} e^{F(\tilde{t_0}(Y^{1/2}),\dots)} - (1),\end{aligned}$$

where $\tilde{t_i}(Y^{1/2}) = 2^{-(2i+1)/3}(2i-1)!!trY^{-i-1/2}$. We can express matrix Airy function in another form.

Lemma 0.13. *If Φ is a conjugacy invariant function on $\mathcal{H}_N$, then for any diagonal real matrix Y,*

$$\begin{aligned}&\int_{\mathcal{H}_N} \Phi(X)e^{-\sqrt{-1}trXY}dX \\ &= \frac{(-2\pi\sqrt{-1})^{N(N-1)/2}}{detY_i^{j-1}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \Phi(D)e^{-\sqrt{-1}trDY}detD_i^{j-1}dD_1\cdots dD_N\end{aligned}$$

Now we have

$$\begin{aligned}A(Y) &= \frac{(-2\pi\sqrt{-1})^{N(N-1)/2}}{detY_i^{j-1}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \prod_i e^{\sqrt{-1}\frac{D_i^3}{3}-D_iY_i}detD_i^{j-1}dD_1\cdots dD_N \\ &\sim \text{const}\cdot \sum_{Y^{\frac{1}{2}}} e^{-\frac{2\sqrt{-1}}{3}trY^{3/2}} \prod_{i=1}^{N} Y_i^{-\frac{3}{4}} \frac{det f_j(Y_i^{-1/2})}{detY_i^{j-1}} - (2).\end{aligned}$$

We can compare (1) and (2). Then we get

$$e^{F(t_0(\Lambda),\dots)} \sim \frac{det\tilde{f}_j(-2^{(2i+1)/3}\Lambda_i)}{det(-2^{(2i+1/3)\Lambda_i})^{-j}}.$$

We further introduce some properties for the τ-functions of the KdV hierarchy.

We say a Sato subspace in $\mathbb{C}((z))$ is an infinite dimensional vector subspace possessing a basis $f_1, f_2, \dots$ such that $g_j(z) = z^{-j}(1 + \circ(1))$ for all j. We then define the τ-function associated to the Sato subspace W as the fraction

$$\tau_W(T_1, T_2, \dots) = \frac{\cdots \wedge Mg_2 \wedge Mg_1 \wedge z^0 \wedge \dots}{\cdots \wedge z^{-2} \wedge z^{-1} \wedge z^0 \wedge \dots}$$

Lemma 0.14. *Let W be a Sato subspace generated by $f_i = z^i(1 + \circ(1))$. Then for any $N \geq 0$,*

$$\frac{det(f_i(z_j))}{det z_j^{-i}} = \tau_W(T_1(z_*), T_2(z_*), \dots),$$

where $T_k(z_*) := \frac{1}{k}\sum_{i=1}^N z_i^k$.

Lemma 0.15. *Let W be a Sato space such that $z^{-2}W \subset W$. Then we have*
(1) $\tau_W(T_1, T_2, \dots)$ does not depend on T_{2i} for $i > 0$.
(2) $2\frac{\partial^2}{\partial T_1^2} \log \tau_W(T_1, T_3, \dots)$ satisfies KdV hierarchy w.r.t. variables $T_1, T_3, \dots$.

By using the above two lemmas, we can see that $e^{F(t_0(\Lambda),\dots)}$ is a τ-function of KdV hierarchy with respect to variables $T_{2i+1} = \frac{2^{(2i+1)/3} t_i}{(2i+1)!!}$. One can check that it is also a τ-function for KdV hierarchy w.r.t. variables $T_{2i+1} = \frac{t_i}{(2i+1)!!}$ and this proves the Witten's Conjecture.

Remark 0.16. The $F(t_0, t_1, \dots)$ is completely determined by $F(t_0, 0, 0, \dots) = \frac{1}{6}t_0^3$, KdV-hierarchy, string equation and dilaton equation.

References

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