1. Mid term 10/24 (四) (25%, 25%, 30~35%)

2. 小考 ch. 1. HW、15%~20%

No.

Date 9/10 (二)

王金龍

office hour : (四) am: 9:00 ~ 10:00 (4F C32)

Mid term、Final 各 40 % 作業 20 %.

微分幾何 (曲線、曲面)

chapter I.

Def: a parametrized curve is a continuous function

$$\alpha : \underset{I}{\underset{\|}{(a, b)}} \longrightarrow \mathbb{R}^3$$

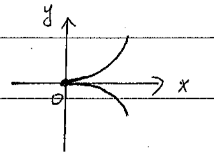
$\alpha(t) = (x(t), y(t), z(t))$

① α is called C^k if $x(t), y(t), z(t) \in C^k$.

($f \in C^k$ if $f', \cdots, f^{(k)}$ all exist and $f^{(k)} \in C^0$)

② α is called differentiable if it is $\underline{\underline{C^\infty}}$

例 ① $\alpha(t) = (t^2, t^3)$

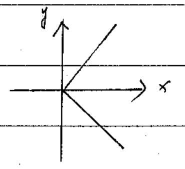


t 很小

⇒ $\alpha'(0) = 0$

⇒ 微分完變零向量

② $\alpha(t) = (|t|, t)$



Let $f(t) = \begin{cases} e^{-\frac{1}{t^2}} & t \neq 0 \\ 0 & t = 0 \end{cases}$

then $f'(0) = f''(0) = \cdots = 0$

$\beta(t) = (|t| f(t), t f(t))$ 圖像和 $\alpha(t)$ 同，只是跑的速度不同

變成 C^∞ curve

~~(or velocity vector)~~

Def : The tangent vector at t is defined to be $\alpha'(t)$
$$\overset{\shortparallel}{(x'(t), y'(t), z'(t))}$$

$\mathbb{T} :=$ unit tangent vector $= \dfrac{\alpha'(t)}{|\alpha'(t)|}$

$\mathbb{T}$ is well-defined at those t s.t. $\alpha'(t) \neq 0$

in ep① $\alpha'(t) = (2t, 3t^2)$ $\begin{cases} t<0, & \nwarrow \\ t>0, & \nearrow \end{cases}$

$\mathbb{T}$ is not continuous at $t=0$

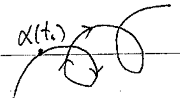
ep② similar

Def : A (parametrized of) C^k curve ($k \geq 1$, $k=\infty$ if not written explicitly)
$\alpha : I \longrightarrow \mathbb{R}^3$ is called <u>regular</u>
if $\alpha'(t) \neq 0$ $\forall\, t \in I$

warming : $\alpha(t) = (t, t, t) \in \mathbb{R}^3$
$\beta(t) = (f(t), f(t), f(t))$, $\beta'(0) = 0$ not regular
參數化後

Def : The arc length of a regular curve
$\alpha : \underset{I}{\underset{\shortparallel}{(a, b)}} \longrightarrow \mathbb{R}^3$ is the function defined by

$$s(t) = \int_{t_0}^{t} |\alpha'(u)|\, du \quad , \quad s'(t) = |\alpha'(t)|$$



$\alpha'(t) \neq 0$ 保証不回跑回來.

Remark : regular $\Rightarrow$ α is <u>locally</u> an $\underset{(1\text{-}1)}{\text{injective}}$ map into $\mathbb{R}^3$

ep $y^2 = x^3 + x^2 = x^2(x+1)$ (想)

Let $y = tx$, $t^2x^2 = x^2(x+1)$

if $x \neq 0$, $t^2 = x+1$

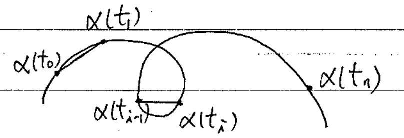
$\Rightarrow \begin{cases} x(t) = t^2 - 1 \\ y(t) = t(t^2-1) \end{cases}$

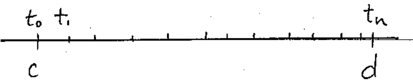
$\alpha(t) = (x(t), y(t)) \subset \mathbb{R}^2$

$\alpha'(t) = (2t, 3t^2-1) \neq 0 \Rightarrow$ regular (C^∞)

but $\alpha(1) = \alpha(-1) = (0, 0)$!

◎Thm : For any C'-curve $\alpha : I \longrightarrow \mathbb{R}^3$ (EXERCISE 1-3 第8題)
$s(t)$ coincide with the following (Apostol ex 7.21)





P : partition of $[c, d]$ into n intervals

$\|P\| = \max \Delta t_i$

$\Delta t_i = t_i - t_{i-1}$, $i = 1, \cdots, n$

$$\sup_{P} \sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})|$$

Remark : A continuous curve α is called "rectifiable" if
$$\sup_{P} \sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})| < \infty$$

ep step 0 : 1

step 1 : $\frac{4}{3}$

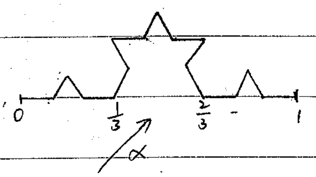
step 2 : $(\frac{4}{3})^2$

step 3 : $(\frac{4}{3})^3$

⋮

step n : $(\frac{4}{3})^n \longrightarrow \infty$

$\overset{\vee}{1}$



$t \in [0,1]$

dragon curve

(Cantor)

α′ does not exist for any t

① for any t, $\lim_{n\to\infty} \alpha_n(t)$ exists

② $\alpha(t) = \lim_{n\to\infty} \alpha_n(t)$ is continuous.

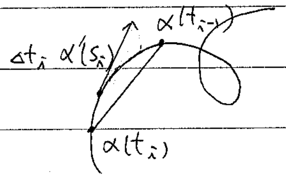
Thm.

<pf> $\left|\int_c^d |\alpha'(t)|\,dt - \sum_{i=1}^{n} |\alpha'(s_i)| \Delta t_i\right| < \frac{\varepsilon}{2}$ $s_i \in [t_{i-1}, t_i]$

is true for any partition with $\|P\| < \delta$

(積分存在的定義)

$\sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})|$



" $\alpha(t_i) - \alpha(t_{i-1}) = \alpha'(u_i)(t_i - t_{i-1})$ " 錯的

for some $u_i \in (t_{i-1}, t_i)$

if true, then " $|\alpha(t_i) - \alpha(t_{i-1})| = |\alpha'(u_i)| \Delta t_i$ " 有可能對 ??

α′ is conti on $\underset{\text{compact}}{[c,d]}$ ⇒ |α′| is unif conti

i.e. $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t.

$|u-s| < \delta \Rightarrow \left||\alpha'(u)| - |\alpha'(s)|\right| < \frac{\varepsilon}{2(d-c)}$

$$\sum_{i=1}^{n} \left||\alpha'(s_i)| - |\alpha'(t_i)|\right| \Delta t_i \leq \sum_{i=1}^{n} \Delta t_i \cdot \frac{\varepsilon}{2(d-c)} = \frac{\varepsilon}{2}$$

$$\Rightarrow \int_c^d \alpha'(t)\,dt - \sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})|\Big| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$ ※.

pick any unit vector $\vec{a} \in \mathbb{R}^3$

$(\alpha(t_i) - \alpha(t_{i-1})) \cdot \vec{a}$

$\alpha(t) \cdot \vec{a} = f(t)$

$f(t_i) - f(t_{i-1}) = f'(s) \Delta t_i$

by MVT, $(\alpha(t_i) - \alpha(t_{i-1})) \cdot \vec{a} = (\alpha'(s) \cdot \vec{a}) \Delta t_i$

may pick $\vec{a} = \frac{\alpha(t_i) - \alpha(t_{i-1})}{|\alpha(t_i) - \alpha(t_{i-1})|}$

($\vec{a}$ 會隨 $(\alpha(t_i) - \alpha(t_{i-1}))$ 改變)

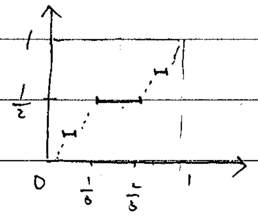
⇔ a.e. C^1

"=" ⇔ f absolutely conti

⇓

($\int_0^1 f' \neq f(1) - f(0)$)

f′ does not exist for an uncountable set (but measure zero)



$f' = 0$ a.e.

$\geq \varepsilon$ ①

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$t \quad n \quad b$

$\mathbb{T} \quad \mathbb{n} \quad \mathbb{b}$

$T \quad N \quad B$

$\underline{(u \times v) \times w = (u \cdot w) v - (v \cdot w) u}$

$F(u, v, w) \overset{?}{=} G(u, v, w)$

$(u \times v) \times w \perp (u \times v)$

$\Rightarrow \underset{\overset{\parallel}{au+bv}}{(u \times v) \times w} \in u, v$ plane.

$0 = a(u \cdot w) + b(v \cdot w) \Rightarrow \frac{a}{b} = -\frac{(v \cdot w)}{(u, w)} = \lambda$ ※

$\alpha : I \longrightarrow \mathbb{R}^3$ regular curve.

arc length s, $s(t) = \int_{t_0}^{t} |\alpha'(u)| du$, $\frac{ds}{dt} = |\alpha'(t)|$

* $\underline{\alpha \text{ is parametrized by arc length} \Leftrightarrow |\alpha'| = 1}$

$$\begin{tikzcd}
 & \mathbb{R}^3 \\
I \arrow[ur, "\alpha"] \arrow[r, shift right, "s"'] & \tilde{I} \arrow[u] \arrow[l, shift right, "s^{-1}"']
\end{tikzcd}$$

$\alpha \circ s^{-1} : \tilde{I} \longrightarrow \mathbb{R}^3$.

Always may assume α is parametrized by arc length.

$\alpha : I \longrightarrow \mathbb{R}^3$.

$\alpha'(s) = T(s)$

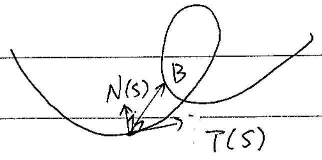
$\alpha''(s) = T'(s)$, define curvature

$\underline{k(s) := |T'(s)| \geq 0}$ 曲率

if $k(s) > 0$,

$\underline{\text{define } \underset{\overset{\uparrow}{\text{unit vector}}}{N(s)} \text{ by } T'(s) = k(s) \cdot N(s)}$

, called the normal of α at s



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observation

$|v(t)| = c \quad \forall t$

$\Rightarrow v(t) \cdot v'(t) = 0$ (解釋 $T \perp N$)

<pf> $v(t) \cdot v(t) = c^2$.

$v'(t) v(t) + v(t) v'(t) = 0$ ※.

define $B(s) := T(s) \times N(s)$ is called the bi-normal vector (of α at s).

The plane spaned by T and N is called the oscalating plane (密切面).

($B(s)$ 是 oscalating plane 的法向量)

To see whether α lies in a plane in $\mathbb{R}^3$, we need to look at $B'(s)$ α 是否落於平面上，可由 $B'(s)$ 知道.

Let $N' = aT + bB$ (because $N' \perp N$)

$a = N' \cdot T = (\cancel{N \cdot T})' - N \cdot T'$

$= -N \cdot kN = \underline{-k}$.

$b = N' \cdot B = (\cancel{N \cdot B})' - N \cdot \underline{B'}$

$= -\tau N \cdot N = -\tau$

$$\left\{ \begin{array}{l} B' = cT + \tau N \\ c = B' \cdot T = (B \cdot T)' - B \cdot T' \\ \quad = 0 - B \cdot kN = 0 \end{array} \right.$$

Def: The torsion 繞率 of α at s is defined by $\underline{B'(s) = \tau(s) \cdot N(s)}$

$$\boxed{\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}}$$

$\leftarrow$ Frenet formula.

T.N.B Serret-Frenet frame.

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$$\begin{cases} k(s) = |\alpha''(s)|^{\text{by def}} \\ \tau(s) = -\dfrac{(\alpha' \times \alpha'') \cdot \alpha'''}{k^2} = -\dfrac{|\alpha', \alpha'', \alpha'''|}{k^2} \end{cases}$$ 行列式

(缺点：參數需取 arc length)

$$\boxed{\begin{array}{l} \text{for general parameter :} \\ \begin{cases} k(t) = \dfrac{|\alpha' \times \alpha''|}{|\alpha'|^3} \\ \tau(t) = -\dfrac{|\alpha', \alpha'', \alpha'''|}{|\alpha' \times \alpha''|^2} \end{cases} \end{array}}$$

習題

$$\tau = B' \cdot N = (B \cdot N)' - B \cdot N'$$

<pf> $\tau = B' \cdot N = - B \cdot N'$ $\qquad N = \frac{T'}{k}$

$$= - B\left(\frac{T'}{k}\right)'$$

$$= - \frac{B \cdot T'' k - \cancel{B \cdot \overset{kN}{\underset{\shortparallel}{T'}} \cdot k'}}{k^2}$$

$$= - k \frac{B \cdot T''}{k^2} \qquad \because \; T = \alpha' \Rightarrow T'' = \alpha'''$$

$$B = T \times N = \left(\alpha' \times \frac{T'}{k}\right) = \left(\alpha' \times \frac{\alpha''}{k}\right)$$

$$= - \frac{(\alpha' \times \alpha'') \cdot \alpha''}{k^2}$$

~~本~~課習題証明是錯的.

Ex §1.3 — 4. 5. 6. 7. ⑧ 9.

§1.4 — 12.

§1.5 — 1. 5. 8. 9. 10. 12. 14. 16.

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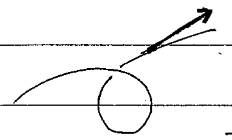
$\alpha : (a, b) \longrightarrow \mathbb{R}^3$ regular curve $\alpha'(t) \neq 0$



$$T(t) = \frac{\alpha'(t)}{|\alpha'(t)|}$$

$$\underset{\substack{\uparrow \\ t = A(u)}}{T'(t)} = \boxed{k(t)} \underset{\substack{\uparrow \\ \text{unit vector}}}{N(t)}$$

多一項

$$◎ \quad \frac{d}{du} T(A(u)) = \frac{dT}{dt}(A(u)) \cdot \frac{dA}{du} = \boxed{k(\underset{\substack{\shortparallel \\ t}}{\underline{A(u)}}) \, N(\underset{\substack{\shortparallel \\ t}}{\underline{A(u)}}) \cdot A'(u)}$$

So we always use arc length $s := \int_{t_0}^{t} |\alpha'(w)| dw$ as the parameter

In all the definitions of k, τ

need to require that $k \neq 0$ (hence may choose N such that $k > 0$)

$T, N \Rightarrow B = T \times N$

Serret – Frenet formula

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

◎Fundamental theorem of curves.

a curve α (parametrized by arc length) is uniquely determined by $k(s)$, $\tau(s)$ and its initial data $\alpha(0)$, $\alpha'(0)$ $\alpha''(0)$

出發点 初始速度

(常微分方程唯一存在定理)

Thm Existence & Uniqueness Theorem in O.D.E.

consider the following O.D.E for $\mathbb{X}(t) \in \mathbb{R}^n$

$(*)$ $\mathbb{X}'(t) = A(t)\,\mathbb{X}(t)$; $A : [a,b] \longrightarrow M_{n\times n}(\mathbb{R})$: continuous , $|A| \le M$

(linear system)

Picard Method if $(*)$ is true , then $\mathbb{X}(t) - \mathbb{X}(a) = \int_a^t A(u)\,\mathbb{X}(u)\,du$

given any $y(t)$, $(Ty)(t) = \int_a^t A(u)\,y(u)\,du + y(a)$, $y \mapsto Ty$

start with $\mathbb{X}_0(t)$ = initial constant vector $\mathbb{X}(a)$

$$\mathbb{X}_1 = T\mathbb{X}_0$$
$$\mathbb{X}_2 = T\mathbb{X}_1$$
$$\vdots$$
$$\mathbb{X}_{n+1} = T\mathbb{X}_n$$

有極限.

$$\mathbb{X}_{n+1}(t) - \mathbb{X}_n(t) = T\mathbb{X}_n(t) - T\mathbb{X}_{n-1}(t)$$
$$= \int_a^t A(u)(\mathbb{X}_n(u) - \mathbb{X}_{n-1}(u))\,du$$

$$\|\mathbb{X}_{n+1}(t) - \mathbb{X}_n(t)\| \le \int_a^t \|A(u)\|\|\mathbb{X}_n(u) - \mathbb{X}_{n-1}(u)\|\,du$$

$\|A(u)\|$ 是什麼?

$$\le M \cdot \sup_{[a,b]} \|\mathbb{X}_n(u) - \mathbb{X}_{n-1}(u)\|\,(b-a)$$

if $(b-a)\cdot M < \frac{1}{2}$, then $\sup\|\mathbb{X}_{n+1}(t) - \mathbb{X}_n(t)\| < \frac{1}{2}\sup\|\mathbb{X}_n(u) - \mathbb{X}_{n-1}(u)\|$ (☆)

$C^0([a,b],\mathbb{R}^n)$ with metric $\|f-g\| = \sup_{t=[a,b]}\|f(t)-g(t)\|$ is a complete metrix space.

↑ compact

☆☆ $\|\mathbb{X}_{n+\ell} - \mathbb{X}_n\| \le \|\mathbb{X}_{n+\ell} - \mathbb{X}_{n+\ell-1}\| + \cdots + \|\mathbb{X}_{n+1} - \mathbb{X}_n\|$

☆ $\Rightarrow \|\mathbb{X}_{n+1} - \mathbb{X}_n\| < (\frac{1}{2})^n\|\mathbb{X}_1 - \mathbb{X}_0\| \le (\frac{1}{2})^{n+1}|\mathbb{X}(a)|$

$\Rightarrow$ ☆☆ $\le [(\frac{1}{2})^{n+\ell} + (\frac{1}{2})^{n+\ell-1} + \cdots + (\frac{1}{2})^{n+1}]|\mathbb{X}(a)| \le (\frac{1}{2})^n|\mathbb{X}(a)|$

$\Rightarrow \lim_{n\to\infty}\mathbb{X}_n$ exists as a continuous function $\mathbb{X}(t)$

$$\mathbb{X}_{n+1}(t) = \int_a^t A(u)\,\mathbb{X}_n(u)\,du + \mathbb{X}(a)$$

take $\lim_{n\to\infty}$ $\Rightarrow$ $\mathbb{X}(t) = \int_a^t A(u)\,\mathbb{X}(u)\,du + \mathbb{X}(a)$

$\Rightarrow \mathbb{X}(a)$ is C^1 and $\mathbb{X}'(t) = A(t)\,\mathbb{X}(t)$, $\mathbb{X}(a) = \mathbb{X}(a)$ 存在性.

uniqueness & continuity

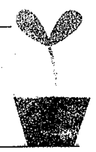
A metric space $(\overset{\text{space}}{S}, \overset{\text{metric}}{d})$ is called complete if every Cauchy seq. in S converges in S.

$$\|\mathbb{X}_1(t) - \mathbb{X}_0(t)\| = \|\int_a^t A(u)\,\mathbb{X}_0(u)\,du\| \le M\int_a^t du\,\|\mathbb{X}(a)\| \le M(b-a)\|\mathbb{X}(a)\|$$
$$\le \frac{1}{2}\|\mathbb{X}(a)\|$$

Apply the same method, one can show that for

$\mathbb{X}'(t) = F(\mathbb{X},t)$, F : conti , Lipchitz in $\mathbb{X}$

$$\|F(\mathbb{X},t) - F(\mathbb{Y},t)\| \le M\cdot(\mathbb{X}-\mathbb{Y})$$



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O.D.E : $\mathbb{X}(t) : I \longrightarrow \mathbb{R}^n$

$F(\mathbb{X}, t) : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$

$$\begin{cases} \mathbb{X}'(t) = F(\mathbb{X}, t) \\ \mathbb{X}(0) \text{ is given.} \end{cases}$$

The case we use : $\mathbb{X}'(t) = \underset{\underset{M_{n\times n}(\mathbb{R})}{\uparrow}}{A(t)} \cdot \mathbb{X}(t)$

Has seen the existence . Now we prove the uniqueness.

If $\mathbb{X}' = A\mathbb{X}$ $\quad \mathbb{X}(0)$

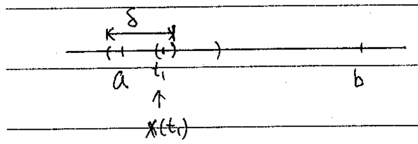
$\mathbb{Y}' = A\mathbb{Y}$ $\quad \overset{\shortparallel}{\mathbb{Y}(0)}$

$\Rightarrow (\mathbb{X} - \mathbb{Y})' = A(\mathbb{X} - \mathbb{Y})$

$(\mathbb{X}(t) - \mathbb{Y}(t)) = \int_0^t A(u)\,(\mathbb{X}(u) - \mathbb{Y}(u))\,du.$

$\leq \underbrace{t \cdot \sup_{u\in[0,t]} \|A(u)\|}_{\leq \frac{1}{2}} \cdot \sup_{u\in[0,t]} \|\mathbb{X}(u) - \mathbb{Y}(u)\|$

take t st.
the LHS = sup. $\qquad \Rightarrow \; \text{-}\!\times\!\text{-}$.



$$\Rightarrow \begin{cases} \mathbb{X}'(t) = A(t)\mathbb{X}(t) \\ \mathbb{X}(0) \text{ is given} \end{cases}$$ 存在唯一解.

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“0” 這是不行的 (?)

Thm Given 2 continuous function $k(s), \tau(s) : \underset{\underset{(a,b)}{\shortparallel}}{I} \longrightarrow \mathbb{R}^3$, there exists an unique curve $\alpha : I \longrightarrow \mathbb{R}^3$ parametrized by arc length st. $k(s)$ = curvature of α at s

$\tau(s)$ = torsion of α at s $\quad$ with $\alpha(0), \overset{\alpha''(0)}{\alpha'(0)}$ the given initial data.

<pf> $$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

$T(0) = \alpha'(0)$ is given

by 常微分方程唯一存在定理

$$\boxed{\mathbb{X} = \begin{pmatrix} T \\ N \\ B \end{pmatrix} : I \longrightarrow \mathbb{R}^9 \Rightarrow \begin{array}{l} \exists!\ \text{solution} \\ \text{if } T(0), N(0), S(0) \text{ are given} \end{array}}$$

claim : $T(s), N(s), B(s)$ are still orthonormal $\forall s$

(1° $|T(s)| = |N(s)| = |B(s)| = 1$
2° 兩兩內積均為零)

$|T|^2, |N|^2, |B|^2$ $\qquad$ with initial data at $s=0$

$\langle T,N\rangle, \langle N,B\rangle, \langle T,B\rangle$ $\qquad (1,1,1,0,0,0)$

① $(|T|^2)' = (\langle T,T\rangle)' = 2\langle T', T\rangle = 2\langle kN, T\rangle = 2k\langle N, T\rangle$

② $(|N|^2)' = (\langle N,N\rangle)' = 2\langle N', N\rangle = 2\langle -kT - \tau B, N\rangle$

$= -2k\langle T,N\rangle - 2\tau\langle B,N\rangle$

③ $(|B|^2)' = (\langle B,B\rangle)' = 2\langle B', B\rangle = 2\tau\langle N,B\rangle$

④ $(\langle T,N\rangle)' = \langle T',N\rangle + \langle T, N'\rangle$

$= k|N|^2 - k|T|^2 - \tau\langle T,B\rangle$

⑤ $\langle T,B\rangle' = \langle T',B\rangle + \langle T,B'\rangle = k\langle N,B\rangle + \tau\langle T,N\rangle$

⑥ $\langle N,B\rangle' = \langle N',B\rangle + \langle N,B'\rangle$

$= -k\langle T,B\rangle - \tau|B|^2 + \tau|N|^2$

Apply the ODE to $\mathbb{V} = \begin{pmatrix} |T|^2 \\ |N|^2 \\ |B|^2 \\ \langle T,N\rangle \\ \langle T,B\rangle \\ \langle N,B\rangle \end{pmatrix}$, $\mathbb{V}' = S\mathbb{V}$, $\mathbb{V}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \underset{\underset{解}{|}}{\mathbb{V}(t)}$

驗証此常数解
是一個解
神奇!!

$\mathbb{V} : I \longrightarrow \mathbb{R}^6.$

$\mathbb{V}(0)$ 代入 ①~⑥ ，果然為解

$\therefore \mathbb{V}(0)$ 就是此 ODE 之解

$$\alpha(0),\ \underset{T(0)}{\underset{\parallel}{\alpha'(0)}},\ \underset{T'(0)=\underset{0}{\underset{\nparallel}{\underline{k(0)}}}N(0)}{\underset{\parallel}{\alpha''(0)}} \qquad \underline{k(s)\neq 0}$$

保證有

$T(0),\ N(0),\ B(0) = T(0)\times N(0)$

Want $\alpha'(s) = T(s)$.

so that define $\alpha(s) = \int_0^s T(u)\,du + \alpha(0)$ (verify 曲、繞率為給定的)

Use the notation $T_\alpha, N_\alpha, B_\alpha, k_\alpha, \tau_\alpha$ to denote those quantities defined by α

By definition of α : $\alpha'(s) = T(s)$

$|T(s)| = 1 \Rightarrow s$ is arc length parameter

$\Rightarrow T_\alpha = T$ ※

$$T_\alpha' = k_\alpha N_\alpha$$
$$\overset{\parallel}{T}' = kN, \quad k(s)\neq 0 \Rightarrow k_\alpha(s)\neq 0 \ \&\ k_\alpha(s) = k(s) \text{ ※}$$

$(\because |N_\alpha| = 1 = |N|)$

$N_\alpha(s) = N(s)$ ※

$B_\alpha = T_\alpha \times N_\alpha = T\times N = B$ ※

$$B' = \tau N \qquad N = N_\alpha \Rightarrow \tau_\alpha = \tau \text{ ※}$$
$$\overset{\parallel}{B_\alpha'} = \tau_\alpha N_\alpha$$

for uniqueness : 課本証法

if α, β both satisfy the condition, then consider (p.20)

$f(s) = |T_\alpha - T_\beta|^2 + |N_\alpha - N_\beta|^2 + |B_\alpha - B_\beta|^2$, $f(0) = 0$ $\Rightarrow f'(s) = 0$

$\underline{T_\alpha = T_\beta}$, $N_\alpha = N_\beta$, $B_\alpha = B_\beta$ $\forall s$

$$\alpha(s) = \int_0^s T_\alpha(u)\,du + \alpha(0)$$
$$\beta(s) = \int_0^s T_\beta(u)\,d\overset{\parallel}{u} + \beta(0) \quad \sim \text{Uniqueness}$$

◎ $\underline{\text{Ep}}$ $\alpha : I \longrightarrow \mathbb{R}^3$: regular curve at least C^3

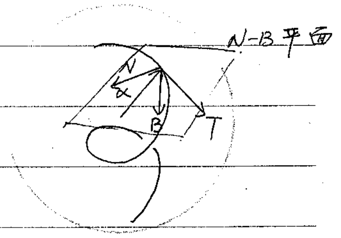
then $\operatorname{Im}\alpha \subset S^2(r) \Longleftrightarrow R^2 + (R')^2 S^2 = C$ $\begin{cases} R = \frac{1}{k} \\ S = \frac{1}{\tau} \end{cases}$

$\uparrow$ some constant r $\qquad$ $\uparrow$ some constant c $= r^2$ 半徑平方.

曲線 α 落在半徑 r 的球面上. (有用於研究地表活動)

<pf> $(\Rightarrow)$ $|\alpha|^2 = r^2 = C$

$0 = \underset{T}{\overset{\parallel}{\alpha'}}\cdot\alpha$, always assume α is parametrized by arc length s



$$\Rightarrow \alpha = fN + gB$$
$$f = \alpha\cdot N = \alpha\cdot\frac{T'}{k} = \frac{1}{k}(-\alpha', T)$$
$$= -\frac{1}{k} = -R$$
$$g = \alpha\cdot B = -\alpha\cdot\frac{N' + kT}{\tau}$$
$$= -S(\alpha, N') = -S\big(\underset{-R}{\overset{\parallel}{(\alpha, N)}}{}' - \cancel{(\underset{T}{\overset{\parallel}{\alpha'}}, N)}\big) = SR'$$
$$\Rightarrow \alpha = -R\vec{N} + SR'\vec{B}$$
$$|\alpha|^2 = R^2 + (SR')^2 = r^2 = C$$

$(\Leftarrow)$ $R^2 + \overset{(R'S)^2}{\overset{\parallel}{(R')^2 s^2}} = c$

consider $\beta(s) := \alpha(s) - (-R(s)\vec{N}(s) + s(s)\underline{R'(s)}\,\vec{B}(s))$

$= \alpha + RN - R'SB$

$\beta'(s) = \cancel{T} + \cancel{R'N} + R(\cancel{-kT} - \tau B) + (R'S)'B - R'\underset{1}{\underset{\parallel}{\cancel{S\tau}}}N$

$= -[R\tau + (R'S)']B = 0$

$\Uparrow$

Differentiate, get $2RR' + 2(R'S)'R'S$

$= 2R'(R + \frac{1}{\tau}(R'S)') = 0$

$\Rightarrow \beta(s) = \overset{p_0}{\cancel{\beta(0)}}$

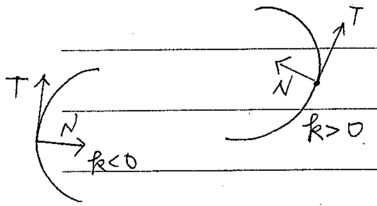
$\Rightarrow \alpha(s) - \underset{p_0}{\cancel{\alpha(0)}} = -RN + (R'S)B$ ※

$|\alpha(s) - p_0|^2 = R^2 + (R')^2 s^2 = r^2$.

Special Case : plane curves

$\alpha : I \longrightarrow \mathbb{R}^2$ parametrized by arc length.

$\alpha' = T$



$1 = \langle T, T \rangle$

$0 = \langle T', T \rangle$, $T' \perp T$, $T' = kN$, k is automatically defined by this equation.

$N' = \lambda T$

$\lambda = N' \cdot T = -N \cdot T' = -k(N, N) = -k$

$$\begin{pmatrix} T \\ N \end{pmatrix}' = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix} \begin{pmatrix} T \\ N \end{pmatrix}$$

In general, $\alpha : I \longrightarrow \mathbb{R}^n$ regular curve s.t. $\alpha'(s), \alpha''(s), \cdots, \alpha^{(n-1)}(s) \neq 0$ may uniquely define principle directions $T_1, T_2, \cdots, T_n$ s.t.

$$\begin{pmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{pmatrix}' = \begin{pmatrix} 0 & k_1 & & 0 \\ -k_1 & 0 & k_2 & \\ & -k_2 & 0 & \ddots \\ & & \ddots & k_{n-1} \\ 0 & & -k_{n-1} & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix}$$

---

§1-6

$y = x$, $y = x^2$, $y = x^3$, $y^2 = x^3$

Ep $\alpha(s) = \alpha(0) + \underset{T(0)}{\underset{\parallel}{\alpha'(0)}}s + \underset{\frac{T'(0)}{2} = \frac{k}{2}N(0)}{\underset{\parallel}{\frac{\alpha''(0)}{2}}}s^2 + \underset{\frac{T''(0)}{6} = \frac{1}{6}(kN)'(0)}{\underset{\parallel}{\frac{\alpha'''(0)}{6}}}s^3 + R$, $\lim_{s \to \infty} \frac{R}{s^3} = 0$

$\frac{1}{6}(kN)'(0) = \frac{1}{6}[k'N + k(-kT - \tau B)](0)$

Let $\alpha(0) = 0$

$T(0) = \vec{e}_1$

$N(0) = \vec{e}_2$

$B(0) = \vec{e}_3$

$\alpha(s) = \underbrace{(s - \frac{s^3}{6}k^2)}_{x(s)} T(0) + \underbrace{(\frac{k}{2}s^2 + \frac{1}{6}k's^3)}_{y(s)} N(0) - \underbrace{\frac{k\tau}{6}s^3}_{z(s)} B(0) + R$

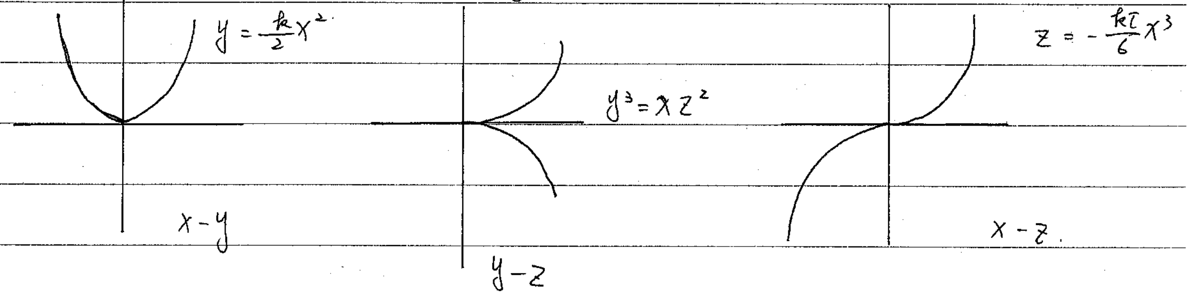
local canonical form

when s small ~ 0

$x \sim s$

$y \sim \frac{k}{2}s^2$, $(k > 0)$

$z \sim -\frac{k\tau}{6}s^3$

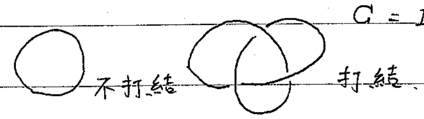


Ex §1.6 — 2.3.

No.

Date 9/4 (三)

$\alpha : [0, l] \to \mathbb{R}^3$: closed curve : $\alpha(0) = \alpha(l)$



simple curve : $\alpha(t_1) = \alpha(t_2) \Rightarrow t_1 \neq t_2$ or $t_1 = a,\ t_2 = b$
(injective)

Jordan curve := simple closed curve

We will study only regular Jordan curves.

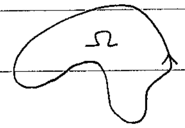
① ~~$\int_C k\,ds \geq 2\pi$, "=" $\Leftrightarrow$ C lies in a plane.~~

② If C does not lie in a plane, then $\int_C k\,ds \geq 4\pi$.

③ If C is knotted, then $\int_C k\,ds > 4\pi$
打結

Galobal :

Today : plane Jordan curves : $\alpha : [0, l] \to \mathbb{R}^2$ parametrized by arc length



Jordan curve thm :
$\mathbb{R}^2 \setminus \alpha([0,l])$ has exactly 2 connected components, one is bounded (interior), one is unbounded, with $\alpha([0,l])$ the common boundary. Moreover, the bounded comp is simply connected

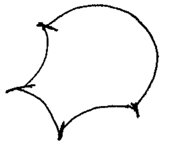
(目前不証)

A = area of Ω

l = total length of C = Im(α)

Thm : Isoperimetric ineq :
1872 (Weierstrass)
$l^2 \geq 4\pi A$, "=" $\Leftrightarrow$ C is a circle (Schmidt 1939)

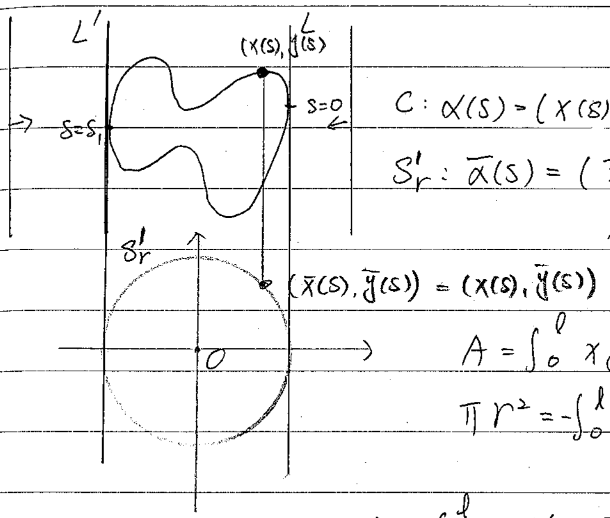
Rm For isoperimethic ineq.
we only need α to be
piecewise C'



<pf> Schmidt 先生.

$A = \int_a^b x(t)\, y'(t)\, dt = -\int_a^b x'(t)\, y(t)\, dt$ for any parametrization.

(Greens thm
$\int_{\Omega} P\,dx + Q\,dy = \int_{\Omega} (Q_x - P_y)\, dx\, dy$)



$C : \alpha(s) = (x(s), y(s)) : [0, l] \to \mathbb{R}^2$ s: arc length

$S_r^1 : \bar{\alpha}(s) = (\underset{x(s)}{\underset{\shortparallel}{\bar{x}(s)}}, \bar{y}(s)) : [0, l] \to \mathbb{R}^2$

s is not arc length for S_r^1

$A = \int_0^l x y'\, ds$

$\pi r^2 = -\int_0^l x' \bar{y}\, ds$

$A + \pi r^2 = \int_0^l (x y' - \bar{y} x')\, ds$

$\boxed{\leq} \int_0^l \underbrace{\sqrt{x^2 + \bar{y}^2}}_{\shortparallel \atop r} \underbrace{\sqrt{(y')^2 + (-x')^2}}_{\shortparallel \atop 1}\, ds$ Apply Cauchy ineq.

∵ parametrized by arc length

$= l r$

$\frac{1}{2} l r \geq \frac{1}{2}(A + \pi r^2) \boxed{\geq} \sqrt{A \cdot \pi r^2}$

i.e. $l^2 \geq 4\pi A$ 尚未 chech "=" 何時成立.

"=" holds $\Leftrightarrow \begin{cases} \dfrac{x}{y'} = \dfrac{\bar{y}}{-x'} \boxed{= \pm \dfrac{\sqrt{x^2 + \bar{y}^2}}{\sqrt{(-x')^2 + y'^2}}} = \pm r \\ A = \pi r^2 \Rightarrow l r = 2\pi r^2 \Rightarrow l = 2\pi r \end{cases}$

重要

◎ r is independent of the choose of L

- $x = \pm r y'$ $\Rightarrow$ ~~$x^2 + y^2 = r^2(x'^2 + y'^2) = r^2$~~
- $y = \pm r x'$

取另一個方向的 L

證明了圖本身就是一個圓

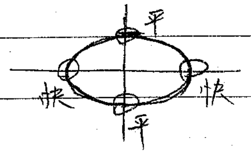
4 Vertex thm : 4頂点定理 convex

$\alpha : [0, \ell] \to \mathbb{R}^2$ simple closed curve (regular, C^3)

Then $\exists$ at least 4 value of s s.t. $k'(s) = 0$

$\leftarrow \alpha(s)$ is call vertex if $k'(s) = 0$

Ep. Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $T'(s) = \underset{|T'(s)|}{\underset{\shortparallel}{k(s)}} N(s)$



$\alpha(t) = (a\cos t, b\sin t)$, t is not arc length.

$$k(t) = \frac{|\alpha' \wedge \alpha''|}{|\alpha'|^3}$$

$$= \frac{1}{(a^2\cos^2 t + b^2\sin^2 t)^{\frac{3}{2}}} \begin{vmatrix} -a\sin t & b\cos t \\ -a\cos t & -b\sin t \end{vmatrix} \begin{matrix} \leftarrow \alpha' \\ \leftarrow \alpha'' \end{matrix}$$

$$= \frac{ab}{(a^2\cos^2 t + b^2\sin^2 t)^{\frac{3}{2}}} \quad ; \quad \cancel{k(t) = 0} \ ?$$

if $a = b = r$, $k(s) = \frac{1}{r} = $ const

$k' \equiv 0$

$C = \operatorname{Im} \alpha$ (trace of α)

Def : C is convex

if $C = \partial\Omega$ then $p, q \in \Omega \Rightarrow \overline{pq} \subset \Omega$ 幾何

Remark : For Jordan curve C (simple closed)

convex $\Leftrightarrow$ $k(s) \geq 0 \ \forall s$ 解析

(or $k(s) \leq 0 \ \forall s$)

$k(0) = k(\ell)$ 光滑的 C^3

$\alpha(0) = \alpha(\ell)$ closed.

<pf> $\int_0^\ell k'(s)\,\alpha(s)\,ds = k(s)\alpha(s)\Big|_0^\ell - \int_0^\ell k(s)\underset{\underset{T}{\uparrow}}{\alpha'(s)}\,ds$

$= \int_0^\ell N'(s)\,ds$ (by frame formula) $N' = -kT - \cancel{\tau B}^{\,0}$

$= N(s)\Big|_0^\ell$

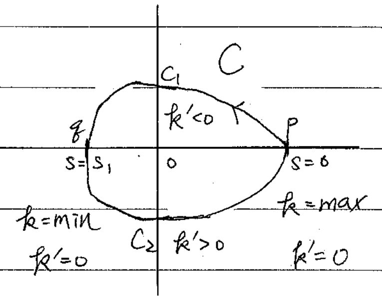
$= N(\ell) - N(0) = 0$

$$\Rightarrow \boxed{\begin{cases} \int_0^\ell k'(s)\,ds = 0 \\ \int_0^\ell k'(s)\,x(s)\,ds = 0 \\ \int_0^\ell k'(s)\,y(s)\,ds = 0 \end{cases}} \qquad \int_0^\ell k' x^2\,ds \neq 0$$

Let p, q be the max and min point of $k(s)$

Let $p = \alpha(0)$, $q = \alpha(s_1)$

May rotate $\mathbb{R}^2$ to put C in the following way :



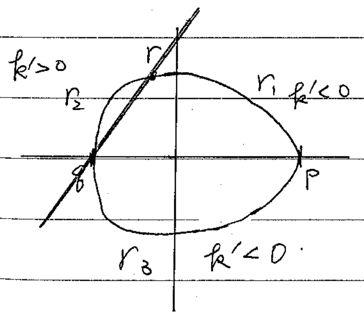
s.t. $\overline{pq} = x$-axis

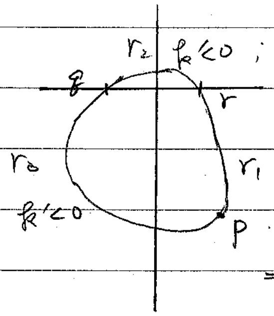
if there are only 2 vertices (p & q)

then ⊙ $\begin{cases} k' < 0 \text{ on } C_1 \quad (y > 0) \\ k' > 0 \text{ on } C_2 \quad (y < 0) \end{cases}$ by convexity

$0 = \int_0^\ell k' y\,ds < 0 \ \longrightarrow\!\times$

If there are only 3 vertices p, q, r s.t $k'(s) = 0$.





$\Rightarrow k' y > 0$

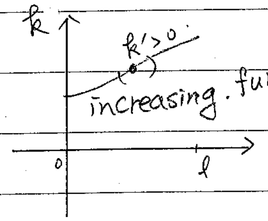
$0 = \int_0^\ell k' y\,ds > 0$

$\Rightarrow k' y > 0$

e^z (ℝ& lim) $\Rightarrow$ $e^z = \sum \frac{z^n}{n!}$ (Taylor) $\Rightarrow$ $e^{iz} = \cos z + i\sin z$ ⇒ 反三角函數
三角函數. ⇒ π

◎ If $k'(s) \geq 0 \quad \forall s \in [0, \ell]$ $\Rightarrow$ $k(s) = \text{constant}$

$\Rightarrow k' \equiv 0$



increasing . func. ⇒ ~~X~~

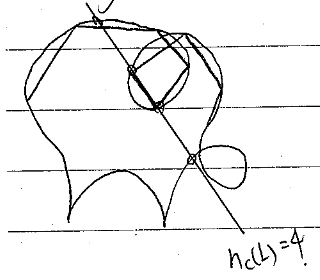
$k(0) = k(\ell)$

(估計 DNA)

Cauchy-Croftan formula : 隨便看到一曲線，求弧長.

方法：拿一條線去量

缺 ：size 和人差太多



rectifiable curve

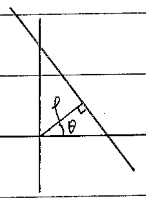
$\alpha : [a, b] \to \mathbb{R}^2$

$\sup_P \sum \ell(c_i) < \infty$

$P = \{c_i\}$

Thm : α , C^1 $\Rightarrow$ rectifiable , $L = \int_a^b |\alpha'(t)| dt$

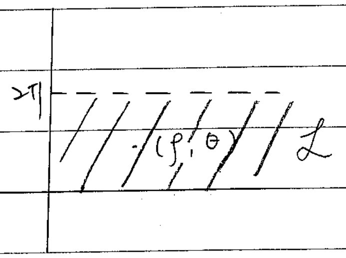
<u>Def</u> The (parameter) space of lines in $\mathbb{R}^2$. $\mathcal{L} = \hat{\mathbb{R}}^2 / \sim$



$L \longleftrightarrow (\rho, \theta) \in \hat{\mathbb{R}}^2$

(New) $0 \leq \theta < 2\pi$

$0 \leq \rho < \infty$



Given any curve $C \subset \mathbb{R}^2$

$n_c : \mathcal{L} \to \mathbb{N} \cup \{0\}$

$L \longmapsto n_c(L) = \#(L \cap C)$

$\frac{1}{2} \int_{\mathcal{L}} n_c(\rho, \theta) d\rho d\theta = \ell$ 想成一塊一塊加起來.

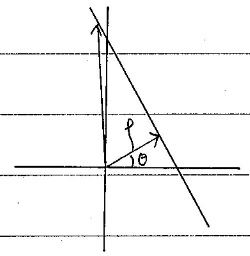
$= \text{length of } C$

Important Remark :

Rigid motion on $\mathbb{R}^2$ induces a rigid motion on $\hat{\mathbb{R}}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

on $\mathbb{R}^2$.



$L \longleftrightarrow (\rho, \theta)$

$\Rightarrow L : (\cos\theta, \sin\theta) \begin{pmatrix} x \\ y \end{pmatrix} = \rho$

$L \longrightarrow \bar{L}$

$(\rho, \theta) \xrightarrow[T]{} (\bar{\rho}, \bar{\theta}) = (\rho - a\cos\theta - b\sin\theta \, , \, \theta - \varphi)$

$$\bar{L} : (\cos\theta, \sin\theta) \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} + a\cos\theta + b\sin\theta = \rho$$

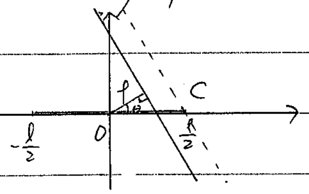
$$(\underbrace{\cos(\theta - \varphi)}_{\overset{\shortparallel}{\bar{\theta}}}, \sin(\theta - \varphi)) \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \underbrace{\rho - a\cos\theta - b\sin\theta}_{\overset{\shortparallel}{\bar{\rho}}}$$

$$J(T) = \det \begin{pmatrix} \frac{\partial \bar{\rho}}{\partial \rho} & \frac{\partial \bar{\theta}}{\partial \rho} \\ \frac{\partial \bar{\rho}}{\partial \theta} & \frac{\partial \bar{\theta}}{\partial \theta} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} = 1.$$

* $d\rho d\theta$ is invariant under rigid motion in $\mathbb{R}^2$

<pf> Case 1 : C = line of length l (in anywhere of $\mathbb{R}^2$) 直線段

May put C in x-axis



$0 \le \theta < 2\pi$

$0 \le \rho \le \frac{l}{2}|\cos\theta|$

$n_C = \begin{cases} 0 \\ 1 \end{cases}$

$$LHS = \frac{1}{2}\int_0^{2\pi}\int_0^{\frac{l}{2}|\cos\theta|} 1 \, d\rho \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{l}{2}|\cos\theta| \, d\theta$$

$$= \frac{l}{4} \cdot 4\int_0^{\frac{\pi}{2}} \cos\theta \, d\theta = l.$$

Case 2 : $C = \bigcup_{i=1}^{r} C_i$ piecewise-polygon 折線.

length $C_i = l_i$

$n_C = \sum_{i=1}^{r} n_{C_i}$

$$\frac{1}{2}\int_{\mathcal{L}} n_C \, d\rho \, d\theta = \sum_{i=1}^{r} \frac{1}{2}\int_{\mathcal{L}} \underline{n_{C_i}} \, d\rho \, d\theta$$ by step 1.

$$= \sum_{i=1}^{r} l_i$$

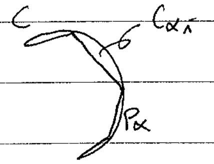
$$= \text{length of } C$$

Case 3 : (general case)

C : rectifiable curve

$P_\alpha = \{C_{\alpha_i}\}$

partition into polygon



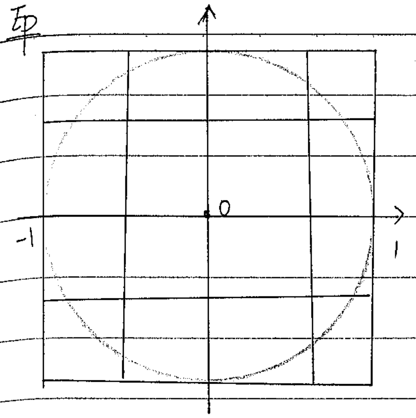
$l_\alpha = \sum_i l(C_{\alpha i}) \longrightarrow l = l(C)$ as $\alpha \longrightarrow \infty$

$\frac{1}{2}\int_{\mathcal{L}} n_{C_\alpha} \, d\rho \, d\theta = l_\alpha \to l(C)$

$\swarrow$ ⊛ 測度論 $\alpha \to \infty$

$\frac{1}{2}\int_{\mathcal{L}} n_C \, d\rho \, d\theta$

Eg



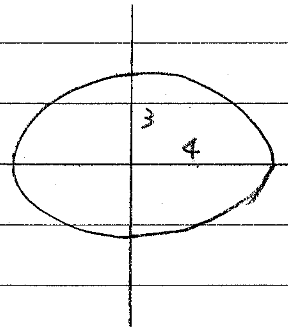
$\frac{1}{2}\int_{\mathcal{L}} n_c \, d\rho \, d\theta$ ① $\begin{pmatrix} 0 \le \theta \le 2\pi \\ 0 \le \rho \le 1 \end{pmatrix}$

$\Delta\rho = \frac{1}{2}$, $\Delta\theta = \frac{\pi}{2}$

$\vee \left(\Delta\rho = \frac{1}{N}, \Delta\theta = \frac{\pi}{4}\right)$ ② $\begin{pmatrix} 0 \le \theta \le \pi \\ -1 \le \rho \le 1 \end{pmatrix}$

$= 2 \cdot \frac{1}{2} \cdot \underset{\substack{\parallel \\ 8}}{n} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = 2\pi$

巧合的黎曼和 ______



$\begin{cases} N = 10 \\ l = 22.1035 \end{cases}$

$L = 22.1455$, $N = 30$.

HW. §1.7 — 6, 11, 13, 14, 15 .

No.

Date 10/1 (三)

Thm : $\mathbb{R}^n$ is a curve !

ie. $\exists\ \alpha : \mathbb{R} \longrightarrow \mathbb{R}^n$ s.t. $Im\,\alpha = \mathbb{R}^n$ (space - filling curve)

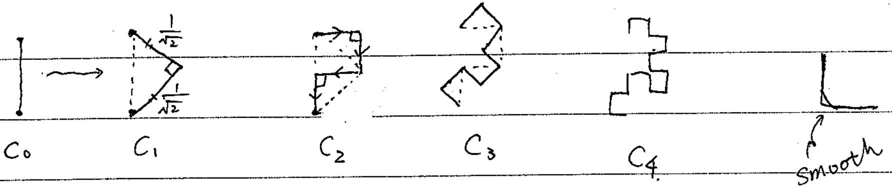
1. Koch - Mandelbrot construction

(fractal geometry)

" Self - similar " construction

自我内似

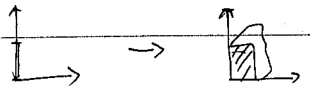
不走直線

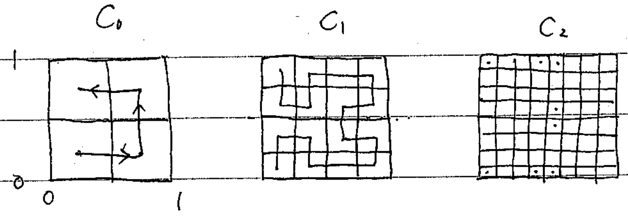


$C_n(t) : \underset{cpt.}{[0,1]} \longrightarrow \mathbb{R}^2$, C^∞ , simple.

let $C(t) = \lim_{n \to \infty} C_n(t)\ \exists$, and C^0

$Im\, C \supset 2\,dim$ box.





$C_n : [0,1] \longrightarrow [0,1]^2$.

$C = \lim_{n \to \infty} C_n$ — 全平面.

is C^0 , $Im\, C = [0,1]^2$.

Exercise : Use this curve α to construct $\alpha_{peano} : \mathbb{R} \longrightarrow \mathbb{R}^2$

No.

Date

Thm : $\underset{\underset{[0,1]}{cpt.}}{f : X} \longrightarrow \underset{\mathbb{R}^n \text{ or Hawsdorff}}{Y}$ conti

f : injective $\Rightarrow f \underset{homeo.}{\simeq} Im(f)$ (why $\mathbb{R} \underset{homeo}{\not\simeq} \overset{(or\ \mathbb{R}^n)}{\mathbb{R}^2}$)

If homeo $\underset{\underset{\times}{disconnected}}{\mathbb{R} \setminus 0} \simeq \underset{connected.}{\mathbb{R}^n \setminus pt}$ ↯

Simple conti curve $\alpha : [0,1] \to \mathbb{R}^2$.

- length of α.

Hawsdorff $\overset{\mathbb{R}>0}{\text{dimension}}$ & Hawsdorff measure

Any set $S \subset \mathbb{R}^n$ $d \in \mathbb{R} \geqslant 0$ fixed.

$\varepsilon > 0$, ε - cover of S $A = \{A_i\}_{i=1}^{\infty}$ $dime\, A_i \leqslant \varepsilon$

s.t. $\bigcup_{A_i \in A} A_i \supset S$ closed or open

$\lim_{\varepsilon \to 0} \inf_{A} \sum_{A_i \in A} (dime\, A_i)^d = m_H^d(S)$

For any S

Thm $\exists !\ d \in \mathbb{R}$, $d \geqslant 0$

s.t. $d_1 > d \Rightarrow m_H^{d_1}(S) = 0$ (理論的核心).

$d_2 < d \Rightarrow m_H^{d_2}(S) = \infty$

$d = d_H(S)$ is called the Hawsdorff dimension of S

~~$m_H^d(S)$~~

$\alpha(t) = (x(t), y(t))$

Weierstrass

$$f(x) = \sum_{n=1}^{\infty} a^n e^{ixb^n} \qquad a \in \mathbb{C},\ |a|<1,\ b \in \mathbb{R}$$

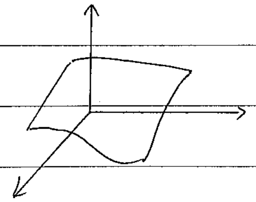
$$= \sum_{n=1}^{\infty} a^n \cos(b^n x) + i b^n \sin(b^n x) = A(x) + iB(x)$$

Hardy : $A(x)$ is nowhere differentiable.

Surface in $\mathbb{R}^3$

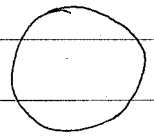
常見曲面

① $z = f(x, y)$



② level set

$F(x, y, z) = C$



③ • $\overset{\in \mathbb{R}^3}{(x, y, z)} = (x(u,v), y(u,v), z(u,v))$

$(u,v) \in U \subset \mathbb{R}^2$. → 採用的定義

(① 是 ③ 的 special case)

Def : $S \subseteq \mathbb{R}^3$ subset

$\forall p \in S$, $\exists$ open nbd V in S $(V = \tilde{V} \cap S)$ ← $\mathbb{R}^3$ 中的 ball.

and a map.

$$\mathbb{R}^2 \supset U \xrightarrow[\mathbb{X}]{} V \subset S \subset \mathbb{R}^3$$

(u,v) $\quad q \longmapsto p$ $\quad (x,y,z)$

$\mathbb{X}(u,v) = (x(u,v), y(u,v), z(u,v))$.

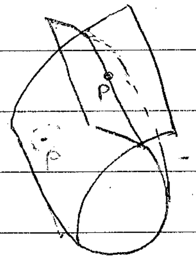
s.t. $\mathbb{X}$ is a homeomorphism.

• S is a $\boxed{\text{regular}}$ surface (of class C^k, $k \geq 1$)

if $\mathbb{X}(u,v)$ is C^k and $d\mathbb{X}_1$ is of rank 2.

$(\mathbb{X}(q) = p)$

被排除



$d\mathbb{X} = \mathbb{X}' = D\mathbb{X} = \mathbb{X}_*$

$q = (u_0, v_0)$

$$d\mathbb{X}_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}_{(u,v) = q = (u_0, v_0)}$$ q 点的微分

$d\mathbb{X}_q : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $d\mathbb{X}_q = [\mathbb{X}_u, \mathbb{X}_v] \in M_{3\times 2}$

rank 2 $\equiv$ image is 2-dim'l.

$\equiv \mathbb{X}_u \neq 0$, $\mathbb{X}_v \neq 0$ and $\mathbb{X}_u \neq \lambda \mathbb{X}_v$ 線性獨立.

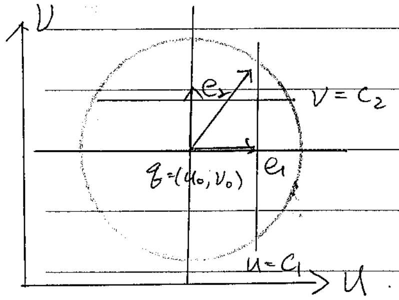
$\equiv$ one of the three minors is $\neq 0$.

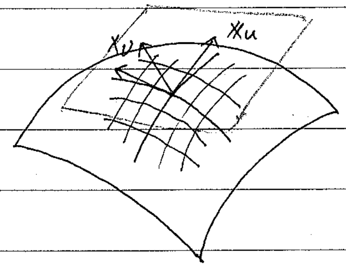
$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$. $\begin{vmatrix} x_u & x_v \\ z_u & z_v \end{vmatrix}$. $\begin{vmatrix} y_u , & y_v \\ z_u , & z_v \end{vmatrix}$ 任一不為零.

convention : a vector is always understood as a column vector

ep. $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ or $\begin{pmatrix} u \\ v \end{pmatrix}$

$(U, \mathbb{X})$ is called a local coordinate system or a local parametrization





$d\mathbb{X}_q(e_1) = \mathbb{X}_u$

$d\mathbb{X}_q(e_2) = \mathbb{X}_v$

Recall (Advanced caculus)

$F : U_{\subseteq \mathbb{R}^m} \longrightarrow \mathbb{R}^n$ is differentiable at $q \in U$ if.

$\exists$ linear map $T : \mathbb{R}^m \longrightarrow \mathbb{R}^n$. $(T \in M_{n\times m}(\mathbb{R}))$

s.t. $F(q+h) - F(q) = Th + o(|h|)$

T is called $F'(q) \equiv dF_q \equiv DF(q) \equiv F_{*q}$

◎ $F'(q)$ exists $\Rightarrow \frac{\partial F}{\partial x_i}(q)$ $i = 1, \cdots, m$ exists $F_i = \partial^i F = F_{x_i}$

$(x_1, \cdots, x_m)$ coor of $\mathbb{R}^m$.

(let $\vec{h} = h \cdot \vec{e}_i$)

and $T = [F_1, \cdots, F_m] \in M_{n\times m}(\mathbb{R})$.

◎ 反過來對不對?

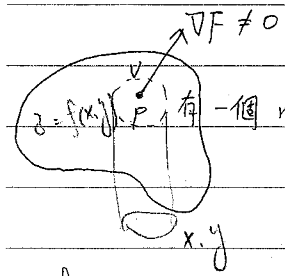
(i.e. F is C^1)

conversely, if F_i exists and is continuous in a nbd U of q, then F' exists in U

locally speaking, any func is a linear func.

Prop Given $\overset{\in C^1}{F(x,y,z)}$, if $\nabla F(p) \neq 0$ then the level set $F(x,y,z) = F(p)^{\shortparallel C}$ is a surface in a nbd V of p.

in fact, V can be realized as a graph.



有一個 nbd 可被參數化.

<pf> $\nabla F(p) \neq 0$ may assume $F_z(p) \neq 0$.

$\overset{\shortparallel}{(F_x, F_y, F_z)}$ 目的：把 F 和 z 互換

consider $G(x,y,z) = (\overset{u}{\overset{\shortparallel}{x}}, \overset{v}{\overset{\shortparallel}{y}}, \overset{w}{\overset{\shortparallel}{F}}(x,y,z))$.

$\mathbb{R}^3 \longrightarrow \mathbb{R}^3$.

$$J_G = \det G' = \det\begin{pmatrix} 1 & 0 & F_x \\ 0 & 1 & F_y \\ 0 & 0 & F_z \end{pmatrix} = F_z$$

$J_G(\xi) \neq 0$, $G \in C^1$ 反函數存在 (by 反函數定理).

Inverse function theorem

$\Rightarrow \exists\ \overline{\xi} = G^{-1}$ locally near p.

if. $(u, v, w) = G(x, y, z)$.

then $\left.\begin{matrix} x = \Phi_1(u,v,w) = u \\ y = \Phi_2(u,v,w) = v \end{matrix}\right\}$ 沒換过.

$z = \Phi_3(u,v,w) = \Phi_3(x,y,w)$

On the level set $w = F(x,y,z) = C$

$\Rightarrow \boxed{z = \Phi_3(x,y,c) =: f(x,y)}$

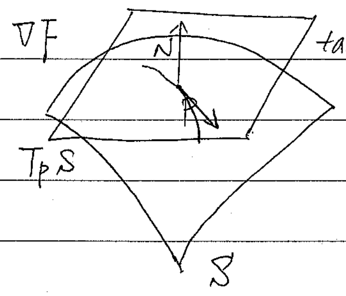
i.e. V is a graph

but a graph is a surface.

$(x, y) \longmapsto (x, y, f(x,y))$ *

$\underset{\overset{\shortparallel}{(u,v)}}{(x,y)} \xrightarrow[\mathbb{X}]{} (x, y, f(x,y))$ $\mathbb{X} = \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}$

$d\mathbb{X}_q = \left(\begin{matrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} \\ \begin{matrix} f_x & f_y \end{matrix} \end{matrix}\right) \rightarrow \text{rank} = 2.$



tangent plane := 切向量的總和.

Def of T_pS : (the tangent plane)

T_pS is the set of all $\alpha'(0)$ with α a curve in S s.t $\alpha(0) = p$

prop : $T_pS \perp \nabla F(p)$ if S is given by the level set $F = C$.

<pf> $\alpha(t) \in S$: $F(x(t), y(t), z(t)) = 0$

$F_x x'(t) + F_y y'(t) + F_z z'(t) = 0 = \boxed{\nabla F \cdot \alpha'(t)}$ 錯了!!

$= \underset{\text{ok!}}{\nabla F(\alpha(t)) \cdot \alpha'(t)}$, let $t = 0$.

In fact, $T_pS = \mathbb{R}\mathbb{X}_u + \mathbb{R}\mathbb{X}_v$ if $\mathbb{X}(u,v) : U \longrightarrow \underset{\overset{\cup}{p}}{V} \subset S$ $\begin{cases} \mathbb{X}_u(p) = \frac{d}{du}\mathbb{X}(u, v_0) \\ \mathbb{X}_v(p) = \frac{d}{dv}\mathbb{X}(u_0, v) \end{cases}$

Rmk : thm : Any "orientable" regular surface is a level set

(下次証).

如何在不同的 def 中換來換去

(Inverse func. thm)

The normal vector is $N = \dfrac{\mathbb{X}_u \times \mathbb{X}_v}{|\mathbb{X}_u \times \mathbb{X}_v|}$

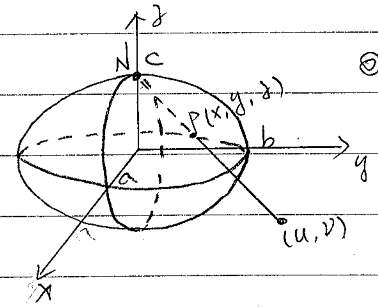
for graph $z = f(x,y)$, $\mathbb{X}_x = \begin{pmatrix} 1 \\ 0 \\ f_x \end{pmatrix}$, $\mathbb{X}_y = \begin{pmatrix} 0 \\ 1 \\ f_y \end{pmatrix}$

$\mathbb{X}_x \times \mathbb{X}_y = (-f_x , -f_y , 1)$

$N = \dfrac{(-f_x , -f_y , 1)}{\sqrt{1+f_x^2+f_y^2}}$ ——o 重要的法向量.

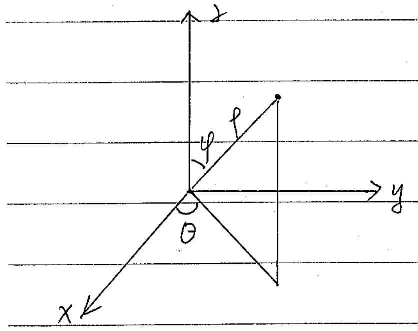
Ep. ① Ellipsoid : $(\frac{x}{a})^2 + (\frac{y}{b})^2 + (\frac{z}{c})^2 = 1$

☆



⊙ "spherical - like" coordinates :

$$\begin{cases} x = a \sin\varphi \cos\theta \\ y = b \sin\varphi \sin\theta \\ z = c \cos\varphi \end{cases} \qquad \begin{matrix} 0 \le \varphi \le \pi \\ 0 \le \theta \le 2\pi \end{matrix}$$



⊙ Euclidean coordinates :

$z = \pm c \sqrt{1-(\frac{x}{a})^2-(\frac{y}{b})^2}$

$\mathcal{U} = \{ (x,y) \in \mathbb{R}^2 \mid (\frac{x}{a})^2 + (\frac{y}{b})^2 < 1 \}$

$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases}$$

⊙ Rational coordinate system : (sterographic projection)

" Key point : $\overline{Np}$ intersects S exactly 2 pts (其一固定為 N) "

$L(t) = (1-t)\underbrace{N + t(u,v)}_{兩個點} = (tu, tv, c(1-t))$.

on S $t^2[(\frac{u}{a})^2+(\frac{v}{b})^2] + \underset{t^2-2t+\cancel{1}}{\overset{\shortparallel}{(1-t)^2}} = \cancel{1}$ 好事 ^^

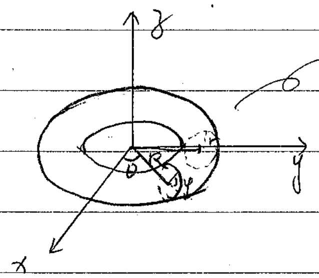
$\Rightarrow t = \dfrac{2}{\Lambda+1}$, where $\Lambda = (\frac{u}{a})^2 + (\frac{v}{b})^2$

or $t = 0$.

$\mathbb{X}(u,v) = \left(\dfrac{2u}{\Lambda+1} , \dfrac{2v}{\Lambda+1} , \dfrac{c(\Lambda-1)}{\Lambda+1} \right)$, $\Lambda = (\frac{u}{a})^2 + (\frac{v}{b})^2$.

all x, y, z are rational funcs in u, v

Ep. ② Torus (2 radius R and r)



$\overset{F(x,y,z)}{\shortparallel}$
o—— $((x^2+y^2)^{\frac{1}{2}} - R)^2 + z^2 = r^2$. 要求 $R > r$.

$\nabla F = \left(\dfrac{(2\sqrt{x^2+y^2} - R) \cdot 2x}{2\sqrt{x^2+y^2}} , (\ )2y , (\ )2z \right)$

$\neq 0$ for $(x,y) \neq (0,0)$.

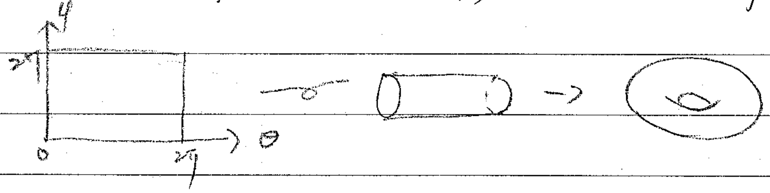
∴ It's a regular surface.

☆

⊙ "spherical - like" $\mathbb{X}(\theta,\varphi) = ((R + r\cos\varphi)\cos\theta , (R + r\cos\varphi)\sin\theta , r\sin\varphi)$

$0 \le \varphi \le 2\pi$

$0 \le \theta \le 2\pi$.



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$$\mathbb{X}(u,v) : \underset{q\in}{U} \overset{\subset \mathbb{R}^2}{} \longrightarrow S \subset \mathbb{R}^3$$

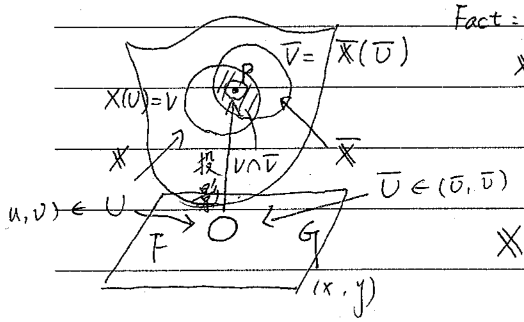
$d\mathbb{X}_q : \mathbb{R}^2 \to \mathbb{R}^3$ has rank 2.

Goal : to define differentiable functions on S

$$\begin{tikzcd} S \arrow[r, "f"] & \mathbb{R} \\ U_{\subset \mathbb{R}^2} \arrow[u, "\mathbb{X}"] \arrow[ur, "f\circ \mathbb{X}"'] & \end{tikzcd}$$

f is differentiable $\overset{(C^k,\ k\geq 1)}{\text{iff}}$ $f\circ \mathbb{X}$ is differentiable

for all coordinates charts $(U, \mathbb{X})$



Fact :

$$\begin{tikzcd} \mathbb{X}^{-1}(V\cap\bar{V}) \arrow[r, leftrightarrow, "\text{homeo}"] & \bar{\mathbb{X}}^{-1}(V\cap\bar{V}) \\ \cap & \cap \\ U & \bar{U} \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{X}^{-1}(V\cap\bar{V}) \arrow[r, shift left, "\bar{\mathbb{X}}^{-1}\circ\mathbb{X}"] & \bar{\mathbb{X}}^{-1}(V\cap\bar{V}) \arrow[l, shift left, "\text{homeo}"', "\mathbb{X}^{-1}\circ\bar{\mathbb{X}}"] \end{tikzcd}$$

$\hookrightarrow$ (diffeomorphisms)

<pf> : take a nbd $\tilde{V}$ of p in S

s.t. $\tilde{V}$ is a graph on (x,y)-plane

that is $\tilde{V}$ is given by $(x, y, f(x,y))$, $(x,y) \in$

$\mathbb{X}(u,v) = (x(u,v), y(u,v), z(u,v))$

$z(u,v) = f(x(u,v), y(u,v))$

$$d\mathbb{X}_q = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ f_x x_u + f_y y_u & f_x x_v + f_y y_v \end{pmatrix} \Rightarrow \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \neq 0 \text{ because } d\mathbb{X}_q \text{ has rank } 2$$

$\mathbb{X}(q) = p$

$(u,v) \xrightarrow[F]{} (x,y)$ by inverse function thm

$\det(DF) \neq 0$ F is a local diffeomorphism

$\Rightarrow \bar{\mathbb{X}}^{-1}\circ\mathbb{X} = G^{-1}\circ F$ is C^k, $k\geq 1$

$(G^{-1}\circ F)(u,v) = (\bar{u}, \bar{v})$

$$\begin{pmatrix} \frac{\partial\bar{u}}{\partial u} & \frac{\partial\bar{u}}{\partial v} \\ \frac{\partial\bar{v}}{\partial u} & \frac{\partial\bar{v}}{\partial v} \end{pmatrix}$$

$f\circ\bar{\mathbb{X}} = f\circ\mathbb{X}\circ\mathbb{X}^{-1}\circ\bar{\mathbb{X}}$

$= \underbrace{(f\circ\mathbb{X})}_{C'} \circ \underbrace{(\mathbb{X}^{-1}\circ\bar{\mathbb{X}})}_{C'}$

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Def : differentiable map between surfaces

$$\begin{tikzcd} p\in S_1 \arrow[r, "f"] & S_2 \subset \mathbb{R}^3 \\ U \arrow[u, "\mathbb{X}"] \arrow[ur, "f\circ\mathbb{X}"'] & \end{tikzcd}$$

is a map f s.t. $f\circ\mathbb{X} : U \longrightarrow \mathbb{R}^3$ is C^k

$\forall$ coordinates $(U, \mathbb{X})$

$df_p : \underset{v\in}{T_p S_1} \longrightarrow T_{f(p)} S_2$

$v \longmapsto (f\circ\alpha)'(0)$ $\alpha(0) = p$

$v = \alpha'(0)$ for some curves $\alpha(t) \in S_1$

$(f\circ\alpha)'(0) = df_p \circ \alpha'(0) = df_p(v)$ 想

重要 chain rule $p\in S_1 \xrightarrow{f} S_2 \xrightarrow{g} S_3$ S_i : curve or surface

$d(g\circ f)_p = dg_{f(p)} \circ df_p$

linear transformation

<pf>

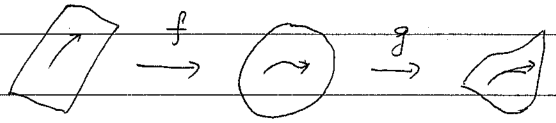
$d(g\circ f)_p\, \alpha'(0)$

$= (g\circ f\circ\alpha)'(0)$

$= (g\circ(f\circ\alpha))'(0)$

$= dg_{f(p)}\, (f\circ\alpha)'(0)$ by def of $dg_{f(p)}$

$= dg_{f(p)} \circ df_p \cdot \alpha'(0)$ by def of df_p.

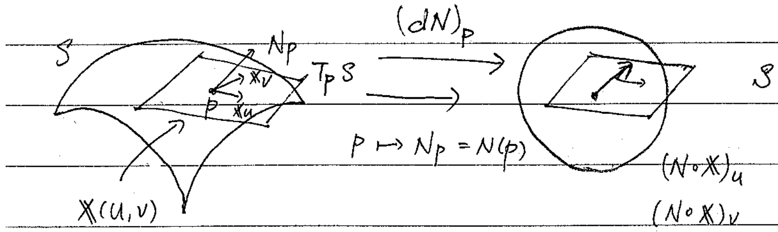


Ex §2.2 — 7. 17.

§2.3 — 10. 16.

§2.4 — 5, 10, 12, 13, 15. 17.

Gauss Map = N

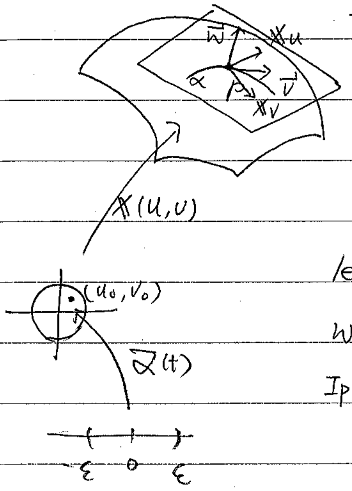


$S^2 = \{ v \in \mathbb{R}^3 ,\ |v| = 1 \}$

$$S_1 \xrightarrow[df_p]{f} S_2$$

$$T_p S_1 \longrightarrow T_{f(p)} S_2 .$$

" first fundamental form " i.e. inner product on T_pS



$T_pS \subset \mathbb{R}^3$, " $\langle \vec{v}, \vec{w} \rangle$ "

$I_p(\vec{v}, \vec{w}) := \vec{v} \cdot \vec{w} = \langle \vec{v}, \vec{w} \rangle_{\mathbb{R}^3}$ in $\mathbb{R}^3$.

Given $\vec{v} \in T_pS$ $\quad \alpha(t) = \mathbb{X}(u(t), v(t))$ $\quad (u(0), v(0)) = (u_0, v_0)$

let α be a curve s.t $\alpha(0) = p$, $\alpha'(0) = \vec{v}$ $\quad \mathbb{X}(u_0, v_0) = p$.

want to compute $I_p(\vec{v}, \vec{v})$

$$I_p(\vec{v}, \vec{v}) = |\vec{v}|^2 = |\alpha'(0)|^2 = \left(\frac{ds}{dt}\right)^2 .$$

$$= |\mathbb{X}_u u'(0) + \mathbb{X}_v v'(0)|^2$$

$$= \underset{E}{\underset{\|}{\langle \mathbb{X}_u , \mathbb{X}_u \rangle}} \underline{u'(0)^2} + 2 \underset{F}{\underset{\|}{\langle \mathbb{X}_u , \mathbb{X}_v \rangle}} \underline{u'(0)}\, \underline{v'(0)} + \underset{G}{\underset{\|}{\langle \mathbb{X}_v , \mathbb{X}_v \rangle}} \underline{v'(0)^2}$$

$$= E \underset{(\frac{du}{dt})^2}{(u')^2} + 2F \underset{(\frac{du}{dt})(\frac{dv}{dt})}{u'v'} + G \underset{(\frac{dv}{dt})^2}{(v')^2}$$

$$= (u', v') \begin{pmatrix} E & F \\ F & G \end{pmatrix} \underset{\vec{a}.}{\underset{\|}{\begin{pmatrix} u' \\ v' \end{pmatrix}}} = \vec{a}^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} \vec{a}$$

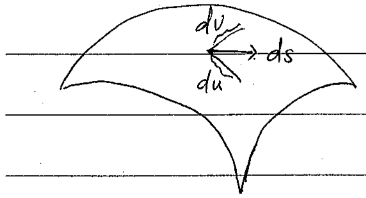
conclusion : for any vector $\vec{v} = a\mathbb{X}_u + b\mathbb{X}_v \in T_pS$

$$\underset{\langle a\mathbb{X}_u + b\mathbb{X}_v ,\ a\mathbb{X}_u + b\mathbb{X}_v \rangle}{\underset{\|}{I_p(\vec{v}, \vec{v})}} = (a, b) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} .$$

☆ " $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$ "

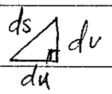
曲面上，計算長度的規則

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2.$$



In Euclidean space $(dt)^2$ (有d* 叫 form)

$$ds^2 = du^2 + dv^2$$



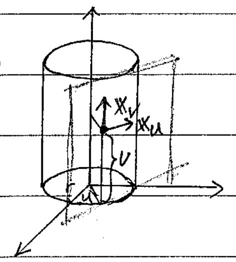
$$S(t) := \int_{t_0}^{t} |\alpha'(*)|\, d*$$

$$\frac{ds}{dt} = |\alpha'(t)|$$

$$I_p(\vec{v}) := I_p(\vec{v}, \vec{v})$$

$$I_p(\vec{v}, \vec{w}) = (a, b)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}.$$

Ep. ①



$$X(u,v) = (r\cos\tfrac{u}{r},\ r\sin\tfrac{u}{r},\ v)$$

$$X_u = (-\tfrac{r}{r}\sin\tfrac{u}{r},\ \tfrac{r}{r}\cos\tfrac{u}{r},\ 0)$$

$$X_v = (0, 0, 1)$$

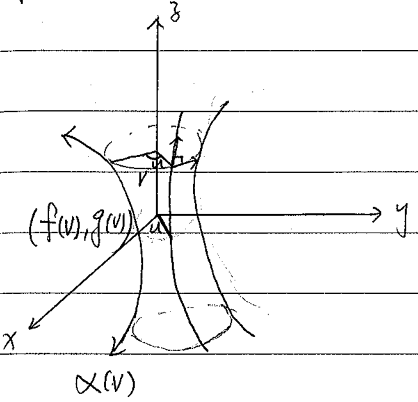
$$E = \langle X_u, X_u\rangle = r^2\sin^2 u + r^2\cos^2 u = r^2$$

$$F = \langle X_u, X_v\rangle = 0$$

$$G = \langle X_v, X_v\rangle = 1$$

$$ds^2 = \underset{\overset{\|}{1}}{r^2}\, du^2 + dv^2.$$ first fundamental form.

Ep② surface of revolution.



Let $\alpha(v) = (f(v), g(v))$ be a curve in x-z plane

Let S be the surface obtained by rotating α wrt the z-axis

$$X(u,v) = (f(v)\cos u,\ f(v)\sin u,\ g(v)).$$

$$X_u = (-f\sin u,\ f\cos u,\ 0)$$

$$X_v = (f'\cos u,\ f'\sin u,\ g')$$

$\langle X_u, X_v\rangle = 0$

$$ds^2 = f^2 du^2 + ((f')^2 + (g')^2)\, dv^2$$

If v = arc length (func.) of α

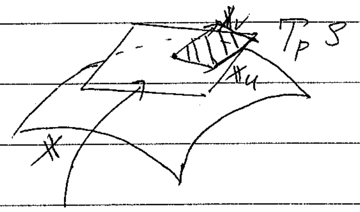
then $(f')^2 + (g')^2 = 1 \Rightarrow ds^2 = f^2 du^2 + dv^2$

Ex §2.5 — 4. 7. 8. 12. 14.

No.

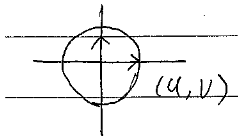
Date 10/17 (四)

1st fundamental form



$E = |\mathbb{X}_u|^2$. $F = \langle \mathbb{X}_u, \mathbb{X}_v \rangle$, $G = |\mathbb{X}_v|^2$

$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$



Simplest — application :

Area of surface

$$|\mathbb{X}_u \times \mathbb{X}_v|^2 + |\mathbb{X}_u \cdot \mathbb{X}_v|^2 = |\mathbb{X}_u|^2 \cdot |\mathbb{X}_v|^2 .$$

$$(a^2 b^2 \sin^2\theta + a^2 b^2 \cos^2\theta = a^2 b^2)$$

$$|\mathbb{X}_u \wedge \mathbb{X}_v|^2 = \sqrt{EG - F^2}$$

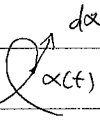
Definition of area :

$\text{area}(\Omega) := \int_Q |\mathbb{X}_u \wedge \mathbb{X}_v| \, du\,dv$

$\Omega \subseteq S$

$\Omega = \mathbb{X}(Q)$

$$\begin{cases} \text{area} : \mathbb{X} : U \longrightarrow S \subset \mathbb{R}^3 , \quad d\mathbb{X} = [\mathbb{X}_u, \mathbb{X}_v]_{3\times 2} . \\ \text{length} : \end{cases}$$



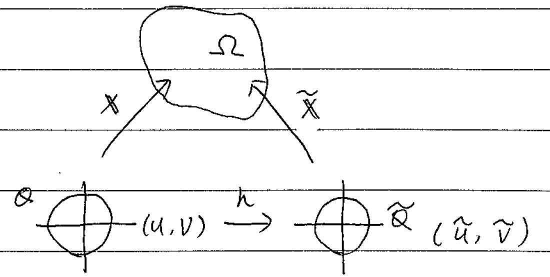
This is independent of the choice of parametrization $\mathbb{X}$.

No.

Date

證明和參數的選擇無關

$$\overset{3\times 2}{d\mathbb{X}} = d\tilde{\mathbb{X}} \circ \tilde{\mathbb{X}}^{-1} \circ \mathbb{X}$$
$$= d(\tilde{\mathbb{X}} \circ h)$$
$$= \underset{3\times 2}{d\tilde{\mathbb{X}}} \circ \underset{2\times 2}{dh}$$



$h = \tilde{\mathbb{X}}^{-1} \circ \mathbb{X}$

$$\text{i.e. } [\mathbb{X}_u, \mathbb{X}_v] = [\tilde{\mathbb{X}}_{\tilde{u}}, \tilde{\mathbb{X}}_{\tilde{v}}] \begin{bmatrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{u}}{\partial v} \\ \frac{\partial \tilde{v}}{\partial u} & \frac{\partial \tilde{v}}{\partial v} \end{bmatrix}$$

you may compute this directly :

硬算 ○ ○ ○ . . . .
$$\mathbb{X}_u = \tilde{\mathbb{X}}_{\tilde{u}} \frac{\partial \hat{u}}{\partial u} + \tilde{\mathbb{X}}_{\tilde{v}} \frac{\partial \tilde{v}}{\partial u}$$
$$\mathbb{X}_v = \tilde{\mathbb{X}}_{\tilde{u}} \frac{\partial \hat{u}}{\partial v} + \tilde{\mathbb{X}}_{\tilde{v}} \frac{\partial \tilde{v}}{\partial v}$$

$$\Rightarrow |\mathbb{X}_u \times \mathbb{X}_v| = |\tilde{\mathbb{X}}_{\tilde{u}} \times \tilde{\mathbb{X}}_{\tilde{v}}| \cdot \left| \frac{\partial(\tilde{u}, \tilde{v})}{\partial(u,v)} \right|$$

(always works for any dimensions)

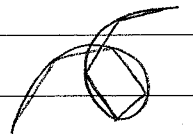
☆ in our case , $\mathbb{X}_u \times \mathbb{X}_v = \tilde{\mathbb{X}}_{\tilde{u}} \times \tilde{\mathbb{X}}_{\tilde{v}} \left(\frac{\partial \tilde{u}}{\partial u} \frac{\partial \tilde{v}}{\partial v} - \frac{\partial \tilde{v}}{\partial u} \frac{\partial \tilde{u}}{\partial v} \right)$

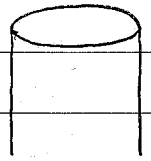
$$\int_Q \underbrace{|\mathbb{X}_u \times \mathbb{X}_v| \, du\,dv}_{dA} \qquad \left(= \int_{\tilde{Q}} |\tilde{\mathbb{X}}_{\tilde{u}} \times \tilde{\mathbb{X}}_{\tilde{v}}| \, d\tilde{u}\,d\tilde{v} \right)$$

$$= \int_Q |\tilde{\mathbb{X}}_{\tilde{u}} \times \tilde{\mathbb{X}}_{\tilde{v}}| \underbrace{\left(\frac{\partial \tilde{u}}{\partial u} \frac{\partial \tilde{v}}{\partial v} - \frac{\partial \tilde{v}}{\partial u} \frac{\partial \tilde{u}}{\partial v} \right)}_{\text{Jac} = \det(dh)} du\,dv$$

$$= \int_{\tilde{Q}} |\tilde{\mathbb{X}}_{\tilde{u}} \times \tilde{\mathbb{X}}_{v}| \, d\tilde{u}\,d\tilde{v} \quad \text{by change of variable formula}$$

Ex 1-7 (15) —(b)





Ep. I graph surface $z = f(x, y)$

$$\mathbb{X}(x, y) = (x, y, f(x, y))$$

$$\mathbb{X}_x = (1, 0, f_x)$$

$$\mathbb{X}_y = (0, 1, f_y)$$

$$\Rightarrow E = 1 + f_x^2, \quad F = f_x f_y, \quad G = 1 + f_y^2$$

$$ds^2 = (1+f_x)^2 dx^2 + 2 f_x f_y \, dx dy + (1+f_y^2) dy^2$$

$$dA = \sqrt{EG - F^2} \, dx dy$$

$$EG - F^2 = (1 + f_x^2)(1 + f_y^2) - f_x^2 f_y^2.$$

$$= 1 + f_x^2 + f_y^2$$

$$dA = \sqrt{1 + f_x^2 + f_y^2}$$

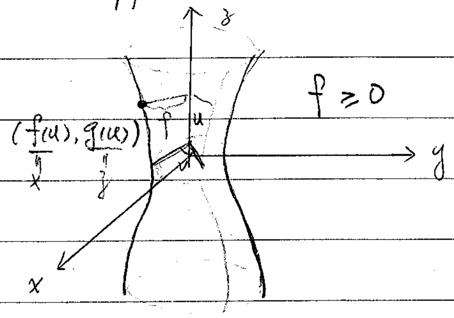
eg

$$z = \overset{f}{\overset{\parallel}{xy}}, \quad \Omega = \mathbb{X}(Q), \quad Q = \{x^2 + y^2 \leq 1\} \subset \mathbb{R}^2.$$

$$\text{area}(\Omega) = \int_Q \sqrt{1 + x^2 + y^2} \, dx dy$$

Ep II. Surface of revolution:

Pappus theorem



$f \geq 0$

$$\text{Area} = 2\pi \int_0^l f(u) \, du$$

where $u \mapsto (f(u), g(u))$ is the curve in $x-z$ plane and u is its arc length.

Recall : $ds^2 = f^2 du^2 + (f'(u)^2 + g'(u)^2) dv^2.$

$$dA = \sqrt{EG - F^2} \, du dv$$

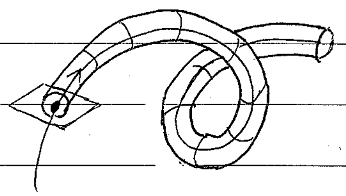
$$= f \sqrt{f'^2 + g'^2} \, du dv$$

$$A = \int_Q f \sqrt{f'^2 + g'^2} \, du dv$$

$$= 2\pi \int_{u_0}^{u_1} f \sqrt{f'^2 + g'^2} \, du \quad ※$$

§2.5 # 11.

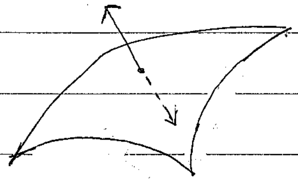
Ep Ⅲ (exercise)



area = $2\pi r \cdot L$

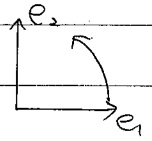
L = Length of the curves

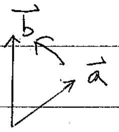
$N(p) = \dfrac{\mathbb{X}_u \times \mathbb{X}_v}{|\mathbb{X}_u \times \mathbb{X}_v|}$



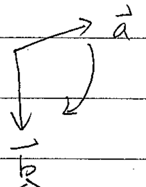
" orientation of S "

on $\mathbb{R}^2$:

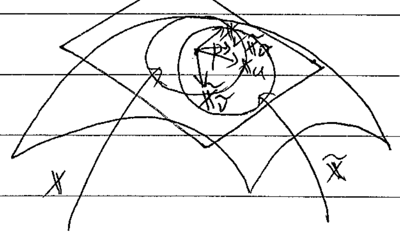




positive $|\vec{a}\ \vec{b}| > 0$

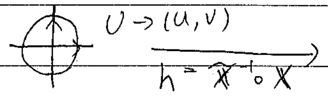


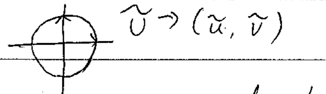
negative $|\vec{a}\ \vec{b}| < 0$



(可定向)

Def: S is orientable if $\exists$ coordinates charts $\mathbb{X}$'s st. their Jacobians are all positive.





$\det(dh) \overset{\shortparallel}{} \dfrac{\partial(\tilde{u},\tilde{v})}{\partial(u,v)}$

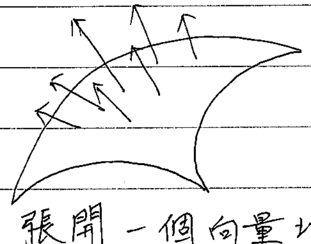
and this particular choice of coor charts are called an "orientation."

"Non-orientable" if no such collection of charts exists.

prop S is orientable $\Leftrightarrow$ (the $\overset{\text{unit}}{\vee}$ normal vector field is well-defined on all S.)

$\Leftrightarrow$ $\exists$ unit normal vector field on S

$N : S \longrightarrow \mathbb{R}^3$



張開一個向量場

<pf> $(\Rightarrow)$ simply define $N(p) = \dfrac{\mathbb{X}_u \times \mathbb{X}_v}{|\mathbb{X}_u \times \mathbb{X}_v|}(p)$

Since $\mathbb{X}_u \times \mathbb{X}_v = \tilde{\mathbb{X}}_{\tilde{u}} \times \tilde{\mathbb{X}}_{\tilde{v}} \cdot \dfrac{\partial(\tilde{u},\tilde{v})}{\partial(u,v)}$, we see that $N(p)$ is well-defined.

$(\Leftarrow)$ $N : S \longrightarrow \mathbb{R}^3$ is given

look at $\langle \underset{\text{on all } S}{N(p)}, \underset{\text{only on some open set in } S}{\dfrac{\mathbb{X}_u \times \mathbb{X}_v}{|\mathbb{X}_u \times \mathbb{X}_v|}} \rangle = \pm 1$

If $= +1$, then keep this chart $\mathbb{X}$

If $= -1$, then replace $\mathbb{X}$ by a new parametrization

$\tilde{\mathbb{X}}(v,u) = \mathbb{X}(u,v)$

done ✗.

p is a critical pt. if $\nabla F(p)=0$

$a \in \mathbb{R}$ is a regular value of f if $f^{-1}(p) \subset \mathbb{R}^3$ contains no critical pt.

Corollary: $F: \mathbb{R}^3 \longrightarrow \mathbb{R}$ differentiable.

with $a \in \mathbb{R}$ a regular value

then $S = F^{-1}(a)$ is orientable.

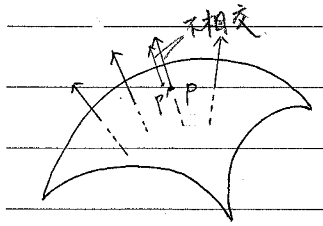
<pf> because $N(p) = \dfrac{\nabla F(p)}{|\nabla F(p)|}$ is well-defined

※

☆ Thm Any compact surface (without boundary) in $\mathbb{R}^3$ is orientable.

(以後再証，有点難)

☆ Thm Any $\overset{\text{orientable}}{\vee}$ surface in $\mathbb{R}^3$ is a level surface of some function $F: \mathbb{R}^3 \longrightarrow \mathbb{R}$, (§2-7). (証明不考).



consider points of the form $p+tN_p$, $t \in (-\varepsilon, \varepsilon)$

$(u, v, t) \longmapsto p+tN_p$
$V \times (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^3$ invertable.

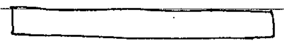
$\Rightarrow$ locally, 不相交.

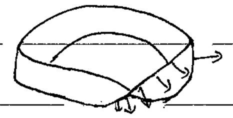
$\because$ compact $\Rightarrow$ galobally, 不相交.

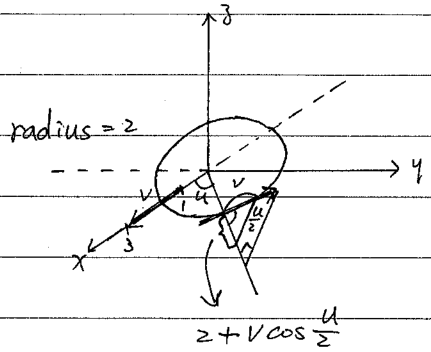
$F: \mathbb{R}^3 \longrightarrow \mathbb{R}$
$\ \ \|$
$\ \ t$

Ep (of a non-orientable surface in $\mathbb{R}^3$)

Mibins band (stripe)







$x(u,v) = \left((2+v\cos\frac{u}{2})\cos u,\ (2+v\cos\frac{u}{2})\sin u,\ \sin\frac{u}{2} \right)$

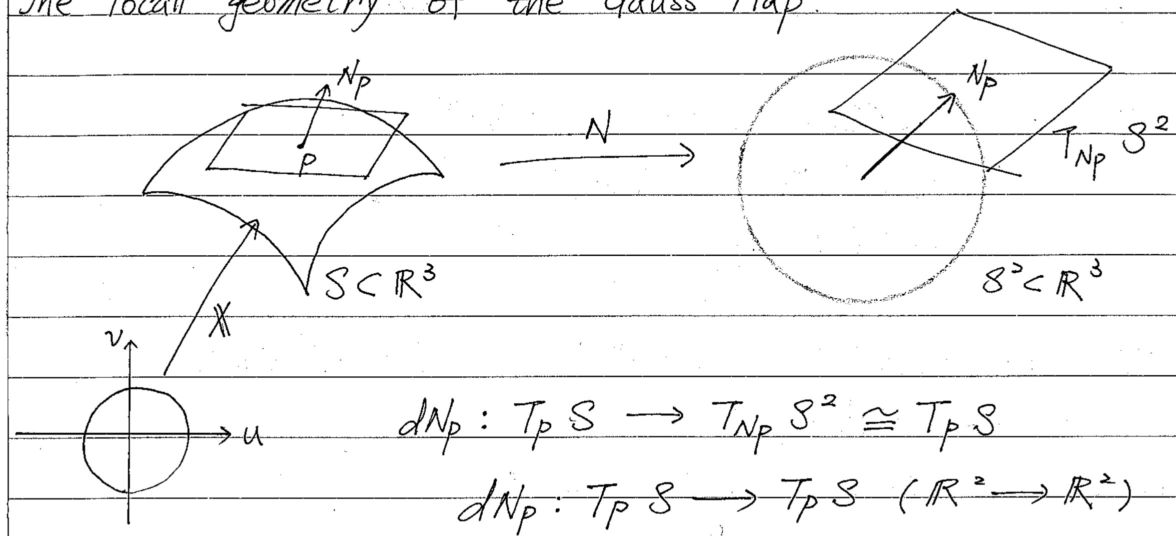
$N(\underline{u}, \underline{v})$ $u=0$, $v=0$
$u=2\pi$, $v=0$

Ex §2.6 — 7.

NO. 10/31 (四)

chapter 3.

The locall geometry of the Gauss Map.



$$dN_p : T_p S \longrightarrow T_{N_p} S^2 \cong T_p S$$

$$dN_p : T_p S \longrightarrow T_p S \quad (\mathbb{R}^2 \longrightarrow \mathbb{R}^2)$$

Prop: dN_p is self-adjoint

i.e. $\langle dN_p(\vec{v}), \vec{w} \rangle = \langle \vec{v}, dN_p(\vec{w}) \rangle$ for $\vec{v}, \vec{w} \in T_p S$

<pf> pick local parametrization $\mathbb{X}(u,v)$

$\Rightarrow$ $T_p S$ has a basis $\{\mathbb{X}_u, \mathbb{X}_v\}$

only have to check $dN_p(\mathbb{X}_u) \cdot \mathbb{X}_v \overset{?}{=} \mathbb{X}_u \cdot dN_p(\mathbb{X}_v)$

$dN_p(\mathbb{X}_u) = N_u$ (denoted by $\frac{\partial}{\partial u}(N \circ \mathbb{X})$)

$dN_p(\mathbb{X}_u) \cdot \mathbb{X}_v = N_u \cdot \mathbb{X}_v = (N \cancel{\cdot \mathbb{X}_v})_u - N \cdot \mathbb{X}_{vu}$ $\left(\mathbb{X}_{vu} = \frac{\partial^2 \mathbb{X}}{\partial v \partial u}\right)$

$dN_p(\mathbb{X}_v) \cdot \mathbb{X}_u = -N \cdot \mathbb{X}_{uv}$

N on use $\mathbb{X}_{uv} = \mathbb{X}_{vu}$

observation : It is important to look at $\overset{\wedge}{\mathbb{X}_{uu}}$ $\mathbb{X}_{uv}, \mathbb{X}_{vu}, \mathbb{X}_{vv} \circ N$

ef : Second fundamental form (為了算 normal curvature)

$\mathrm{I\!I}_p(\vec{v}, \vec{w}) = -\langle dN_p(\vec{v}), \vec{w} \rangle$ (symmetric)

Notice that $\mathrm{I\!I}_p(\mathbb{X}_u) = \mathrm{I\!I}_p(\mathbb{X}_u, \mathbb{X}_u) = N\cdot\mathbb{X}_{uu} = e$

$\mathrm{I\!I}_p(\mathbb{X}_u, \mathbb{X}_v) = N\cdot\underset{=\mathbb{X}_{vu}}{\underbrace{\mathbb{X}_{uv}}} = f$

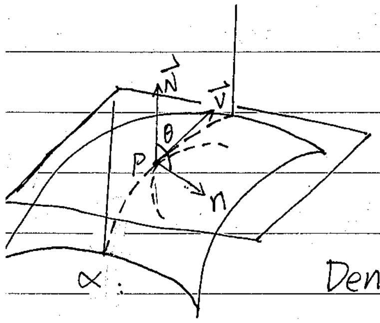
$\mathrm{I\!I}_p(\mathbb{X}_v) = \mathrm{I\!I}_p(\mathbb{X}_v, \mathbb{X}_v) = N\cdot\mathbb{X}_{vv} = g$

$\mathrm{I\!I}_p(\vec{v}) = \mathrm{I\!I}_p(a\mathbb{X}_u + b\mathbb{X}_v, a\mathbb{X}_u + b\mathbb{X}_v)$, $\vec{v} = a\mathbb{X}_u + b\mathbb{X}_v$

$= ea^2 + 2fab + gb^2$

where e, f, g are functions in p

rmal curvature :



$|\vec{v}| = 1$, $C = E \cap S$ is the curve, parametrize C by arc length α , $\alpha(0) = p$

$\alpha'(0) = \vec{v}$, $\alpha''(0) = T'(0) = k(0)n(0)$

$n(0) = \pm N_p$

Denote $k_n(\vec{v}) = N\cdot\boxed{\alpha''(0)}$ — Normal curvature

$= k(0)\underset{=\pm 1}{\underbrace{n(0)\cdot N_p}}$

re generally, for any curve $\alpha : I \to S$ with $\alpha(0) = p$

$\alpha'(0) = T(0) = \vec{v}$, parametrized by arc length

$/\cdot\alpha'' = N\cdot T' = N\cdot kn = k\cos\theta$

$\overset{\|}{\underset{0}{\cancel{(N\cdot T)'}}} - N'\cdot T = -dN_p(\vec{v})\cdot\vec{v} \qquad \because T(0) = \vec{v}$

$= \mathrm{I\!I}_p(\vec{v}) = \underset{\text{normal curvature}}{k_n(\vec{v})}$

Thm (Meusnier) : $k = \dfrac{k_n}{\cos\theta}$ (在 Gauss 之前，就有的結果).

$\alpha(t) = \mathbb{X}(u(t), v(t))$, $\alpha' = \mathbb{X}_u u' + \mathbb{X}_v v'$, $\vec{v} = \alpha'(0)$

◎ $\mathrm{I\!I}_p(\vec{v}) = N\cdot\alpha'' = N\cdot(\mathbb{X}_{uu}u'^2 + \mathbb{X}_{uv}u'v' + \cancel{\mathbb{X}_u}u'' + \mathbb{X}_{vu}u'v' + \mathbb{X}_{vv}v'^2 + \cancel{\mathbb{X}_v}v'')$

$= eu'^2 + 2fu'v' + gv'^2$ ($edu^2 + 2fdudv + gdv^2$)

$\mathrm{I\!I}_p(\vec{v}) = \mathrm{I\!I}_p(a\mathbb{X}_u + b\mathbb{X}_v, a\mathbb{X}_u + b\mathbb{X}_v) = ea^2 + 2fab + gb^2$

$\vec{v} = a\mathbb{X}_u + b\mathbb{X}_v$

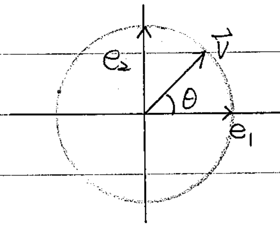
一組固有向量為垂直正交基底

◎ let e_1, e_2 be the eigenvector of $-dN_p$

$-dN_p(e_1) = \underset{\substack{\downarrow\\ \text{normal curvature}}}{k_1} e_1 \quad k_1 \geq k_2$, $-dN_p(e_2) = k_2 e_2$, $e_1 \perp e_2$.

$\mathrm{I\!I}_p(ae_1 + be_2) = \mathrm{I\!I}_p(ae_1 + be_2, ae_1 + be_2)$

$= \mathrm{I\!I}_p(e_1)a^2 + \underset{= -dN_p(e_1)e_2 = -ke_1e_2 = 0}{\underbrace{2ab\,\mathrm{I\!I}_p(e_1, e_2)}} + \mathrm{I\!I}_p(e_2)b^2 = k_1a^2 + k_2b^2$



$\vec{v} = \cos\theta e_1 + \sin\theta e_2 = T_pS$

$\mathrm{I\!I}_p(\cos\theta e_1 + \sin\theta e_2)$

$\|$ $\qquad\qquad\qquad\qquad \|$

Euler's Thm : $k_n(\cos\theta e_1 + \sin\theta e_2) = k_1\cos^2\theta + k_2\sin^2\theta$

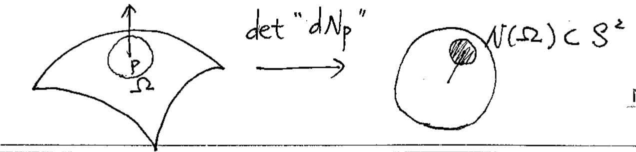
$\Rightarrow k_1$ = max of normal curvature

k_2 = min of normal curvature

Application : $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} k_n(\theta)\,d\theta = \dfrac{1}{2\pi}\cdot 2\pi\cdot\dfrac{1}{2}(k_1 + k_2)$

$= \underset{\swarrow}{2k_2} + \underset{\searrow}{\dfrac{k_1 - k_2}{2}}$

近視 閃光



NO.

Gauss's theory of surface | 在p点附近，法向量散開的程度

Gauss curvature $K_p =$ "$\det(dN_p)$" | 擴散的面積差異

(eg. in the basis e_1, e_2 $K_p = k_1 k_2$, $-dN_p \sim \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$)

Mean curvature $H = \frac{1}{2}(k_1 + k_2)$

Prop If $K_p \neq 0$, then $K_p = \lim_{\Omega \to p} \frac{A(N(\Omega))}{A(\Omega)}$

<pf>

pick parametrization $\mathbb{X}(u,v) : U \subset \mathbb{R}^2 \xrightarrow{\mathbb{X}} S \xrightarrow{N} S^2$

$N_u \times N_v = dN_p(\mathbb{X}_u) \times dN_p(\mathbb{X}_v)$

suppose $= (a\mathbb{X}_u + b\mathbb{X}_v) \times (c\mathbb{X}_u + d\mathbb{X}_v)$

$= (ad - bc)\mathbb{X}_u \times \mathbb{X}_v$ $\qquad N_u = \frac{\partial}{\partial u}(N \circ \mathbb{X})$

$= \det(dN_p)\, \mathbb{X}_u \times \mathbb{X}_v$

$$\frac{A(N(\Omega))}{A(\Omega)} = \frac{\frac{1}{|B|}\int_B |N_u \times N_v|\, du dv}{\frac{1}{|B|}\int_B |\mathbb{X}_u \times \mathbb{X}_v|\, du dv} \underset{\overrightarrow{M.V.T}}{\overset{相似}{\sim}} \frac{|N_u \times N_v|(b')}{|\mathbb{X}_u \times \mathbb{X}_v|(b)}$$

$\Omega = \mathbb{X}(B)$, $B \subset \mathbb{R}^2$

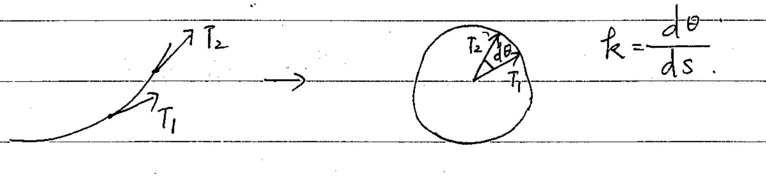
$\int f' = f'(c) \boxed{}$

$N \circ \mathbb{X} : U \longrightarrow S^2$

$\Omega \to P$, $b', b \to$ some pt q

$\mathbb{X}(q) = P$

$$\lim_{\Omega \to p} \frac{A(N(\Omega))}{A(\Omega)} = \det(dN_p) = K_p .$$

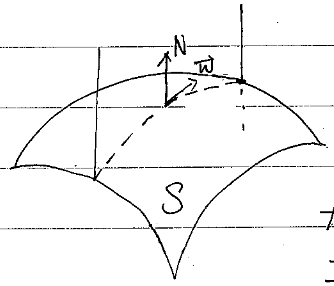


$k = \frac{d\theta}{ds}$.

NO. 11/5 (二)

⊙ 2nd fundamental form :

the function of normal curvature



$k_n(\vec{w}) = II_p(\vec{w})$

$= -dN_p(\vec{w}) \cdot \vec{w}$

for any curve $\alpha(t) = \mathbb{X}(u(t), v(t))$

$II_p(\alpha'(0)) = -\underline{dN_p(\alpha'(0))} \cdot \alpha'(0)$

$\left(\frac{d}{dt}(N \circ \alpha)\big|_{t=0}\right)$ by def $\alpha(0) = p$.

$\alpha(t) = \mathbb{X}(u(t), v(t))$ $\qquad = -N'(0) \cdot \alpha'(0)$

$\alpha'(t) = \mathbb{X}_u u' + \mathbb{X}_v v'$ $\qquad = N_p \cdot \alpha''(0)$

$= N_p \cdot (\mathbb{X}_{uu} u'^2 + 2\mathbb{X}_{uv} u'v' + \mathbb{X}_{vv} v'^2)$

$+ \underset{0}{\cancel{N_p \cdot \text{tangent}}}$

$= (N_p \cdot \mathbb{X}_{uu}) u'^2 + 2(N_p \cdot \mathbb{X}_{uv}) u'v' + (N_p \cdot \mathbb{X}_{vv}) v'^2$

$= (e du^2 + 2f du dv + g dv^2) / dt^2$

$\mathbb{X}_1 \sim \mathbb{X}_u$; $\mathbb{X}_2 \sim \mathbb{X}_v$

local formula for "$-dN_p$" : $T_pS \longrightarrow T_pS$

$☆_1 : -dN_p(\mathbb{X}_1) = a_{11}\mathbb{X}_1 + a_{12}\mathbb{X}_2 = -N_1 \Leftarrow N_1 = \frac{\partial N}{\partial u} = \frac{\partial}{\partial u}(N \circ \mathbb{X}(u,v))$

$☆_2 : -dN_p(\mathbb{X}_2) = a_{21}\mathbb{X}_1 + a_{22}\mathbb{X}_2 = -N_2$

$☆_1 \cdot \mathbb{X}_1$: $-N_1 \cdot \mathbb{X}_1 = N \cdot \mathbb{X}_{11} = e = a_{11}E + a_{12}F = II_p(\mathbb{X}_1)$

$☆_1 \cdot \mathbb{X}_2$: $-N_1 \cdot \mathbb{X}_2 = N \cdot \mathbb{X}_{21} = f = a_{11}F + a_{12}G = II_p(\mathbb{X}_1, \mathbb{X}_2)$

$$\begin{cases} a_{11} = \dfrac{\begin{vmatrix} e & F \\ f & G \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}} = \dfrac{eG - fF}{EG - F^2} \\[3ex] a_{22} = \dfrac{\begin{vmatrix} E & e \\ F & f \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}} = \dfrac{Ef - eF}{EG - F^2} \end{cases}$$

$$\star_2 \cdot \mathbb{X}_1 = f = a_{21}E + a_{22}F = \mathrm{II}_p(\mathbb{X}_2, \mathbb{X}_1) = \mathrm{II}_p(\mathbb{X}_1, \mathbb{X}_2)$$

$$\star_2 \cdot \mathbb{X}_2 = g = a_{21}F + a_{22}G = \mathrm{II}_p(\mathbb{X}_2)$$

$$a_{21} = \frac{fG - gF}{EG - F^2}, \quad a_{22} = \frac{Eg - Ff}{EG - F^2}.$$

$$-dN_p = \frac{1}{EG-F^2}\begin{pmatrix} eG - fF & Ef - Fe \\ fG - gF & Eg - Ff \end{pmatrix} \sim \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$$

wrt to the basis $\{\mathbb{X}_1, \mathbb{X}_2\}$.

○ $H = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}\mathrm{tr}(-dN_p) = \frac{1}{2}\frac{eG - 2fF + Eg}{EG - F^2}.$

平均曲率

○ $K = k_1 k_2 = \det(-dN_p) = \frac{eg - f^2}{EG - F^2}.$ 高斯曲率.

$$\left(\begin{aligned} &-(N_1 \ N_2) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} (\mathbb{X}_1 \ \mathbb{X}_2) \\ &N_1 \wedge N_2 = \det(-dN_p) \cdot \mathbb{X}_1 \wedge \mathbb{X}_2. \end{aligned} \right)$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} \Rightarrow \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{\det\begin{pmatrix} e & f \\ f & g \end{pmatrix}}{\det\begin{pmatrix} E & F \\ F & G \end{pmatrix}} = K.$$

Think: $\mathbb{X}(u,v)$: $\mathbb{X}_1$, $\mathbb{X}_2$, E, F, G.

$\mathbb{X}_{11}$, $\mathbb{X}_{12}$, $\mathbb{X}_{22}$; $N = \frac{\mathbb{X}_1 \times \mathbb{X}_2}{|\mathbb{X}_1 \times \mathbb{X}_2|}$

$$e = N \cdot \mathbb{X}_{11} = \frac{|\mathbb{X}_1 \ \mathbb{X}_2 \ \mathbb{X}_{11}|}{|\mathbb{X}_1 \times \mathbb{X}_2|} = \frac{|\mathbb{X}_1 \ \mathbb{X}_2 \ \mathbb{X}_{11}|}{\sqrt{EG - F^2}}$$

$$f = N \cdot \mathbb{X}_{12} = \frac{|\mathbb{X}_1 \ \mathbb{X}_2 \ \mathbb{X}_{12}|}{\sqrt{EG - F^2}}$$

$$g = N \cdot \mathbb{X}_{22} = \frac{|\mathbb{X}_1 \ \mathbb{X}_2 \ \mathbb{X}_{22}|}{\sqrt{EG - F^2}}$$

Ep ① Surface of revolution.

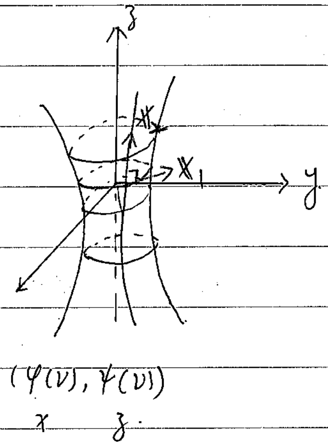
$\mathbb{X}(u,v) = (\varphi(v)\cos u, \varphi(v)\sin u, \psi(v))$

$\mathbb{X}_1 = (-\varphi \sin u, \varphi(v)\cos u, 0)$

$\mathbb{X}_2 = (\varphi' \cos u, \varphi' \sin u, \psi')$

$E = \varphi^2$, $F = 0$, $G = \varphi'^2 + \psi'^2$

$= 1$



$ds^2 = \varphi^2 du^2 + dv^2.$

$\sqrt{EG - F^2} = \varphi$

$v \mapsto (\varphi(v), \psi(v))$

$\quad\ \ x \quad\ \ z.$

$\overset{\prime\prime}{\alpha}(v)$, v = arc length

$\mathbb{X}_{11} = (-\varphi \cos u, -\varphi(v)\sin u, 0)$

⊚ $\underline{\mathbb{X}_{12} = (-\varphi' \sin u, \varphi' \cos u, 0)}$ // $\mathbb{X}_1$

$\mathbb{X}_{22} = (\varphi'' \cos u, \varphi'' \sin u, \psi'')$

$$e = \frac{|\mathbb{X}_1 \ \mathbb{X}_2 \ \mathbb{X}_{11}|}{\sqrt{EG - F^2}} = \frac{1}{\varphi}\begin{vmatrix} -\varphi \sin u & \varphi \cos u & 0 \\ \varphi' \cos u & \varphi' \sin u & \psi' \\ -\varphi \cos u & -\varphi \sin u & 0 \end{vmatrix}$$

$$= \frac{-1}{\varphi}\psi' \varphi^2 (\sin^2 u + \cos^2 u) = -\varphi \psi'$$

$f = 0$

$$g = \frac{|\mathbb{X}_1 \ \mathbb{X}_2 \ \mathbb{X}_{22}|}{\sqrt{EG - F^2}} = \frac{1}{\varphi}\begin{vmatrix} -\varphi \sin u & \varphi \cos u & 0 \\ \varphi' \cos u & \varphi' \sin u & \psi' \\ \varphi'' \cos u & \varphi'' \sin u & \psi'' \end{vmatrix}$$

$$= \frac{1}{\varphi}(\psi' \varphi \varphi'' - \psi'' \varphi \varphi') = \varphi'' \psi' - \varphi' \psi''$$

NO.

$$K = \frac{eg-f^2}{EG-F^2} = \frac{-\varphi\psi'(\varphi''\psi'-\varphi'\psi'')}{\varphi^4} = -\frac{1}{\varphi}(\varphi''\psi'^2-\varphi'\psi'\psi'') \quad (*)$$

since $\varphi'^2+\psi'^2=1$

$\Rightarrow \varphi'\varphi''+\psi'\psi''=0$

$$(*) = \frac{-1}{\varphi}(\psi'^2\varphi''-\varphi'(-\varphi'\varphi''))$$

$$= \frac{-1}{\varphi}(\varphi''(\psi'^2+\varphi'^2)) = \boxed{-\frac{\varphi''}{\varphi} = K}$$ 重要

$K = -\frac{\varphi''}{\varphi}$, $\varphi''+K\varphi=0$. (ODE)

球 $\underset{\not\Longleftarrow}{\Longrightarrow}$ $K=1$

Ep② graph $\mathbb{X}(x,y) = (x, y, \overset{z}{\overset{\shortparallel}{h(x,y)}})$

$\mathbb{X}_1 = (1, 0, h_x)$ $\quad N = \frac{(-h_x, -h_y, 1)}{\sqrt{1+h_x^2+h_y^2}}$

$\mathbb{X}_2 = (0, 1, h_y)$

$\mathbb{X}_{12} = (0, 0, h_{xy})$

$\mathbb{X}_{11} = (0, 0, h_{xx})$

$\mathbb{X}_{22} = (0, 0, h_{yy})$

$e = \frac{h_{xx}}{\sqrt{1+h_x^2+h_y^2}}$, $f = \frac{h_{xy}}{\sqrt{1+h_x^2+h_y^2}}$, $g = \frac{h_{yy}}{\sqrt{1+h_x^2+h_y^2}}$

$$K = \frac{eg-f^2}{EG-F^2} = \frac{h_{xx}h_{yy}-h_{xy}^2}{1+h_x^2+h_y^2}$$

$$H = \frac{1}{2}\frac{eG-2fF+gE}{EG-F^2} = \frac{1}{2}\frac{(1+h_x^2)h_{xx}-2h_xh_yh_{xy}+(1+h_y^2)h_{yy}}{(1+h_x^2+h_y^2)^{\frac{3}{2}}}$$

NO. 11/7 (四)

$S \xrightarrow{N} S^2$

$T_pS \xrightarrow{-dN} T_pS \cong T_pS^2$ self-adjoint

eigenvalues k_1 , k_2

$K = \det(-dN_p) = k_1 \cdot k_2$

$H = \frac{1}{2}\operatorname{tr}(-dN_p) = \frac{1}{2}(k_1+k_2)$

classification of points / curves

① $p \in S$ umbilical pt if $k_1 = k_2$ ("spherical")

prop: If all pts in S are umbilical, then $S \subset S_r^2$ or a plane

<pf> $dN_p(\vec{v}) = \lambda(p)\vec{v}$ for all $\vec{v} \in T_pS$

claim: λ is indep of p ie. a constant on S

$$(*) \begin{cases} N_1 = dN_p(\mathbb{X}_1) = \lambda\mathbb{X}_1 \\ N_2 = dN_p(\mathbb{X}_2) = \lambda\mathbb{X}_2 \end{cases} \qquad N_1 = \frac{\partial N}{\partial u} = \frac{\partial}{\partial u}(N \circ \mathbb{X})$$

$N_{12} = N_{21} \Rightarrow \lambda_2\mathbb{X}_1 + \lambda\mathbb{X}_{12} = \lambda_1\mathbb{X}_2 + \lambda\mathbb{X}_{21}$

$\Rightarrow \lambda_2\mathbb{X}_1 = \lambda_1\mathbb{X}_2$

(Notice: $\lambda = -\mathrm{II}_p(\vec{v})$ is C^∞)

but $\mathbb{X}_1$, $\mathbb{X}_2$ are linear indep. $\Rightarrow \lambda_1 = \lambda_2 = 0$

$\Rightarrow \lambda$ is a constant

if $\lambda = 0 \Rightarrow \mathbb{X}\cdot N$ is constant $\qquad N_1 = 0$, $N_2 = 0 \Rightarrow N =$ constant vect

$(\mathbb{X}\cdot N)_1 = \underset{0}{\cancel{\mathbb{X}_1 \cdot N}} + \mathbb{X}\cdot N_1 = 0 \quad (\because N_1 = \lambda\mathbb{X}_1 = 0)$

$(\mathbb{X}\cdot N)_2 = 0$

$\Rightarrow \mathbb{X}(u,v) \subset$ some plane $*$.

if $\lambda \neq 0$, $\boxed{\mathbb{X} - \frac{N}{\lambda} =: \mathbb{Y}}$ $\qquad \begin{cases} \mathbb{X}_1 - \frac{1}{\lambda}N_1 = 0 \\ \mathbb{X}_2 - \frac{1}{\lambda}N_2 = 0 \end{cases}$ by (*)

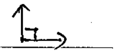
$\mathbb{Y}_1 = \mathbb{Y}_2 = 0$ $\Leftarrow$

$\Rightarrow \mathbb{Y}(u,v) =$ constant vector

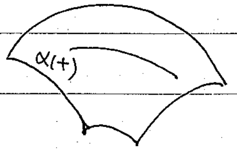
$\Rightarrow |\mathbb{X} - \mathbb{Y}| = \left|\frac{N}{\lambda}\right| = \frac{1}{\lambda}$

$\Rightarrow \mathbb{X}(u,v) \in$ sphere of radius $\frac{1}{\lambda}$ centered at $\mathbb{Y}$.

NO.

② Principal directions = "eigen" – directions 

line of curvature
(curve)
:= $\alpha'(t)$ is a principal direction in $T_{\alpha(t)}S$ $\forall t$



$$N'(t) = dN(\alpha'(t)) = \lambda(t)\alpha'(t)$$

Important : Is that possible to have a coor. system $\mathbf{x}(u,v)$ st. the coor. curves $\underset{\alpha(u)}{\overset{\shortparallel}{\mathbf{x}(u,v_0)}}$, $\underset{\alpha(v)}{\overset{\shortparallel}{\mathbf{x}(u_0,v)}}$ are all lines of curvatures ?

(ie. $\mathbf{x}_u, \mathbf{x}_v$ are exactly in the principal directions ?)

(Answer is Yes, locally near an non-umbilical pt)

$F = 0$

$dN_p(\mathbf{x}_1) = \lambda \mathbf{x}_1$

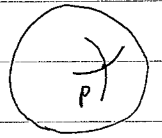
$f = N \cdot \mathbf{x}_{12} = -N_2 \cdot \mathbf{x}_1 = -dN_p(\mathbf{x}_2)\cdot \mathbf{x}_1 = -\lambda \underbrace{\mathbf{x}_2 \cdot \mathbf{x}_1} = 0$

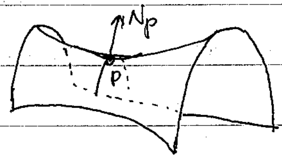
$ds^2 = E\,du^2 + G\,dv^2$
$= e\,du^2 + g\,dv^2$

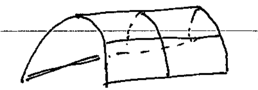
NO.

③ Semi – intrinsic classification of pts

$$\left\{\begin{array}{lll} p \in S, \text{ is} & \text{elliptic} & \text{if } k_p > 0 \\ & \text{hyperbolic} & \text{if } k_p < 0 \\ & \text{parabolic} & \text{if } k_p = 0 \\ & (\text{planner}) \hookrightarrow & \text{if } dN_p = 0 \end{array}\right.$$







$\mathbf{x}(0,0) = p$

$$\mathbf{x}(u,v) = p + \mathbf{x}_u u + \mathbf{x}_v \cdot v + \frac{1}{2}(\mathbf{x}_{uu}u^2 + 2\mathbf{x}_{uv}uv + \mathbf{x}_{vv}v^2) + R(u,v$$

$$\lim_{u,v\to 0} \frac{R}{\sqrt{u^2+v^2}} = 0$$

height function :

$d(u,v) := (\mathbf{x}(u,v) - p)\cdot N_p$ (N_p — z軸 , $d(u,v)$: z座標)

$= \boxed{\begin{array}{l}\frac{1}{2}\mathrm{II}_p(\vec{w}) \\ + R\cdot N_p\end{array}}$, $\vec{w} = \mathbf{x}_u u + \mathbf{x}_v v$
$(\mathbf{x}_1 u + \mathbf{x}_2 v)$

easy consequence

① If p elliptic, then $\left\{\begin{array}{l} d>0 \\ (\text{or } d<0 \end{array}\right\}$ in a nbd of p.

② If p is hyperbolic, then for any nbd U of p
∃ pts q st. $d>0$ and q' st. $d<0$ simultaneously

for parabolic / planner pts no such criterion

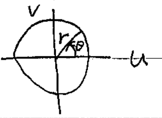
Ep graph $X(u,v) = (u, v, u^3 - 3uv^2)$

$h_{uu}(0) = h_{uv}(0) = h_{vv}(0) = 0$.

$dN_p \equiv 0$, $(u,v) = (0,0)$.

$u = r\cos\theta$

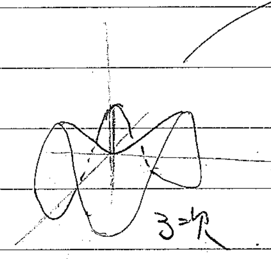
$v = r\sin\theta$



$f(u,v) = u^3 - 3uv^2 = r^3(\cos^3\theta - 3\cos\theta \cdot \sin^2\theta)$

$= r^3(4\cos^3\theta - 3\cos\theta)$

$= r^3 \cos(3\theta)$



$\nearrow 0$

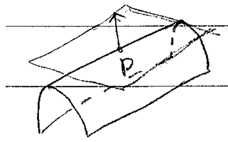
$X(u,v) = (u, v, 4u^2v^2)$

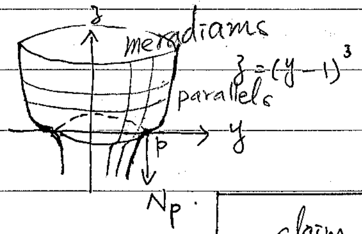
$f(u,v) = 4u^2v^2 = 4r^4\cos^2\theta \sin^2\theta$

$= r^4(\sin 2\theta)^2$

$= r^4 \frac{1-\cos 4\theta}{2}$ ← 振盪4次

Ep parabolic.





$T_p S = (x, y)$ - plane

> claim: for surface of revolution
> the line of curvature are exactly
> the meradians & parallels.

Actually, the diff eq'n for line of curvature is given by

$N'(t) = dN(\alpha'(t)) = \lambda(t)\,\alpha'(t)$, $\alpha(t) = X(u(t), v(t))$

$N_u u' + N_v v' = \lambda(X_u u' + X_v v')$

$-N_1 = a_{11} X_1 + a_{12} X_2$

$-N_2 = a_{21} X_1 + a_{22} X_2$. (exercise)

Exercise: the diff equation in (u,v) - coordinate

is $\begin{vmatrix} (u')^2 & -u'v' & (v')^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0$

Cor. A coor system has its coor. curves to the line of curvature

$\Leftrightarrow F = 0 = f$

$(\Leftarrow)$ the determinant $= u'v' \begin{vmatrix} E & G \\ e & g \end{vmatrix} = 0$

so $u' = 0$ or $v' = 0$ are solutions i.e. $u =$ const or $v =$ const.

④ Asymptotic directions

i.e. vector $\vec{w} \in T_pS$ st. $II_p(\vec{w}) = 0$

asymptotic curve: $\alpha(t) = X(u(t), v(t))$ s.t. $II(\alpha'(t)) = 0 \quad \forall t$

$\qquad \| $

$e u'^2 + 2 f u'v' + g v'^2 = 0$

* Fact: for p a hyp pt,

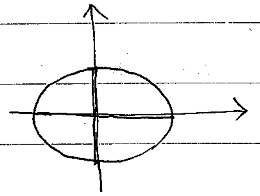
the coor. curves are asymp curves $\Leftrightarrow e = g = 0$

Hilbert: any complete abstract surface with $K = -1$ can not be isometrically embeded in $\mathbb{R}^3$

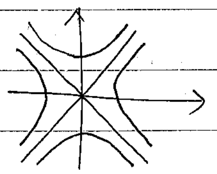
Def Dupin indicatrix "approximation graph" (level curve)

$D_p \subseteq T_pS$ st. $II_p(\underset{\shortparallel}{\vec{w}}) = \pm 1$

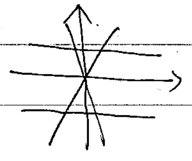
$\overset{\psi}{\vec{w}} = u\vec{e}_1 + v\vec{e}_2$ $\quad k_1u^2 + k_2v^2 = \pm 1$



elliptic



hyperbolic



$\boxed{\text{parabolic}}$

$$d(u,v) = (\mathbb{X}(u,v) - p)\cdot N_p$$
$$= \frac{1}{2}II_p(\vec{w}) + R\cdot N_p$$

Ex §3.2 — 3. 7. 9. 12. 13. 15. 17. 19. 20

§3.3 — 4. 5. 7. 10. 16. 20. 21. 22. 23. 24.

§3.4 — 1. 5. 7. 9

☆ $E\,du^2 + 2F\,du\,dv + G\,dv^2$.

$$K = \frac{eg - f^2}{EG - F^2}$$

$$e = N\cdot\mathbb{X}_{11} = \frac{\mathbb{X}_1\times\mathbb{X}_2}{|\mathbb{X}_1\times\mathbb{X}_2|}\cdot\mathbb{X}_{11} = \frac{|\mathbb{X}_1\ \mathbb{X}_2\ \mathbb{X}_{11}|}{|\mathbb{X}_1\times\mathbb{X}_2|}$$

$$f = N\cdot\mathbb{X}_{12} = \frac{|\mathbb{X}_1\ \mathbb{X}_2\ \mathbb{X}_{12}|}{|\mathbb{X}_1\times\mathbb{X}_2|}$$

$$g = N\cdot\mathbb{X}_{22} = \frac{|\mathbb{X}_1\ \mathbb{X}_2\ \mathbb{X}_{22}|}{|\mathbb{X}_1\times\mathbb{X}_2|} = \frac{|\mathbb{X}_1\ \mathbb{X}_2\ \mathbb{X}_{22}|}{\sqrt{EG - F^2}}$$

$$K = \frac{1}{(EG-F)^2}\left\{|\mathbb{X}_1\ \mathbb{X}_2\ \mathbb{X}_{11}|\cdot|\mathbb{X}_1\ \mathbb{X}_2\ \mathbb{X}_{22}| - |\mathbb{X}_1\ \mathbb{X}_2\ \mathbb{X}_{12}|^2\right\}$$

$|AB| = |A||B|$

$|A^t| = |A|$

$$\begin{bmatrix}\mathbb{X}_1 & \mathbb{X}_2 & \mathbb{X}_{22}\end{bmatrix}$$

$$\begin{bmatrix}\mathbb{X}_1{}^t\\ \mathbb{X}_2{}^t\\ \mathbb{X}_{11}{}^t\end{bmatrix}\begin{bmatrix}\mathbb{X}_1\cdot\mathbb{X}_1 & \mathbb{X}_1\cdot\mathbb{X}_2 & \mathbb{X}_1\cdot\mathbb{X}_{22}\\ \mathbb{X}_1\cdot\mathbb{X}_2 & \mathbb{X}_2\cdot\mathbb{X}_2 & \mathbb{X}_2\cdot\mathbb{X}_{22}\\ \mathbb{X}_{11}\cdot\mathbb{X}_1 & \mathbb{X}_{11}\cdot\mathbb{X}_2 & \mathbb{X}_{11}\cdot\mathbb{X}_{22}\end{bmatrix}$$

$$\mathbb{X}_1\cdot\mathbb{X}_{22} = (\mathbb{X}_1\cdot\mathbb{X}_2)_2 - \mathbb{X}_{12}\cdot\mathbb{X}_2$$
$$= F_2 - \frac{1}{2}(\mathbb{X}_2\cdot\mathbb{X}_2)_1$$
$$= F_2 - \frac{1}{2}G_1$$

$$\mathbb{X}_2\cdot\mathbb{X}_{22} = \frac{1}{2}(\mathbb{X}_2\cdot\mathbb{X}_2)_2 = \frac{1}{2}G_2$$

$$\det{}_1 = \begin{vmatrix} E & F & F_2 - \frac{1}{2}G_1\\ F & G & \frac{1}{2}G_2\\ \frac{1}{2}F_1 & F_1 - \frac{1}{2}G_2 & \boxed{\mathbb{X}_{11}\cdot\mathbb{X}_{22}}\end{vmatrix}$$

$\mathbb{X}_{11}\cdot\mathbb{X}_{22} = (\mathbb{X}_1\cdot\mathbb{X}_{22})_1 - \underset{\sim\sim\sim}{\mathbb{X}_1\cdot\mathbb{X}_{221}}$ 沒希望了..

$= F_{21} - \frac{1}{2}G_{11} - \mathbb{X}_1\cdot\mathbb{X}_{221}$

變成 $\mathbb{X}_{11}\cdot\mathbb{X}_{22} - \mathbb{X}_{12}\cdot\mathbb{X}_{12} = F_{21} - \frac{1}{2}G_{11} - (\mathbb{X}_1\cdot\mathbb{X}_2$

$+ \mathbb{X}_{12}\cdot\mathbb{X}_{12}$ $\overset{\shortparallel}{\frac{1}{2}E_2}$

$$\boxed{\mathbb{X}_{11}\cdot\mathbb{X}_{22}} = F_{21} - \frac{1}{2}G_{11} - \frac{1}{2}E_{22}$$

$$\begin{bmatrix}\mathbb{X}_1 & \mathbb{X}_2 & \mathbb{X}_{12}\end{bmatrix}$$

$$\begin{bmatrix}\mathbb{X}_1{}^t\\ \mathbb{X}_2{}^t\\ \mathbb{X}_{12}{}^t\end{bmatrix}\begin{bmatrix}E & F & \mathbb{X}_1\cdot\mathbb{X}_{12}\\ F & G & \mathbb{X}_2\cdot\mathbb{X}_{12}\\ \mathbb{X}_1\cdot\mathbb{X}_{12} & \mathbb{X}_2\cdot\mathbb{X}_{12} & \mathbb{X}_{12}\cdot\mathbb{X}_{12}\end{bmatrix}$$

$$\mathbb{X}_1\cdot\mathbb{X}_{12} = \frac{1}{2}(\mathbb{X}_1\cdot\mathbb{X}_1)_2 = \frac{1}{2}E_2$$

$$\mathbb{X}_2\cdot\mathbb{X}_{12} = \frac{1}{2}(\mathbb{X}_2\cdot\mathbb{X}_2)_1 = \frac{1}{2}G_1$$

神奇地移過去

$$\det{}_2 = \begin{vmatrix} E & F & \frac{1}{2}E_2\\ F & G & \frac{1}{2}G_1\\ \frac{1}{2}E_2 & \frac{1}{2}G_1 & \boxed{\mathbb{X}_{12}\cdot\mathbb{X}_{12}}\end{vmatrix}$$

變成 0

$\det_1 - \det_2$

1827 General Investgation
of curved surface

NO.

1827 Thm (Gauss)   (Theorema Egregium)
A most excellent theorem

$K = \frac{1}{(EG-F^2)^2}(\det_1 - \det_2)$

K depends only on the first fundamental form, not on $\mathbb{X}$, N or II

Intrinsic (内在)
extrinsic (外在)
* Riemann 1850

NO. 11/14 (四)

How to construct coordinate systems?

eg. $p \in S$, not an umbilic pt
principal directions $e_1 \perp e_2$ $\overset{e_2}{\llcorner} e_1$

Q: Is there a coor system $\mathbb{X}(u,v)$ st.
$\mathbb{X}_1 = e_1$, $\mathbb{X}_2 = e_2$ 不可能

If true, $\Rightarrow E = \mathbb{X}_1 \cdot \mathbb{X}_1 = 1$
$F = \mathbb{X}_1 \cdot \mathbb{X}_2 = 0$ $\Rightarrow ds^2 = du^2 + dv^2$
$G = \mathbb{X}_2 \cdot \mathbb{X}_2 = 1$ $\Rightarrow K \equiv 0$ 跟平面一樣

10中最想要的是 $F=0$

修正 $\mathbb{X}_1 = \lambda e_1$
$\mathbb{X}_2 = \mu e_2$, λ, μ are func.

$$\Rightarrow \begin{cases} E = \lambda^2 \\ G = \mu^2 \\ F = 0 \end{cases}$$ 真的可以做到.

in this special case, we want line of curvature to be the coordinate line

$$-dN_{\alpha(t)}\alpha'(t) = k_1 \alpha'(t)$$

$\alpha'(t) = \lambda e_1$ $\alpha(t) = \mathbb{X}(u(t), v(t))$

$\alpha(t)$ $-dN = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\alpha'(t) = \mathbb{X}_1 u' + \mathbb{X}_2 v'$

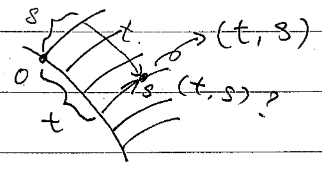
wrt the basis $\mathbb{X}_1, \mathbb{X}_2$

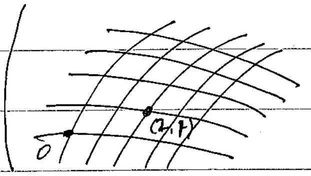
$$\begin{cases} a_{11}u' + a_{12}v' = k_1 u' \\ a_{21}u' + a_{22}v' = k_1 v' \end{cases}$$

$$(a_{11}u' + a_{12}v')v' = (a_{21}u' + a_{22}v')u'$$

$$a_{11}u'^2 + (a_{11} - a_{22})u'v' + a_{12}v'^2 = 0$$

$$\hookrightarrow \begin{vmatrix} v'^2 & -u'v' & u'^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0 \quad \text{(ODE)}$$





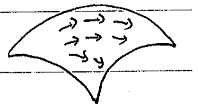
直觀

Frobenius integrability thm

Main theorem

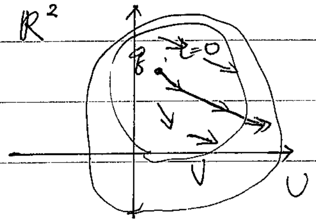
☆ Thm Let w_1, w_2 be 2 (C^∞) vector fields on S
st. $w_1(p), w_2(p)$ are linear independently,
then $\exists \underset{\text{nbd}}{U} \ni p$ and a $\mathbb{X}$: coor system on U
st. $\mathbb{X}_1 \parallel W_1$, $\mathbb{X}_2 \parallel W_2$

vector field
$w: S \to \mathbb{R}^3$ C^∞ map
$p \mapsto w(p) \in T_pS \subseteq \mathbb{R}^3$



C^∞, ie. $W \circ \mathbb{X}$
$W(u,v)$ is C^∞

on $\mathbb{R}^2$



v.f. $F(x,y) : \underset{\mathbb{R}^2}{\overset{\cap}{U}} \to \mathbb{R}^2$

ODE : Find $\alpha(t) = (X(t), y(t))$
s.t. $\frac{d\alpha}{dt} = F(\alpha(t))$

initial $\alpha(0) = \xi$

Thm : $\exists !$ solution for some $(t \in (-\varepsilon, \varepsilon))$ maximal time interval

Want to solve the "flow" of F

$$\begin{cases} \frac{\partial \alpha(\xi, t)}{\partial t} = F(\alpha(t)) \\ \alpha(\xi, 0) = \xi \end{cases}$$

☆ Thm : $\alpha(\xi, t)$ is C^∞ in ξ · smooth dependence on initial data
(Hirsh - smale)

$\alpha(\xi, t)$ is defined on some $\underbrace{V \times (-\varepsilon, \epsilon)}_{\to U}$ for $V \Subset U$

存在，唯一，光滑性． α is called trajectory of F

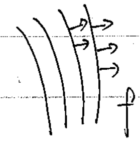
Basic construction

if $F : U \to \mathbb{R}^2$, $F(p) \neq 0$

then $\exists \underset{p}{\overset{\Psi}{V}} \subset U$ and $f : V \to \mathbb{R}$

st. f = constant along any trajectoriy of F (ie. level set)

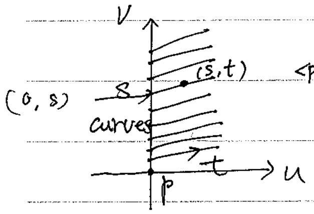
Moreover, $df_\xi \neq 0$ $\forall \xi \in V$



$\nabla f_\xi \neq 0$

Let $F(p) = (\underset{0}{\overset{}{F^1}}{}_{\#}, F^2)$

will show that (t, s) is a coor. system.



<pf> Let $p = (0, 0)$

consider $\alpha((0, s), t) = G(t, s)$

$$dG = \left(\frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial s} \right)\Big|_{(0,0)}$$

$$= \begin{pmatrix} F_1 & 0 \\ F_2 & 1 \end{pmatrix} = F_1 \neq 0$$

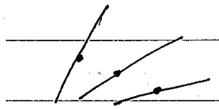
by inverse func. thm , $(t, s) = (t(u,v), s(u,v))$
$\alpha((0, s(u,v)), t(u,v)) = (u, v)$

So , let $f(u,v) = s(u,v)$

$\underset{c}{\overset{}{f = \text{constant}}}$ gives the trajectory $\alpha(\underset{(0,c)}{\overset{}{\xi}}, t)$

f is called the first integral of F (首次積分)

r: field of directions /vector field.
(line distribution)



r defined on $U \subset \mathbb{R}^2$

$p \longmapsto r(p)$ a line in $\mathbb{R}^2$

r is differentiable (C^∞) if

locally, $\exists$ v.f. $\underset{C^\infty}{F} \neq 0$ s.t. $\mathbb{R}F(p') = r(p')$

$\forall p'$ near p

line of curvature $\begin{vmatrix} v'^2 & -u'v' & u'^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0$

$$A\,du^2 + B\,du\,dv + C\,dv^2 = 0$$

$$\Rightarrow A\left(\frac{du}{dv}\right)^2 + B\left(\frac{du}{dv}\right) + C = 0$$

equivalently, $a(u,v)\,du + b(u,v)\,dv = 0$

$$\rightarrow (a,b)\cdot(du,dv) = 0$$

From $\mathbb{R}^2$ to S, a tangent vector field $\vec{w}$ on $U \subset S$

i.e. $\vec{w}(p) \in T_pS \quad \forall p \in U$

$$\vec{w} = a(u,v)\,\mathbf{x}_1 + b(u,v)\,\mathbf{x}_2$$

$$= d\mathbf{x}(\underbrace{ae_1 + be_2}_{\begin{bmatrix} a \\ b \end{bmatrix}})$$

still have the notion of trajectory.

i.e. $\alpha(t) \subset S$ st. $\frac{d\alpha}{dt} = \vec{w}$

if $\frac{dc}{dt} = F$ ($\exists !$)

Apply $d\mathbf{x} \Rightarrow \underset{\overset{\parallel}{\frac{d}{dt}(\mathbf{x}\circ c) = \frac{d\alpha}{dt}}}{d\mathbf{x}\left(\frac{dc}{dt}\right)} = d\mathbf{x}(F) = \vec{w}$

Main theorem

<pf> w_1, w_2 2 v.f. on $U \subset S$

w_1, w_2 linearly indep at p.

consider f_1, f_2 1st. integrals of w_1, w_2 in $V \subset U$

Let $\varphi: V \rightarrow \mathbb{R}^2$

$$q \mapsto (f_1(q), f_2(q)) = (u,v) \in \mathbb{R}^2$$

$d\varphi_p(w_1) = (\underset{\overset{\parallel}{0}}{\underline{df_{1p}(w_1)}}, \underset{\overset{\nparallel}{0}}{\underline{df_{2p}(w_1)}})$ because $df_{2p} \neq 0$ and w_1 is l. indep of w_2, $df_{2p}(w_2) = 0$

$df(w) = \frac{d}{dt}(f\circ\alpha)\Big|_{t=0}$

$\alpha'(0) = w$.

$$d\varphi_p(w_2) = (\neq 0, 0)$$

$d\varphi_p$ is nonsingular, ie $d\varphi_p$ is rank 2

$\therefore \varphi$ is a local diffeomorphism

$\Rightarrow \exists\, \mathbf{x} = \varphi^{-1}$ onto a nbd of p in S

st. the coor. curves and $f_1 = \text{const}$.

$f_2 = \text{const}$. ※

Notice! $\mathbf{x}_1 \neq w_1$, $\mathbf{x}_2 \neq w_2$

① $\frac{d\alpha}{dt} = F(\alpha(t))$

α: trajectory.

② $\underline{a\,du + b\,dv = 0}$

sol: this C is called an integral curve

<u>Cor</u> 2 fields of directions $r_1 \neq r_2$

$\Rightarrow \exists\ \mathbb{X}$ st. the coor curves

= integral curves of r_1, r_2.

<u>Cor</u> $\exists$ orthogonal paramatrization in any point

$\mathbb{X}_1, \mathbb{X}_2$, $\widetilde{\mathbb{X}}_2 := \mathbb{X}_2 - \frac{\mathbb{X}_2 \cdot \mathbb{X}_1}{\mathbb{X}_1 \cdot \mathbb{X}_1} \mathbb{X}_1 = \mathbb{X}_2 - \frac{F}{E} \mathbb{X}_1$

$\widetilde{\mathbb{X}}_1 := \mathbb{X}_1$

$\widetilde{\mathbb{X}}_1 \perp \widetilde{\mathbb{X}}_2$

by Main theorem $\Rightarrow$ New coor system $\underset{\substack{\uparrow \\ 不行}}{\cancel{\mathbb{X}}}$ $\mathbb{Y}$

<u>Cor</u> If $K_p < 0$ (hyperbolic pt)

$\mathrm{II}_p(ae_1 + be_2) = k_1 a^2 + k_2 b^2 = (\qquad)(\qquad)$

$\exists$ 2 families of asymptotic directions.

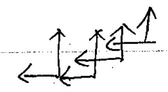
apply maim thm get $\mathbb{X}(u,v)$

$$e u'^2 + 2 f u'v' + g v'^2 = 0$$

◎ <u>Cor</u> p is not an umbilic pt

$k_1 \neq k_2$ at p.

$k_1 \neq k_2$ on $\underset{p\in}{U}$



$e_1 \perp e_2$

$$\begin{vmatrix} v'^2 & -u'v' & u'^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0$$

$\mathbb{X}(u,v)$ has coor. curve

= line of curvature.

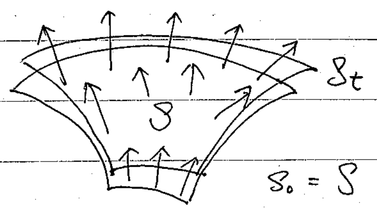
$\Longleftrightarrow F = \underline{\underline{f}} = 0$

NO. 11/19(三)

Minimal surfaces

K : Gauss curvature

$\rightsquigarrow$ H : mean curvature



Def : S is minimal surface if $H \equiv 0$

S : surface in R^3 given by $\mathbb{X} : \underset{\subset \mathbb{R}^2}{\overline{D}} \longrightarrow \mathbb{R}^3$

$\mathbb{X}(t;u,v) : (-\varepsilon, \varepsilon) \times D \to \mathbb{R}^3$ is a normal variation if

$\mathbb{X}(t;u,v) = \mathbb{X}(u,v) + t \cdot \underline{h} N(u,v)$

$h(u,v) \sim$ any func.

Define S_t = image of $\mathbb{X}(t;\cdot,\cdot)$

(或 $\mathbb{X}^t$)

有興趣的是面積怎麼變化

Area of $S_t = A(t) = \int_{\overline{D}} \sqrt{E(t)G(t) - F^2(t)}\ dudv$

$$\mathbb{X}_1^t = \mathbb{X}_1 + th_1 N + th N_1$$

$$\mathbb{X}_2^t = \mathbb{X}_2 + th_2 N + th N_2$$

$-N \cdot \mathbb{X}_{11} = -e$

$$E(t) = \mathbb{X}_1^t \cdot \mathbb{X}_1^t = E + 2th \boxed{\mathbb{X}_1 \cdot N_1} + t^2(h_1^2 + h^2 N_1 \cdot N_1)$$

$$F(t) = \mathbb{X}_1^t \cdot \mathbb{X}_2^t = F + th(\underline{\mathbb{X}_1 \cdot N_2} + \underline{\mathbb{X}_2 \cdot N_1}) + t^2(h_1 h_2 + h^2 N_1 \cdot N_2)$$

$=-f \qquad =-f$

$$G(t) = \mathbb{X}_2^t \cdot \mathbb{X}_2^t = G + 2th \underset{=-g}{\underline{\mathbb{X}_2 \cdot N_2}} + t^2(h_2^2 + h^2 N_2 \cdot N_2)$$

~~$E(t)G(t) - F^2(t) = EG - F^2 - 2th(Eg + Ge - 2Ff) + t^2$ 及高次~~

$$= (EG - F^2)(1 - 4th H + t^2 \tilde{R})$$

$$A(t) = \int_D \sqrt{EG-F^2}\sqrt{1-4thH+t^2\widehat{R}}\,dudv$$

$$A'(t) = \int_D \sqrt{EG-F^2}\,\frac{-4hH+2t\widehat{R}}{2\sqrt{1-4thH+t^2\widehat{R}}}\,dudv$$

$$A_h'(0) = -2\int_D hH\sqrt{EG-F^2}\,dudv$$

$\uparrow$
和 h 有關

$A_h'(0) = 0 \quad \forall h \iff H \equiv 0$

<pf> (⇐) ok! (⇒) Let $h = H$ if $H \neq 0$

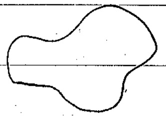
$A_h'(0) \leq 0$ ⨯

$\therefore H \equiv 0$

Prop : S is a critical pt for the area func. in normal variation
$\iff$ S is a minimal surface
(i.e. $H \equiv 0$)

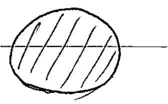
Remark : plateau problem

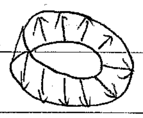
1935 Douglas Rado



$\Gamma \subseteq \mathbb{R}^3$
curve (piecewise C^1)

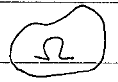
prove $\exists$ a surface S with $\partial S = \Gamma$ st. S homeo D^2 and
$A(S)$ is the smallest among all S' with $\partial S' = \Gamma$ and $S' \cong D^2$





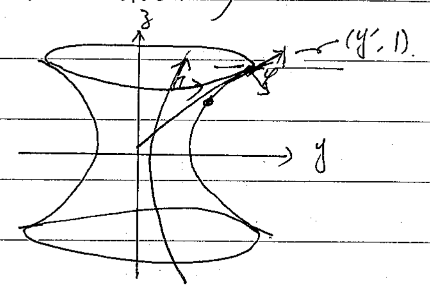
Mobius band

$A(S)$ under the restriction that $S = \partial\Omega$ $Vol(\Omega) = $ fixed
Ω can vary



$S \iff H$ is constant.

Ep (Catenoid) = surface of revolution of "catenary"



$y = a\cosh\frac{z}{a}$, $a \in \mathbb{R}^+$

is the unique surface of revolution
which is a minimal surface.

$y = f(z)$ $\qquad k_1 = \dfrac{y''}{(\sqrt{1+y'^2})^3}$

$$k_2 = -\frac{1}{y}\cdot\cos\varphi = -\frac{1}{y}\frac{1}{\sqrt{1+y'^2}}$$

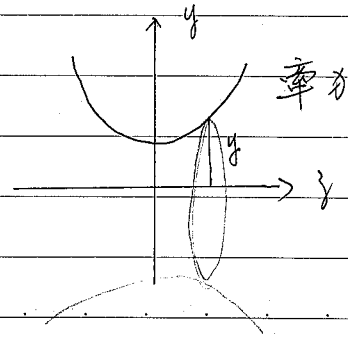
$$\frac{y''}{(\sqrt{1+y'^2})^3} = \frac{1}{y}\frac{1}{\sqrt{1+y'^2}} \Rightarrow \frac{y''y}{1+y'^2} = 1$$

$$\Rightarrow \frac{2y'y''}{1+y'^2} = \frac{2y'}{y}$$

$\therefore \log(1+y'^2) = 2\log y + C$

$1+y'^2 = k^2y^2$

$y = \frac{1}{k}(\cosh(kz)+\tilde{C})$



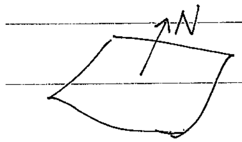
牽狗線.

$\int dm\,gy = \int ds\,\rho gy$

$= \rho g\int y\sqrt{1+y'^2}\,dz = V(y(z))$ func. of z

旋轉體面積

Minimal Surfaces ($H \equiv 0$ 的 surface)



$X^t = X + thN$, $A(t)$ = Area S_t.

$A'(0) = -2\int_D hH \cdot dA$

mean curvature vector $\vec{H} = H \cdot N$

$A \downarrow$ along the $\vec{H}$ direction.

$A \nearrow$ along the $-\vec{H}$ direction.

($A'(0) = 0$ $\forall$ normal variation $\Leftrightarrow$ $H \equiv 0$)

☆☆ In fact, if $H \equiv 0$, then $A''(0) \geq 0$ $\forall$ normal variation

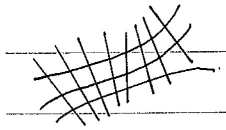
i.e. S is local minimum (一定要做)
(stable)

[Hint : $A(t) = \int_D \sqrt{1 - \overset{0}{\cancel{4thH}} + t^2 \underbrace{A(t)}_{\text{算清楚}}}\, dA$]

Ruled Surface = 1 parameter family of straight lines in R^3
↑
line

i.e. $X(t,v) = \underset{\substack{\uparrow \\ \text{curve in } R^3}}{\alpha(t)} + v \cdot \underset{\substack{\uparrow \\ \text{vector field on } \alpha(t)}}{w(t)}$, $|w| = 1$

i.e. $w : [a,b] \to R^3$, C^∞



There are ∞ many choice of α.

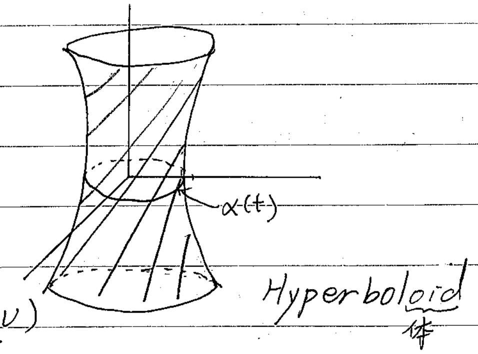
$x^2 + y^2 - z^2 = 1$

$\Rightarrow \alpha(t) = (\cos t, \sin t)$

$w(t) = \alpha'(t) + e_3$

$X(t,v) = \alpha + vw$

$= (\underbrace{\cos t - v\sin t}_{x}, \underbrace{\sin t + v\cos t}_{y}, \underset{z}{v})$



$x^2 + y^2 \doteq 1 + v^2 \doteq z^2 + 1$

Remark: "This" is the only surface with both ruled and

2 basis facts :

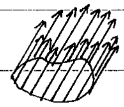
- $\exists!$ line of striction i.e. a section $\beta(t)$ st. $\beta' \cdot w' = 0$ 每個線上只有一点.

<pf> . $\beta(t) = \vec{\alpha}(t) + f(t)\vec{w}(t)$

$\beta' \cdot w' = (\alpha' + f'\underbrace{w + fw') \cdot w'}_{\text{垂直}} = \alpha' \cdot w' + f \cdot |w'|^2 = 0$.

$\Longleftrightarrow f = -\frac{\alpha' \cdot w'}{|w'|^2}$ $(w' \neq 0)$

(if $w' = 0$ $\Rightarrow$ cyclindrical



不討論)

- $K \leq 0$: Assume that $\alpha \equiv \beta$ = line of striction

$X = \beta + vw$

$X(u,v) = \beta(u) + v \cdot w(u)$

$X_1 = \beta' + vw'$

$X_2 = w$

$X_1 \wedge X_2 = \underset{\substack{\| \\ \lambda(u)w'}}{\underbrace{\beta' \times w}} + vw' \times w$, $|X_1 \wedge X_2|^2 = \lambda^2 |w|^2 + v^2 |w' \wedge w|^2 = (\lambda^2 + v^2)|w$

ruled surface S is a parametrized surface.

P is singular $\Leftrightarrow$ $v = 0$ (i.e. pt $\mathrm{Im}\beta$) & $\lambda = 0$

$\begin{cases} X_{11} = \beta'' + vw'' \\ X_{12} = w' \\ X_{22} = 0 \end{cases}$ $\Rightarrow$ $g = 0$

$f = \frac{|X_1\ X_2\ X_{12}|}{|X_1 \wedge X_2|} = \frac{|\beta' + \cancel{vw'}\ \ w\ \ w'|}{\sqrt{EG - F^2}}$

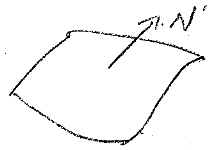
$= \frac{(\beta' \times w) \cdot w' = \lambda w' \cdot w'}{\sqrt{EG - F^2}}$

$K = \frac{\overset{0}{\cancel{eg}} - f^2}{EG - F^2} = -\frac{\lambda^2 |w'|^4}{(EG - F^2)^2} = -\frac{\lambda^2}{(\lambda^2 + v^2)^2} \leq 0$

$K_p = 0$

NO.

Minimal Surfaces. ($H \equiv 0$ 的 surface) 11/21 (四)



$X^t = X + thN$, $A(t) = \text{Area } S_t$.

$A'(0) = -2\int_D hH \cdot dA$.

mean curvature vector $\vec{H} = H \cdot N$

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☆☆ In fact, if $H \equiv 0$, then $A''(0) \geqslant 0$ $\forall$ normal variation

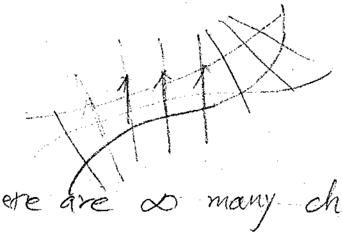
ie. S is $\underset{\text{(stable)}}{\text{local}}$ minimum (一定要做)

[Hint : $A(t) = \int_D \sqrt{1 - \underset{\overset{\|}{0}}{\cancel{4thH}} + t^2 \underset{算清楚}{\underline{A(t)}}}\, dA$.]

$\underset{\overset{\uparrow}{\text{(line)}}}{\underline{\text{Ruled}}}$ Surface = 1 parameter family of straight lines in $\mathbb{R}^3$

ie. $X(t,v) = \underset{\overset{\nearrow}{\text{curve in } \mathbb{R}^3}}{\alpha(t)} + v \cdot \underset{\overset{\uparrow}{\text{vector field on } \alpha(t)}}{w(t)}$, $|w| = 1$

ie. $w : [a,b] \to \mathbb{R}^3$, C^∞



There are ∞ many choose of α.

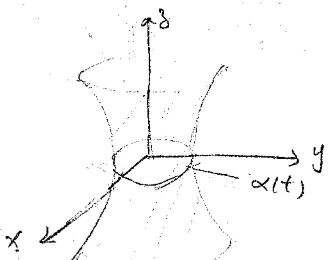
Remark: "This" is <u>the only</u> surface with is both ruled and a surface revolution

$x^2 + y^2 - a^2 z^2 = b^2$.

$x^2 + y^2 - z^2 = 1$

$\Rightarrow \alpha(t) = (\cos t, \sin t)$.

$w(t) = \alpha'(t) + e_3$.



$\text{Hyperb}\overbrace{\text{oloid}}^{体}$.

$X(t,v) = \alpha + vw$

$= (\underbrace{\cos t - v\sin t}_{x}, \underbrace{\sin t + v\cos t}_{y}, \underbrace{v}_{z})$ $x^2 + y^2 = 1 + v^2 = 1 + z^2$.

2. basic facts :

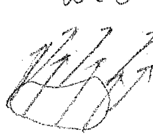
- $\exists !$ line of <u>striction</u> ie. a section $\beta(t)$ s.t. $\beta' \cdot w' = 0$ 　每個線上只有一点

<pf> $\beta(t) = \vec{\alpha}(t) + f(t)\,\vec{w}(t)$

$$\beta' \cdot w' = (\alpha' + f'\underbrace{w + fw') \cdot w'}_{垂直} = \alpha' \cdot w' + f \cdot |w'|^2 = 0$$

$$\Leftrightarrow \quad f = -\frac{\alpha' \cdot w'}{|w'|^2} \qquad (w' \neq 0)$$

$w'=0$ 不討論



cyclindrical

- $K \leq 0$: Assume that $\alpha \equiv \beta$ = line of striction.

$\mathbb{X} = \beta + vw$

$\mathbb{X}(u,v) = \beta(u) + v \cdot w(u)$

$$\begin{cases} \mathbb{X}_1 = \beta' + vw' \\ \mathbb{X}_2 = w \end{cases}$$

$$\mathbb{X}_1 \times \mathbb{X}_2 = \underbrace{\beta' \times w}_{\underbrace{\lambda(u)}\, w'} + v\, w' \times w$$

$$|\mathbb{X}_1 \times \mathbb{X}_2|^2 = \lambda^2 |w'|^2 + v^2 |w' \times w|^2 = (\lambda^2 + v^2)|w'|^2$$

ruled surface S is a parametrized surface

<u>p singular $\Leftrightarrow$ $v=0$ (ie. pt. $\mathrm{Im}\,\beta$) & $\lambda = 0$.</u>

$$\begin{cases} \mathbb{X}_{11} = \beta'' + vw'' \\ \mathbb{X}_{12} = w' \\ \mathbb{X}_{22} = 0 \end{cases}$$

$$f = \frac{|\mathbb{X}_1\ \mathbb{X}_2\ \mathbb{X}_{12}|}{|\mathbb{X}_1 \times \mathbb{X}_2|} = \frac{|\beta' + \cancel{vw'}\ \ w\ \ w'|}{\sqrt{EG - F^2}}$$

$$(\beta' \times w) \cdot w' = \lambda w'$$

$\Rightarrow g = 0$

$$K = \frac{\overset{0}{\cancel{e}}g - f^2}{EG - F^2} = -\frac{\lambda^2 |w'|^4}{(EG-F^2)^2} = -\frac{\lambda^2}{(\lambda^2+v^2)^2} \leq 0$$

$K_p = 0 \Leftrightarrow p \in$ singular pt.

$\mathrm{II}_p(w) \overset{?}{\doteq} 0$ 　 $\mathrm{II}_p(\mathbb{X}_2, \mathbb{X}_2) = N \cdot \mathbb{X}_{22} = g = 0$

$\mathbb{X} = \alpha + vw$ $\Rightarrow$ the ruling is a asymptotic curve

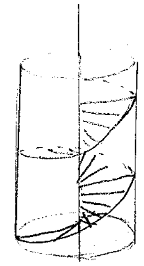
$\mathbb{X}_2 = w$

- w is the principal normal of α

$\alpha = T$, $\alpha'' = T' = k\vec{n}$.

The only ruled surface which is minimal is the Helicoid

Helix = the "unique" curve with costant K & T



<pf>. S ruled, $K<0$.

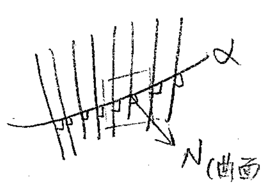
so every pt. has 2 asymptotic directions.

at any such pt. may obtain coor system st.

the coor. lines are asymptotic lines.

S minimal, $H = 0$.

Prop (in exercise) $\boxed{\S 3.2-7}$ $H=0 \Rightarrow$ asymp directions are $\perp$.



- the ruling is a asymptotic curve.
- take α be another asymptotic curve.

claim : w is the principal normal of α.

$\alpha' = T$

$\alpha'' = T' = k\vec{n}$

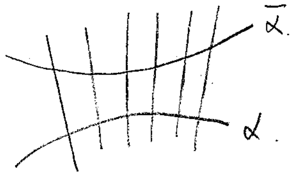
<pf> $\mathrm{II}(\alpha') = 0$

$\overset{\|}{N} \cdot \alpha'' = N \cdot (k\vec{n}) \Rightarrow N \cdot \vec{n}$. 　 $\alpha' = T \perp N$

$\Rightarrow \vec{n} \,/\!/\, w$.

$\#$.

That is, the oscilating plane is the tangent plane.



for any other section $\bar{\alpha}$

$\alpha, \bar{\alpha}$ have the same principal normal vectors.

(Bertrand mate)

(Helix $\Leftrightarrow$ 只有一個 k & τ.)

Let $\alpha, \bar{\alpha}$ be a pair of Bertrand mate,

$$☆\begin{cases} \bar{\alpha} = \alpha + rn & \alpha(s)\text{, s arc length of } \alpha. \\ \bar{\alpha}' = \alpha' + r'n & \text{" , " wrt to s.} \\ \quad + r(-kT - \tau B) & \\ = (1 - rk)T + r'n - r\tau B & \end{cases}$$

since $\bar{n} = n$, $\bar{n} \perp \bar{\alpha}'$

$\Rightarrow r' = 0$ ie. r = constant.

Let $\bar{s}$ be the arc length of $\bar{\alpha}$

$\bar{T}$ be the tangent of $\bar{\alpha}$.

$$(T\cdot\bar{T})' = T'\cdot\bar{T} + T\cdot\bar{T}' = \underbrace{T'\cdot\bar{T}}_{kn=k\bar{n}\ \perp} + T\cdot\underset{\substack{\| \\ k\bar{n} \\ \| \\ n}}{\frac{d\bar{T}}{d\bar{s}}}\cdot\frac{d\bar{s}}{ds} = 0$$

$\Rightarrow T\cdot\bar{T} = \cos\theta$ = constant angle

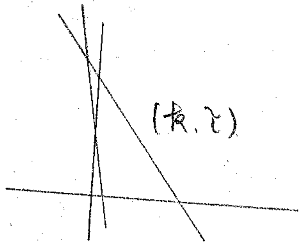
$$= T\cdot\underset{\substack{\uparrow \\ \bar{\alpha}'}}{\frac{d\bar{\alpha}}{ds}}\cdot\frac{ds}{d\bar{s}} = (1-rk)\frac{ds}{d\bar{s}} \text{ by } ✡$$

$$|\sin\theta| = |\bar{T}\times T| = \left|\frac{ds}{d\bar{s}}\right|\cdot\left|[(1-\cancel{rk})T+\overset{0}{\cancel{rn'}}-r\tau B]\times T\right|$$

$$= \left|r\tau\frac{ds}{d\bar{s}}\right|$$

So $\left|\frac{1-rk}{r\tau}\right| = \left|\frac{\cos\theta}{\sin\theta}\right|$ = constant $= c$

$1-rk = cr\tau$. i.e. $rk + cr\tau = 1$



Now, suppose we have more than one such $\bar{\alpha}$, call α_1, α_2.

$$\Rightarrow \begin{cases} r_1 k + c_1 r_1 \tau = 1 \\ r_2 k + c_2 r_2 \tau = 1 \end{cases}$$

by def of γ, $\alpha_1 \neq \alpha_2$, $r_1 \neq r_2$.

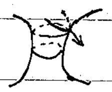
- $c_1 = c_2$, $\Rightarrow$ get // lines. (無解) —✗—
- $c_1 \neq c_2$ $\Rightarrow$ get unique k, τ.

Ex §3.5 — 2, 3, 11, 12, 13, 14.

Minimal Surface :

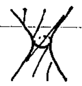
- surface of revolution + minimal.

  ⇒ catenoid



- surface of revolution + ruled

  ⇒ Hyperboloid

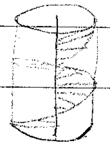


$K \le 0$, $X(u,v) = \beta(u) + v\vec{w}(u)$

↑ line of striction.

- ruled + minimal.

  ⇒ Helicoid



$K_p < 0$

$\underline{H \equiv 0}$

(Try).

Weierstrass Representation of minimal surface in $\mathbb{R}^3$
(using cpx analytic functions)

Def : Isothermal coordinates

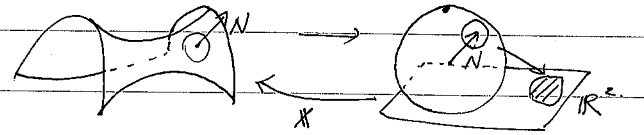
$X : U \longrightarrow S \subset \mathbb{R}^3$

if $ds^2 = \lambda^2(du^2 + dv^2)$ i.e. $E = G$, $F = 0$

we called (u,v) an isothermal coordinate.

isothermal coordinates always exists but the proof is hard.

- But for minimal surface, the proof is easy.



$H = k_1 + k_2 = 0$

$K = k_1 k_2 \le 0$

~~prop~~ (1) $\mathbb{X}$ is isothermal $(E=G=\lambda^2,\ F=0)$

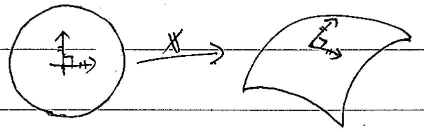
$\Leftrightarrow \mathbb{X}$ is a conformal map. $(\langle d\mathbb{X}_p(V_1), d\mathbb{X}_p(V_2)\rangle = \lambda^2(p)\langle V_1, V_2\rangle)$

(2) $\mathbb{X}$ is isothermal $\Rightarrow \underset{\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)\mathbb{X}}{\underset{\parallel}{\Delta\mathbb{X}}} = 2\lambda^2\vec{H}$

<pf> (1) is trivial, $E = \mathbb{X}_1\cdot\mathbb{X}_1 = |\mathbb{X}_1|^2$

$G = \mathbb{X}_2\cdot\mathbb{X}_2 = |\mathbb{X}_2|^2$

$F = \mathbb{X}_1\cdot\mathbb{X}_2$



(2) $\mathbb{X}_{11}\cdot\mathbb{X}_1 = \frac{1}{2}E_1 = \frac{1}{2}G_1 = \mathbb{X}_{12}\cdot\mathbb{X}_2$

$= \underset{F=0}{\underset{\parallel}{(\mathbb{X}_1\cdot\mathbb{X}_2)_2}} - \mathbb{X}_1\cdot\mathbb{X}_{22}$.

$\underbrace{(\mathbb{X}_{11}+\mathbb{X}_{22})}_{\Delta\mathbb{X}}\cdot\mathbb{X}_1 = 0$

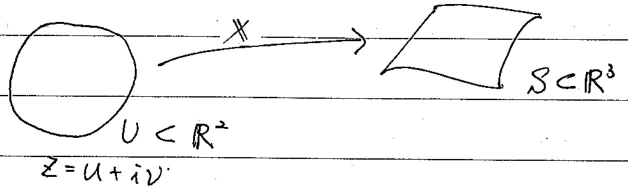
similarly $\Delta\mathbb{X}\cdot\mathbb{X}_2 = 0$.

$\Rightarrow \Delta\mathbb{X} = \underline{f}N$

$H = \frac{1}{2}\frac{Eg - 2\cancel{F}f + Ge}{EG - \cancel{F^2}} = \frac{1}{2}\frac{\lambda^2(g+e)}{\lambda^4} = \frac{g+e}{2\lambda^2}$

$\Rightarrow 2\lambda^2 H = \underset{N\cdot\mathbb{X}_{11}}{\underset{\parallel}{g}} + \underset{N\cdot\mathbb{X}_{22}}{\underset{\parallel}{e}} = N\cdot\Delta\mathbb{X} = \underline{f}$

$\therefore \Delta\mathbb{X} = 2\lambda^2\underbrace{H\vec{N}}_{\vec{H}}$ ※



$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)$

$\frac{\partial}{\partial\bar{z}} = \frac{1}{z}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$

Fact : $\frac{\partial f}{\partial\bar{z}} = 0 \Leftrightarrow f$ cpx analytic

<pf> $f = F + iG$.

$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)(F+iG)$

$= \frac{1}{2}[(F_u - G_v) + i(F_v + G_u)] = 0 \Leftrightarrow \overset{柯\ 黎}{C\text{-}R}$ eq'n.

under isothermal coor.

$\varphi = \frac{\partial\mathbb{X}}{\partial z}$, φ is cpx analytic (holomorphic)

$\Leftrightarrow$ the surface S parametrized by $\mathbb{X}$ is minimal

<pf> $\frac{\partial}{\partial\bar{z}}\underset{\varphi}{\underset{\parallel}{\left(\frac{\partial\mathbb{X}}{\partial z}\right)}} = \frac{\partial}{\partial\bar{z}}\frac{\partial}{\partial z}\mathbb{X} = \frac{1}{4}\Delta\mathbb{X} = \frac{1}{2}\lambda^2\vec{H}$.

$\varphi \longleftrightarrow \mathbb{X}$ 互相決定.

$\varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix}$

prop : $\mathbb{X}$ is isothermal $\Leftrightarrow \varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0$.

<pf> $\varphi = \frac{\partial\mathbb{X}}{\partial z} = \frac{1}{2}(\mathbb{X}_1 - i\mathbb{X}_e)$.

$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = \frac{1}{4}(\mathbb{X}_1 - i\mathbb{X}_2)\cdot(\mathbb{X}_1 - i\mathbb{X}_2)$

$= \frac{1}{4}[\underset{0}{\underset{\parallel}{(E-G)}} - 2i\underset{6}{\underset{\parallel}{F}}]$ ※.

Recipe. find 3 analytic functions $\varphi_1, \varphi_2, \varphi_3$ st. $\varphi_1^2+\varphi_2^2+\varphi_3^2=0$

let $X = 2\,\mathrm{Re}\int \varphi\, dz$. check $\boxed{\frac{\partial X}{\partial z} = \varphi}$

$\Rightarrow X$ is a parametrized minimal surface in $\mathbb{R}^3$.

$$(\varphi_1 + i\varphi_2)(\varphi_1 - i\varphi_2) = -\varphi_3^2$$

let $\frac{\varphi_1 + i\varphi_2}{\varphi_3} = \frac{-\varphi_3}{\varphi_1 - i\varphi_2} = \frac{1}{g}$, g meromorphic func.

$f = \varphi_1 + i\varphi_2$: holomorphic

$\varphi_1 + i\varphi_2 = f$

$\varphi_1 - i\varphi_2 = -\varphi_3 g = -f g^2$, $f = \frac{\varphi_3}{g}$

$$\Rightarrow \begin{cases} \varphi_1 = \frac{1}{2} f(1-g^2) \\ \varphi_2 = \frac{1}{2i} f(1+g^2) \\ \varphi_3 = fg \end{cases}$$

Ep. $f(z)=1$, $g(z)=z$

gives Enneper's surface

$$\begin{cases} \varphi_1 = \frac{1}{2}(1-z^2) \\ \varphi_1 = \frac{1}{2i}(1+z^2) \\ \varphi_3 = z \end{cases} \xrightarrow{\int dz} \begin{pmatrix} \frac{1}{2}(z - \frac{1}{3}z^3) & \\ \frac{1}{2i}(z + \frac{1}{3}z^3) & \\ \frac{1}{2}z^2 & z = u+iv \end{pmatrix}$$

$$X = 2\,\mathrm{Re}\int \varphi\, dz = \left(u - \tfrac{1}{3}u^3 + uv^2,\ v - \tfrac{1}{3}v^3 + vu^2,\ u^2 - v^2\right)$$

Ep. Sherk's surface

$f = \frac{z^2}{1-z^4}$, $g = \frac{1}{z}$ p. 209

consider $X_\theta = 2\,\mathrm{Re}\int e^{i\theta} \varphi\, dz$

Fact, all X_θ has the same 1st fundamental form

$$|\varphi|^2 = \left|\tfrac{1}{2}(X_1 - iX_2)\right|^2$$

$$= \frac{1}{4}(\underset{E}{\underset{\parallel}{|X_1|^2}} + \underset{G}{\underset{\parallel}{|X_2|^2}}) = \frac{1}{2}\lambda^2$$

$|e^{i\theta}\varphi|^2 = |\varphi|^2$ $ds^2 = \lambda^2(du^2 + dv^2)$.

把 catenoid 和 helicoid 連起來.

catenoid $f(z) = \frac{1}{z^2}$, $g(z) = z$. φ.

helicoid $\theta = \frac{\pi}{2}$. $i\varphi$.

Ex §3.5 — 2. 3. 11. 12. 13. ⑭.

NO. 12/12 (四)

§4.1 isometry and conformal maps (暫略)

§4.2. Gauss – Codazzi Equation :

$S \subseteq \mathbb{R}^3$, $\begin{matrix} E & F & G \\ e & f & g \end{matrix}$

Q : What is the necessary and sufficient condition for the existence of S with $\mathbb{X} : U \longrightarrow S$ s.t. $I = E\,du^2 + 2F\,du\,dv + G\,dv^2$

$I\!I = e\,du^2 + 2f\,du\,dv + g\,dv^2$

$\mathbb{X}(u,v)$, $\mathbb{X}_1$, $\mathbb{X}_2$, N

$$\begin{cases} \mathbb{X}_{11} = \Gamma^1_{11}\mathbb{X}_1 + \Gamma^2_{11}\mathbb{X}_2 + \overset{\overset{\mathbb{X}_{11}\cdot N = e}{\downarrow}}{e}N \quad \text{—} ① \\ \mathbb{X}_{12} = \mathbb{X}_{21} = \Gamma^1_{12}\mathbb{X}_1 + \Gamma^2_{12}\mathbb{X}_2 + fN \\ \mathbb{X}_{22} = \Gamma^1_{22}\mathbb{X}_1 + \Gamma^2_{22}\mathbb{X}_2 + gN \\ N_1 = a_{11}\mathbb{X}_1 + a_{12}\mathbb{X}_2 \\ N_2 = a_{21}\mathbb{X}_1 + a_{22}\mathbb{X}_2 \end{cases}$$

$① \cdot \mathbb{X}_1 : \quad \mathbb{X}_{11}\cdot\mathbb{X}_1 = \Gamma^1_{11}E + \Gamma^2_{11}F$

$① \cdot \mathbb{X}_2 : \quad \mathbb{X}_{11}\cdot\mathbb{X}_2 = \Gamma^1_{11}F + \Gamma^2_{11}G$

$\mathbb{X}_{11}\cdot\mathbb{X}_1 = \frac{1}{2}(\mathbb{X}_1\cdot\mathbb{X}_1)_1 = \frac{1}{2}E_1$

$\mathbb{X}_{11}\cdot\mathbb{X}_2 = (\mathbb{X}_1\cdot\mathbb{X}_2)_1 - \mathbb{X}_1\cdot\mathbb{X}_{21} = F_1 - \frac{1}{2}E_2$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \Gamma^1_{11} \\ \Gamma^2_{11} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}E_1 \\ F_1 - \frac{1}{2}E_2 \end{pmatrix}$$

* conclusion : Γ^k_{ij} can be written in terms of E. F. G and their derivatives up to 1st order.

We have compatibility relations

$(\mathbb{X}_{11})_2 = \mathbb{X}_{112} = \mathbb{X}_{121} = (\mathbb{X}_{12})_1$

$(\partial_1 = \frac{\partial}{\partial u} \quad \partial_2 = \frac{\partial}{\partial v})$

$\partial_2\Gamma^1_{11}\mathbb{X}_1 + \Gamma^1_{11}\mathbb{X}_{12} + \partial_2\Gamma^2_{11}\mathbb{X}_2 + \Gamma^2_{11}\mathbb{X}_{22} + e_2N + eN_2$

$= \partial_1\Gamma^1_{12}\mathbb{X}_1 + \Gamma^1_{12}\mathbb{X}_{11} + \partial_1\Gamma^2_{12}\mathbb{X}_2 + \Gamma^2_{12}\mathbb{X}_{21} + f_1N + fN_1$

$\mathbb{X}_1$ term : $\partial_2\Gamma^1_{11} + \cancel{\Gamma^1_{11}\Gamma^1_{12}} + \Gamma^2_{11}\Gamma^1_{22} + e\,a_{21}$

$= \partial_1\Gamma^1_{12} + \cancel{\Gamma^1_{11}\Gamma^1_{12}} + \Gamma^2_{12}\Gamma^1_{12} + f\,a_{11}$

$a_{11} = \dfrac{Ff - eG}{EG - F^2}$

$a_{21} = \dfrac{gF - fG}{EG - F^2}$

$e\,a_{21} - f\,a_{11} = \partial_1\Gamma^1_{12} - \partial_2\Gamma^1_{11} + \Gamma^2_{12}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22}$

$\qquad\|$

$\dfrac{1}{EG-F^2}(egF - e\cancel{fG} - ffF + e\cancel{fG})$

$\qquad\|$

$F \cdot \dfrac{eg - f^2}{EG - F^2} = F \cdot K$

Theorema Egrejium (Gauss)

K can be written in terms of E, F, G up to 2nd derivatives.

$\mathbb{X}_2$ term will give equivalent equation

N term : $\Gamma^1_{11}f + \Gamma^2_{11}g + e_2 = \Gamma^1_{12}e + \Gamma^2_{12}f + f_1$

共3條方程式

$$\left[\begin{array}{l} \left.\begin{array}{l} e_2 - f_1 = \Gamma^1_{12}e + (\Gamma^2_{12} - \Gamma^1_{11})f - \Gamma^2_{11}g \\ g_1 - f_2 = \Gamma^2_{21}g + (\Gamma^1_{21} - \Gamma^2_{22})f - \Gamma^1_{22}e \end{array}\right\} \sim \text{Codazzi eq'n. (from } \mathbb{X}_{112} = \mathbb{X}_{221}\text{)} \; (\mathbb{X}_{12} \,\&\, \mathbb{X}_{22}) \\ \\ K = \frac{1}{F}(\partial_1\Gamma^1_{12} - \partial_2\Gamma^1_{11} + \Gamma^2_{12}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22}) \sim \text{Gauss eq'n} \end{array}\right.$$

☆ Theorem (Bonnet) :

Given E, F, G, e, f, g satisfy Gauss – Codazzi eq'n

(正定 $EG - F^2 > 0$) $\Leftrightarrow \exists$ a surface with these E. F. G. e. f.
$E > 0$, $G > 0$

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

$$= \sum g_{ij}\,du^i du^j \qquad (u,v) = (u^i, u^j)$$

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

$$(g^{ij}) = (g_{ij})^{-1} = \frac{1}{EG-F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$

$$\sum_l g^{il} g_{lj} = \delta^i_j$$

Formula for the Christoffel Symbol

(背) $\Gamma^k_{ij} = \sum_l \frac{1}{2} \underbrace{g^{kl}}_{\text{[illegible]}} (\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij})$

Thm (Frobenine) $\frac{\partial x_k}{\partial u^\alpha} = U^k_\alpha(u, \mathbb{X})$ $\quad \mathbb{X} = (x_1, x_2, \cdots, x_n)$

$k = 1, \cdots, n$ $\quad \mathbb{X} : \underset{u_1,\cdots,u_m}{\mathbb{R}^m} \longrightarrow \underset{x_1,\cdots,x_n}{\mathbb{R}^n}$

$\alpha = 1, \cdots, m$

$U^k_\alpha : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}.$

$(\underbrace{1, \cdots, m}_{\alpha,\beta,\gamma}, \underbrace{\cdots\cdots}_{i,j,k})$

\# of functions $= n$

\# of equations $= n \cdot m$

compatibility condition :

$$\frac{\partial^2 x_k}{\partial u^\alpha \partial u^\beta} = \frac{\partial}{\partial u^\beta}\left(U^k_\alpha(u, \mathbb{X})\right) = U^{k\,\frac{\partial u}{\partial u}}_{\alpha,\beta} + \sum_j U^k_{\alpha,j} \frac{\partial x_j}{\partial u^\beta}$$

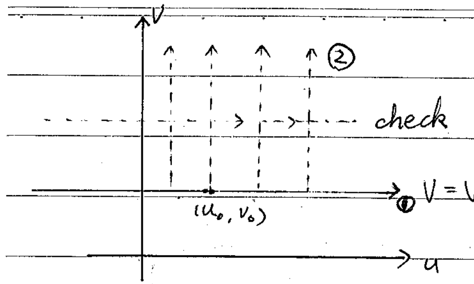
$$\| \qquad\qquad /// \quad \forall\, \alpha, \beta, k$$

$$\frac{\partial^2 x_k}{\partial u^\beta \partial u^\alpha} = U^k_{\beta,\alpha} + \sum_j U^k_{\beta,j} \frac{\partial x_j}{\partial u^\alpha}$$

It's enough to prove it for $m = 2$. $\quad (u^1, u^2) = (u, v)$

$$\frac{\partial x_k}{\partial u} = U^k(u, v, \mathbb{X})$$

$$\frac{\partial x_k}{\partial v} = V^k(u, v, \mathbb{X})$$



① fix $V = V_0$ varies u

solve $\frac{dy_k}{du} = U^k(u, v_0, y)$

with initial condition $y_k(u_0, v_0) = x_{k_0}$

② take y_k as initial data solve for each fixed u

$$\frac{dx_k}{dv} = V^k(u, v, \mathbb{X}) \qquad (\text{initial} : y_k(u))$$

Need to check $\overset{\text{LHS}}{\frac{\partial x_k}{\partial u}} = \overset{\text{RHS}}{U^k(u, v, \mathbb{X})}$ $\quad **$

when $V = V_0$ it is ok !

$$\left(\frac{\partial x_k}{\partial u}\right)\Big|_{V=V_0} = \frac{dy_k}{du} = U^k(u, v_0, \underset{\underset{y}{\|}}{\mathbb{X}}) \quad \text{ok!}$$

In general, ☆ will show LHS & RHS satisfy the same ODE

A. $\frac{\partial}{\partial v}\left(\frac{\partial x_k}{\partial u}\right) = \frac{\partial}{u}\left(\frac{\partial x_k}{\partial v}\right) = \frac{\partial}{\partial u}\left(V^k(u, v, \mathbb{X})\right) = \frac{\partial V^k}{\partial u} + \sum_i \frac{\partial V^k}{\partial x_i}\left(\frac{\partial x_i}{\partial u}\right)$

B. $\frac{\partial}{\partial v}\left(U^k(u, \overset{\circ}{v}, \mathbb{X})\right) = \frac{\partial U^k}{\partial v} + \sum_i \frac{\partial U^k}{\partial x_i} \cdot \underset{\underset{V^i}{\|}}{\underline{\underline{\frac{\partial x_i}{\partial v}}}} \overset{\uparrow}{=} \frac{\partial V_k}{\partial u} + \sum_i \frac{\partial V^k}{\partial x_i} \cdot \underset{\underset{O_i}{\|}}{\underline{\underline{U_i}}}$

by compatibility.

同一個 $\begin{cases} \frac{\partial \star_k}{\partial v} = \frac{\partial V^k}{\partial u} + \sum_i \frac{\partial V^k}{\partial x_i} \star_i \\ \frac{\partial O_k}{\partial v} = \frac{\partial U^k}{\partial u} + \sum_i \frac{\partial V^k}{\partial x_i} O_i \end{cases}$

Uniqueness of ODE $\Longrightarrow \frac{\partial x_k}{\partial u} = U^k(u, v, \mathbb{X})$ ※.

Back to Bornet's thm

frame $\{\mathbb{X}_1, \mathbb{X}_2, N\}$ with initial data at (u_0, v_0)

$\mathbb{R}^2 \longrightarrow \mathbb{R}^9$

the compatibility relation is exactly the Causs - Codazzi eq'n.

$F^1, \cdots, F^9$

$\frac{\partial F^i}{\partial u}$ = ~~functions in~~

$\frac{\partial F^i}{\partial v}$ = $\mathbb{X}_u$ $\mathbb{X}_v$ (not $\mathbb{X}$) in fact, on E, F, G, e, f, g.

重要 Christoffel symbol Γ_{ij}^k, $1 \le i, j, k \le 2$.

$$\mathbb{X}_{11} = \underbrace{\Gamma_{11}^1 \mathbb{X}_1 + \Gamma_{11}^2 \mathbb{X}_2}_{\text{on } T_pS} + \underbrace{eN}_{\text{normal part}}$$

$g_{ij} = \mathbb{X}_i \cdot \mathbb{X}_j$ (ie. $E = g_{11}$, $F = g_{12}$, $G = g_{22}$)

$$\partial_k g_{ij} = \mathbb{X}_{ik} \cdot \mathbb{X}_j + \mathbb{X}_i \cdot \mathbb{X}_{jk}$$

$$= \sum_{l=1}^{2} \Gamma_{ik}^l \mathbb{X}_l \cdot \mathbb{X}_j + \sum_{l=1}^{2} \Gamma_{jk}^l \mathbb{X}_i \cdot \mathbb{X}_l$$

$\left(\begin{matrix} \partial_1 = \frac{\partial}{\partial u} \\ \partial_2 = \frac{\partial}{\partial v} \end{matrix}\right)$

$$\boxed{\partial_k g_{ij} = \sum_{l=1}^{2} \Gamma_{ik}^l g_{lj} + \sum_{l=1}^{2} \Gamma_{jk}^l g_{li}}$$

$\mathbb{X}_{ij}^{T} = \sum_l \Gamma_{ij}^l \mathbb{X}_l$

↑ the tangent component

有 l 出現，就是要 sum 起來

(以後不寫 Σ)

cyclic trick

$$\partial_k g_{ij} = \Gamma_{ik}^l g_{lj} + \Gamma_{jk}^l g_{li}$$

$$\partial_i g_{kj} = \Gamma_{ki}^l g_{lj} + \Gamma_{ij}^l g_{lk}$$

$$\partial_j g_{ki} = \Gamma_{kj}^l g_{li} + \Gamma_{ij}^l g_{lk}$$

$$\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij} = 2\Gamma_{ij}^l g_{lk}$$

$$\Rightarrow \boxed{\Gamma_{ij}^s = \frac{1}{2}\sum_k g^{sk}(\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij})}$$

$$\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij} = 2\Gamma_{ij}^l g_{lk}$$

$$= 2\begin{pmatrix} \Gamma_{ij}^1 & \Gamma_{ij}^2 \end{pmatrix}\begin{pmatrix} g_{1k} \\ g_{2k} \end{pmatrix}$$

$k=1$ $\quad 2\begin{pmatrix} \Gamma_{ij}^1 & \Gamma_{ij}^2 \end{pmatrix}\begin{pmatrix} g_{11} \\ g_{21} \end{pmatrix} = \star$

$k=2$ $\quad 2\begin{pmatrix} \Gamma_{ij}^1 & \Gamma_{ij}^2 \end{pmatrix}\begin{pmatrix} g_{12} \\ g_{22} \end{pmatrix} = \star\star$

$$\Rightarrow \begin{pmatrix} \star & \star\star \end{pmatrix} = 2\begin{pmatrix} \Gamma_{ij}^1 & \Gamma_{ij}^2 \end{pmatrix}\begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \star & \star\star \end{pmatrix}\begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix}^{-1} = 2\begin{pmatrix} \Gamma_{ij}^1 & \Gamma_{ij}^2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

NO.

<u>ep</u>. surface of revolution

$X(u,v) = (f(v)\cos u\ ,\ f(v)\sin u\ ,\ g(v))$. (not necessary $(f')^2+(g')^2=1$)

$$ds^2 = \underset{f^2}{\overset{\|}{E}}du^2 + 2\underset{0}{\overset{\|}{F}}dudv + \underset{(f')^2+(g')^2}{\overset{\|}{G}}dv^2$$

$$(g_{ij}) = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} f^2 & 0 \\ 0 & (f')^2+(g')^2 \end{pmatrix},\quad (g^{ij}) = \begin{pmatrix} f^{-2} & 0 \\ 0 & ((f')^2+(g')^2)^{-1} \end{pmatrix}$$

$$\Gamma^1_{11} = \frac{1}{2}g^{11}(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}) + \frac{1}{2}\underset{0}{\cancel{g^{12}}}$$
$$= \frac{1}{2}g^{11}\partial_1 g_{11}$$
$$= \frac{1}{2}g^{11}\underbrace{\partial_u f^2(v)} = 0$$

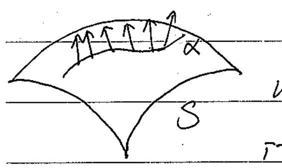
$$\Gamma^2_{11} = \frac{1}{2}g^{22}(\partial_1 \cancel{g_{12}} + \partial_1 \cancel{g_{21}} - \partial_2 g_{11}) + \frac{1}{2}\underset{0}{\cancel{g^{21}}}$$
$$= -\frac{1}{2}g^{22}\,2ff' = \frac{-ff'}{(f')^2+(g')^2}$$

$$\Gamma^1_{12} = \frac{ff'}{f^2},\quad \Gamma^2_{12} = 0,\quad \Gamma^1_{22} = 0,\quad \Gamma^2_{22} = \frac{f'f''+g'g''}{(f')^2+(g')^2}$$

* covariant derivative (共變微分)

given a vector field $v(t)$ along $\alpha(t)$

ie. $v(t) \in T_{\alpha(t)}S$



what is the reasonable notion of $v'(t)$?

if think $v(t) \in R^3$, $v'(t) \notin T_{\alpha(t)}S$ in general

so we define $\frac{Dv}{dt} = \left(\frac{dv}{dt}\right)^T \to$ orthogonal projection to $T_{\alpha(t)}S$

<u>Def</u> (parallel translation)

V is a parallel v.f. $\underset{def}{\iff} \frac{Dv}{dt} = 0$

(平移)

NO.

Formula for $\frac{Dv}{dt}$

$V(t) = a(t)X_1 + b(t)X_2 \in T_{\alpha(t)}S$ $\alpha(t) \in S$ is given

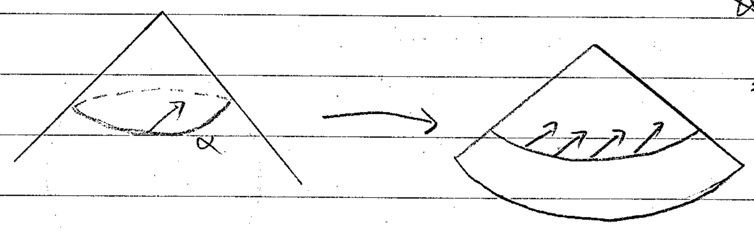
$$\frac{Dv}{dt} = a'X_1 + b'X_2 + a(X_{11}u' + X_{12}v')^T + b(X_{21}u' + X_{22}v')^T$$

$$= (a' + \Gamma^1_{11}u'a + \Gamma^1_{12}v'a + \Gamma^1_{21}u'b + \Gamma^1_{22}v'b)X_1 +$$
$$(b' + \Gamma^2_{11}u'a + \Gamma^2_{12}v'a + \Gamma^2_{21}u'b + \Gamma^2_{22}v'b)X_2$$

$$\begin{pmatrix} a \\ b \end{pmatrix}' = A\begin{pmatrix} a \\ b \end{pmatrix} \iff \frac{Dv}{dt} = 0 \quad \begin{pmatrix} a \\ b \end{pmatrix}' = \begin{pmatrix} \Gamma^1_{11}u' + \Gamma^1_{12}v' & \Gamma^1_{21}u' + \Gamma^1_{22}v' \\ \Gamma^2_{11}u' + \Gamma^2_{12}v' & \Gamma^2_{21}u' + \Gamma^2_{22}v' \end{pmatrix}\Big($$

$\Rightarrow \exists!$ parallel translation of any $v(t)$ along α.

☆ Γ^k_{ij} 只和 E, F, G 有関



$\Rightarrow$ 圓錐和平面的E, F

相同

$\Rightarrow$ 在球上做平移和在圓錐上做平移相同

∵ 切平面完全相同

<u>Def</u> geodesics (測定線) (平移最重要的應用)

令 $\underline{V = \alpha'}$, $\frac{DV}{dt} = \frac{D\alpha'}{dt} = (\alpha'')^T$

(測定線是距離最短的線)

α is a geodesic $\overset{def}{\iff} \frac{D\alpha'}{dt} = 0$

連接兩点的最短直

<u>Cor</u>. α : geodesic

$\Rightarrow t \sim$ arc length α的參数和 arc length 成正比

$\langle pf\rangle \frac{d(\alpha'\cdot\alpha')}{dt} = 2\alpha''\cdot\alpha'$

$$= 2\underset{0}{\underset{\|}{\frac{D\alpha'}{dt}}}\cdot\alpha' = 0 \quad + \underline{法向量\cdot\alpha'} = 0$$

$\Rightarrow |\alpha'| = $ constant

NO.

$\alpha(t) = \mathbb{X}(u(t), v(t))$

$V(t) = \alpha'(t) = u'(t)\,\mathbb{X}_1 + v'(t)\,\mathbb{X}_2 \in T_{\alpha(t)}S$

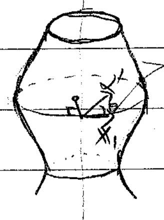
$$\frac{D\alpha'}{dt} = (u'' + \Gamma^1_{11}(u')^2 + 2\Gamma^1_{12}u'v' + \Gamma^1_{22}(v')^2)\,\mathbb{X}_1 +$$
$$(v'' + \Gamma^2_{11}(u')^2 + 2\Gamma^2_{12}u'v' + \Gamma^2_{22}(v')^2)\,\mathbb{X}_2$$ 測定線方程

$$\boxed{\begin{array}{l}(u, v) = (u^1, u^2) \\ u^{k''} + \sum_{i,j=1,2}\Gamma^k_{ij}\,u^{i'}u^{j'} = 0 \quad , \quad k = 1, 2\end{array}}$$

$(\alpha'')^T = 0$

加速度垂直曲面
那就会是最短距離

ep. surface of revolution 的測地線



① $f' = 0$

② $u = \text{const.}$

$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$ ～ 第三條方程式

$\Rightarrow 1 = E\left(\frac{du}{ds}\right)^2 + G\left(\frac{dv}{ds}\right)^2$

$\Gamma^1_{11} = 0 = \Gamma^1_{22} = \Gamma^2_{12}$

$$\begin{cases} u'' + 2\Gamma^1_{12}u'v' = 0 \\ v'' + \Gamma^2_{11}(u')^2 + \Gamma^2_{22}(v')^2 = 0 \end{cases}$$

$u'' + 2\frac{f'}{f}u'v' = 0$ ie. $f u'' + 2 \overset{f'(v)}{f'}\, \underset{v'(t)}{\underset{u'(t)}{u'v'}} = 0$

$(f^2u')' = 2f f'(v)\,v'u' + f^2u'' = 0$

对"t"

ie. $f^2\frac{du}{ds} = \text{constant}$, $\left(\begin{array}{l}\text{一開始的假設} \\ t = s = \text{arc length}\end{array}\right)$

$\Updownarrow$

clairant relation

$r\cos\theta = \text{constant}$ $(f = r)$

$\because f^2\frac{du}{ds} = r \cdot r\frac{du}{ds}$

$\cos\theta = \frac{\alpha' \cdot \mathbb{X}_1}{|\mathbb{X}_1|} = \frac{(u'\mathbb{X}_1 + v'\mathbb{X}_2)\cdot\mathbb{X}_1}{|\mathbb{X}_1|} = u'|\mathbb{X}_1| = ru'$

$(\because |\mathbb{X}_1|^2 = r^2)$

§4.3 — 1. 2. 3. 5. 7

§4.4 — 1. 2. 5. 12. 13. 14. 15. 17. 21. 23.

NO. 12/19 (四)

Parallel translation

$\alpha : I \longrightarrow S$

$V(t) \in T_{\alpha(t)}S$ (tangent vector field along the curve α)

$\frac{DV}{dt} := \left(\frac{dv}{dt}\right)^T$ (covariant differentiation)

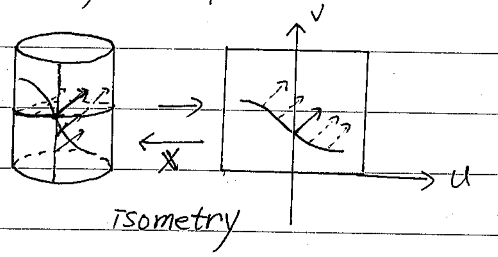
V is called parallel if $\frac{DV}{dt} \equiv 0$

Given $V_0 \in T_{\alpha(t)}S$, the parallel translation of V_0 along α is the unique solution of the ODE

解 $\frac{DV}{dt} \equiv 0$, $V(0) = V_0$

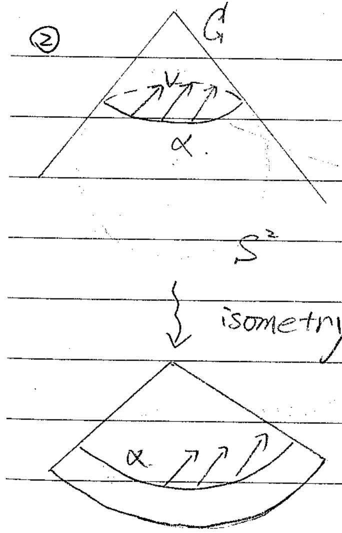
Some easy examples :

①



$K \equiv 0$

②



$\frac{DV}{dt}$ is the same in S^2 and in C properties of $\frac{DV}{dt}$:

$* \ \frac{d}{dt} V\cdot W = \frac{DV}{dt}\cdot W + V\cdot \frac{DW}{dt}$

$\frac{dV}{dt}\cdot W$ (normal 的部分為零)

$*$ $V(t)$ parallel

$\Rightarrow |V(t)| = $ constant

<pf> $\frac{d}{dt} V\cdot V = \frac{DV}{dt}\cdot V + V\cdot\frac{DV}{dt} = 0.$

Geodesics:

$\alpha : I \longrightarrow S$, $V = \alpha'$

Def $\underset{\overset{\|}{(\alpha'')^T}}{\frac{D\alpha'}{dt}} \equiv 0 \Longleftrightarrow \alpha$ is a geodesic.

i.e. $\alpha'' \,/\!/\, N$

Basic property of geodesic.

$\alpha(t)$ is a geodesic $\Rightarrow$ $t = \frac{s}{\lambda} + c$ (λ, c : constant)

<pf> $\frac{D\alpha'}{dt} \equiv 0 \Rightarrow |\alpha'| = \text{constant} = \lambda$

then consider $\alpha(\frac{s}{\lambda}) = \beta(s)$

$\left|\frac{d\beta}{ds}\right| = \left|\frac{1}{\lambda}\alpha'(\frac{s}{\lambda})\right| = 1$

i.e. s is the arc length $t = \frac{s}{\lambda} + C$ 只是參數不一樣而已

Convention: All geodesics are assumed to be parametrized by its arc length.

Ep S = surface of revolution

$\alpha(t) = X(u(t), v(t))$

$$\begin{cases} u'' + \Gamma_{11}^{1} u'^2 + 2\Gamma_{12}^{1} u'v' + \Gamma_{22}^{1} v'^2 = 0 \\ v'' + \Gamma_{11}^{2} u'^2 + 2\Gamma_{12}^{2} u'v' + \Gamma_{22}^{2} v'^2 = 0 \end{cases}$$

by 常微分方程存在唯一

可"局部"解出 geodesic.

$$\Rightarrow \begin{cases} u'' + \frac{2f'}{f} u'v' = 0 \quad \text{———} \ ① \\ v'' - \frac{ff'}{(f')^2+(g')^2} u'^2 + \frac{ff''+gg''}{(f')^2+(g')^2} v'^2 = 0 \end{cases}$$

和"parallel"不同

平移的話，只要α有多長

就可从解多長.

① $\Rightarrow$ clairant relation $f^2 u' = $ const.

$\underset{\overset{\|}{f}}{r}\cos\theta = $ const.

$1 = E(\frac{du}{dt})^2 + 2\underset{0}{\cancel{F\frac{du}{dt}\frac{dv}{dt}}} + G(\frac{dv}{dt})^2$ (t = arc length)

$\because F = 0$

$E = f^2$, $F = 0$, $G = (f')^2 + (g')^2$.

$f^2\frac{du}{dt} = c$ $\quad (1 = f^2(\frac{du}{dt})^2 + (g'^2 + f'^2)(\frac{dV}{dt})^2)$

3rd 方程式

interpret u as a func in V

$= f^2\frac{du}{dv}\frac{dv}{dt}$

$= f^2\frac{du}{dv}\sqrt{\frac{1 - f^2(\frac{du}{dt})^2}{f'^2 + g'^2}}$, $u' = \frac{c}{f^2}$.

$f^2\frac{du}{dv} = c\sqrt{\frac{(f')^2 + (g')^2}{f^2 - c^2}}\cdot f$

測定線方程

只給了一個方向

$\frac{du}{dv} = \frac{c}{f}\sqrt{\frac{f'^2 + g'^2}{f^2 - c^2}}$

☆ 在這麼簡單的情況下，

算 geodesic 都這麼麻煩，

$\Rightarrow$ 不可能這樣繼續下去

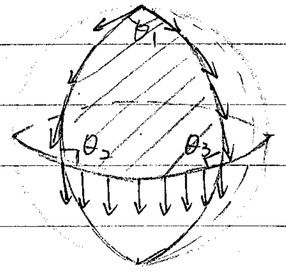
$u = c\int \frac{1}{f}\sqrt{\frac{f'^2 + g'^2}{f^2 - c^2}}\, dv$

為什麼不令 v 為 arc length 讓 $f'^2 + g'^2 = 1$?

∵ 如果這麼做，又要去解 arc length 的方程式

很累的 ……

Gauss - Bonnet thm



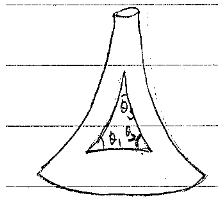
geodesic triangle (測地三角形)
由三條測地線構成的三角形

$$\begin{cases} \text{内角和 } 270^\circ \\ A = \frac{\pi}{2} \left(\frac{4\pi}{8}\right) \text{ (可想像成平移回來的誤差)} \end{cases}$$

K = 1.

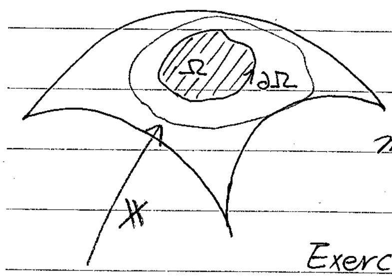
$$\theta_1 + \theta_2 + \theta_3 - \pi = A = \int_\Omega K\,dA$$

$\Rightarrow \int_\Omega K\,dA$ 似乎和某個角度有關



$$\theta_1 + \theta_2 + \theta_3 - \pi = -A = \int_\Omega K\,dA$$

K = -1



$(u, v) \in U$

$$\int_\Omega K\,dA$$

may pick coordinate system st. $X_1 \perp X_2$ (i.e. F

Exercise 1. $K = \frac{-1}{2\sqrt{EG}}\left(\left(\frac{E_2}{\sqrt{EG}}\right)_2 + \left(\frac{G_1}{\sqrt{EG}}\right)_1\right)$ $\begin{cases} \text{① Theorema Egregi} \\ \text{② } \Gamma_{ij}^{k} \\ \text{③ 黎曼几何 (以} \end{cases}$

$$\Rightarrow \int_\Omega K\,dA = -\frac{1}{2}\int_U \Big(\underbrace{\left(\frac{E_2}{\sqrt{EG}}\right)_2}_{-P} + \underbrace{\left(\frac{G_1}{\sqrt{EG}}\right)_1}_{Q.}\Big)\,du\,dv \qquad dA = \sqrt{EG-F^2}\,du\,dv$$

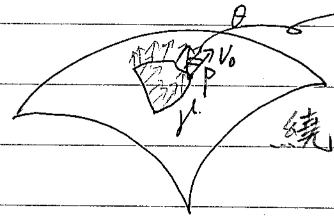
$$= \sqrt{EG}\,du\,dv$$

Green's thm $\int_{\partial U} P\,dx + Q\,dy = \int_U (Q_x - P_y)\,dx\,dy$

$$= -\frac{1}{2}\int_{\partial U}\left(\left(-\frac{E_2}{\sqrt{EG}}\right)du + \left(\frac{G_1}{\sqrt{EG}}\right)dv\right)$$

整個 Gauss-Bonnet 定理就是要了解這個裡面是什麼

Def Holonomy angle :



$\theta \rightarrow$ Holonomy angle.
(只和 γ 有關) 隨便一條 closed γ
繞边界做平移 隨便一個點 p
向量 V_0
V_0 做平移, 回來可能
差了一個夾角

① θ is indep of V_0

<pf> given $V_0, W_0 \in T_pS$, $|V_0| = 1$, $|W_0| = 1$

get v.f. V , W

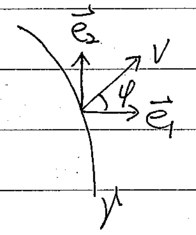
$$\underset{\cos\psi}{\overset{}{\underset{\parallel}{\frac{d}{dt}(V\cdot W)}}} = \frac{DV}{dt}\cdot W + V\cdot\frac{DW}{dt} = 0.$$

θ 只和 γ 有關

☆ Let $\vec{e}_1$, $\vec{e}_2$ be an orthonormal basis in $T.S$ in Ω

$$\left(\text{e.g. } e_1 = \frac{X_1}{|X_1|} , \quad e_2 = \frac{X_2}{|X_2|} \right)$$

☆ Let V be any parallel v.f. along γ, $|V|=1$



$$\frac{D\vec{e}_1}{dt} = \Big(\underbrace{\frac{d\vec{e}_1}{dt}}_{\perp \vec{e}_1}\Big)^T = \lambda \vec{e}_2 , \quad \lambda = ? \qquad \perp N$$

λ : algebraic value of $\frac{D\vec{e}_1}{dt}$

目標：想知道平行向量場角度的變化，所以跟隨便的一個東西做比較看它角度的變化（不然不知道自己的角度是從哪裡開始的）

$$\cos\varphi = V \cdot \vec{e}_1$$

$$-\sin\varphi \cdot \varphi' = \cancel{\frac{DV}{dt}} \cdot \vec{e}_1 + V \cdot \frac{D\vec{e}_1}{dt}$$

$$= \lambda V \cdot \vec{e}_2$$

$$= \lambda \cos\left(\frac{\pi}{2} - \varphi\right) = \lambda \sin\varphi .$$

$$\Rightarrow \boxed{\varphi' = -\lambda}$$

The actual computation of $\left[\frac{D\vec{e}_1}{dt}\right] \equiv \lambda$

曲線不需是光滑的

至少 piecewise C^1

$$\frac{D}{dt}\left(\frac{X_1}{\sqrt{E}}\right) = \frac{-1}{2\sqrt{E}^3}\left(E_1 \frac{du}{dt} + E_2 \frac{dv}{dt}\right) X_1$$

$$+ \frac{1}{\sqrt{E}}\left(X_{11}^T \frac{du}{dt} + X_{12}^T \frac{dv}{dt} \right)$$

$$\lambda = \frac{D}{dt}\left(\frac{X_1}{\sqrt{E}}\right) \cdot \frac{X_2}{\sqrt{G}} = \frac{1}{\sqrt{EG}}\Big(\underset{\underset{\underset{-\frac{1}{2}E_2}{\parallel}}{\cancel{(X_1 \cdot X_2)_1}^{\,0} - X_1 \cdot X_{12}}}{\underset{\parallel}{X_{11} \cdot X_2}} \frac{du}{dt} + \underset{\underset{\frac{1}{2}G_1}{\parallel}}{X_{12} \cdot X_2} \frac{dv}{dt} \Big)$$

似乎真的太巧了一点

$$-\frac{1}{2}\int_{\partial U}\left(\frac{-E_2}{\sqrt{EG}} du + \frac{G_1}{\sqrt{EG}} dv \right) = -\int_{\partial U} \left[\frac{D\vec{e}_1}{dt}\right] dt$$

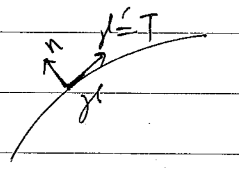
$$= \underset{\underset{\text{holonomy angle}}{\parallel}}{\int_{\partial U} \varphi' \, dt} = \varphi \Big|_{t=0}^{t=l}$$

Theorem (Gauss-Bonnet)

① ~~$\int_\Omega K \, dA = +\theta$~~ , ~~$\theta$: holonomy angle along $\partial\Omega$~~

（第一個形式）

Def. geodesic curvature

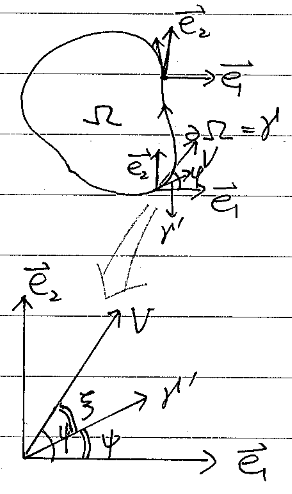


γ'的 algebraic value

$\left[\frac{D\gamma'}{dt}\right] := k_g$ called the geodesic curvature along

~~(t = arc length)~~

$$\frac{DT}{dt} = \lambda n$$



first, assume γ is smooth

if γ is not smooth.

$$\int_\Omega K \, dA + \int_{\partial\Omega} k_g \, ds + \sum \theta_i = 2\pi .$$

$$\varphi = \xi + \psi$$

$$\int_{\gamma} \varphi' \, dt = \int_{\gamma} (\xi' + \psi') \, dt$$

$$= \underset{\underset{k_g}{\parallel}}{\underbrace{\int_{\gamma} \xi' \, dt}} + \underset{\underset{\underset{\underset{1}{\parallel}}{\underline{k}}}{\parallel}}{\int_{\gamma} \psi' \, dt} \quad 2k\pi , \; k \in \mathbb{Z} .$$

② $\int_\Omega K \, dA + \int_{\partial\Omega} k_g \, ds + \underline{\qquad\qquad} = 2\pi$.

Ex — § 4.2 — 4.6, 14, 17, 19.

NO. 12/24 (二)

Gauss – Bonnet

pick F=0

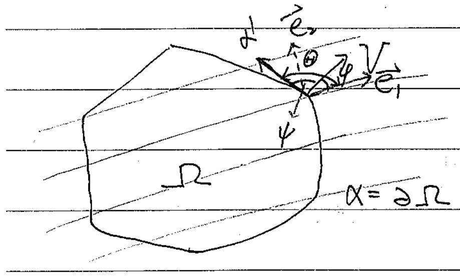
① $\int_\Omega K\,dA = \int_\Omega -\frac{1}{\cancel{2\sqrt{EG}}}\left(\left(\frac{E_2}{\sqrt{EG}}\right)_2 + \left(\frac{G_1}{\sqrt{EG}}\right)_1\right)\cancel{\sqrt{EG}}\,du\,dv$

Green's thm $= \int_{\partial\Omega}\frac{1}{2}\left(\frac{E_2}{\sqrt{EG}}du - \frac{G}{\sqrt{EG}}dv\right) = -\int_{\partial\Omega}\left[\frac{D\vec{e}_1}{dt}\right]dt$

$= \int_{\partial\Omega}\varphi'\,dt$

$= $ holonomy angle.

$\vec{e}_1 = \frac{X_1}{|X_1|}$, $\frac{D\vec{e}_1}{dt} = \underset{\left[\frac{De_1}{dt}\right]}{\underset{\uparrow}{\lambda}}\vec{e}_2$



$\cos\varphi = V\cdot\vec{e}_1$

$-\sin\varphi\cdot\varphi' = V\cdot\frac{D\vec{e}_1}{dt}$

$= \lambda V\cdot\vec{e}_2 = \lambda\sin\varphi.$

$\Rightarrow \varphi' = -\lambda \equiv -\left[\frac{D\vec{e}_1}{dt}\right]$

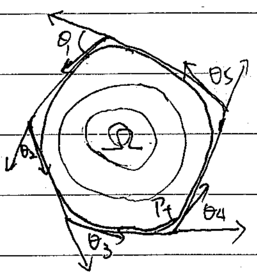
曲率的積分只是在衡量 φ 轉一圈回來的變化

② $\underset{\text{angle}(\alpha',\vec{e}_1)}{\underset{\uparrow}{\varphi = \Theta}} - \psi \leftarrow \text{angle}(\alpha', V)$

$\int_{\partial\Omega}d\varphi = \underbrace{\int_{\partial\Omega}d\Theta}_{一定差2\pi的倍數} - \int_{\partial\Omega}d\psi$

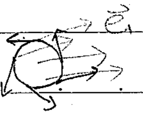
Hopf : $\underbrace{\int_{\partial\Omega}d\Theta = 2\pi - \sum_i\theta_i}$, θ_i : outer angle

(Thm of turning tangents)



<pf> step 1. Assume $\partial\Omega$ smooth

- $\int_{\Gamma_t}d\Theta = 2\pi k$ for some $k\in\mathbb{Z}$
- $\int_{\Gamma_t}d\Theta$ is continuous in t



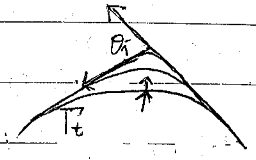
$\Rightarrow \int_{\Gamma_t}d\Theta$ is constant in t $\Rightarrow \underbrace{\int_{\partial\Omega}d\Theta = 2\pi}$

(k is fixed $\forall t$)

NO.

step 2 : $\partial\Omega = \cup C_i$, piecewise smooth.

Γ_t = smooth , $\lim_{t\to0}\Gamma_t = \partial\Omega$



$\lim_{t\to0}\int_{\Gamma_t}d\Theta = 2\pi$

$\|$

$\int_{\partial\Omega}d\Theta + \sum_i\theta_i \Rightarrow \int_{\partial\Omega}d\Theta + \sum_i\theta_i = 2\pi$ #

$\int_{\partial\Omega}d\psi$, $\cos\psi = \alpha'\cdot V$

$-\sin\psi\,\psi' = \frac{D\alpha'}{dt}\cdot V$, $\frac{D\alpha'}{dt} = (\alpha'')^T$

pick $t=s$: arc length

$|\alpha'| = 1$, $\frac{D\alpha'}{dt} = k_g\vec{n}$

$\Rightarrow -\sin\psi\,\psi' = k_g\vec{n}\cdot V$

$= k_g\underline{\cos(\psi + \frac{\pi}{2})}$

$= -\sin\psi\,k_g$ $\therefore \psi' = k_g$

為什麼不會是 $\cos(\psi - \frac{\pi}{2}) = \sin\psi$

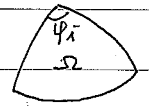
$\int_\Omega K\,dA = 2\pi - \sum_i\theta_i - \int_{\partial\Omega}k_g\,ds.$

$\Rightarrow$ locally Gauss – Bonnet thm.

2-dim 1-dim 0-dim 的曲率修正

$\int_\Omega K\,dA + \int_{\partial\Omega}k_g\,ds + \sum\theta_i = 2\pi.$

eg. if $\partial\Omega$ = geodesic triangle. $\Rightarrow k_g = 0$

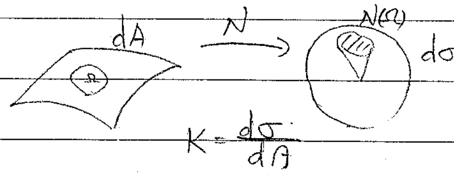


let $\varphi_i = \pi - \theta_i$ interior angle.

$\underbrace{\varphi_1 + \varphi_2 + \varphi_3 - \pi = \int_\Omega K\,dA}$ Gauss 發明微分幾何

原始的原因

非歐几何的源頭



$K = \frac{d\sigma}{dA}$

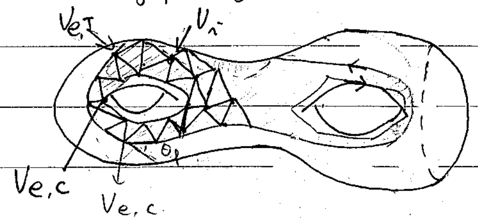
$\int_\Omega K\,dA = \underbrace{\int_{N(\Omega)}d\sigma}$ 有時候叫球面角

③ $R \subseteq S$ oriented surface

$\partial R = \cup C_i$ C_i : smooth curve

邊邊是一條曲線

$R = \bigcup_{j=1}^{F} T_j$, T_j : topological triangle $\subseteq$ in one cood. chart



sum together the local G−B on T_j

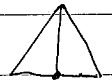
$$\int_R K\,dA + \int_{\partial R} k_g\,ds + \sum_{\substack{j=1,\cdots,F\\ k=1,2,3}} \theta_{j,k} = 2\pi \cdot F$$

Euler (characteristic) number

$$\chi(R) = \underset{\substack{\uparrow\\ \text{number of}\\ \text{vertexes}}}{V} - \underset{\substack{\uparrow\\ \#\text{ of}\\ \text{edge}}}{E} + \underset{\substack{\uparrow\\ \#\text{ of}\\ \text{faces}}}{F}$$

$\chi(R)$ 和看到的洞有關係

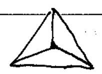
Fact : $\chi(R)$ is independ of triangulations.



$V = +1$

$E = +2$ $\Delta\chi = 0$

$F = +1$



$V = +1$

$E = +3$ $\Delta\chi = 0$

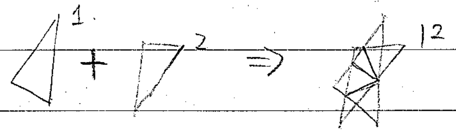
$F = +2$

induction

$\Rightarrow \chi$ is the same under any further triangulation

Two triangulation Δ_1 , Δ_2

$\chi(\Delta_1) = \chi(\Delta_{12}) = \chi(\Delta_2)$



Exercise : if M is a oriented compact surface ($\partial M = \emptyset$) with genus g , then $\chi(M) = 2-2g$

g : surface 的洞的個數

Global Gauss − Bonnet

$$\int_R K\,dA + \int_{\partial R} k_g\,ds + \sum_l \theta_l = 2\pi\,\chi(R)$$

$$\sum \theta_{j,k} = \pi \cdot 3F - \sum \underset{\substack{\uparrow\\ \text{interior}\\ \text{angle}}}{\varphi_{j,k}}$$

$V = V_i + V_e$

$= V_i + V_{e,T} + V_{e,c}$

$$= \pi\,(2\underset{\substack{\uparrow\\ \text{interior}}}{E_i} + \underset{\substack{\uparrow\\ \text{exterior}}}{E_e}) - \Big(2\pi V_i + \pi V_{e,T} + \sum_l \underbrace{(\pi - \theta_l)}_{R\text{的外角}}\Big)$$

$$= 2\pi\,(E_i + E_e) + 2\pi\,(V_i + V_e) + \sum_l \theta_l$$

∂R : closed curve $\Rightarrow E_e = V_e$

$$= 2\pi\,(E - V) + \sum_l \theta_l .$$

$$\Rightarrow \int_R K\,dA + \int_{\partial R} k_g\,ds + \sum_l \theta_l = 2\pi\,(V - E + F)$$

$$= 2\pi\,\chi(R)$$

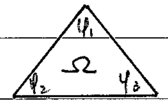
Cor M orientable surface ($\partial M = \emptyset$)

$$\int_M K\,dA = 2\pi\,\chi(M) = 4\pi\,(1-g) = 2\pi\,(\underbrace{2-2g}_{=\chi(M)})$$

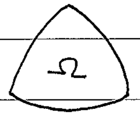
NO. 12/26 (四)

Applications of Gauss - Bonnet :

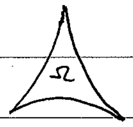
① Geodesic triangle



$K=0$



$K>0$



$K<0$

$\because k_g = 0$

$$\int_\Omega K\,dA + 0 + \sum_{i=1}^{3} (\pi - \varphi_i) = 2\pi$$

$$\text{ie. } (\varphi_1 + \varphi_2 + \varphi_3) - \pi = \int_\Omega K\,dA = \int_{N(\Omega)} d\sigma \quad ※$$

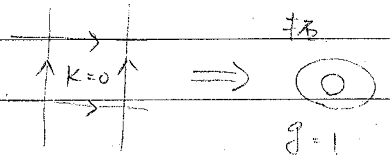
② Any compact surface without boundary with $K \geqslant 0$ and not $\equiv 0$ is homeomorphic to S^2

$$\langle \text{pf} \rangle \quad \underset{0}{\underset{\vee}{\int_M K\,dA}} = 2\pi\, \underset{2-2g}{\underset{\parallel}{\chi(M)}} \quad , \ g = \text{genus} = \#\text{ of holes}$$

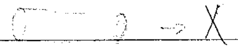
$$\Rightarrow g = 0$$

Remark : A cpt orientable surface is uniquely determined by its genus up to homeomorphism.

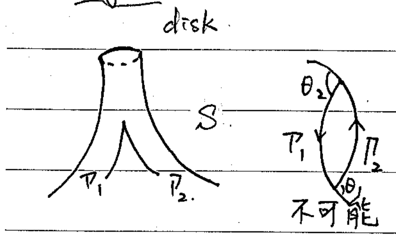
(cf. Massey's book algebraic topology)



$\mathbb{R}^2/\mathbb{Z}^2 = M \simeq T^2$



③ If $K \le 0$, then no two geodesics Γ_1, Γ_2 can bound a simple rigion in S.



在高維度中，有個一樣的定理

<pf> $\int_\Omega \underset{\le 0}{K}\, dA + \theta_1 + \theta_2 = 2\pi$

$\Rightarrow \theta_1 + \theta_2 \ge 2\pi$

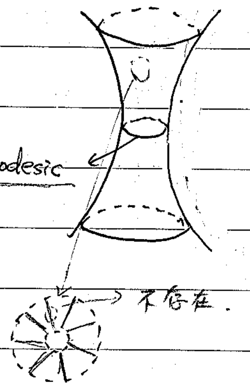
$\Rightarrow \theta_1 = \pi,\ \theta_2 = \pi$, $\left(\int_\Omega K\, dA \overset{!!}{=} 0\right)$

$\Rightarrow \Gamma_1 \equiv \Gamma_2$ —×— $(\because \Gamma_1 \ne \Gamma_2)$

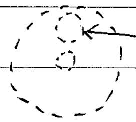
(a geodesic is determined by $\gamma(0), \gamma'(0)$)

④ If S is homeomorphic to a cylinder and $K<0$, then $\exists$ at most one simple closed geodesic. (i.e. smooth closed curve)

If fact, $\exists!$ geodesic

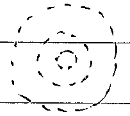


<pf> Topologically, the surface looks like

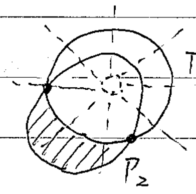


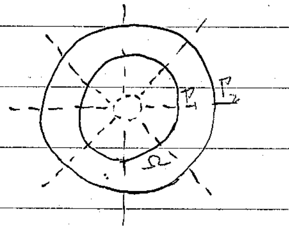
Γ 不會長這樣

$\because \underbrace{\int_\Gamma K\, dA}_{<0} + 0 + 0 = 2\pi$ —×—

$\Rightarrow$  Γ 一定長這樣

if there is second closed geodesic Γ_2, there is 2 case:





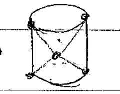
case 1: $\Gamma_1 \cap \Gamma_2 \ne \emptyset$.

take at one simple region bounded by Γ_1, Γ_2

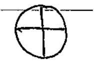
—×— ③

case 2: $\Gamma_1 \cap \Gamma_2 = \emptyset$.

Let Ω be the region bounded by Γ_1, Γ_2

Then $\chi(\Omega) = 0$. 用算的 $\Rightarrow$ 

(Remark: $\chi(0) = 1$

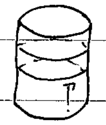
 $5 - 8 + 4 = 1$)

$6V - 14E + 8F$

by global G-B:

$\underbrace{\int_\Omega K\, dA}_{<0} + 0 + 0 = 2\pi \underset{\overset{!!}{0}}{\chi(\Omega)}$ —×—

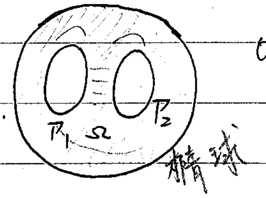
* if $K \equiv 0$, the cylinder is the standard one.



⑤ If M is cpt , $K>0$, then ∴ any two simple closed geodesics P_1 , P_2 must intersect

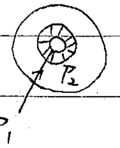
we must have $\chi(\Omega)=0$, where Ω is the region bounded by P_1 , P_2.

<pf>



if $P_1 \cap P_2 = \phi$

$$0 < \int_\Omega K\,dA + 0 + 0 = 2\pi\,\chi(\Omega) \;\; \cancel{\;}$$



$$\chi(M) = 2 - 2\underset{0}{g} = 2$$

$$\chi(\Omega) = 2 - 2 = 0 \;\#$$

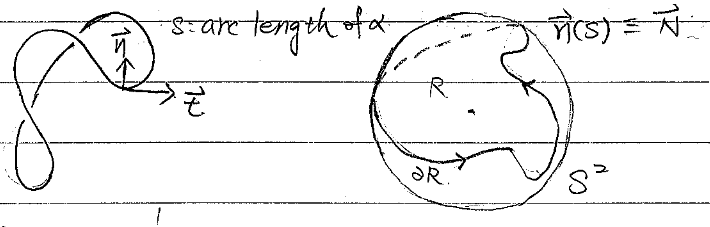
☆ ⑥ Jacobi thm (有什麼推廣?)

Let $\alpha : I \longrightarrow \mathbb{R}^3$ be a closed curve with $\underline{\underline{k>0}}$

(this is not necessary , just for convenience)

If $\vec{n}(s) \in S^2$ is a simple curve in S^2 (其實也不需要)

then $\vec{n}(s)$ separate S^2 into two parts with equal area !



<pf> $\underbrace{\int_R \overset{1}{K}\,dA}_{\text{area}(R)} + \int_{\partial R} k_g\,d\bar{s} + 0 = 2\pi$

∂R is the curve $\vec{n}$

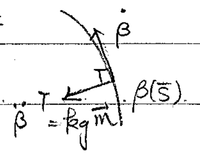
$\bar{s}$ = arc length of $\vec{n}$.

Jacobi's thm is equivalent to $\int_{\partial R} k_g\,d\bar{s} = 0$

$$\cdot = \frac{d}{d\bar{s}} \;;\; ' = \frac{d}{ds}$$

$$k_g = \left[\frac{D\dot{n}}{d\bar{s}}\right] = \ddot{n}\cdot(n \times \dot{n}) = |n \;\; \dot{n} \;\; \ddot{n}|$$

recall :



$\ddot{\beta}^T = k_g\,\vec{m}$, $\vec{m} = \underline{\underline{\vec{N}}} \times \dot{\beta}$

$$k_g = \ddot{\beta}^T \cdot (\vec{N} \times \dot{\beta})$$
$$= \ddot{\beta} \cdot (\vec{N} \times \dot{\beta})$$

$$n'' \cdot \frac{ds}{d\bar{s}} + n'\frac{ds}{d\bar{s}}$$

$$\dot{n} = \frac{dn}{d\bar{s}} = \frac{dn}{ds}\cdot\frac{ds}{d\bar{s}} = n'\,\frac{ds}{d\bar{s}}$$
$$= (-kt - \tau b)\,\frac{ds}{d\bar{s}}$$

$$\ddot{n} = (-kt - \tau b)\frac{d^2s}{d\bar{s}^2} + (-k't - \tau' b)\left(\frac{ds}{d\bar{s}}\right)^2 + (-k\underset{kn}{\underline{t'}} - \tau\underset{\tau n}{\underset{\sim}{b'}})\left(\frac{ds}{d\bar{s}}\right)^2$$

$$|\dot{n}| = 1 \;\Rightarrow\; (k^2+\tau^2)\left|\frac{ds}{d\bar{s}}\right|^2 = 1$$

$$k_g = |n,\; -kt-\tau b,\; -k't-\tau' b|\,\left|\frac{ds}{d\bar{s}}\right|^3$$
$$= (|n,\, -kt,\, -\tau' b| + |n,\, -\tau b,\, -k't|)\left(\frac{ds}{d\bar{s}}\right)^3$$
$$= (-k\tau' + \tau k')\left(\frac{ds}{d\bar{s}}\right)^3 \qquad , \quad |t . n . b| = 1$$
$$= \left(\frac{\tau k' - k\tau'}{k^2+\tau^2}\right)\frac{ds}{d\bar{s}}$$

$$k_g\,d\bar{s} = \left(\frac{\tau k' - k\tau'}{k^2+\tau^2}\right)ds$$
$$= \frac{-d}{ds}\left(\tan^{-1}\left(\frac{\tau}{k}\right)\right)ds \qquad \Rightarrow \int_{\partial R} k_g\,d\bar{s} = 0$$

⑦ Hopf Poincare' index thm :

S cpt surfaces, $\vec{v}$: smooth tangent vector field

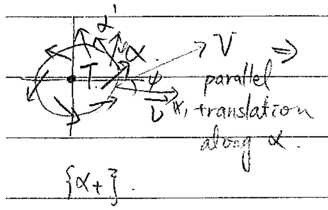
Define the singular points of $\vec{v}$ to be those $p \in S$

s.t. $\vec{v}(p) = 0$.

We consider only $\vec{v}$ with isolated singular pts.

S cpt $\Rightarrow$ $\vec{v}$ has only a finite number of singular pts

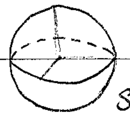
Def $I_p\vec{v}$: $\int_\alpha \varphi' dt = 2\pi \underbrace{I_p}_{\text{index}}$



$\Rightarrow \int_\alpha \varphi' dt$ is indep of α.

$\alpha = \partial T$

Thm $\sum_p I_p(\vec{v}) = \chi(S)$



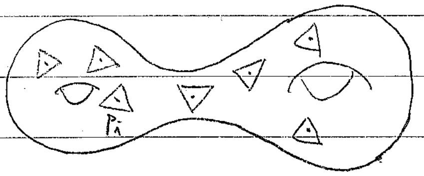
$\chi(S^2) = 2$

$\int_T K dA = \int_\alpha \psi' dt$

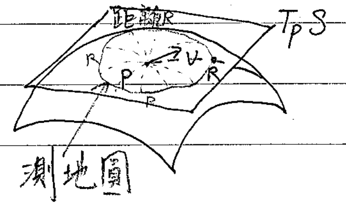
$2\pi I_p = \int_\alpha \varphi' dt$

$\int_{T_i} K dA - 2\pi I_{p_i} = \int_{\alpha_i} (\psi - \varphi)' dt$

$\sum_i (\quad) = 0 \Rightarrow \int_S K dA = 2\pi \sum I_{p_i}$



geodesic polar coordinates



exponential map

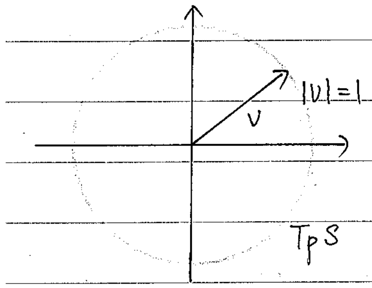
$$\begin{aligned} \exp_p : \; & T_pS \longrightarrow S \\ & \;\; \overset{\in}{v} \;\longmapsto\; \overset{\in}{\gamma(1)} \end{aligned}$$

, γ is the unique geodesic s.t.

$av \longmapsto \tilde{\gamma}(1) \equiv \gamma(a)$ $\quad \gamma(0) = p$, $\gamma'(0) = v$, $|\gamma'(0)| = |v|$

$\tilde{\gamma}(t) = \gamma(at)$ $\quad$ so, in fact $t = |v| s$

$\Rightarrow \tilde{\gamma}'(0) = a\gamma'(0) = av$



$\forall\, v \longleftrightarrow \gamma_v$ is the geod. st.

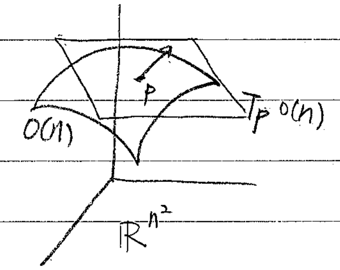
$\gamma_v(0) = p$, $\gamma_v'(0) = v$, $|v| = 1$ 單位速度

$\exp_p(w) = \gamma_{\frac{w}{|w|}}(|w|)$
$\quad\quad\quad\quad \overset{\Uparrow}{S^1}$

$|\gamma'(t)| = 1 \Rightarrow p \sim \gamma(|w|) = |w|$

$\forall t$

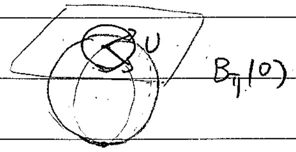
$O(n) \subset \mathbb{R}^{n^2}$



Proposition

① $\exp_p$ is C^∞ in a neighborhood of U of p

② $\exp_p$ is a diffeomorphism in a neighborhood U of p



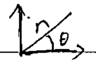
in fact $d(\exp_p)_0 : \underset{\overset{|||}{T_pS}}{T_0(T_pS)} \longrightarrow T_pS \equiv id.$

$dF_p : T_pS_1 \to T_qS_2$

$$\begin{tikzcd} S_1 \arrow[r, "F"] & S_2 \\ p \arrow[r, mapsto] & q \end{tikzcd}$$

☆ ③ $ds^2 = dr^2 + G(r,\theta)\, d\theta^2$, $\sqrt{G}(0,\theta) = 0$

$\exp_p : \underset{\overset{\text{S||}}{\mathbb{R}^2}}{\overset{\subset U}{T_pS}} \longrightarrow S$ $\qquad \sqrt{G}_r(0,\theta) = 1$



<pf>. ① follows from theory of ODE ※.

② $d(\exp_p)_0(w)$.

pick the curve to be tw , $t \in \mathbb{R}$.

$d(\exp_p)_0(w) = w$ $\forall w$ by def. $\Rightarrow d(\exp_p)_0 = id.$

$$w = \frac{d(\exp_p(tw))}{dt}\Big|_{t=0}$$

$$\begin{tikzcd} \exp_p(w) \arrow[rr, bend left] & \gamma(t) & \gamma(1) \end{tikzcd}$$

$\gamma(0) = p$

$\gamma'(0) = w$

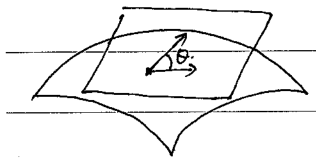
$\exp_p(tw) = \gamma(t)$

<pf> ③ (r,θ) as our coordinates.

· $ds^2 = \underset{\overset{||}{1.}}{E}\, dr^2 + F\, dr d\theta + G\, d\theta^2$. , $X = \exp_p$

$E = X_r \cdot X_r$ $\qquad X(r, \underset{\theta_0}{\overset{||}{\theta}})$ is a geodesic

$= 1.$ $\qquad r$ = arc length. $\Rightarrow |X_r| = 1$



$F_r = (X_r \cdot X_\theta)_r$

$= \underset{0\ \frac{DX_r}{dr} = 0.}{\cancel{X_{rr}}} \cdot X_\theta + X_r \cdot X_{\theta r}$ (還是可以固定θ)

$X_r \cdot X_{\theta r} = \frac{1}{2}(\underbrace{X_r \cdot X_r}_{\overset{||}{1}})_\theta \equiv 0.$ $\qquad \Rightarrow F_r = 0$

$\Rightarrow F(r,\theta) = F(\theta)$ is indep. of r.

but when $r = 0$. , $X_r \cdot X_\theta = 0$

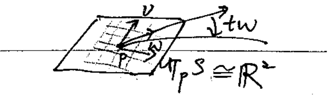
$d(\exp_p)_0 = id.$

$ds^2|_{T_pS} = dr^2 + r^2 d\theta^2.$ $\Rightarrow F = 0$.

exponential map

$\exp_p(tw) = \gamma(t)$ is a geodesic with $\gamma(0) = p$, $\gamma'(0) = w$

(the speed $|\gamma'(t)| = |w|$)



normal coordinates (u, v)

polar coordinates (r, θ)

$\rightarrow \exp_p(u\vec{e}_1 + v\vec{e}_2) : U_{\subset T_pS} \xrightarrow{X} S$

$\rightarrow \begin{cases} u = r\cos\theta , \ v = r\sin\theta \\ \exp_p(r\cos\theta\,\vec{e}_1 + r\sin\theta\,\vec{e}_2) = \tilde{X}(\underline{r}, \underline{\theta}) \end{cases}$

$ds^2 = \tilde{E}du^2 + 2\tilde{F}dudv + \tilde{G}dv^2.$

$\rightarrow ds^2 = dr^2 + G(r,\theta)\,d\theta^2$

$\sqrt{G}(0,\theta) = 0$, $\sqrt{G}_r(0,\theta) = 1$

$\sqrt{G} = \sqrt{\underset{\overset{\shortparallel}{1}}{E}G - \underset{\overset{\shortparallel}{0}}{F^2}}$

$dA = \sqrt{EG-F^2}\,drd\theta = \sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}\,dudv$

$= \sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}\,rdrd\theta$

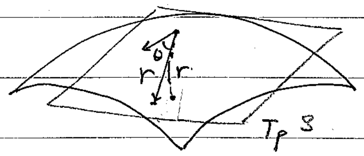
$= r\underbrace{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}_{} = A$

$r = 0$, $(u,v) = (0,0)$, $\tilde{E} = 1$, $\tilde{F} = 0$, $\tilde{G} = 1$

$\Rightarrow \sqrt{\tilde{E}\tilde{G}-\tilde{F}^2} = 1.$

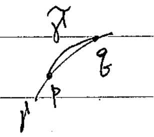
$\sqrt{G}_r = A_r r + A \cdot 1$

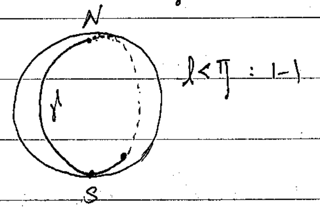
$r = 0$, $\sqrt{G}_r(0,\theta) = 1$

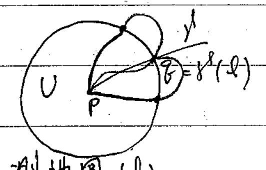


corolloary ①.

if γ is a geodesic with $\gamma(0)$, then for $|t|$ small enough $(\underset{\overset{\shortparallel}{p}}{\gamma(0)}, \underset{\overset{\shortparallel}{q}}{\gamma(t)})$, Any curve $\underset{\ne\gamma}{\tilde{\gamma}}$ connect p, q has bigger length than γ.







測地圓 (l)

<pf> $|\tilde{\gamma}| = \int_0^l ds$

$= \int_0^l \sqrt{dr^2 + \underset{\overset{\vee}{0}}{G}d\theta^2} \geq \int_0^l dr = l$

"=" $\Leftrightarrow$ $d\theta = 0$ ie. θ = constant.

※. 局部最短 唯一最短.

cor ② in general K does not determine 1st fund form.

But , if $K \equiv$ const. then this is true! (Minding)

ie. Any 2 surfaces S_1 , S_2 with the same constant curvature K are locally isometric.

lemma. $K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{\cancel{E_2}}{\sqrt{EG}}\right)_2 + \left(\frac{G_1}{\sqrt{EG}}\right)_1\right)$ in (r,θ) coor. $E = 1$, $F = 0$, G

$\qquad 0$

$K = -\frac{1}{2\sqrt{G}}\left(\frac{G_r}{\sqrt{G}}\right)_r$

let $\varphi = \sqrt{G}$ $\Rightarrow K = -\frac{1}{\cancel{2}\varphi}\left(\frac{\cancel{2}\cancel{\varphi}\varphi_r}{\cancel{\varphi}}\right)_r = -\frac{\varphi_{rr}}{\varphi}$

$K = -\frac{\varphi_{rr}}{\varphi}$

<pf>. $\varphi_{rr} + \overset{\substack{\text{const.}\\ \downarrow}}{K}\varphi = 0$ Let's solve this for fixed θ.

- $K=0$, $\varphi_{rr}=0$

$\Rightarrow \varphi(r,\theta) = a_1(\theta)\,r + a_2(\theta)$ $(\varphi=\sqrt{G})$

$\varphi(0,\theta)=0 \Rightarrow a_2(\theta)\equiv 0$

$\sqrt{G}_r(0,\theta)=1 \Rightarrow a_1(\theta)\equiv 1$

$\Rightarrow ds^2 = dr^2 + r^2 d\theta$

ie. S can be isometrically mapped into a plane locally.

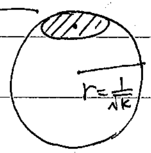
- $K>0$, $\varphi(r,\theta) = a_1(\theta)\cos\sqrt{K}r + a_2(\theta)\sin\sqrt{K}r$

$\varphi(0,\theta)=0 \Rightarrow a_1\equiv 0$

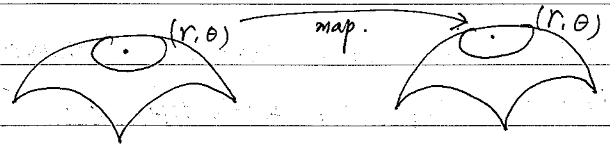
$\varphi_r(0,\theta)=1 \Rightarrow a_2(\theta)\sqrt{K}\equiv 1$

$\varphi(r,\theta) = \frac{1}{\sqrt{K}}\sin\sqrt{K}r = \sqrt{G}$.

$\Rightarrow ds^2 = dr^2 + \frac{\sin^2\sqrt{K}r}{K}d\theta^2$.



$\Rightarrow$ for any two surface S_1, S_2.



- $K<0$, $\varphi(r,\theta) = a_1(\theta)\cosh\sqrt{-K}r + a_2(\theta)\sinh\sqrt{-K}r$

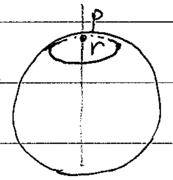
$\Rightarrow \begin{cases} a_1(\theta)\equiv 0 \\ a_2(\theta)\cdot\sqrt{-K}\equiv 1 \end{cases}$

$\sqrt{G} = \varphi(r,\theta) = \frac{\sinh\sqrt{-K}r}{\sqrt{-K}}$

$ds^2 = dr^2 + \frac{\sinh^2\sqrt{-K}r}{-K}d\theta^2$

---

corollary③(Gauss)



① $K(p) = \lim_{r\to 0}\frac{3}{\pi}\frac{2\pi r - L}{r^3}$

L = length of $\partial B_r(p)$ ← geodesic circle of radius r

② $K(p) = \lim_{r\to 0}\frac{12}{\pi}\frac{\pi^2 - A}{r^4}$

<pf> for any fixed θ,

ie. $\varphi(r,\theta) = \underbrace{\varphi(0,\theta)}_{0} + \underbrace{\varphi_r(0,\theta)}_{1}r + \frac{1}{2}\underbrace{\varphi_{rr}(0,\theta)}_{-K(p)\underbrace{\varphi(0,\theta)}_{0}}r^2 + \frac{1}{6}\varphi_{rrr}(0,\theta)r^3 + o(r^3)$

$\frac{1}{6}\varphi_{rrr}(0,\theta) = -\frac{1}{6}(K\varphi)_r(0,\theta)$

$= -\frac{1}{6}(K_r\underbrace{\varphi}_{0} + K\underbrace{\varphi_r}_{1})(0,\theta)$

$\Rightarrow \varphi = r - \frac{1}{6}K(p)r^3 + o(r^3)$ —— 重要.

$ds^2 = dr^2 + \varphi^2 d\theta^2$

($\varphi = r$ 就是欧氏空間)

$L = \int_0^{2\pi} ds = \int_0^{2\pi}\sqrt{\underset{0}{\cancel{dr^2}} + G\,d\theta^2}$

$= \int_0^{2\pi}\varphi\,d\theta$

$= 2\pi r - \frac{\pi}{3}K(p)r^3 + \tilde{o}(r^3)$

$K(p) = \frac{2\pi r - L}{r^3}\frac{3}{\pi} + \underset{0}{\cancel{\frac{\tilde{o}(r^3)}{r^3}}}$, $r\to 0$

下 學 期 。。。

NO. 2003 2/18

Rigidity of S^2

Fact : M : surface , $K \equiv 1$. If M is not compact then M is not necessarity a part of S^2.

Thm : S cpt in $\mathbb{R}^3$ with $K \equiv 1 \Rightarrow S = S^2$

Lemma : If $\exists\ p \in S$ st. $K_p > 0$ and $\begin{cases} k_1 \text{ is a local max} \\ k_2 \text{ is a local min} \end{cases}$ at p

then p is umbilical (ie. $k_1 = k_2$ at p) . ($k_1 \geq k_2$)

<pf> If not , $\exists$ nbd (of p) with lines of curvature = coor. lines

so $F = 0 = f$ with $k_1 = \frac{e}{E}$, $k_2 = \frac{g}{G}$.

$$\Rightarrow \text{Codazzi eq} : \begin{cases} e_2 = \frac{E_2}{2}(k_1 + k_2) \\ g_1 = \frac{G_1}{2}(k_1 + k_2) \end{cases}$$

$$e_2 = (E k_1)_2 = E(k_1)_2 + E_2 k_1$$

$$(k_1)_2 = \frac{1}{2}\frac{E_2}{E}(k_2 - k_1) ; \quad (k_2)_1 = \frac{1}{2}\frac{G_1}{G}(k_1 - k_2)$$

$$K = -\frac{-1}{\sqrt{EG}}\left(\left(\frac{E_2}{\sqrt{EG}}\right)_2 + \left(\frac{G_1}{\sqrt{EG}}\right)_1 \right)$$

$$\Rightarrow -2K \cdot EG = E_{22} + G_{11} + E_2\sqrt{EG}\left(-\frac{1}{2}\right)\frac{(EG)_2}{\sqrt{EG}^3}$$

$$+ G_1\sqrt{EG}\left(-\frac{1}{2}\right)\frac{(EG)_1}{\sqrt{EG}^3}.$$

$$= E_{22} + G_{11} + E_2 M(u,v) + G_1 N(u,v).$$

$$\begin{cases} E_{22} = \left(\frac{2(k_1)_2 E}{k_2 - k_1}\right)_2 = \frac{2E(k_1)_{22}}{k_2 - k_1} + (k_1)_2\, \alpha(u,v) \\ G_{11} = \cdots \end{cases}$$

$$\underbrace{-(k_1-k_2)2K\cdot EG}_{<0} = -2E(k_1)_{22} + 2G(k_2)_{11} + \tilde{M}(u,v)\overset{\overset{0 \text{ at } p}{\|}}{(k_1)_2} + \tilde{N}(u,v)\underset{\underset{0 \text{ at } p}{\|}}{(k_2)_1}$$

$\geqslant 0$.

<pf of Rigidity thm>

$K = \text{const.} > 0 = k_1 k_2$.

let k_1 have its max at p (S cpt).

then clearly, k_2 has its min at p

so p is umbilical ie. ($k_2(p) = k_1(p)$)

$$k_1(x) \leq k_1(p) = k_2(p) \leq k_2(x).$$

ie. all $x \in S$ are umbilical pt..

hence $S \equiv S^2$ (of some radius) ※

Remark We use only the fact that $k_1 \nearrow \Rightarrow k_2 \searrow$

Thm: $\underset{\text{cpt.}}{S} \subseteq \mathbb{R}^3$ and $\overset{(K>0)}{H} = \text{const.} \Rightarrow S = S^2$.

NO. 2/20.

The congruence theorem of ovaloids $\to$ def: $K>0$.
(convex body)

↓

the same up to rotations and translations

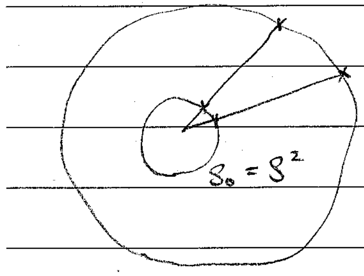
Bonnet thm: 2 isometric surface are congruent

⇔ They have the same II.

Given S_0, the underlying topological surface ($\cong S^2$)

S, S^* are two surfaces in $\mathbb{R}^3$ homeo to S_0

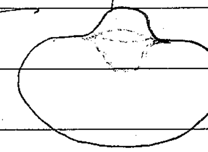
st. $K>0$ & $(E, F, G) = (E^*, F^*, G^*)$ (ie. isometric)



let $\begin{cases} \lambda = L^* - L \\ \mu = M^* - M \\ \nu = N^* - N \end{cases}$

wish to show $\lambda = 0 = \mu = \nu$.

wunter-ep.



$$\frac{LN-M^2}{EG-F^2} = K = K^* = \frac{L^*N^*-M^{*2}}{E^*G^*-F^{*2}}.$$

$\Rightarrow LN - M^2 = L^*N^* - M^{*2}$

in this case $\lambda\nu - \mu^2 = 0$ ⇕

claim: $\lambda du^2 + 2\mu du dv + \nu dv^2$ is either 0 or indefinite (非正)

(point-wise). $\begin{bmatrix} \lambda & \mu \\ \mu & \nu \end{bmatrix}(p)$.

<pf> notice the area of $Ax^2 + 2Bxy + Cy^2 = 1$ ellipse.

area $= (AC - B^2)^{-\frac{1}{2}}\pi$.

(because $\lambda_1 \bar{x}_1^2 + \lambda_2 \bar{x}_2^2 = 1$ has area $\pi\sqrt{\lambda_1\lambda_2}^{-1}$)

~~at one point~~

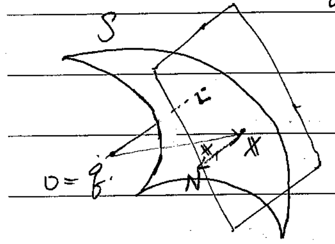
Now $Lx^2 + 2Mxy + Ny^2$

$L^*x^2 + 2M^*xy + N^*y^2$ can't dominate each other

since their level set at 1 have the same area.

Def. Supporting function

$q \in \mathbb{R}^3 \setminus S$, define $P(X) := \operatorname{dist}(q, T_X S)$



~~N = inner normal~~

when S is ovaloid . let q be in its interior

then $P(X) = -X \cdot N > 0$

Prop for isometric closed surfaces S, S^*.

$$\Rightarrow \int_S \frac{\lambda\nu - \mathcal{M}^2}{EG - F^2} \underbrace{(p + p^*)}_{\text{supporting func.}} dA = 0 .$$

claim 1 : $\int_S (KP - H)\, dA = 0$

2 : $\int_S (K'P - H^*)\, dA = 0$.

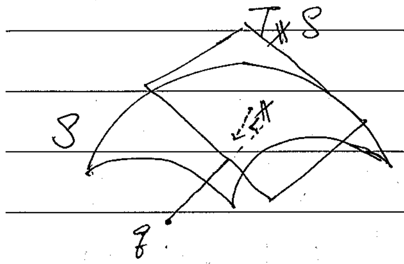
Ovaloid : the boundary of a convex body in $\mathbb{R}^3$

Hadamard characterization :

S smooth surface without boundary in $\mathbb{R}^3$, $\underline{\underline{K > 0}}$

$\Rightarrow S$ is a ovaloid.

Supporting function :



$P(X) = \operatorname{dist}(q, T_X S)$.

when S ovaloid , $q \equiv 0$ in the interior o

$P(X) = -X \cdot N > 0$

Main theorem : If S, S^* are isometric ovaloid, then S and S^* are the same up to a Euclidean motion in $\mathbb{R}^3$.

$S \xrightarrow{f} S^*$, f is an isometry

if $\langle df(v), df(w) \rangle = \langle v, w \rangle \quad \forall\, v, w \in T_p S , \ \forall p \in S$

Proposition : $\int_S \frac{\lambda\nu - \mathcal{M}^2}{EG - F^2} (p + p^*)\, dA = 0$

We use the same coordinate system on S and S^*

II on S is $L\,du^2 + 2M\,dudv + N\,dv^2$. (A)

S^* is $L^* du^2 + 2M^* dudv + N^* dv^2$ (B)

$$\underset{E^*}{\underset{\shortparallel}{E}}\, du^2 + 2\underset{F^*}{\underset{\shortparallel}{F}}\, dudv + \underset{G^*}{\underset{\shortparallel}{G}}\, dv^2$$

$$K = K^* \Rightarrow \frac{LN - M^2}{EG - F^2} = \frac{L^*N^* - M^{*2}}{E^*G^* - F^*}$$

$$\Rightarrow LN - M^2 = L^*N^* - M^{*2}.$$

(B) − (A) : difference $= \lambda du^2 + 2\mu\, du dv + \nu dv^2$.

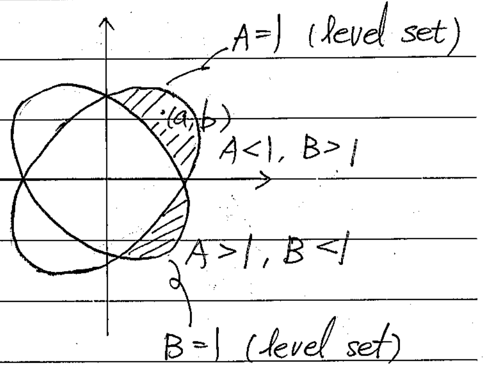
Lemma : This is indefinite or 0 $(\Leftrightarrow \lambda\nu - \mu^2 = 0)$

re-explain :

$v = a\mathbb{X}_u + b\mathbb{X}_v = \mathbb{X}_u \frac{du}{dt} + \mathbb{X}_v \frac{dv}{dt}$

$(a, b)\begin{bmatrix} L & M \\ M & N \end{bmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$

$(a, b)\begin{bmatrix} L^* & M^* \\ M^* & N^* \end{bmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$



<pf> $\lambda\nu - \mu^2 = (L^* - L)(N^* - N) - (M^* - M)^2$

(1947) $= (L^*N^* - M^{*2}) + (LN - M^2) - (LN^* - 2M\cdot M^* + NL^*)$

$$\frac{\lambda\nu - M^2}{EG - F^2} = 2K - \frac{LN^* - 2MM^* + NL^*}{EG - F^2} \equiv \underset{\sim\sim\sim}{2K - 2K'}$$

$$\int_S (K - K')(P + P^*)\, dA$$

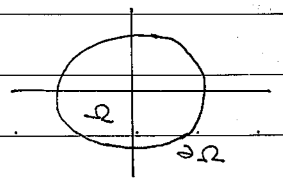
$$= \int_S \{ \underset{①}{(KP - H)} - \underset{②}{(K'P - H)} + \underset{③}{(KP^* - H^*)} - \underset{④}{(K'P^* - H^*)} \} dA$$

will show ① = 0 = ④ (Carton)

then by symmetry then get ② = 0 = ③

differential form :
- external product $e_1 \wedge e_2 = -e_2 \wedge e_1$
- $d(Pdx + Qdy) := dP \wedge dx + dQ \wedge dy$
  $= (P_x dx + P_y dy) \wedge dx + (Q_x dx + Q_y dy) \wedge dy$
  $= (Q_x - P_y) dx \wedge dy$

· Green's thm.

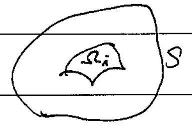


$\int_{\partial\Omega} \boxed{Pdx + Qdy} = \int_\Omega (Q_x - P_y) dx dy$

α : 1-form

$\int_{\partial\Omega} \alpha = \int_\Omega d\alpha$

① $\int_S \overset{d\alpha}{\underline{(KP - H)}}\, dA$



cut S into piece $\{\Omega_i\}$ s.t.
each Ω_i is contained in one coor. chart.

$dN := N_u du + N_v dv$

$\oint_{\partial\Omega_i} \underbrace{|\mathbb{X}\ N\ dN|}_{\alpha}$

$= \oint |\mathbb{X}\ N\ N_1| du + |\mathbb{X}\ N\ N_2| dv$

$= \int_{\Omega_i} (|\mathbb{X}\ N\ N_2|_1 - |\mathbb{X}\ N\ N_1|_2) du dv$

$$= \int_{\Omega_i} \begin{pmatrix} |\mathbb{X}_1\ N\ N_2| + |\mathbb{X}\ N_1\ N_2| + |\mathbb{X}\ \cancel{N}\ N_{21}| \\ \underbrace{-|\mathbb{X}_2\ N\ N_1|}_{①} - \underbrace{|\mathbb{X}\ N_2\ N_1|}_{③} - |\mathbb{X}\ \cancel{N}\ N_{12}| \end{pmatrix} du dv$$

To evaluate ① and ② may pick special coordinates

say use line of curvature $\begin{cases} N_1 = -k_1 \mathbb{X}_1 \\ N_2 = -k_2 \mathbb{X}_2 . \end{cases}$

① $= |\mathbb{X}_1\ N\ -k_2\mathbb{X}_2| - |\mathbb{X}_2\ N\ -k_1\mathbb{X}_1|$

$= -k_2 |\mathbb{X}_1\ N\ \mathbb{X}_2| + k_1 |\mathbb{X}_2\ N\ \mathbb{X}_1| = k_1 + k_2 = 2H$.

② $= 2|\mathbb{X}\ N_1\ N_2|$

since $N_i \in T S$, $\mathbb{X}$ only has contribution in N direction

i.e. $(\mathbb{X}\cdot N)N = -PN$.

So ② $= 2|\mathbb{X}\cdot N_1\ N_2| = -2PK|\mathbb{X}_1 \times \mathbb{X}_2|$

i.e. the integral $= \int_{\Omega_i} (-2PK + 2H) dA$.

in total $-2\int_S (PK - H) dA = \sum_i \oint_{\partial\Omega_i} |\mathbb{X}\ N\ dN| = 0$

∵ dN 不交換

$$\underbrace{|d\mathbb{X}\ N\ dN|}_{2H} + \underbrace{|\mathbb{X}\ \overset{\neq 0}{dN}\ dN|}_{-2PK} + |\mathbb{X}\ \underset{0}{\cancel{N\ ddN}}|$$

NO. 92.2.27.

Last time : $-2\int_{\Omega_i} (KP - H)\,dA = \oint_{\partial\Omega_i} |\mathbb{X}\ N\ dN|$

$\qquad\qquad\qquad\qquad\overset{\shortparallel}{d\alpha}\qquad\qquad\qquad\overset{\shortparallel}{\alpha}$

Today : $-2\int_{\Omega_i} (K'P - H^*)\,dA \overset{?}{=} \oint_{\partial\Omega_i} |\mathbb{X}\ N\ P|$

$\uparrow$ $\frac{1}{2}\frac{LN^* - 2MM^* + L^*N}{EG-F^2}$ $\qquad$ $\uparrow$ some 1-form

Recall : $dN = N_1 du + N_2 dv = -\sum_{i,j=1}^{2} l_i^j \mathbb{X}_j du^i \quad (l_i^j = -a_{ji})$

in Do Carmo p.155 $\begin{cases} N_1 = a_{11}\mathbb{X}_1 + a_{12}\mathbb{X}_2 \\ N_2 = a_{21}\mathbb{X}_1 + a_{22}\mathbb{X}_2 \end{cases}$

$$a_{11} = \frac{MF-LG}{EG-F^2} \qquad a_{12} = \frac{NF-MG}{EG-F^2}$$

$$a_{21} = \frac{LF-ME}{EG-F^2} \qquad a_{22} = \frac{MF-NE}{EG-F^2}$$

if $\mathbb{X}_1$, $\mathbb{X}_2$ are an ONB at $p \in S$

then get $-\begin{pmatrix} L & M \\ M & N \end{pmatrix}$

if $\mathbb{X}_1$, $\mathbb{X}_2$ are ONB at $p \in S$, then $l_i^j = L, M, N$.

let $P := -\sum_{i,j=1}^{2} \underbrace{l_i^{j*}}_{S^*} \underbrace{\mathbb{X}_j}_{S} du^i$!

notice that $dP \neq 0$ in general !

NO.

$$\oint_{\partial\Omega_i} |\mathbb{X}\ N\ P| = \int_{\Omega_i} \underset{\substack{\shortparallel ? \\ ①\ 2H^*}}{|d\mathbb{X}\ N\ P|} + \underset{\substack{\shortparallel ? \\ ②\ -2PK'}}{|\mathbb{X}\ dN\ P|} + \underset{\substack{\shortparallel ? \\ ③\ 0}}{|\mathbb{X}\ N\ dP|}$$

③ : $|\mathbb{X}\ N\ dP|$

↖ the only contribution from $\mathbb{X}$ is of the form $a\mathbb{X}_1 + b\mathbb{X}_2 (+\cancel{c}N)$

$$|\mathbb{X}_{k'}\ N\ dP| = (\mathbb{X}_{k'} \times N)\cdot dP = \pm \mathbb{X}_k \cdot dP.$$

$$= \pm\, d(l_i^{j*}\mathbb{X}_j)\cdot\mathbb{X}_k du^i$$

$$= \pm(\cdots d\mathbb{X}_j^* \cdot \mathbb{X}_k^*)$$

$$= \pm\, \underset{\sim}{ddN^*} \cdot \mathbb{X}_k^* = 0.$$

① $|d\mathbb{X}\ N\ P| = (-l_i^{*j}\mathbb{X}_j du^i) \times N \cdot d\mathbb{X}$

the non-zero terms are of the form $|\mathbb{X}_1\ \mathbb{X}_2\ N|\cdot(\cdots)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad |\mathbb{X}_1^*\ \mathbb{X}_2^*\ N^*|$

$\Rightarrow |d\mathbb{X}\ N\ P| = |d\mathbb{X}^*\ N^*\ dN^*|$

$= 2H^*$

② $|\mathbb{X}\ dN\ P| = |\mathbb{X}\ \ N_1 du + N_2 dv\ \ -l_1^{*j}\mathbb{X}_j du - l_2^{*j}\mathbb{X}_j dv|$

$$= (|\mathbb{X}\ \ \underset{\substack{\shortparallel \\ l_1^{i}\mathbb{X}_i}}{N_1}\ \ -l_2^{*j}\mathbb{X}_j| + |\mathbb{X}\ \ \underset{\substack{\shortparallel \\ -l_2^{i}\mathbb{X}_i}}{N_2}\ \ -l_1^{*j}\mathbb{X}_j|)\,du\,dv$$

projection in N direction
$\mathbb{X}\cdot N$

at a point p, pick coor. system s.t. coor. lines are lines of curv.

$= -2K'P\sqrt{EG-F^2}\,du\,dv$

$$\begin{array}{ccc} |N\ \mathbb{X}_1\ \mathbb{X}_2|LN^* & - & |N\ \mathbb{X}_1\ \mathbb{X}_2|\ MM^* \\ |N\ \mathbb{X}_1\ \mathbb{X}_2|MM^* & & |N\ \mathbb{X}_1\ \mathbb{X}_2|-NL^* \end{array}$$

$$= \underset{\overset{\|}{\sqrt{EG-F^2}}}{|N\ \mathbb{X}_1\ \mathbb{X}_2|}\,(LN^*-2MM^*+L^*N)$$

---

invariance of $\left\{\begin{array}{l} \text{domain}\quad \underset{\mathbb{R}^n}{U_{\text{open}}} \xrightarrow{f} \mathbb{R}^n \quad f(U) \text{ open?} \\ \text{dim.}\quad \mathbb{R}^n \overset{?}{\simeq} \mathbb{R}^m \Leftrightarrow n=m. \end{array}\right.$

92. 3. 4.

Def: A topological space M is called a manifold if it is locally homeomorphic to $\mathbb{R}^n$.

i.e. for any $p \in M$, $\exists$ U nbd of p st. $U \underset{\uparrow \mathbb{X}}{\simeq} \mathbb{R}^n$ (for a fixed

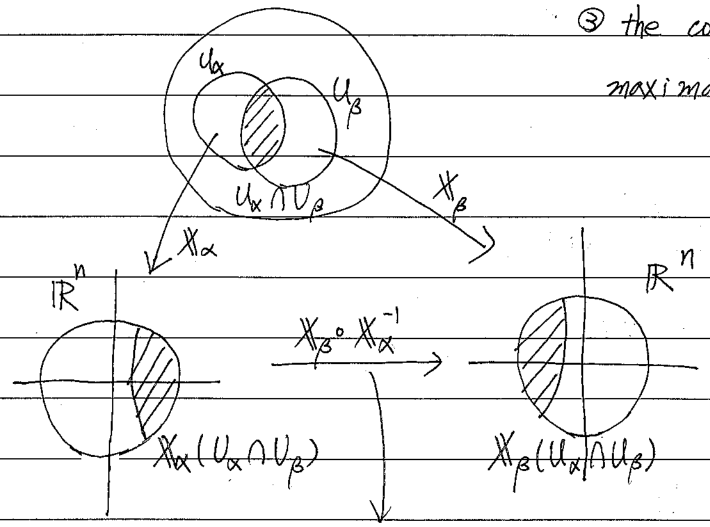
homeo

U is called a coor. chart, $\mathbb{X}$: coor. (func)

Def: a C^k differentiable structure on a manifold M is a collection of open charts $(U_\alpha, \mathbb{X}_\alpha)_{\alpha\in\Lambda}$ s.t. ① $\bigcup_{\alpha\in\Lambda} U_\alpha = M$

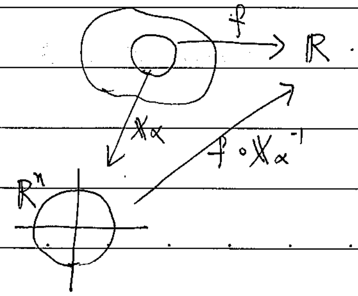
② $\mathbb{X}_{\beta\alpha}$ are C^k $\forall\ \alpha, \beta \in \Lambda$

③ the collection $\{(U_\alpha, \mathbb{X}_\alpha)\}$ is maximal with properties ① & ②



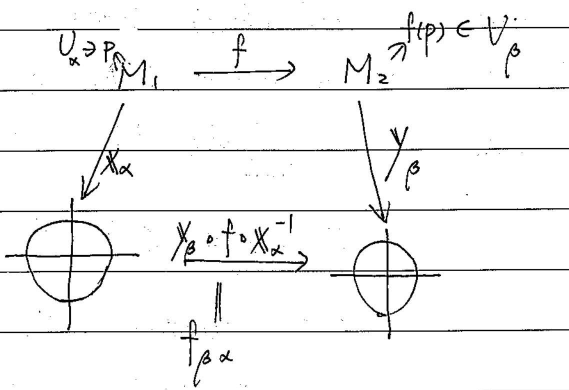
$$\mathbb{X}_\alpha(U_\alpha \cap U_\beta) \xrightarrow{\mathbb{X}_{\beta\alpha}} \mathbb{X}_\beta(U_\alpha \cap U_\beta)$$

Def: A continuous function $f: M \to \mathbb{R}$ is called C^k if M is at least C^k and $f \circ \mathbb{X}_\alpha^{-1}$ is C^k, $\forall\alpha$



M_1, M_2, C^k manifolds

A C^k map. $f: M_1 \longrightarrow M_2$. is a conti map st. $f_{\beta\alpha}$ is C^k $\forall \alpha, \beta$ / $f(U_\alpha) \subset V_\beta$



(abstract non-sense)

Category and functors:

Boubaki (德) Hilbert Gödel

Elements of Mathematics

$\mathcal{C}$ category

ob $\mathcal{C}$ objects (物件)

A, B in ob$\mathcal{C}$

Mor(A, B) set of "morphisms" from A to B

st. $\exists$ $Mor(A,B) \times Mor(B,C) \longrightarrow Mor(A,C)$

$\qquad f \qquad\qquad g \qquad\qquad g \circ f$.

$(g \circ f) \circ h = g \circ (f \circ h)$

examples: ① the category of groups. Group

ob Group = all groups

Mor(G, H) = group homomorphisms from A to B

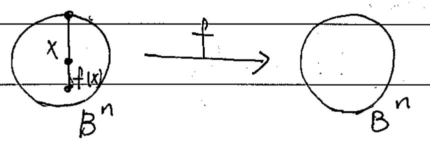
---

② Top : the category of topological spaces

ob Top = all top spaces.

Mor(X, Y) = the set of all conti map from X to Y.

Brauwer fixed point thm.



$\exists$ $p \in B^n$ st. $f(p) = p$.

if no fixed point then

let $g(x) = \overrightarrow{f(x)x} \cap S^{n-1}$ $\forall x \in B^n$.

clearly $g|_{S^{n-1}}$ = identity

(topological spaces) Top $\xrightarrow{H^{n-1}}$ Ab (abelian groups), H : a functor.

$$\begin{tikzcd}
S^{n-1} \arrow[rr, "id"] \arrow[dr, "i"'] & & S^{n-1} \arrow[r, bend left, "H^{n-1}"] & H^{n-1}(S^{n-1}) \arrow[rr, "\overset{0}{\neq}\ id"] \arrow[dr] & & H^{n-1}(S^{n-1}) \\
& B^n \arrow[ur] & & & H^{n-1}(B^n) \arrow[ur] \arrow[d, phantom, "\|"] & \\
& & & & 0 &
\end{tikzcd}$$

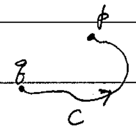
$e \xrightarrow{F} e'$

$$\begin{tikzcd}
A \arrow[d, "\varphi"] \arrow[dd, bend right] & F(A) \arrow[d, "F(\varphi)"] \arrow[dd, bend right] \\
B \arrow[d, "\psi"] & F(B) \arrow[d, "F(\psi)"] \\
C & F(C)
\end{tikzcd}$$

$F(\psi \circ \varphi) = F(\psi) \circ F(\varphi)$.

n=2

H^1 = 1-cycle / 1-boundary



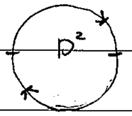
$\partial c = p - q$.

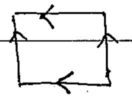
Thm : Any 2 compact 2-dim manifold $\overset{M}{\vee}$ is a connected sum

$T^2 \# T^2 \# \cdots \# T^2$

or $Rp^2 \# Rp^2 \# \cdots \# Rp^2$.

$D^2/\sim$ $\quad x \sim -x$ if $x \in S^1$





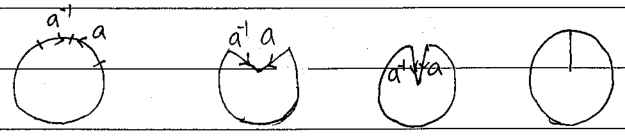
<pf> step 0 Assume M has a triangulation

$$M = \bigcup_{i=1}^{N} Q_i \quad Q \simeq \triangle \quad , Q_i \cap Q_j = \text{edge}$$

step 1 將三角形並置於平面上.

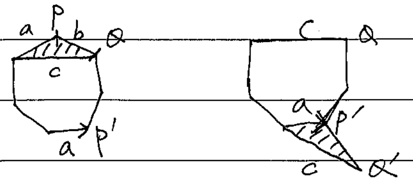
step 2 eliminates adjacent edges* of 1st kind ie. aa^{-1}

may assume no *.



step 3. Transform to a polygon st. all vertices and identified to one point.

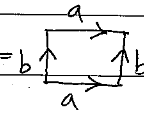
if $P \not\sim Q$ with $\overline{PQ} = b$



$\begin{cases} P \text{ disappear} \\ Q' \text{ appear} \end{cases}$

Q: classify compact topological (triangulable) surface.

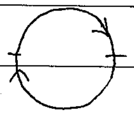
Expected result : $S \cong T \# \cdots \# T$, $T =$



$(\equiv T^2)$

or $\cong RP^2 \# \cdots \# RP^2$.

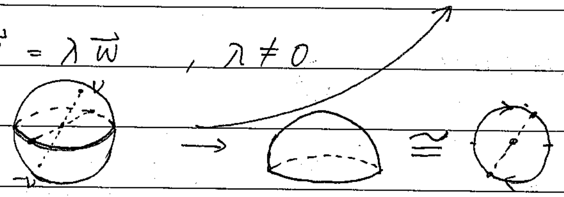
$RP^2 =$



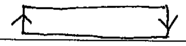
$$\mathbb{R}P^2 = (\mathbb{R}^3 \setminus 0)/\sim$$

$$\vec{v} \sim \vec{w} \Leftrightarrow \vec{v} = \lambda \vec{w} \quad , \lambda \neq 0$$

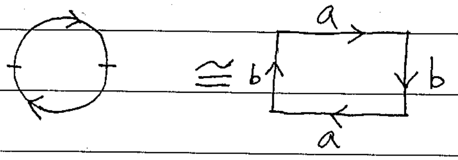
$$= S^2/\{\pm 1\}$$

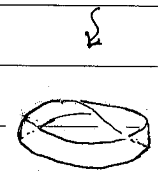


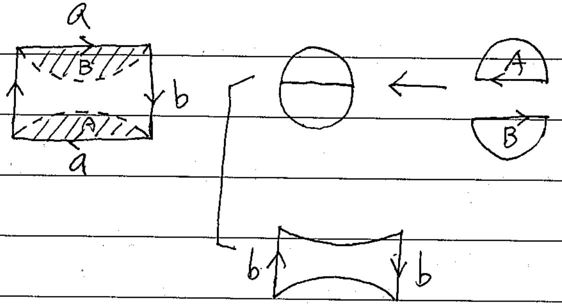
Fact : $\mathbb{R}P^2 \setminus D^2 \cong$ Möbius band $M (\equiv M^2)$



non-orientable





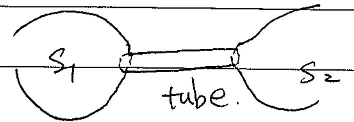


Recall : any compact $\underset{\text{closed}}{\wedge}$ surface in $\mathbb{R}^3$ is orientable.
(will be proved later).

Fact : $S_1 \supset S_2$, S_2 non-orientable
$\Rightarrow S_1$ non-orientable

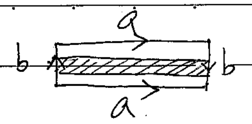
Def of # (connected sum)

$$S_1 \# S_2 = (S_1 \setminus D^2) \cup \text{tube} \cup (S_2 \setminus D^2)$$



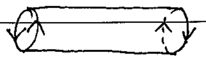
* orientation to do the #

Klein Bottle :

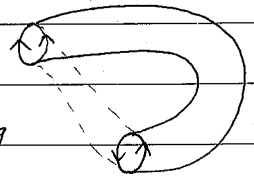


$a^{-1} b^{-1} a b^{-1}$

$K \equiv K^2$

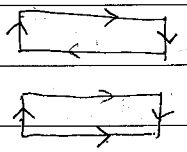


Fact : $K = M \cup M$.

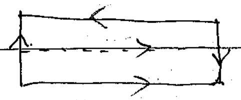


immersion
but not embedding

$K \setminus M =$



$\Rightarrow$



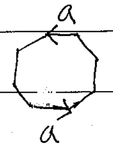
continue the proof:

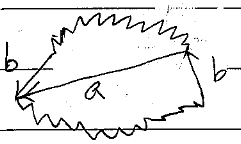
recall steps 1-3 $\Rightarrow$ only one class of vertexe

step 4 : make any pair of edges of 2nd kind adjacent

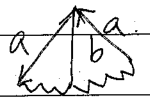
$(\cdots a \cdots a^{-1} \cdots)$ 1st kind

$(\cdots a \cdots a \cdots)$ 2nd kind





$\longrightarrow$



Notice that : If there are no edges of 1st kind,

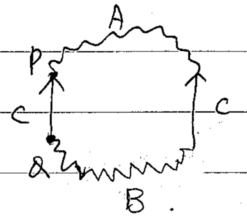
then get $a_1 a_1 a_2 a_2 a_3 a_3 \cdots$

$\underbrace{(a_1 a_1)}_{\mathbb{R}P^2} \# (a_2 a_2) \# \cdots$

If $\exists$ 1st kind $\cdots c \cdots c^{-1} \cdots$

claim $\exists\ d, d^{-1}$ s.t. $\cdots c \cdots d \cdots c^{-1} \cdots d^{-1}$

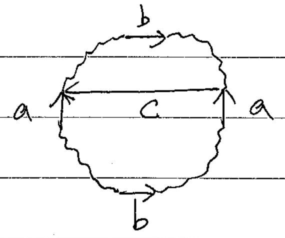
<pf> If not, then get



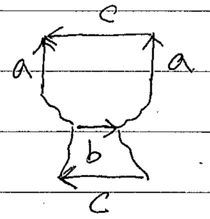
edges in A $\nsim$ edges in B.

but this $\Rightarrow P \nsim Q$ ✗

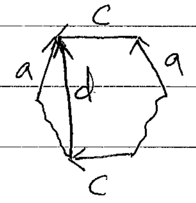
step 5.



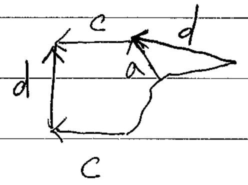
$\Rightarrow$



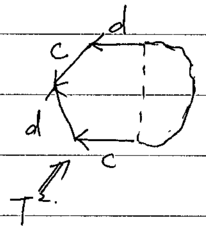
$\Rightarrow$



$\Rightarrow$



$d c d^{-1} c^{-1}$ $\Rightarrow$



Now already proved

Any surface (cpt closed) = connected sum of $\mathbb{R}P^2$ & T^2

☆ step 6 $T^2 \# \mathbb{R}P^2 \cong \underbrace{\mathbb{R}P^2 \# \mathbb{R}P^2}\# \mathbb{R}P^2$

lemma : $\mathbb{R}P^2 \# \mathbb{R}P^2 \cong K$

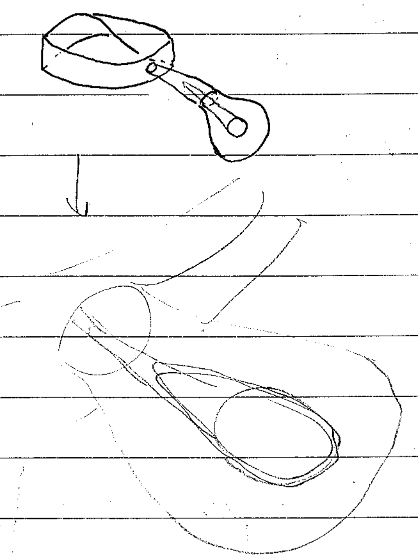
<pf> $\mathbb{R}P^2 \setminus D^2 \cong M$ and since $M \cup M \cong K$

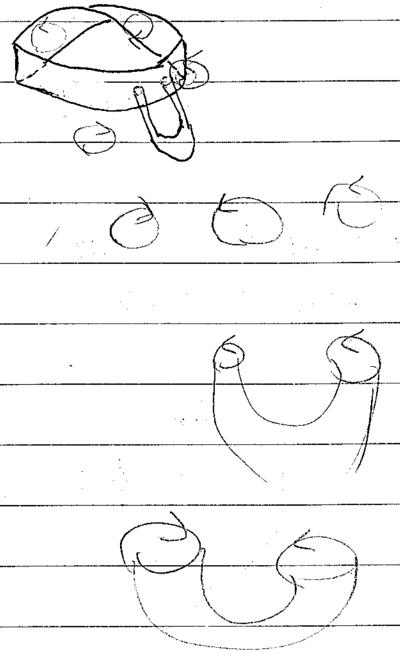
$\mathbb{R}P^2 \setminus D^2 \cong M$.

$\therefore \mathbb{R}P^2 \# \mathbb{R}P^2 \cong K$. ※

It's enough to prove that $T^2 \# M \cong K \# M$

<pf>. $\Rightarrow T^2 \cong \mathbb{R}P^2 \# \mathbb{R}P^2$.





92. 3. 11.

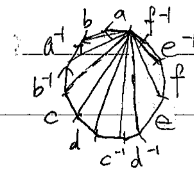
M cpt closed surface

$\Rightarrow M \underset{\substack{\uparrow \\ \text{homeo}}}{\cong}$ ① $T \# T \# \cdots \# T$ or ② $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2$.

the number of the summands is called the genus of M

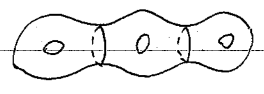
- $g(M)$ is an invariant

①



↙?

$$\chi(M) = V - E + F = \underbrace{1}_{V} - \underbrace{(2g + (g-1)\cdot 4 + 1)}_{E} + \underbrace{(g-2)4 \cdot}_{F}$$

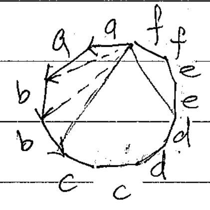


$$= 2 - 2g.$$

$1 - (2g + (g-2)\cdot 4 + 5)$ $\quad 1 + 8 - 5 - 8 + 6$

$+ (g-2)4 + 3 \cdot 2$ $\quad = -2g - 4g + 4g = 2 - 2g$

②



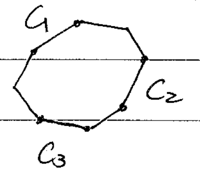
$$\chi(M) = 1 - (g + 2g - 3) + 2g - 2. = 2 - g.$$

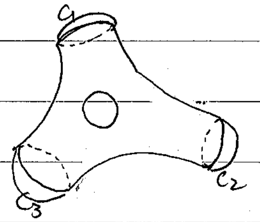
$2g$

$1 - (g + 2g - 3)$

(CW-complex ―)

Remark : If M is cpt with $\partial M = \coprod_{i=1}^{r} C_i$





① orientable $T \# T \# \cdots \# T \setminus \coprod_{i=1}^{r} D_i$, $\partial D_i = S_i$

② non-orientable.

For non-compact surface (Massey)

A Riemannian manifold is a differentiable manifold (ie. C^∞ mfd) with a given first fundamental form.

A 2-dim'l Riemannian manifold is also called an (abstract) geometric surface.

The simplest type of examples :

$M = \mathbb{R}^2$, $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$,

E, F, G any C^∞ functions s.t. $\begin{pmatrix} E & F \\ F & G \end{pmatrix}(u_0, v_0)$ is positive definite.

$M = H$ ($\overset{\text{Poincaré}}{\text{Upper half plane}}$) $= \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$.

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \quad \left(= \frac{|dz|^2}{|\operatorname{Im} z|^2} \right)$$

$$K = \frac{-1}{2\sqrt{EG}} \left(\cancel{\left(\frac{G_1}{\sqrt{EG}}\right)_1} + \left(\frac{E_2}{\sqrt{EG}}\right)_2 \right) \quad , \quad x \leftrightarrow 1 \ , \ y \leftrightarrow 2$$

$$= \frac{-1}{2\frac{1}{y^2}} \left(\frac{2}{y^2} \right) = -1 \ ※.$$

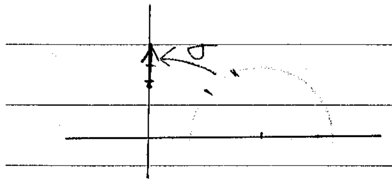
$x = $ const.

$$\int ds = \int \frac{dy}{y} = \ln y \nearrow \infty$$

(Poincaré ↔ Möbius transformation)

The model for non-Euclidean geometry (平行公設)

Then all geodesics are either vertical lines or circle with center on x-axis.



$$w = \sigma(z) = \frac{az+b}{cz+d} \quad , \ a, b, c, d \in R.$$

$$ds^2 = \frac{dz\,d\bar{z}}{(\operatorname{Im} z)^2} = \frac{dx^2 + dy^2}{(\operatorname{Im} z)^2}.$$

複変

Möbius 変換

$z \xrightarrow{\sigma} \dfrac{az+b}{cz+d}$

$$czw + dw = az + b$$

$$\Rightarrow (cw - a)z = b - dw \quad , \quad z = \frac{b - dw}{cw - a}$$

$$\frac{1}{2}\left(\frac{b - dw}{cw - a} + \frac{b - d\bar{w}}{c\bar{w} - a} \right)$$

$$\operatorname{Im}(z) = \frac{1}{2i}\left(\frac{b - dw}{cw - a} - \frac{b - d\bar{w}}{c\bar{w} - a} \right)$$

$$= \frac{1}{2i}\left(\frac{bc\bar{w} - \cancel{d|w|^2} - \cancel{ab} + adw - cbw + \cancel{ab} + \cancel{d|w|^2} - ad}{|cw - a|^2} \right.$$

$$= \frac{(ad - bc)\operatorname{Im} w}{|cw - a|^2}$$

$$ds^2 = \frac{dx^2 + dy^2}{|\operatorname{Im} z|^2} = \frac{dz\,d\bar{z}}{|\operatorname{Im} w|^2} \cdot \frac{|cw - a|^4}{(ad - bc)^2}.$$

$$dz\,d\bar{z} = \underbrace{\frac{-d(c\cancel{w} - a) - (b - d\cancel{w})c}{(cw - a)^2}\,dw}_{dz} \cdot \underbrace{\frac{-d(c\cancel{\bar{w}} - a) - (b - d\cancel{\bar{w}})c}{(c\bar{w} - a)}\,d\bar{w}}_{d\bar{z}}$$

$$= \frac{(ad - bc)}{|cw - a|^4}\,dw\,d\bar{w}.$$

73 . 3 . 13 (四)

Do Carmo 倒數第二節

"Riemann Geometry"

△ M n-manifold, $M \xrightarrow{f} \mathbb{R}$ C^∞ function.

What is "different." act on f

△ Recall, $v \in \mathbb{R}^n$, f,

directional derivative : $\frac{d}{dt} f(p+tv)|_{t=0} = Df \cdot v$
$\overset{\parallel}{D_v}$

△ The notation of 1st order differential operator

$p \in \mathbb{R}^n$, C_p^∞ the "germ" of C^∞ functions near p

$f, g \in C_p^\infty$, $f \sim g \Leftrightarrow \exists\, U \in p$ st. $f|_U = g|_U$

$X : C_p^\infty \to \mathbb{R}$

① $\mathbb{R}$-linear map $X(af) = aX(f)$, $a \in \mathbb{R}$

② Lebnitz rule : $X(fg) = X(f)g(p) + f(p)X(g)$

Thm: $\exists$ isomorphism $\left\{\begin{matrix}\text{1st order} \\ \text{diff. operator}\end{matrix}\right\} \cong T_p\mathbb{R}^n$ $\left(\begin{matrix}\triangle v \to v \\ X \to ?\end{matrix}\right)$

△ M n-dim manifold, $M \xrightarrow{f} \mathbb{R}$, C^∞ function

$p \in M$ "C_p^∞" T_pM is defined to be the vector space of all 1st order diff operators on C_p^∞

claim $\mathbb{R}^n$ $(x_1, \cdots, x_n)$ $X = \sum_{i=1}^{n} X(x_i)\frac{\partial}{\partial x_i}$

$X \longmapsto (X(x_1), \cdots, X(x_n)) \in \mathbb{R}^n$

<pf> (此証明不嚴謹, 假設已有 Taylor expan.)

W.L.O.G. may take $p=0$

$$X(f) = X(f(0) + \frac{\partial f}{\partial x_i}(0)x_i + \frac{1}{2!}\frac{\partial^2 f}{\partial x_i \partial x_j} x_i x_j$$

$$= \frac{\partial f}{\partial x_i}(0)X(x_i) + \frac{1}{2!}\frac{\partial^2 f}{\partial x_i \partial x_j} \underbrace{X(x_i x_j)}_{X(x_i)\overset{\parallel}{\underset{0}{x_j(0)}} + \overset{\parallel}{\underset{0}{x_i(0)}} X(x_j)} + \cdots$$

$$= (\underset{\uparrow\,\mathbb{R}}{X(x_i)} \frac{\partial}{\partial x_i}|_0) f$$

Show : $f(x) = f(0) + \sum_{i=1}^{n} g_i(x) x_i$

<pf> $f(x) - f(0)$

$$= \int_0^1 \frac{d}{dt} f(tx)\, dt$$

$$= \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(tx) x_i\, dt$$

$$= \sum_{i=1}^{n} \left[\int_0^1 \frac{\partial f}{\partial x_i}(tx)\, dt\right] x_i$$

$$= \sum_{i=1}^{n} g_i(x) x_i \qquad \begin{cases} g_i(x) & C^\infty\text{-func.} \\ g_i(0) = \frac{\partial f}{\partial x_i}(0) \end{cases}$$

NO. 73 . 3 . 18 (=)

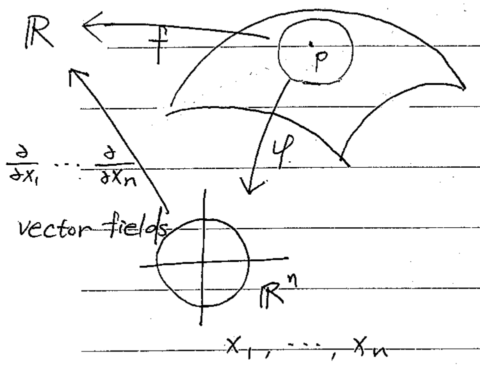
M: manifold (C^∞ mfd)   dim M = n

T_pM : tangent space at $p \in M$

$\mathbb{R}^2$

abstractly, it is the vector space of all 1st order linear differential equations on C_p^∞.

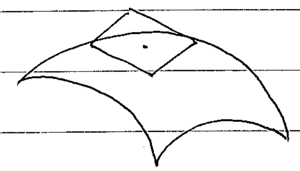
if U is a chart near p



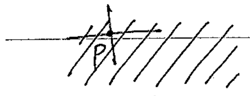
Recall : $X \in T_pM$

$$X(f) = \left[\sum_{i=1}^{n} X(x_i) \underbrace{\frac{\partial}{\partial x_i}\Big|_p}_{\frac{\partial (f \circ \varphi^{-1})}{\partial x_i}(p)} \right] f$$

Riemannian metric $\overset{\overset{ds^2}{\parallel}}{g}$ is a family of inner products $g(p)$ on T_pM which depends smoothly in $p \in M$.



Recall : Poincaré upper half plane,



$\mathbb{R}^2_+ \equiv H$, $\{y > 0\}$, $ds^2 = \frac{dx^2 + dy^2}{y^2}$

$T_pH \cong \mathbb{R}^2$

$\underset{\frac{\partial}{\partial x_1}}{\overset{e_1}{\parallel}}$, $\underset{\frac{\partial}{\partial x_2}}{\overset{e_2}{\parallel}}$

$E = \langle \frac{\partial}{\partial x_1} , \frac{\partial}{\partial x_1} \rangle$

$F = \langle \frac{\partial}{\partial x_1} , \frac{\partial}{\partial x_2} \rangle$

$G = \langle \frac{\partial}{\partial x_2} , \frac{\partial}{\partial x_2} \rangle$

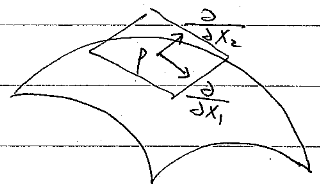
the only reqirement is that $\begin{pmatrix} E & F \\ F & G \end{pmatrix}(p)$ is positive definite $\forall\ p \in M$.

3/18 (=)

let $(U; x_1, \cdots, x_n)$ be a coor. system near p

$$ds^2 = g(p) = \sum_{i,j=1}^{n} g_{ij}(p)\, dx_i\, dx_j$$

$[g_{ij}(p)] > 0 \quad \forall\ p$, C^∞ in p



Recall (Linear algebra) :

V : vector space / $\mathbb{R}$

$V^* := \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ the vector space of all linear functionals.

e.g. If V has a basis $e_1, \cdots, e_n$

then V^* has a dual basis $e_1^*, \cdots, e_n^*$ (coor. func.) defined by $e_i^*(e_j) = \delta_{ij}$

$f : V \to \mathbb{R}$ $v = x_1 e_1 + \cdots + x_n e_n$

$f(e_1)$ $v \longleftrightarrow (x_1, \cdots, x_n)$

$\vdots$ $e_i^*(e_j) = \begin{cases} 0 & \text{if } j \neq i \\ 1 & \text{if } j = i \end{cases}$ $x_i = e_i^*(v)$

$f(e_n)$

- alternating functional (form)

$\varphi : V \times V \longrightarrow \mathbb{R}$ bilinear map

$\varphi(u,v) = -\varphi(v,u)$

- symmetric functional (form)

$\varphi(u,v) = \varphi(v,u)$

Any bilinear form

P.76 Calculus on Manifold $\quad \varphi = \sum_{i,j=1}^{n} \varphi(e_i, e_j) \underline{e_i^* \otimes e_j^*} \qquad \varphi \in J^2(V)$

$(f \otimes g)(u,v) = f(u) \cdot g(v)$

$f(v) = \sum_{i=1}^{n} f(x_i e_i)$

$= \sum_{i=1}^{n} x_i f(e_i) = \left(\sum_{i=1}^{n} (f(e_i) e_i^*)\right)(v)$

$\underset{u_i e_i}{\underset{\shortparallel}{\varphi(u,}} \underset{v_j e_j}{\underset{\shortparallel}{v)}} = u_i v_j \cdot \varphi(e_i, e_j)$

$= \varphi(e_i, e_j) e_i^*(u) e_j^*(v)$

$\varphi : \underbrace{V \times V \times \cdots \times V}_{k} \longrightarrow \mathbb{R}$

$\varphi = \sum_{i_1, \cdots, i_j = 1}^{n} \varphi(e_{i_1}, \cdots, e_{i_k}) e_{i_1}^* \otimes e_{i_2}^* \otimes \cdots \otimes e_{i_k}^*$

alternating, we get $\Lambda^k V^*$

the space of k-forms (on V)

$e^* \wedge e_2^* \wedge e_3^* = \sum (-1)^{|\sigma|} e_{i_1}^* \otimes e_{i_2}^* \otimes e_{i_3}^*$

$\sigma : \{1, 2, 3\}$ purmutation

$a \wedge b = a \otimes b - b \otimes a$

- symmetric $\quad \varphi = \sum_{i,j=1}^{n} \varphi(e_i, e_j) e_i^* \otimes e_j^* \qquad \varphi(e_i, e_j) = \varphi_{ij}$

$\left(\sum_{i,j}^{n} \varphi_{ij} x_i x_j\right) \qquad \varphi_{ij} = \varphi_{ji}$

φ is a inner product if $\varphi(v,v) \geq 0$ and $= 0 \Leftrightarrow v = 0$

then we define $\langle u, v \rangle := \varphi(u,v)$

<u>Def</u> $dx_i|_p$ is the dual basis of $\frac{\partial}{\partial x_i}\Big|_p$

$\{dx_i\}$ dual of $\left\{\frac{\partial}{\partial x_i}\right\} \quad \left(dx_i : \frac{\partial}{\partial x_i}^{*}\right)$

- arc length, geodesics, exponential map

$\gamma : [a,b] \to M$, C^∞

$$\int_a^b |\gamma'|\,dt = \int_a^b \sqrt{\underbrace{g_{ij}(\gamma(t))\underset{\llcorner\lrcorner}{\frac{dx_i}{dt}}\underset{\llcorner\lrcorner}{\frac{dx_j}{dt}}}_{\left(\frac{ds}{dt}\right)^2}}\,dt$$

$$\gamma'(f) = \frac{d(f\circ\gamma)}{dt}$$

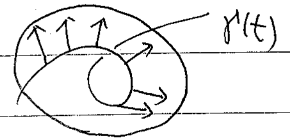
$$\begin{tikzcd}[a,b] \arrow[r, "\gamma"] & M \arrow[d, "f"] \\ & \mathbb{R}\end{tikzcd}$$

geodesic $M \subset \mathbb{R}^3$, N

$$\frac{D\gamma'}{dt} = (\gamma'')^T \underset{\substack{\uparrow \\ \text{expect}}}{=} 0$$

The key is to define covariant derivatives "$\frac{D}{dt}$"

How to differentiate a vector field along a curve

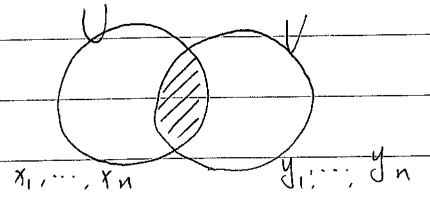


$X(t) \in T_{\gamma(t)}M \quad \forall t$

inside a coor. system

$(x_1, \cdots, x_n)$, $\gamma(t) \longleftrightarrow (x_1(t), \cdots, x_n(t))$

$X(t) = \sum_{i=1}^{n} a_i(t) \frac{\partial}{\partial x_i}$ 行不通…



v vector field on $U \cap V$

$v(p) \in T_pM \quad p \in U\cap V$

in x $\sum a_i \frac{\partial}{\partial x_i} \overset{\text{in } y}{=\!=\!=} \sum b_j \frac{\partial}{\partial y_j}$

座標變換

$y_i(x_1, \cdots, x_n)$

$$\boxed{\frac{\partial}{\partial x_i} = \sum_j \frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j}}$$

$$\sum \left(a_i \frac{\partial y_j}{\partial x_i}\right)\frac{\partial}{\partial y_j} = \sum b_j \frac{\partial}{\partial y_j}$$

$$\Rightarrow \sum_{i=1}^{n} a_i \frac{\partial y_j}{\partial x_i} = b_j$$

$\gamma : [a,b] \longrightarrow M$

$$\left.\begin{aligned} \text{"}\frac{dv}{dt}\text{"} \text{ in } x &\overset{?}{=} \sum \frac{da_i}{dt}\frac{\partial}{\partial x_i} \\ \text{in } y &\overset{?}{=} \sum \frac{db_j}{dt}\frac{\partial}{\partial y_j}\end{aligned}\right\} \text{this means } b_j' = \sum_i a_i' \frac{\partial y_j}{\partial x_i}$$

不能這樣定

$$b_j' = \sum a_i' \frac{\partial y_j}{\partial x_i} + \sum a_i \left(\frac{\partial y_j}{\partial x_i}(\gamma(t))\right)'$$

Levi-Civita 1860~

Christoffel (Γ_{ij}^{k})

E. Cartan

→ Koszul ∇ 倒三角形

$$\text{Levi-Civita} \quad \frac{D}{dt} b_j = b_j' + \underbrace{\{ b\ g_{ij} \}}_{\Gamma_{ji}^{k} b_k x_i'}$$

微幾

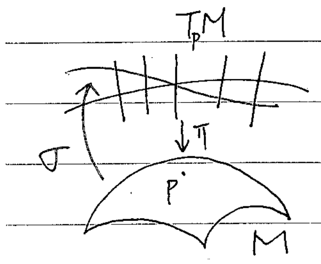
(M, g) Riemannian manifold.

Covariant derivatives

$\Gamma(U, TM)$ = the space of all smooth vector fields on U

(U open set in M)

vector bundle



$\sigma \in \Gamma(U, TM)$, $\sigma : U \to TM$

st. $\sigma(p) \in T_pM$

(ie. $\pi(\sigma(p)) = p$)

if U is a chart with coordinates $(x_1, \cdots, x_n)$

$$\sigma = \sum_{i=1}^{n} \underline{a_i(x)}\frac{\partial}{\partial x_i}, \quad a_i(x) \in C^\infty$$

given a vector field σ on $U \ni p$

given a vector $v \in T_pM$

what is the "directional derivative" $\nabla_v \sigma$?

∇ must satisfy

⓪ $\nabla_v \sigma$ is $\mathbb{R}$-linear in σ

① ∇_v satisfies Lebnitz rule

Lebnitz rule : $\nabla_v(f\sigma) = \underline{v(f)}\,\sigma(p) + f(p)\nabla_v \sigma$

傳統的方向導數

(a number)

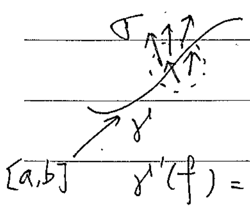
$D_v\sigma := (D_v^{\mathbb{R}^3}\sigma)^T$, $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{R}^3$

$D_v^{\mathbb{R}^3}\sigma = (\nabla\sigma_1 \cdot v, \nabla\sigma_2 \cdot v, \nabla\sigma_3 \cdot v)$

② compatible with metric

$$v\langle\sigma, \tau\rangle = \langle\nabla_v\sigma, \tau\rangle + \langle\sigma, \nabla_v\tau\rangle$$

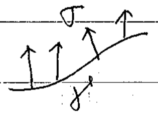
If such "∇" exists, then we may define the covariant derivative along a curve γ'



$\xrightarrow{f} R$

$\gamma'(f) = (f \circ \gamma')'$

We say σ is parallel along γ' if $\nabla_{\gamma'}\sigma \equiv 0$



Facts

① If σ and τ are parallel along γ', then $\langle\sigma, \tau\rangle$ = constant along γ'

<pf> $\gamma'\langle\sigma, \tau\rangle = 0$ by ②

$\frac{d}{dt}[\langle\sigma \circ \tau\rangle \circ \gamma']$ ※

② $|\sigma|$ = const.

Riemann (1855) ~ paper

Thm (Levi − Civita)

(Fundamental theorem of Riem. Geom.)

Given (M, g) $\exists ! \nabla$ satisfy ⓪, ①, ②, ③ ← torsion free $(\Gamma^k_{ij} = \Gamma^k_{ji})$

Now called the Levi-Civita connection.

$$\Updownarrow$$

$$\nabla_{\partial_i} \partial_j = \nabla_{\partial_j} \partial_i$$

<pf> Let $(U, x_1, \cdots, x_n)$ be a chart,

denote $\frac{\partial}{\partial x_i}$ by ∂_i, $i = 1, \cdots, n$

Let $\nabla_{\partial_i} \partial_j = \sum_{k=1}^{n} \Gamma^k_{ij} \partial_k$.

$$g = \sum_{i,j=1}^{n} g_{ij}\, dx^i \otimes dx^j, \quad g_{ij} = \langle \partial_i, \partial_j \rangle$$

$$\partial_k g_{ij} \overset{\text{by ⓪}}{=} \langle \nabla_{\partial_k} \partial_i, \partial_j \rangle + \langle \partial_i, \nabla_{\partial_k} \partial_j \rangle$$

$$= \langle \Gamma^l_{ki} \partial_l, \partial_j \rangle + \langle \partial_i, \Gamma^l_{kj} \partial_l \rangle$$

$$= \Gamma^l_{ki}\, g_{lj} + \Gamma^l_{kj}\, g_{il}.$$

$$\partial_k g_{ij} = \Gamma^l_{ki} g_{lj} + \Gamma^l_{kj} g_{il} \quad \text{— ①}$$

$$\partial_i g_{kj} = \Gamma^l_{ik} g_{lj} + \Gamma^l_{ij} g_{kl}. \quad \text{— ②}$$

$$\partial_j g_{ki} = \Gamma^l_{ji} g_{lk} + \Gamma^l_{jk} g_{li} \quad \text{— ③}$$

③ + ② − ① (assume $\Gamma^r_{pq} = \Gamma^r_{qp}$)

$$2\Gamma^l_{ij} g_{lk} = (\partial_i g_{kj} + \partial_j g_{kj} - \partial_k g_{ij})$$

let (g^{ij}) be the inverse matrix of (g_{ij})

$$\Rightarrow \Gamma^s_{ij} = \frac{1}{2} g^{sk} (\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij})$$

(所有存在性定理都一樣).

Def: a curve γ is a geodesic iff $\nabla_{\gamma'} \gamma' \equiv 0$

- diff eq'n for parallel vector fields

Thm: given γ, and $v \in T_{\gamma(0)} M$

$\exists !$ parallel vector field $\tilde{v}$ st. $\tilde{v}(0) = v$

<pf> pick a coor. chart $(U, \underset{\underset{x_1, \cdots, x_n}{\uparrow}}{X})$

let $\hat{v} = \sum a_i(t) \frac{\partial}{\partial x_i}$.

$$\nabla_{\gamma'} (a_j \partial_j) = 0 \qquad \gamma(t) = (x_1(t), \cdots, x_n(t))$$

$$\gamma'(t) = \sum_{i=1}^{n} x_i'(t)\, \partial_i$$

$$0 = \nabla_{\gamma'} (a_j \partial_j)$$

$$= \gamma'(a_j) \partial_j + a_j \nabla_{\gamma'} \partial_j$$

$$= \frac{d a_j}{dt} \partial_j + a_j x_i' \Gamma^k_{ij} \partial_k.$$

$$= \sum_{k=1}^{n} \left(\frac{d a_k}{dt} + \sum_{i,j=1}^{n} \Gamma^k_{ij} x_i' a_j \right) \partial_k$$

$$\underbrace{\frac{d a_k}{dt}}_{\text{未知}} + \underbrace{\sum_{i,j=1}^{n} \Gamma^k_{ij} x_i'}_{\text{已知}} a_j = 0, \quad k = 1, 2, \cdots, n. \quad \leftarrow \text{Linear system.}$$

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \rightarrow A' = MA \quad \exists ! \text{ sol.}$$

※

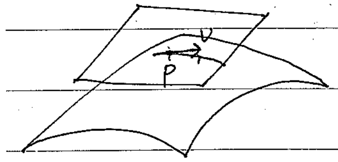
diff - eq'n for geodesic.

Thm : Given any $p \in M$, any $V \in T_pM$
$\exists !$ geodesic $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ st. $\gamma(0) = p$, $\gamma'(0) = V$

<pf>
Apply the previous ODE to $\tilde{v} = \gamma'$
i.e. $a_k = x_k'$ i.e. $x_k'' + \sum_{i,j} \Gamma_{ij}^k x_i' x_j' = 0$, $k = 1, \cdots, n$

Fact : If $\gamma(t)$ is a geodesic , then $t = cs$, $c =$ const.
$\because \gamma' \langle \gamma', \gamma' \rangle = 0 \Rightarrow |\gamma'| = C$ $s =$ arc length

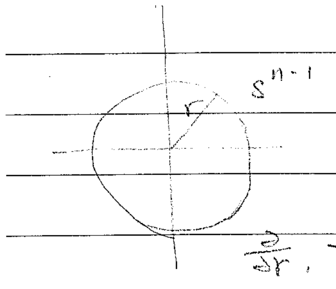
Exponential map.



$\exp_p : U^{\subset T_pM} \longrightarrow M$

that is $\left[\begin{array}{l} \exp_p(tv) \text{ is the geodesic } \gamma(t) \text{ with} \\ \gamma'(0) = V , \gamma(0) = p . \end{array} \right]$

$\exp_p(V) = \gamma(1)$



coor on S^{n-1}

$\theta_1, \cdots, \theta_{n-1}$

$\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_1}, \cdots, \frac{\partial}{\partial \theta_{n-1}}$

---

§5.3 Completeness

Hopf - Rinow thm :
If M is a geodesically complete Riemannian manifold, then
any $p.q \in M$ can be joined by a minimum geodesic.

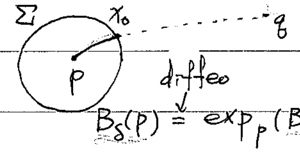
(M, g) $d(p.q) = \inf_{\gamma \text{ joins } p.q} l(\gamma)$ $l(\gamma) = \int_a^b |\gamma'| dt$
$\downarrow$
≥ 0 , $= 0 \Leftrightarrow p = q$

$d(\cdot,\cdot)$ defines a metric on M.
i.e. (M, d) is a metric space.

Recall : a metric space is complete if every Cauchy seq converge

Def : (M, g) geodesically complete if any geodesic can be extended infinitely , i.e. $\exp_p$ is defined in whole T_pM $\forall p$.

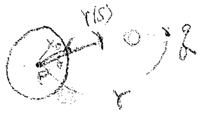
<pf> $p.q \in M$, $r := d(p.q)$



diffeo
$B_\delta(p) \overset{\downarrow}{=} \exp_p(B_\delta(0))$
$\Sigma = \partial B_\delta(p)$

$d(x, q) : \Sigma \longrightarrow \mathbb{R}$ conti.
let $d(x_0, q)$ be the minimum
$x_0 = \exp_p(\delta v)$, $|v| = 1$

$\gamma(s) := \exp_p(sv)$



NO.

(continuity method)

claim: $s \in [\delta, r]$
$\Rightarrow d(\gamma(s), q) = r - s$

<pf> $A = \{ s \in [\delta, r] \mid$ s.t the claim is true $\}$

want to show that $A = [\delta, r]$

- $A \neq \phi$: $\delta \in A$ because

$\inf l(\alpha_{p,q}) = \delta + \inf_{x \in \Sigma} l(\alpha_{x,q}) \Rightarrow r = \delta + d(\underset{\overset{\parallel}{x_0}}{\gamma(\delta)}, q)$.

α : any curve joins p.q.

$x_k \in [\delta, r]$

(証 A open & closed) $d(\gamma(x_k), q) = r - x_k$

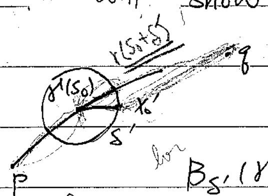
$d(\gamma(x), q) = r - x$

- A is closed since d is a continuous function. trivial!

$\{x_k\} \subset A$
$x_k \to x \in A$ $\quad x \in A$

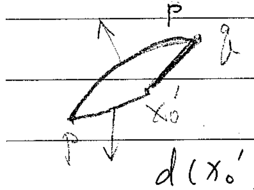
- A is open: let $\delta_0 \in A$

will show that $\delta_0 + \delta' \in A$ for δ' small enough



$B_{\delta'}(\gamma(\delta_0)) = \exp(B_{\delta'}(\gamma(\delta_0))$

Ball in $T_{\gamma(0)}M$



$d(x_0', q)$ min on $\Sigma' \longrightarrow R$.

as before, $d(\gamma(\delta_0), q) = \delta' + d(x_0', q)$

$\overset{\parallel}{r - \delta_0} \Rightarrow d(\underset{\underset{\gamma(\delta_0+\delta')}{\parallel}}{x_0'}, q) = r - \delta_0 - \delta'$

$\underset{\underset{\delta_0+\delta'}{\parallel}}{d(p,q) - d(x_0', q)} \leq d(p, x_0') \leq \delta_0 + \delta''$ $\quad d(p,q) \leq d(p, x_0') + d(x_0', q)$

$d(p, x_0') = \delta_0 + \delta' = d(p,q) - d(x_0', q)$

$\Rightarrow \gamma$ from $\delta = 0$ to $\delta = \delta_0$.

$\cup$ geodesic from $\gamma(\delta_0)$, x_0' is a smooth geodesic

with shortest length.

NO.

$\Rightarrow x_0' = \gamma(\delta_0 + \delta')$

$\Rightarrow \delta_0 + \delta' \in A$, $A = [\delta, r]$

§5.2 — 1, 2, 3, ④ $\qquad$ ④ M is complete.

§5.3 — ①, 6, 7, 10, ⑪ $\qquad$ ①~④ ⇒ geod. convex :

§5.10 — 2, 5, 6, 8, 9, 10.

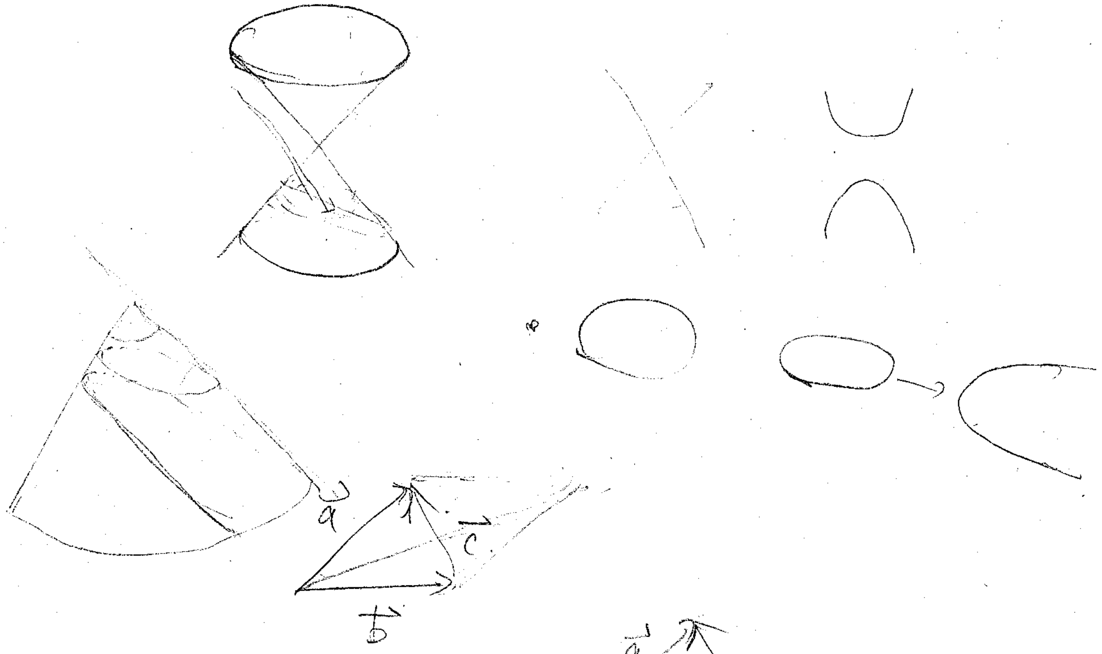
plus: Hoff — Rinow. TFAE:

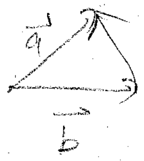
① M is geodesically complete.

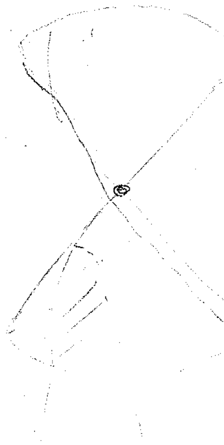
② M is geod. complete at one $p \in M$.

(ie. closed + bdd ⇒ cpt). ③ M satisfies the Heine-Borel property

4/7(四) 期中考.







$\vec{a}-\vec{b}=\vec{c}$

$\vec{a}=\vec{b}+\vec{c}$

此結果
換去左边 借用Taylor
可成嚴格pf.

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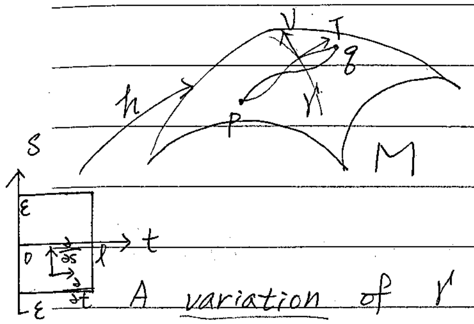
Variations of geodesics :



$\gamma:[0,l]\longrightarrow M$ parameter = t is the arc length

l = length of γ

$p=\gamma(0)$, $q=\gamma(l)$

A variation of γ is a map $h:\overset{t}{[0,l]}\times\overset{s}{(-\varepsilon,\varepsilon)}\longrightarrow M$

s.t. $h(t,0)=\gamma(t)$

let $\gamma_s(t)=h(t,s)$, $\gamma_0=\gamma$

define the length function $L(s)$ = length of γ_s

$$=\int_0^l\left|\frac{\partial h}{\partial t}\right|(s,t)\,dt=\int_0^l\sqrt{\langle T,T\rangle}\,dt$$

$\frac{\partial h}{\partial t}$ is $\gamma_s'(t)$ 代表切向量 denoted by $T(s,t)$
(vector field)

$\frac{\partial h}{\partial s}=:V$ is called the variation vector field (變分向量場)

1st variation formula

$$L'(s)=\int_0^l\frac{\partial}{\partial s}\sqrt{\langle T,T\rangle}\,dt$$

V, T 可交換!!

$$=\int_0^l\frac{\frac{\partial}{\partial s}\langle T,T\rangle}{2\sqrt{\langle T,T\rangle}}\,dt=\int_0^l\frac{\langle\nabla_V T,T\rangle+\langle T,\nabla_V T\rangle}{2\sqrt{\langle T,T\rangle}}\,dt$$

$$=\int_0^l\frac{\langle\nabla_T V,T\rangle}{\sqrt{\langle T,T\rangle}}\,dt$$

Recall the Levi-Civita ∇ is the unique operator

$\nabla_{\partial_i}\partial_j=\nabla_{\partial_j}\partial_i$ $\partial_i=\frac{\partial}{\partial x_i}$

here we use the fact that $\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t} = \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s}$. (Do Carmo)

$L'(0) = \int_0^l \langle \nabla_T V, T \rangle dt$

$= \int_0^l \left(\frac{\partial}{\partial t} \langle V, T \rangle - \langle V, \nabla_T T \rangle \right) dt$

$= \langle V, T \rangle \Big|_{t=0}^{t=l} - \int_0^l \langle V, \nabla_T T \rangle dt$ ~ 1st variation formula

Condition ①: end points are fixed

i.e. $\gamma_s(0) = p$, $\gamma_s(l) = q$ $\forall s \in (-\varepsilon, \varepsilon)$

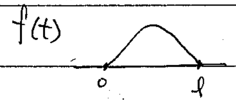
$(\Rightarrow V(0) = V(l) = 0)$

then $L'(0) = -\int_0^l \langle V, \nabla_T T \rangle dt$

$= -\int_0^l \langle V, \nabla_{\gamma'} \gamma' \rangle dt$

$L'(0) = 0$ $\forall$ variation $\Leftrightarrow$ $\nabla_{\gamma'} \gamma' = 0$ i.e. γ is a geodesic

<pf> $\Rightarrow$ Let $V = (\nabla_{\gamma'} \gamma') f(t)$



$\Rightarrow \int_0^l \langle V, \nabla_{\gamma'} \gamma' \rangle = 0$

除非 $\nabla_{\gamma'} \gamma' = 0$

$h(s,t) = \exp_{\gamma(t)} (s V(t))$

2nd variation formula

Let γ be a geodesic

$$L''(0) = \int_0^l \left[\frac{\langle \nabla_V \nabla_T V, T \rangle + \langle \nabla_T V, \nabla_V T \rangle}{\sqrt{\langle T, T \rangle}} - \frac{1}{\cancel{2}} \frac{\langle \nabla_T V, T \rangle \cancel{2} \langle \nabla_V T, T \rangle}{\sqrt{\langle T, T \rangle}^3} \right] dt$$

$$\langle \nabla_T V, T \rangle^2 = \left(\frac{\partial}{\partial t} \langle V, T \rangle - \langle V, \overset{0}{\cancel{\nabla_T T}} \rangle \right)^2$$

Condition ②: normal variation i.e. $V(t,0) \perp \gamma'(t)$ $\forall t \in [0, l]$

then $\langle V, T \rangle \Big|_{s=0} = 0$

get $L''(0) = \int_0^l \left(|\nabla_T V|^2 + \underset{\boxed{-K|V|^2}}{\underset{\|}{\langle \nabla_V \nabla_T V, T \rangle}} \right) dt$ in 2-dim case

$\Rightarrow L''(0) = \int_0^l \left(|\nabla_T V|^2 - K|V|^2 \right) dt$ ~ 2nd variation formula

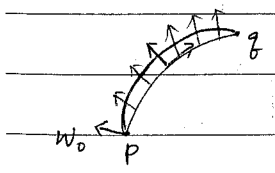
$*$ Theorem (Bonnet) 非常重要

Let M be a complete surface s.t. $K \geq \overset{\text{const.}}{K_0} > 0$

$\Rightarrow$ diam $M \leq \frac{\pi}{\sqrt{K_0}}$, diam $M = \sup\limits_{p,q \in M} d(p,q)$

Cor. M is cpt, $\partial M = \phi$

<pf> for any $p, q \in M$ let γ be a minimal geodesic joins p, q.



Let $W \perp T$ be a parallel vector field along γ, $|W| = 1$

l = length of γ

Let $V(t) = (\sin \frac{\pi t}{l}) W(t)$

If $l > \frac{\pi}{\sqrt{K_0}}$, then $L''(0) = \int_0^l \left[\frac{\pi^2}{l^2} \cos^2 \left(\frac{\pi t}{l} \right) - K \sin^2 \left(\frac{\pi t}{l} \right) \right] dt$

$\nabla_T V = \nabla_T (\sin (\frac{\pi t}{l}) W(t))$ $\leq \frac{1}{2} \left(\frac{\pi^2}{l^2} - K_0 \right) < 0$ $\times$

$= \frac{\pi}{l} \cos \left(\frac{\pi t}{l} \right) W$ (γ min geod. $\Rightarrow L''(0) \geq 0$)

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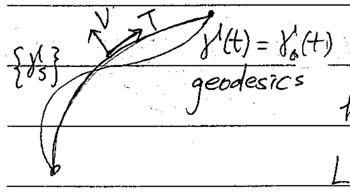
2nd variation formula

$$h:[0,l]\times(-\varepsilon,\varepsilon)\longrightarrow M$$

$\downarrow$

t $\quad$ t: arc length for $\gamma(t)$



$$h(t,s)=\gamma_s(t)$$

$$L(s)=\int_0^l |T|\,dt \quad , \quad T=\text{"}\frac{\partial}{\partial t}\text{"} \quad , \quad V=\text{"}\frac{\partial}{\partial s}\text{"}$$

$$\left(\frac{\partial h}{\partial t} \qquad \frac{\partial h}{\partial s} \quad \text{if } M\subset\mathbb{R}^N\right)$$

$$(L'(0)=0 \quad \forall\, V \Leftrightarrow \nabla_T T=0)$$

$$L''(0)=-\int_0^l |\nabla_T V|^2+K.$$

assume normal variation

$\downarrow$ i.e. $\langle V,T\rangle\big|_{s=0}=0$

$$L'(s)=\int_0^l \frac{d}{ds}\sqrt{\langle T,T\rangle}\,dt$$

$$=\int_0^l \frac{\cancel{2}\langle\nabla_V T,T\rangle}{\cancel{2}\sqrt{\langle T,T\rangle}}\,dt$$

$$\nabla_V T=\nabla_T V$$

$$\langle\nabla_T V,T\rangle=\frac{d}{dt}\langle V,T\rangle-\langle V,\cancelto{0}{\nabla_T T}\rangle$$

$\downarrow$

$$L''(0)=\int_0^l \frac{\langle\nabla_V T,\nabla_V T\rangle+\langle\nabla_V\nabla_V T,T\rangle}{\underbrace{\sqrt{\langle T,T\rangle}}_{=1}}+\frac{\langle\nabla_V T,T\rangle^2}{\underbrace{\sqrt{\langle T,T\rangle}^{3}}_{=1}}\Big)\,dt\Big|_{s=0}$$

$$=\int_0^l |\nabla_T V|^2+\langle\nabla_V\nabla_T V,T\rangle\,dt.$$

積分之後變成曲率

$$=\int_0^l |\nabla_T V|^2+\langle\underbrace{(\nabla_V\nabla_T-\nabla_T\nabla_V)}_{曲率}V,T\rangle+\langle\nabla_T\nabla_V V,T\rangle\,dt$$

(微分的不可交換性)

$$\|$$

$$\frac{d}{dt}\langle\nabla_V V,T\rangle-\langle V,\cancelto{0}{\nabla_T T}\rangle$$

$$\int_0^l\frac{d}{dt}\langle\nabla_V V,T\rangle\,dt$$

$$=\langle\nabla_V V,T\rangle\Big|_{s=0,\,t=0}^{t=l}$$

$$V(0)=V(l)=0$$

$$=0-0=0$$

$$=\int_0^l |\nabla_T V|^2+\langle(\nabla_V\nabla_T-\nabla_T\nabla_V)V,T\rangle\,dt$$

~~(last time)~~(in the surface case)

$$=\int_0^l\left(|\nabla_T V|^2-K|V|^2\right)dt$$

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"Riemann's curvature tensor"

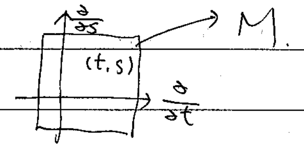
Let $\boxed{R(V,T):=\nabla_V\nabla_T-\nabla_T\nabla_V}$

under the condition

$\langle V,T\rangle\big|_{s=0}=0$, $|T|=1$, $V(0)=V(l)=0$

$\downarrow$

want to show that $\langle R(V,T)V,T\rangle = -K|V|^2.$



Write $T=\partial_i,\ (i=1)$, $V=\partial_j,\ (j=2)$

$$R(\partial_j,\partial_i)\partial_j:=\nabla_{\partial_j}\nabla_{\partial_i}\partial_j-\nabla_{\partial_i}\nabla_{\partial_j}\partial_j$$

$$=\nabla_{\partial_j}(\Gamma_{ij}^k\partial_k)-\nabla_{\partial_i}(\Gamma_{jj}^k\partial_k)$$

$$=(\partial_j\Gamma_{ij}^k)\partial_k+\Gamma_{ij}^k\nabla_{\partial_j}\partial_k-(\partial_i\Gamma_{jj}^k)\partial_k-\Gamma_{jj}^k\nabla_{\partial_i}\partial_k$$

general $\longrightarrow$

$$=\sum_{l=1}^{n}\left(\partial_j\Gamma_{ij}^l-\partial_i\Gamma_{jj}^l+\Gamma_{ij}^k\Gamma_{jk}^l-\Gamma_{jj}^k\Gamma_{ik}^l\right)\partial_l$$

(只用到 1st fund. form)

In the surface case, $i=1$, $j=2$.

$$\langle R(\partial_j,\partial_i)\partial_j,\partial_i\rangle=\langle\underline{R(\partial_2,\partial_1)\partial_1},\partial_2\rangle \qquad \partial_1\perp\partial_2.$$

$\downarrow$

only needs the $l=2$ case.

$$\text{i.e.}\left(\partial_2\Gamma_{12}^1-\partial_1\Gamma_{22}^1+\Gamma_{12}^1\Gamma_{21}^1+\Gamma_{12}^2\Gamma_{22}^1-\Gamma_{22}^1\Gamma_{11}^1-\Gamma_{22}^2\Gamma_{12}^1\right)\underset{\overset{\|}{1}}{\underline{\langle\partial_1,\partial_1\rangle}}$$

By the formula of K (p. 234).

$$-EK=\partial_1\Gamma_{12}^2-\partial_2\Gamma_{11}^2+\Gamma_{12}^1\Gamma_{12}^2+\Gamma_{12}^2\Gamma_{12}^2-\Gamma_{11}^2\Gamma_{22}^2-\Gamma_{11}^1\Gamma_{12}^2.$$

change $1\leftrightarrow2$.

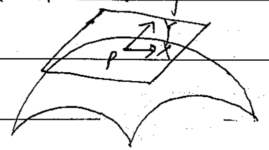
$$-GK=\partial_2\Gamma_{12}^1-\partial_1\Gamma_{22}^1+\Gamma_{12}^1\Gamma_{21}^1+\Gamma_{12}^2\Gamma_{22}^1-\Gamma_{22}^1\Gamma_{11}^1-\Gamma_{22}^2\Gamma_{12}^1$$

$$\Rightarrow\langle R(\partial_2,\partial_1)\partial_1,\partial_2\rangle=-GK=\underline{-K|V|^2}.\ ※$$

generalization of K into higher dimensions:

- for any two vectors $X, Y \in T_pM$

extend X, Y into vector field $\tilde{X}, \tilde{Y}$ in a nbd of p



then define

$$R(X,Y) := \nabla_X \nabla_{\tilde{Y}} - \nabla_Y \nabla_{\tilde{X}} - \nabla_{[\tilde{X},\tilde{Y}]}$$

$\tilde{X}, \tilde{Y}$ 1st order diff opperator

$[\tilde{X},\tilde{Y}]f = \tilde{X}\tilde{Y}f - \tilde{Y}\tilde{X}f$

(need to check that $[\tilde{X},\tilde{Y}]$ is 1st order)

If $\tilde{X} = \Sigma a_i \frac{\partial}{\partial x_i}$ $\tilde{Y} = \Sigma b_j \frac{\partial}{\partial x_j}$

$$[\tilde{X},\tilde{Y}]f = a_i \frac{\partial}{\partial x_i}\left(b_j \frac{\partial}{\partial x_j} f\right) - b_j \frac{\partial}{\partial x_j}\left(a_i \frac{\partial}{\partial x_i} f\right)$$

$$= a_i b_j \cancel{\frac{\partial^2 f}{\partial x_i \partial x_j}} - a_i b_j \cancel{\frac{\partial^2 f}{\partial x_j \partial x_i}}$$

$$+ a_i \frac{\partial b_j}{\partial x_i} \cdot \frac{\partial}{\partial x_j} f - b_j \frac{\partial a_i}{\partial x_j} \cdot \frac{\partial}{\partial x_i} f$$

$$\Rightarrow [\tilde{X},\tilde{Y}] = a_i \frac{\partial b_j}{\partial x_i} \cdot \frac{\partial}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j} \cdot \frac{\partial}{\partial x_i}$$

~ Lie bracket of vector fields

$[\partial_i, \partial_j] = 0$

驗証 $R(X,Y)$ 和 extend 無關

To show that $R(X,Y)$ is indep. of it's enough to show that

- $\begin{cases} R(f\tilde{X},\tilde{Y}) = fR(X,Y) & \text{for } f \text{ any } C^\infty \text{ functions (func. linear)} \\ R(\tilde{X},f\tilde{Y}) = fR(X,Y) & \end{cases}$ 逐点定義的技巧

- $R(\tilde{X},\tilde{Y})$ is $\mathbb{R}$-bilinear

if $\tilde{X}(p) = \hat{X}(p)$

want $R(\tilde{X},\tilde{Y})(p) \overset{?}{=} R(\hat{X},\tilde{Y})(p)$

$\tilde{X} = \tilde{a}_i \frac{\partial}{\partial x_i}$

$\hat{X} = \hat{a}_i \frac{\partial}{\partial x_i}$

$\hat{a}_i(p) = \tilde{a}_i(p)$

$$\Rightarrow \sum_i \tilde{a}_i R\left(\frac{\partial}{\partial x_i}, \tilde{Y}\right)(p)$$

$$\parallel$$

$$\sum_i \hat{a}_i R\left(\frac{\partial}{\partial x_i}, \tilde{Y}\right)(p)$$

① $R(f\tilde{X},\tilde{Y}) \overset{?}{=} fR(\hat{X},\tilde{Y})$

$$\parallel$$

$$\nabla_{f\tilde{X}}\nabla_{\tilde{Y}} - \nabla_{\tilde{Y}}\nabla_{f\tilde{X}} - \nabla_{[f\tilde{X},\tilde{Y}]}$$

$$= f\nabla_{\tilde{X}}\nabla_{\tilde{Y}} - \nabla_{\tilde{Y}}(f\nabla_{\tilde{X}}) - \nabla_{f\tilde{X}\tilde{Y} - \tilde{Y}(f\tilde{X})}$$

$$= f\nabla_{\tilde{X}}\nabla_{\tilde{Y}} - \tilde{Y}(f)\nabla_{\tilde{X}} - f\nabla_{\tilde{Y}}\nabla_{\tilde{X}} - \nabla_{f\tilde{X}\tilde{Y} - \underbrace{(\tilde{Y}f)}_{C^\infty}\tilde{X} - f\tilde{Y}\tilde{X}}$$

$$= f\left(\nabla_{\tilde{X}}\nabla_{\tilde{Y}} - \nabla_{\tilde{Y}}\nabla_{\tilde{X}} - \nabla_{\tilde{X}\tilde{Y} - \tilde{Y}\tilde{X}}\right)$$

$$= fR(\tilde{X},\tilde{Y})$$

② By $-R(B,A) = R(A,B)$

$R(X,Y)(Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

will show that $R(X,Y)(fZ) = f R(X,Y)(Z)$ (exercise).

$R(X,Y,Z,W) := \langle R(X,Y)\underset{\frown}{W}, \underline{Z} \rangle$ Gauss curvature or sectional curvature.

$R_{ijkl} := R(\partial_i, \partial_j, \partial_k, \partial_l) = \langle R(\partial_i, \partial_j)\partial_l, \partial_k \rangle$

We have seen in the 2-dim'l case

If $\partial_1 \perp \partial_2$, $|\partial_1| = |\partial_2| = 1$ at p.

then $R_{1212} = \langle R(\partial_1, \partial_2)\partial_2, \partial_1 \rangle = \underset{\sim}{K}$

in general, given $X, Y \in T_pM$.

$$K(X,Y) := \frac{R(X,Y,X,Y)}{|X \wedge Y|^2} = K(E), \quad E = \mathbb{R}\langle X, Y \rangle \subseteq T_pM.$$

Ricci curvature tensor

Given $X, Y \in T_pM$

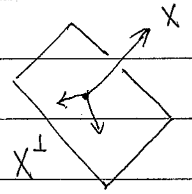
$Ric(X,Y) = \sum_{i=1}^{n} R(X, e_i, Y, e_i)$, $e_1, \cdots, e_n$ ONB of T_pM.

(check : indep of the choice of $\{e_i\}$)

is symmetric in $X \cdot Y$ (check it if you want).

$X = e_1$, $e_2, \cdots, e_n$ ONB.

$Ric(X,X) = \sum_{i=2}^{n} (e_1, e_i, e_1, e_i)$



Bonnet - Meyer's thm

M complete Riemannian mfd. $n = \dim M$.

$\underset{\sim}{\boxed{Ric} \geq (n-1)\underset{\uparrow \text{const.}}{\delta} > 0}$

then M is compact and $\dim M \leq \frac{\pi}{\sqrt{\delta}}$

<pf>: 回家想.

$R_{ij} = Ric(\partial_i, \partial_j)$, $R_{ij} - \frac{R}{2} g_{ij} = T_{ij}$ $\quad R \nearrow$ scalar curvature = tr Ric

Jacobi's thm
- conjugate points
- Jacobi fields §5.5 §5.6 §5.9

Galobal curves 4/17

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Exercise §5.4 — $3. \overset{\frown}{④.⑤}$

§4.7 — 4.5.

§4.7 Existence of convex nbd

Thm. Given $p \in M$ $\exists$ $U \ni p$ s.t. $\forall$ $q_1, q_2 \in U$

$\exists !$ shortest geodesic γ joins q_1, q_2 & $\gamma \subset U$

<pf>

If the coor. are (u, v)

$$\ddot{u} + \Gamma^1_{11} \dot{u}^2 + 2 \Gamma^1_{12} \dot{u}\dot{v} + \Gamma^1_{22} \dot{v}^2 = 0 \quad (u(t), v(t))$$

Now we use $(u, v) = (r, \theta)$ the geodesic polar coor.

then $ds^2 = dr^2 + G\, d\theta^2$

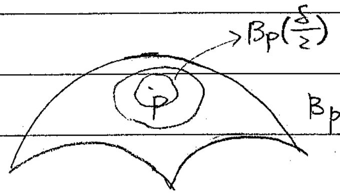
$$\Gamma^1_{11} = \frac{1}{2} g^{11} (\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}) = 0 \quad , \quad g_{11} = 1$$

$$\Gamma^1_{12} = \frac{1}{2} g^{11} (\partial_1 g_{12} + \partial_2 g_{11} - \partial_1 g_{12}) = 0$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11} (- \underbrace{\partial_1 g_{22}}_{\overset{\shortparallel}{G}}) = - \frac{1}{2} G_r$$

get equation : $\ddot{r} - \frac{1}{2} G_r \dot{\theta}^2 = 0$ $\quad$ $\varphi = \sqrt{G}$, $\lim_{r \to 0} \varphi = 0$

so $\ddot{r} = \frac{1}{2} G_r \dot{\theta}^2 \geq 0$ when $|r|$ is small $\quad$ $\lim_{r \to 0} \varphi_r = 1$.



$B_p(\delta) = \exp_p(B_0(\delta))$
$\uparrow$
1-1 onto . δ is small enough st. $G_r > 0$

$G_r = (\varphi^2)_r = 2 \varphi \varphi_r > 0$

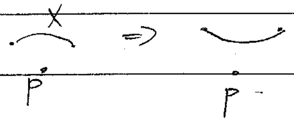
when $|r|$ is small

any $q \in B_p(\frac{\delta}{2})$ has a "1-1 , onto" exp nbd of radius $\frac{\delta}{2}$

Let $\bar{\delta} = \inf \delta_q$, $q \in B_p(\frac{\delta}{2})$

If $\bar{\delta} < \frac{\delta}{2}$, then replace $\frac{\delta}{2}$ by $\bar{\delta}$

$\ddot{\gamma} \geq 0$ 極值發生在邊界



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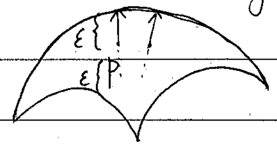
§2.7. Tubular neighborhood.

$S \subseteq \mathbb{R}^3$ given by $X(u, v)$



$$F(u, v, t) := X(u, v) + t N(u, v)$$

Q: Is F a diffeomorphism for $|t| < \varepsilon$?

$$F_u = X_u + t N_u$$
$$F_v = X_v + t N_v$$
$$F_t = N$$

$(|N| = 1 \Rightarrow N_u, N_v \perp N)$

$\Rightarrow$ Jac $F = | X_u + t N_u \,,\, X_v + t N_v \,,\, N |$

when $\underset{\text{at } p}{t = 0}$, Jac $F = | X_u \,,\, X_v \,.\, N | \neq 0$

Given $p \in S$, $\exists$ nbd of the form $D_p \times (-\varepsilon, \varepsilon)$ s.t. F is diffeo.

(Jac F is a conti func. on $D_p \times \underset{\overset{\downarrow}{t}}{\mathbb{R}}$)

this set $F(D_p \times (-\varepsilon, \varepsilon)) := N_\varepsilon(D_p)$ is called the ε - tubular nbd of D_p
$\quad D_p \subset S$

If S is compact,

$\forall p \in S$, $\exists$ D_p , $N_\varepsilon(D_p)$

$\{D_p\}$ is a open cover of S , $\exists$ finite subcover $D_1, \cdots, D_k$
$\qquad \varepsilon_1 \cdots \varepsilon_k$

let $\varepsilon < \min \{ \varepsilon_1, \cdots, \varepsilon_k, \frac{\delta}{2} \}$

δ = the Lebesque number of the open cover $\{D_i\}$

ie. a number $\delta > 0$ st. $d(p, q) < \delta \Rightarrow \exists$ i st. $p, q \in D_i$

Thm : For any compact metric space , the Lebesque number for any open cover exists.

<pf> If not, for any $n \in N$, $\exists$ p_n, q_n st. $d(p_n, q_n) < \frac{1}{n}$

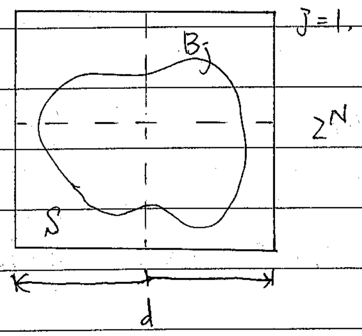
but p_n, q_n do not lie in the same D_i $\forall i$

S cpt $\Rightarrow$ $\{p_n\}$ $\{q_n\}$ both has at least one limit point.

Actually, may choose $\begin{matrix} p_{n_i} \to p \\ q_{n_i} \nearrow \end{matrix}$

in case $S \subseteq \mathbb{R}^N$

we may argue in the "following way"



$j = 1, \cdots, 2^N$ $\lambda = \frac{d}{10}$ st. its λ-nbd

claim: $\exists$ at least one box B_j contains ∞ many pairs $\{p_n, q_n\}$

If this is not true,

then, let $B_{j_1} \supset \infty$-many p_n but only finite q_n

$B_{j_2} \supset$ " q_n " p_n

$|p_n - q_n| \to 0$ $\Rightarrow$ any limit pts of p_n, q_n lies in $B_{j_1} \cap B_{j_2}$.

the λ-nbd of B_{j_1} will contain ∞-many q_n $\times$

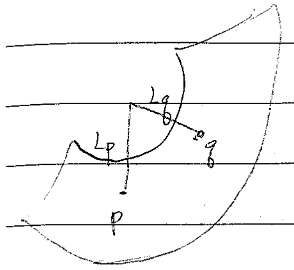
Next step, the box now has size $\frac{d}{2} + \frac{d}{10} \cdot 2 = \frac{7}{10} d$.

⋮

induction, get same limit points of $\{p_n\}$ & $\{q_n\}$ ※

let $p \in D_i$ for some i, then $\exists$ ∞ $\begin{matrix} p_{n_i} \to p \\ q_{n_i} \nearrow \end{matrix}$

for $n_i \gg 0$, get $\begin{matrix} p_{n_i} \in D_i \\ q_{n_i} \in D_i \end{matrix}$ $\times$.



L_p = normal line through p of radius ε.

If $L_p \cap L_q \neq \phi$ $\Rightarrow$ $|p - q| < \varepsilon + \varepsilon < \delta$

but then $p, q \in D_i$

but this $\times$ to the existence of ε_i-tubular nbd.

Cor If S is orientable and compact in $\mathbb{R}^3$

then $\exists$ open set $V \supset S$ and $f: V \to \mathbb{R}$, C^∞

st. $S = f^{-1}\{0\}$ and 0 is a regular value of f.

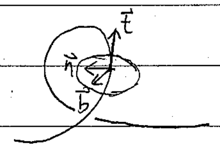
<pf> pick V = tubular nbd $N_\varepsilon(S)$, $f = t$ ※.

(in fact, we may choose $V = \mathbb{R}^3$.)

Apply the same technique to curves in $\mathbb{R}^3$

$\alpha: [0, l] \longrightarrow \mathbb{R}^3$, the tube along α. fix $r > 0$, $r < \frac{1}{k_0}$,

$$X(s,v) = \alpha(s) + r(\cos v \, \vec{n} + \sin v \, \vec{b}) \qquad k_0 = \max k.$$



X_s

X_v

$$EG - F^2 = |X_s \times X_t|^2 = r^2(1 - rk\cos v)^2$$

$$X_s \times X_v = r(1 - rk\cos v) N$$

$$N = -(\cos v \, \vec{n} + \sin v \, \vec{b})$$

$$N_s \times N_v = -k\cos v \, N$$

$$\Rightarrow K(s,v) = -\frac{k\cos v}{r(1 - rk\cos v)}$$

Thm (Fenchel)

$k_{tot}(\alpha) := \int_0^l |k(s)|\, ds$ total curvature

$k_{tot}(\alpha) \geq 2\pi$, "=" $\Leftrightarrow$ α is a plane convex curve.

$\uparrow$ simple closed curve

<pf>. let T be the tube of α of radius r.

$T = R \cup Q$, $R = \{ p \in T , K(p) \geq 0 \}$

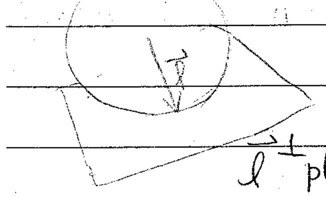
$Q = \{ p \in T , K(p) \leq 0 \}$.

$$\underset{4\pi}{\underset{\parallel}{\text{area}(S^2)}} \leq \int_R K\, dA = \int_R \frac{-k \cos v}{r(1-\cancel{rk}\cos v)} \cancel{r(1-k\cos v)}\, ds\, dv \ , \quad R := [0,l] \times [\tfrac{\pi}{2}, \tfrac{3\pi}{2}]$$

$$= \int_R -k \cos v \, ds\, dv.$$

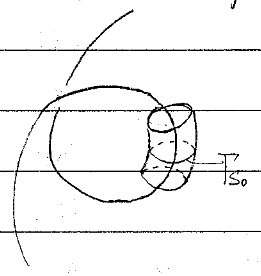
$$= -\int_0^l k(s) \left(\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos v \, dv \right) ds = 2 \int_0^l k\, ds$$

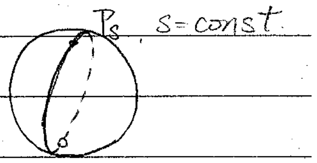
Any unit vector $\vec{l} \in S^2$ is the normal vector of some pt in R



$\Rightarrow \int_0^l k\, ds \geq 2\pi$

$\vec{l}^{\perp}$ plane.





let P_s be the image of N restrict to s = constant which is a "great circle" in S^2

$P_s^+ \subset P_s$ the half circle with $K \geq 0$

If α is a plane convex curve

$\partial P_s^+ = \{ \underset{\text{north}}{\underset{\uparrow}{p}} , \underset{\text{south}}{\underset{\uparrow}{q}} \}$ is independent of $s \in [0, l]$

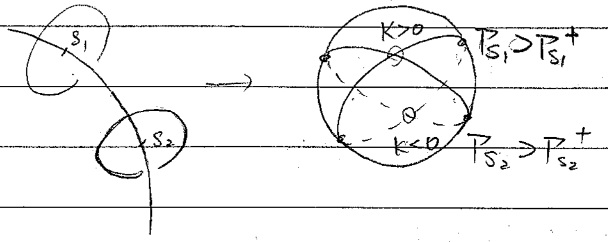
So $2k_{tot}(\alpha) = 2\int_0^l k\, ds = \int_R K\, dA = 4\pi$

---

conversely, if $k_{tot} = 2\pi$, then $\int_R K\, dA = 4\pi$ ...

claim : P_s^+ has the same end pts $\forall s$

if not, $\exists\ s_1 , s_2$



$\Rightarrow \exists\ 2$ pts P_1 , P_2 with $K > 0$ and $N(P_1) = N(P_2)$

$\Rightarrow \int_R K\, dA > 4\pi$ $\cancel{X}$.

$d(q_{n_i}, q_{n_j}) < d(p_{n_i}, q_{n_i}) + d(p_{n_i}, p_{n_j}) + d(p_{n_j}, q_{n_j})$

No.

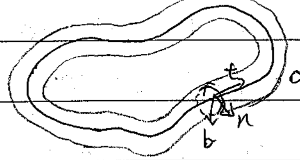
Date 92 : 4 : 10.

Thm (Fenchel) (重要的証明)

α simple closed curve in $\mathbb{R}^3$

$k_{tot}(\alpha) \geqslant 2\pi$, "=" $\Leftrightarrow$ α is a plane convex curve

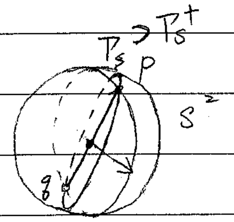
<pf> ($\Rightarrow$)



$\alpha \in \mathbb{R}^3$

$$T = \underset{\underset{K \geqslant 0}{\uparrow}}{R} \cup \underset{\underset{K \leq 0}{\uparrow}}{Q}$$

Gauss map $\xrightarrow{N}$



$K = 0 \leftrightarrow$ bi-normal vector of α

Recall last time we proved

if $\int_0^l k\,ds$ then $\partial T_s^+ = \{p, q\}$ $\forall s \in [0, l]$

$\overrightarrow{pq}$ // the bi-normal vector $\vec{b}(s)$ $\forall s$

$\Rightarrow \alpha$ is a plane curve $\subset E \simeq \mathbb{R}^2$ ~ plane curve

$$\underset{\underset{2\pi}{\parallel}}{k_{tot}} = \int_0^l \underset{\underset{k \text{ in } \mathbb{R}^3}{\uparrow}}{k}\, ds \geqslant \int_0^l \underset{\underset{k \text{ in } E}{\uparrow}}{k^E}\, ds \geqslant 2\pi.$$

$\Rightarrow k^E \geqslant 0$ this is our current definition of convexity.

~convexity ※

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No.

Date : :

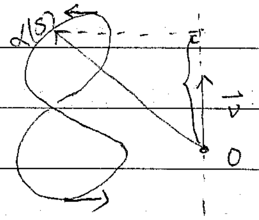
Thm (Fary - Milnor) (重要的証明)

If $C = \alpha([0, l]) \subset \mathbb{R}^3$ is a simple closed knotted curve,
(k > 0)

then $k_{tot} \geqslant 4\pi$

<pf> pick $\vec{v} \in S^2$, not a bi-normal vector of α

Let $h_v : [0, l] \to \mathbb{R}$ be the hight func. in the $\vec{v}$-direction



i.e. $h_v(s) = \alpha(s) \cdot \vec{v}$

s is a critical point of $h_v \Leftrightarrow 0 = h_v'(s) = \alpha'(s) \cdot \vec{v} = \vec{t} \cdot \vec{v}$

at each critical point,

$$h_v''(s) = \vec{t}'(s) \cdot \vec{v} = \underset{\underset{0}{\vee}}{k}\, \vec{n}(s) \cdot \vec{v} \neq 0$$

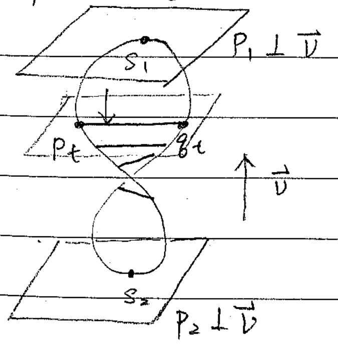
because if $\vec{n} \cdot \vec{v} = 0$ ($\vec{t} \cdot \vec{v} = 0$), then $\vec{v} \parallel \vec{b}$ ※

so, all critical points corresponds to local max or min of h_v.

If $k_{tot}(C) \leqslant 4\pi$, then $\int_R K\,dA = 2\int_0^l k\,dA \leqslant 8\pi$.

claim : $\exists\, v$, not any bi-normal st. h_v has exactly 2 critical points.

If the claim is true, then we move the plane from P_1 to P_2 through P_t parallelly.



Then $P_t \cap C$ into two points p_t, q_t

consider the set $D = \bigcup_{t \in (1,2)} \overline{p_t q_t}$

D is homeomorphic to a disk D^2

then C is un-knotted ※

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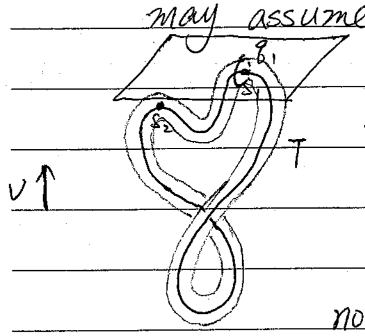
claim :

<pf>. Morse theory.

If not, then $\forall\, v \notin \overline{b}([0,l])$

h_v has at least 3 critical points.

may assume h_v has max at s_1, s_2



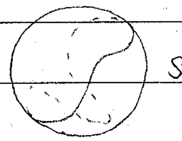
let P be the plane $\perp \vec{v}$ and intersects the highest point of T which corresponds to $s=s_1$

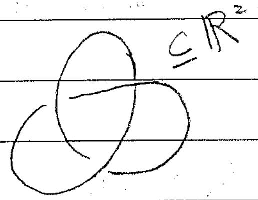
$N(q_1)=\vec{v} \Rightarrow K(q_1)>0$

now we move p parallelly to P′ which intersect the highest point q_2 corresponds to $s=s_2$

Again, $K(q_2)>0$

$$\int_R K\,dA \geqslant 2\cdot 4\pi = 8\pi$$





$$\int_0^l k\,ds = k_{1+t} = 2\pi\cdot 2 = 4\pi$$

---

§ 5.5 Jocabi fields

Recall, in the derivation of 2nd variation formula

$$L''(0) = \int_0^l \left(|\nabla_T V|^2 + \langle R(T,V)T, V\rangle \right) ds = 0$$

$$\overset{\parallel}{\langle \nabla_T V, \nabla_T V\rangle} \qquad \overset{\parallel}{-K|V|^2}$$

$$\overset{\parallel}{T\langle V, \nabla_T V\rangle - \langle V, \nabla_T\nabla_T V\rangle}$$

$\nearrow$

after $\int_0^l \Rightarrow \langle V, \nabla_T V\rangle\Big|_0^l = 0.$ $\quad \langle \nabla_T V, \nabla_T V\rangle$

$$= \frac{\partial}{\partial t}\langle \nabla_T V, V\rangle - \langle \nabla_T^2 V, V\rangle.$$

Another form : $\qquad R(V,T) := \nabla_V\nabla_T - \nabla_T\nabla_V$

$$L''(0) = \int_0^l -\langle \underline{\nabla_T^2 V - R(T,V)\,T}, V\rangle\, ds$$

Rmk : in Do Carmo's book, $\nabla_T = \frac{D}{\partial s}$; $\nabla_V = \frac{D}{\partial t}$, $\nabla_T^2 = \frac{D^2}{\partial s^2}$

in the 2-dim'l case, $\langle R(T,V)T, V\rangle = -K|V|^2$.

$\langle T, V\rangle = 0$, $|T| = 1$

A vector field V along a geodesic $\gamma(s)$ is called a Jacobi field if $\nabla_T^2 V - R(T,V)T = 0$

Jacobi's eq'n.

[in the 2-dim'l case, $R(T,V)W = -K(T\times V)\times W$ ~ hw]

$$\frac{D^2}{\partial t^2}V - K(T\times V)\times T = 0 \quad \text{in Do Carmo}$$

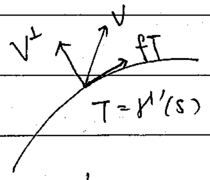
the simplest Jacobi field :

$V = fT$ is Jacobi field $\Leftrightarrow f''(s) = 0$

$\nabla_T \nabla_T (fT) + 0 = 0$ ie. $f(s) = as + b$. for some $a, b \in R$

$\quad \parallel$

$\nabla_T (f'T) = f''T$



$V = V^\perp + fT$, $V^\perp \perp T$

$\nabla_T^2 V \pm R(T, V)T = 0 \Leftrightarrow V^\perp , fT$ both are Jacobi fields.

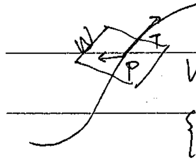
$\quad \parallel$

$\nabla_T^2 V^\perp \pm (R(T, V^\perp)T + f''T)$

$<\nabla_T V^\perp, T> = T \underset{0}{\underset{\parallel}{<V^\perp, T>}} - \underset{0}{\underset{\parallel}{<V^\perp, \nabla_T T>}} = 0$

$<\nabla_T^2 V^\perp, T> = 0$

Note that: Jacobi eq'n is a 2nd order ODE system,
It has $2n$ degree of freedom if $n = \dim S$



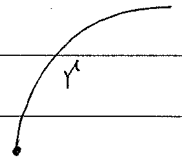
$V = \sum f_i e_i \Rightarrow \sum_{i=1}^{n} f_i''(s) e_i - R(T, \sum_j f_j \underset{\sum_{j=1}^{n} A_{ij}(s) f_j e_i}{\underset{\parallel}{R(T, e_i) T}}$

$\underset{T}{\underset{\parallel}{\{e_1, \cdots, e_n\}}}$, $\nabla_T e_i = 0$

$F = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}(s)$, $F''(s) - A(s)F(S) = 0$.

initial condition $\begin{cases} V(0) \\ V'(0) = \nabla_T V(0) \end{cases}$

Thm A vector field J along a geodesic $\gamma(s)$ is Jacobi $\Leftrightarrow$
J is a variation field of a " variation through geodesics "
ie. $h(s,t) : [0, l] \times (-\varepsilon, \varepsilon) \to S$ (geodesic variation)
st. $\gamma_t : [0, l] \longrightarrow S$ (para. proportional to arc length
is a geodesic $\forall t$.



$= dh(\frac{\partial}{\partial t})$

the variation vector field is $V(s) := \frac{\partial h}{\partial t}(s, 0) = h_*(\frac{\partial}{\partial t})$

<pf>

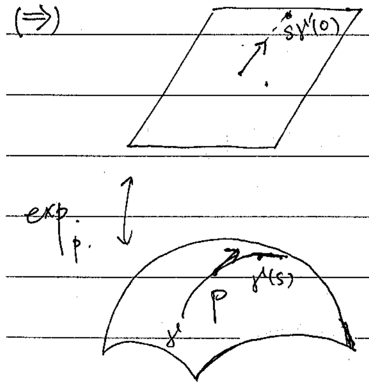
(⇐) $\{\gamma_t\}$ geodesics

$\nabla_T^2 V = \nabla_T \nabla_T V = \nabla_T \nabla_V T$

$= \underset{R(T,V)T.}{\underset{\parallel}{(\nabla_T \nabla_V - \nabla_V \nabla_T)}} T + \underset{0}{\underset{\parallel}{\underbrace{\nabla_V \nabla_T}}} T$

first, let's assume $J(0) = 0$.

(⇒)



$\left.\begin{matrix} J \\ \gamma'(s) \end{matrix}\right\}$ is given , $J(0)$, $J'(0) = \nabla_T J(0)$

$\gamma(s) = \exp_p(s\gamma'(0))$.

Let $h(s,t) = \exp_p(\underset{J(0)}{\underline{0}} + s(\gamma'(0) + tJ'(0)))$.

$V = \frac{\partial h}{\partial t}(S, 0) = \underbrace{(d\exp_p)_{s\gamma'(0)}}_{\text{linear map.}} (sJ'(0))$

$V(0) = 0$

$\nabla_T V(0) = \frac{D}{\partial s} V(s)\Big|_{s=0} = ??\underset{0}{\underset{\parallel}{(0J'(0))}} + \underset{\text{id.}}{\overset{}{\underset{\parallel}{(d\exp_p)_0}}}(J'(0)) = J'(0)$

$\left(= \frac{\partial^2 h}{\partial s \partial t}(0,0). = \frac{\partial^2 h}{\partial t \partial s}(0,0) = d\exp_p \cdots \right)$

$\Rightarrow J = V$ ※.

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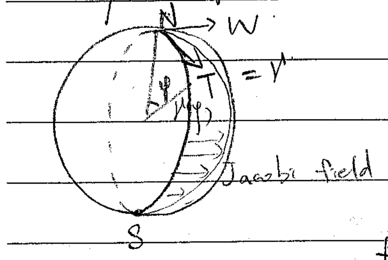
In general, Let $\beta(t)$ be a curve.

st. $\beta(0)=p$ $\beta'(0)=J(0)$. let $V(t)$ be the parallel translation of $\gamma'(0)$ along β

$h(s,t)=\exp_{\beta(t)}(s(V+tW))$, $W(t)$ … $J'(0)$ along β

check this works *

example of Jacobi field:



$$J(\varphi)=f(\varphi)W(\varphi)$$

$$\nabla_T^2 J-R(T,J)T=0$$

$$+K(T\times J)\times T$$

$$f''W+Kf\underbrace{(T\times W)\times T}_{\overset{\shortparallel}{W}}$$

$$\boxed{f''W+Kfw=0}$$

- $K=1$, take $f=\sin\varphi$. 有2個零点(正曲率的特性).

If $K=$ const. $K<0$

$f(\varphi)=ae^{\sqrt{-K}\varphi}+be^{-\sqrt{-K}\varphi}$.

if $f(0)=0$ (ie. $b=-a$).

then $f(\varphi)$ will never be zero again.

$K=0$

$f(\varphi)=a\varphi+b$ 只會一点為零

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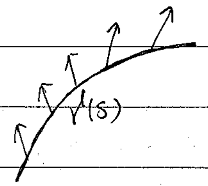
Jacobi field, Hadamard's thm.



$J(s)\in T_{\gamma(s)}S$, $T=\gamma'(s)$.

$\nabla_T^2 J-R(T,J)T=0$ for general Riemannian manifold

$\frac{D^2}{ds^2}J+K(\gamma'\times J)\times\gamma'=0$ surface case.

"$'$" covariant diff. $J''(s)\perp T=\gamma'$.

$$J=J^{\perp}+(as+b)T. \qquad (f(s)T)''=\underset{\substack{\uparrow\\ as+b}}{f''(s)}T=0$$

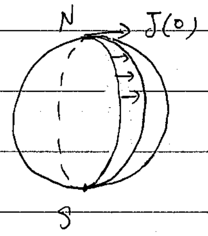
let $J(0)=0$, $J'(0)=W$

$J\longleftrightarrow$ variation vector field of "variations of geod."

In fact, this variation is

$$h(s,t)=\exp_p(s\vec{v}+ts\vec{w})$$

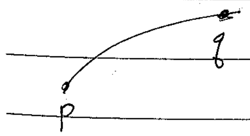
$$J(s):=\left.\frac{\partial h}{\partial t}\right|_{t=0}=(d\exp_p)_{s\vec{v}}(s\vec{w})\ *$$



$J(\theta)=J(0)\sin\theta$.

q is a conjugate point of $p=\gamma(0)$ along γ

iff $\exists$ J Jacobi $\not\equiv 0$,



$J(p)=0=J(q)$

Thm 1 : $K \leq 0$, then any p does not have any conjugate points

Fact 2 : $q = \exp_p(l\gamma'(0))$ is conj to p

$\Leftrightarrow l\gamma'(0)$ is a critical pt of $\exp_p : T_pS \longrightarrow S$

Cor : $K \leq 0 \Rightarrow \exp_p$ is a local diffeomorphism.

"covering maps" $\pi : \tilde{B} \longrightarrow B$ $\overset{\text{(onto)}}{\text{surjective}}$ continuous map of top spaces.

is called a covering map if

$\forall p \in B$, $\exists U \ni p$ st.

$\pi^{-1}(U) = \coprod_\alpha V_\alpha$ and $V_\alpha \xrightarrow{\sim} U$ homeomorphic $\forall \alpha$.

Fact : If $\tilde{B}$ is compact , then any local diffeo. $\tilde{B} \twoheadrightarrow B$ is a covering map.

Thm (Hadamard)

If $S \subseteq \mathbb{R}^3$ is a compact surface with $K > 0$ then the Gauss map $N : S \longrightarrow S^2$ is a diffeo.

<pf>

① N is surjective

② local diffeo. $K(p) = \det(dN_p) > 0$

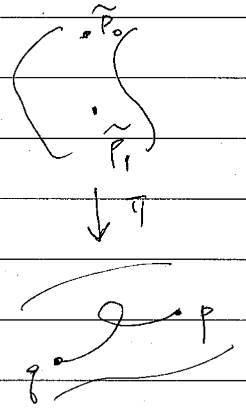
S cpt $\Rightarrow N$ is a covering map.

Thm :

* Covering maps satisfy the
- arc lifting property
- homotopy lifting property.

Given γ with $\gamma(0) = p$ for any $\tilde{p}_i$ st. $\pi(\tilde{p}_i) = p$

$\exists ! \ \tilde{\gamma} \subset \tilde{B}$ st. $\pi(\tilde{\gamma}) = \gamma$.



homotopy : $f, g : A \longrightarrow X$ 2 conti func. are called homotopic (同倫) if

$\exists \ H : A \times [0,1] \longrightarrow X$ conti.

st. $H(a, 0) = f(a)$

$H(a, 1) = g(a)$.

homotopy lifting of curves

Given $\gamma_0 , \gamma_1 : \underset{\overset{\parallel}{A}}{[0,1]} \longrightarrow B$. with $H : \underset{s}{[0,1]} \times \underset{t}{[0,t]} \longrightarrow B$, $H(0,t) = p$

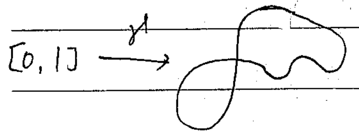
then for any $\tilde{p}$ with $\pi(\tilde{p}) = p$

$\exists ! \ \tilde{H} : [0,1] \times [0,1] \longrightarrow \tilde{B}$ st. $\pi \circ \tilde{H} = H$

$$\begin{tikzcd} & \tilde{B} \arrow[d, "\pi"] \\ {[0,1] \times [0,1]} \arrow[ru, "\tilde{H}"] \arrow[r, "H"'] & B \end{tikzcd}$$

Simply connected :

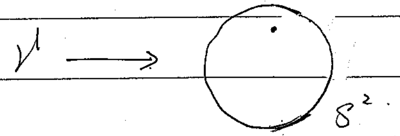
. A topological space is simply connected if any closed curve is homotopic to a constant map.



$H_0 = \gamma$

$H_1 = p$

$\left.\begin{matrix} \mathbb{R}^2 \\ S^2 \end{matrix}\right\}$ is simply connected.



Hadamard's thm

$N : S \longrightarrow S^2$ covering map.

$\uparrow$

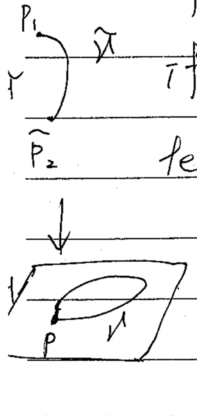
simply connected

Fact : $\tilde{B} \longrightarrow B$ covering map , B simply connected

$\Rightarrow \tilde{B} \underset{\pi}{\cong} B$

<pf> π is a local homeo. so only need to show that

π is 1-1



if $\tilde{p}_1 , \tilde{p}_2 \xrightarrow[\pi]{} p$

let $\tilde{\gamma}$ be a curve in $\tilde{B}$ joins $\tilde{p}_1 , \tilde{p}_2$.

$\gamma = \pi \circ \tilde{\gamma}$, γ is homotopic to (a point)
const map.

$H_0 = \gamma$, $H_1 = p$

$\Rightarrow \tilde{p}_1 = \tilde{p}_2$

∘ $K \leq 0$

know that $\exp_p : T_pS \longrightarrow S$ is a local diffeo.

⇓ ?

covering map

Thm (Hadamard)

If S is complete , $K \leq 0$, then $T_pS \xrightarrow[\exp_p]{} S$ is a diffeo.

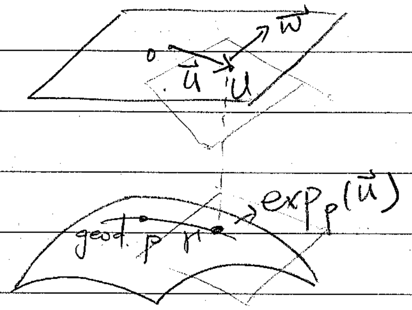
if S is simply connected

($\not\Rightarrow$).

length increasing property. (this $\Rightarrow$ covering map)

ie.

Lemma (Gauss lemma)



① $\langle (d\exp_p)_{\vec{u}} \vec{u} , (d\exp_p)_{\vec{u}} \vec{w} \rangle$
$= \langle \vec{u} , \vec{w} \rangle$

② If $K \leq 0$, then.
$|(d\exp_p)_{\vec{u}} \vec{w}|^2 \geq |\vec{w}|^2$

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Hadamard's thm:

. $K \le 0 \Rightarrow \exp_p : T_pS \longrightarrow S$ is a local diffeo.

($\because K \le 0 \Rightarrow \nexists$ conjugate pts along any geod. γ)

We want to show in fact it is a covering map.

if so, then

→ if $\underset{\substack{\uparrow \\ \text{complete, } K \le 0.}}{S}$ is simply connected, then $T_pS \simeq S$ diffeo.

Gauss lemma:

① $\langle (d\exp_p)_u u, (d\exp_p)_u w \rangle = \langle u, w \rangle$

② If $K \le 0$, $|(d\exp_p)_u w|^2 \ge |w|^2$

<pf>:

① $\gamma(s) = \exp_p(sv)$, $v = \frac{u}{|u|}$, $l = |u|$

let J be a Jacobi field, $J(0) = 0$, $J'(0) = w$

$',\, '' := \frac{D}{ds}$ $(\gamma', J)'' = \gamma' \cdot J'' = 0$.

$\Rightarrow \gamma'(s) \cdot J(s) = cs + c_0$, $c, c_0 =$ const.

$\because J(0) = 0 \quad \therefore c_0 = 0$. $\Rightarrow \gamma'(s) \cdot J'(s) = c$

Recall, $J(s) = (d\exp_p)_{sv}(sw)$

$s = l$, $cl = \gamma'(l) \cdot J(l)$

$= \langle (d\exp_p)_u v, (d\exp)_u lw \rangle$

$\langle u, w \rangle = l\gamma'(0) \cdot J'(0)$

$= lc = \langle (d\exp_p)_u v, (d\exp)_u lw \rangle$

$= \langle (d\exp_p) u, (d\exp)_u w \rangle$. ※

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②: by ①, may assume $w \perp u$.

let $J(s) = (d\exp_p)_{sv}(sw)$, $J(0) = 0$, $J'(0) = w$, $\gamma' \cdot J \equiv 0$

$(|J|^2)'' = (2J \cdot J')' = 2(|J'|^2 + J \cdot J'') \longrightarrow \ge 2|J'|^2 = 2C$

$= 2(|J'|^2 - K|J|^2) \ge 0$. $\quad C = |J'|^2 = |w|^2$

$\Rightarrow J \cdot J' \nearrow$ so $J \cdot J' \ge J(0) \cdot J'(0) = 0$

$\Rightarrow |J|^{2\,\prime} \ge 2Cs + \underset{\substack{\| \\ 2J \cdot J' = 0}}{|J|^{2\,\prime}_{s=0}} = 2Cs$

$|J|^2 \ge Cs^2 + \underset{\substack{\| \\ 0}}{|J|^2_{s=0}} = Cs^2$

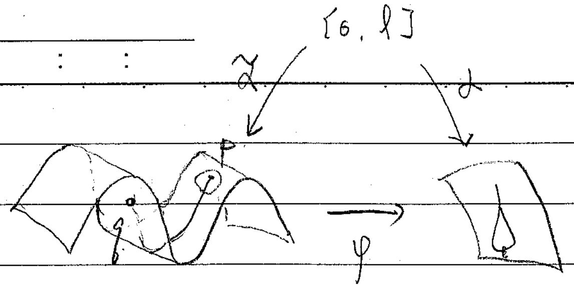
$\therefore s = 1$, $\Rightarrow \langle J(1), J(1) \rangle \ge C$

i.e. $\langle (d\exp)_u w, (d\exp)_u w \rangle \ge |w|^2$

Note that "=" ⟺ $K \equiv 0$

Lemma: If $\varphi : S_1 \longrightarrow S_2$ is a length increasing local diffeo. S_1 complete surface, S_2 : surface then φ is a covering map.

<pf> We prove that φ has the curve lifting property here.



let $\alpha : [0, l] \longrightarrow S_2$ be a curve para. by arc length.

want to find $\tilde{\alpha} \subset S_1$ st. $\varphi \tilde{\alpha} = \alpha$

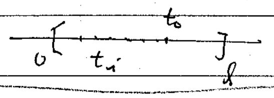
continuity method.

let $A = \{ t \in [0, l] \mid \tilde{\alpha}$ is defined in $[0, t] \}$

$A \neq \phi$

A is open.

$\overset{?}{\longrightarrow}$ A is closed



claim : A is closed. let $t_i \longrightarrow t_0$

(不知道可不可以 lift)

$\{ \tilde{\alpha}(t_i) \} \subset S_1$ is bounded.

if not, then $d(\underset{p}{\underset{\shortparallel}{\tilde{\alpha}(0)}}, \tilde{\alpha}(t_i)) \longrightarrow \infty$

claim:

A is closed,

$t_i \to t_0$ Cauchy seq. in S_2.

$\Rightarrow \tilde{\alpha}(t_i)$: Cauchy seq.

$\Rightarrow \tilde{\alpha}(t_i)$ conv. in S_1, q.

let V be a nbd of q in S_1 st. $\varphi|_V$ is a diffeo.

but $V \ni \tilde{\alpha}(t_n)$ for n large.

so, $\tilde{\alpha}$ can be defined over this V.

---

§5.9 Elementary property of conjugate points

Thm A. A geod. without conju pts. is of minimal length in a nbd of γ.

Notice : this is not necessarily true globaly !

Cut (p) := cut locus.

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Theorem (Jacobi)

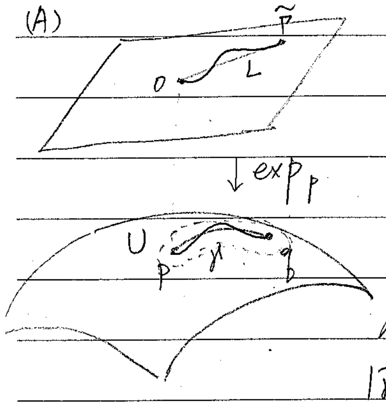
A. A geod. without conj. pt. is (locally) of minimal length in $U \supset \gamma$.

B. If a geod. $\gamma : [0, l] \to S$ has a conj. pt $\gamma(s_0)$, $s_0 < l$, then $\exists$ "broken" variation $h : [0, l] \times (-\varepsilon, \varepsilon) \to S$ with end pts fixed st. $L(t) < L(0)$, $\forall t \neq 0$

($\Rightarrow \exists$ smooth curve $\tilde{\gamma}$ with smaller length)

<pf>.

(A)



$\nexists$ conj pt. $\Rightarrow \exp_p$ is a local diffeo.

by compactness of γ,

$\exists$ $U \supset \gamma$ st. $\exp_p$ is liftible

$\forall$ curve $\tilde{\gamma}$ in U $\tilde{\gamma}(0) = p$, $\tilde{\gamma}(l) = q$

Given $\tilde{\gamma}$, $\exists$ $\tilde{P} \subset T_pS$ st. $\exp_p(\tilde{P}) = \tilde{\gamma}$

Now, use polar coord. on U

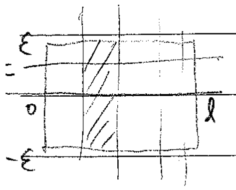
$|\tilde{\gamma}| = \int \sqrt{dr^2 + G\,d\theta^2} \geqslant dr = l$

"=" $\Leftrightarrow$ $d\theta = 0$ ie. θ = const. along $\tilde{P}$

ie. $\tilde{P} \equiv L$ ✲.

Def : $h : [0, l] \times (-\varepsilon, \varepsilon) \to S$ is a broken variation if

- h continuous
- $\exists$ $0 = s_0 < s_1 < \cdots < s_n = l$ st $h : [s_{i-1}, s_i] \times (-\varepsilon, \varepsilon) \to S$, C^∞
- $V = \frac{\partial h}{\partial t}\Big|_{t=0}$ variation field is continuous.



$h_t(s) := h(s, t) : [0, l] \to S$ is a piece wise C^∞ curve

$L_V(t)$ = length of h_t

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Thm (2nd variation)

$L_V''(0) = \int_0^l (|\nabla_T V|^2 - K|V|^2)\, ds$

if V is proper, normal, broken variation.

end pt fixed $V \perp \gamma'$

$\mathcal{V}$ $\mathcal{V}$ $\perp \gamma'$

Def : $\overset{0}{\mathcal{V}}$ the space of all piecewise smooth vector field along γ.

② "Index form"

$I : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$: $I(V, W) = \int_0^l (\langle \nabla_T V, \nabla_T W \rangle - K\langle V, W \rangle)\, ds$

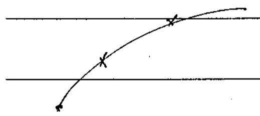
Thm (Morse Index thm)

conj. pts along γ (count with multiplicities)

= max dim of nagative subspace of I.

dim of ker of

$(d\exp_p)_u$ =: multiplicity



We only need

lemma : if $V \in \mathcal{V}$ is a Jacobi field, then $\forall$ $W \in \mathcal{V}$

$I(V, W) = \int_0^l T\langle \nabla_T V, W \rangle - \langle \nabla_T^2 V, W \rangle - K\langle V, W \rangle\, ds$

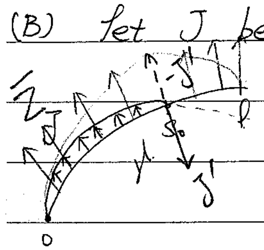
$= \langle \nabla_T V, W \rangle \Big|_0^l - \int_0^l \langle \underbrace{\nabla_T^2 V + KV}_{\overset{\parallel}{0}}, W \rangle\, ds$

Jacobi field

$\to I(V, W) = \langle \nabla_T V, W \rangle \Big|_0^l$

<pf>

(B) let J be Jacobi with $J(0)=0$, $J(s_0)=0$, $J \neq 0$



then $J \cdot \gamma' = 0$

Also, $\nabla_T J(s_0) \neq 0$ (by ODE)

let $\bar{Z}$ be a parallel vector field along γ st.

$\bar{Z}(s_0) = -J'(s_0) \neq 0$

$Z := f\bar{Z}$, f C^∞ func. ≥ 0 ,

$f(0) = f(l) = 0$, $f(s_0) = 1$

Pick. $\eta \in \mathbb{R}$, $\eta > 0$,small

let $Y(s) = \begin{cases} \eta Z(s) + J(s) & s \in [0, s_0] \\ \eta Z(s) & s \in [s_0, l] \end{cases}$ be a piecewise C^∞ v.f. $\perp \gamma'$

$\Rightarrow$ Y determines a proper normal broken variation of γ

$$I_{s_0}(Y,Y) = I_{s_0}(J + \eta Z, J + \eta Z)$$

$$= \underset{0}{I_{s_0}(\cancel{J,J})} + 2\eta I_{s_0}(Z, J) + I_{s_0}(Z,Z)\eta^2.$$

$J(0) = J(s_0) = 0.$

$$= 2\eta \langle \nabla_T J , Z \rangle \Big|_0^{s_0} + \eta^2 I_{s_0}(Z,Z)$$

$$= -2\eta \underset{\overset{\#}{0}}{|J'(s_0)|^2} + \eta^2 I_{s_0}(Z,Z) < 0 \quad \text{for small } \eta.$$

$$L_Y''(0) = I(Y,Y) = I_{s_0}(Y,Y) + I_{[s_0,l]}(Y,Y)$$

$$= -2\eta |J'(s_0)|^2 + \eta^2 I(Z,Z) < 0 \quad \text{for small } \eta$$

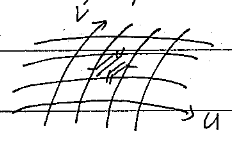
ie. $\gamma \longleftrightarrow L(0)$ is a strictly local maximal.

Hilbert's theorem :

Let S be a complete surface , $\underset{\sim\sim}{K \equiv -1}$. then $\nexists$ isometric immersion $\varphi : S \looparrowright \mathbb{R}^3$

<pf> ① Asymptotic curve form a Tschebyshev net.

asymptotic coord (u, v) $\longleftrightarrow$ $e = 0 = g$



$\Leftrightarrow$ $E_2 = 0$ $E = |X_1|^2$

$G_1 = 0$. $G = |X_2|^2$

$$N_1 \times N_2 = K(X_1 \times X_2) = KDN \quad , \quad D = \sqrt{EG - F^2}$$

$$K = \frac{-f^2}{EG - F^2} = -\left(\frac{f}{D}\right)^2 = -1. \quad \Rightarrow \left(\frac{f}{D}\right) = \pm 1$$

$$\cdot\ N \times N_1 = \frac{1}{D}(X_1 \times X_2) \cdot N_1$$

$$= \frac{1}{D}\{ \underset{\overset{\|}{e=0}}{\underline{(X_1 \cdot N_1)}} X_2 - \underset{\sim\sim\sim}{(X_2 \cdot N_1)} X_1 \}$$

$$= \frac{f}{D} X_1$$

Similarly , $N \times N_2 = -\frac{f}{D} X_2$.

$$(N \times N_1)_2 - (N \times N_2)_1 = N_2 \times N_1 + \cancel{N \times N_{12}} - N_1 \times N_2 - \cancel{N \times N_{21}}$$

$$= -2 N_1 \times N_2$$

$\Rightarrow -2KDN = -2(N_1 \times N_2)$

$= \pm \mathbb{X}_{12} \pm \mathbb{X}_{12} = \pm 2\mathbb{X}_{12}$

ie. $\mathbb{X}_{12} \parallel \vec{N}$

$\Rightarrow E_2 = (\mathbb{X}_1 \cdot \mathbb{X}_1)_2 = 2\mathbb{X}_{12} \cdot \mathbb{X}_1 = 0$.

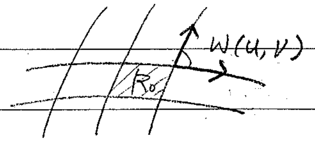
$G_1 = (\mathbb{X}_2 \cdot \mathbb{X}_2)_1 = 2\mathbb{X}_{12} \cdot \mathbb{X}_2 = 0$.

② curvature in T-net. (we do not need $K \equiv -1$)

$ds^2 = du^2 + 2\cos w\, du\, dv + dv^2$

$E_2 = 0$ ie. $E(u)$ change coord. $\tilde{u} = \int \sqrt{E}\, du \Rightarrow \tilde{E} = 1$

$G_1 = 0$ ie $G(v)$ - - - - $\tilde{v} = \int \sqrt{G}\, dv \Rightarrow \tilde{G} = 1$

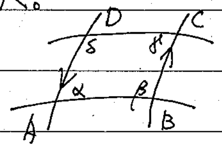


§4.3 ex5 $\Rightarrow K = \dfrac{-W_{uv}}{\sin W}$

ie. $W_{uv} + K \sin w = 0$

corollary : in any rectangular region R_0

$\left| \int_{R_0} K\, dA \right| < 2\pi$.



<pf> $\int_{R_0} K\, dA = \int_{R_0} K \sin w\, du\, dv$.

$= \int_{R_0} -W_{uv}\, du\, dv$.

$= -\int_B^C W_v\, dv - \int_D^A W_v\, dv$.

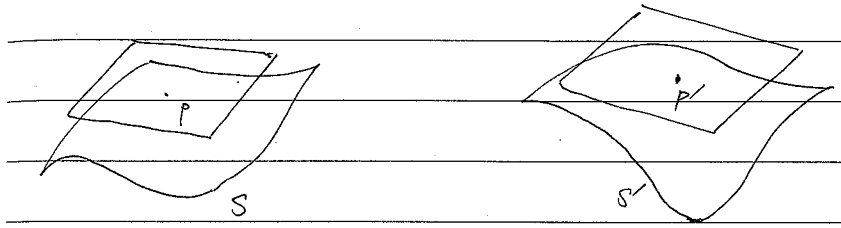
$0 < \alpha, \beta, \gamma, \delta < 2\pi$ $= -\left[(\gamma - (\pi - \beta)) + (\alpha - (\pi - \delta)) \right]$

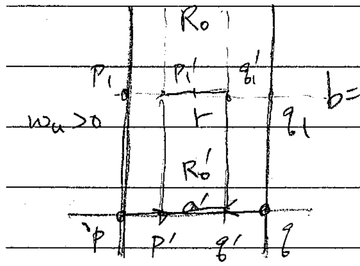
$= 2\pi - (\alpha + \beta + \gamma + \delta) \in (-2\pi, 2\pi)$

$$\begin{tikzcd} \tilde{S} \arrow[d, phantom, "\parallel"] & \\ (T_pS, ds^2) \arrow[d, "\pi"', "\text{covering map}"] \arrow[dr, "\times"] & \\ p \in S \arrow[r, "\varphi"] & \mathbb{R}^3 \end{tikzcd}$$

$\Rightarrow$ asymptotic curve is defined on the whole $\tilde{S}$

$K \equiv -1$





$w_u > 0$

$w_{uv} = -K \sin W = \sin W > 0$
$\overset{\parallel}{-1}$

$W_u \nearrow$ in V

let $W_u(p) > 0$,

$\exists\, q$ in the u-line st. $W_u > 0$ on $\overline{pq}$

$w \nearrow$ in u-line $w(p') - w(p) > \varepsilon$

pick p', q', st. $w(q) - w(q') > \varepsilon$

$w(p_1') - w(p_1) = \int_{p_1}^{p_1'} w_u\, du > \int_p^{p'} w_u\, du = w(p') - w(p) > \varepsilon$

$w(q_1) - w(q_1') > \varepsilon$

$\Rightarrow w(p_1') > \varepsilon$

$w(q_1') < \pi - \varepsilon \Rightarrow \sin r > \sin \varepsilon$

$\left| \int_{R_0'} K\, dA \right| < 2\pi$

$\parallel$

$\int_{R_0'} \sin w\, du\, dv \Rightarrow b < \dfrac{2\pi}{a' \sin \varepsilon}$ ~~X~~.

$\vee$

$\sin \varepsilon \cdot a' b$