Motivic Integration

— Introduction & Applications

2000, 8/30 NCTS. 王金龍

Ref. Nash, Kontsevich, Denef-Loeser

I. THE GROTHENDIECK RING

k — field char $k = 0$

Sch_k — Cat. of schemes $/k$ with reduced str.
i.e. alg. v. $/k$

① $M = K_0(Sch_k)$ — Grothendieck ring of alg. v $/k$

ring gen. by symbols $[S]$, $S \in Sch_k$

s.t. $[S] = [S']$ if $S \cong S'$

$[S] = [S - S'] + [S']$ if $S' \hookrightarrow S$ closed

$[S \times S'] = [S][S']$.

Let $\mathbb{L} = [\mathbb{A}^1_k]$ — Lefschetz motive

② $M[\mathbb{L}^{-1}]$ — localization of M at $\mathbb{L}$ (or M_{loc})

$p \in \mathbb{Z}$, $F^p := \langle \{ [S] \cdot \mathbb{L}^{-i} \mid \dim S - i \leq -p \} \rangle \subset M[\mathbb{L}^{-1}]$

think as coeff   think as q (or p)

$\cdots F^{-1} \supset F^0 \supset F^1 \supset \cdots$ decreasing filtration

③ $\hat{M} = \varprojlim_p M[\mathbb{L}^{-1}]/F^p$ — the completion

with kernel $\bigcap F^p$.

why $\mathcal{M}$, $\mathcal{M}[\mathbb{L}^{-1}]$ and $\hat{\mathcal{M}}$? or More ?

examples

① easy flop in dim 3

$$\begin{tikzcd} & \mathbb{P}^1\times\mathbb{P}^1 \simeq E \subset Y \arrow[dl, "\varphi"'] \arrow[dr, "\varphi'"] & \\ \mathbb{P}^1 \simeq Z \subset X \arrow[rr, dashed, "f"] & & X' \supset Z' \simeq \mathbb{P}^1 \end{tikzcd}$$

contraction in another direction

with $N_{Z/X} = \mathcal{O}(-1)\oplus\mathcal{O}(-1)$

$\Rightarrow \quad [X] = [X-Z]+[Z] = [X'-Z'] + [Z'] = [X']$

② Blow up along smooth center

$$\begin{tikzcd} Y \arrow[d, "h"] & \supset E = \mathbb{P}_Z(N) \arrow[d] \\ X & \supset Z \end{tikzcd}$$

: $\mathbb{P}^{d-1}$ bundle

$N = N_{Z/X}$ $\quad$ codim$(Z,X)=d$

Notice that: $[\mathbb{P}^r] = [A^r]+[A^{r-1}]+\cdots+1$
$= \mathbb{L}^r + \mathbb{L}^{r-1}+\cdots+1$

$\Rightarrow [E] = [Z](1+\mathbb{L}+\cdots+\mathbb{L}^{d-1}) = [Z]\cdot\dfrac{1-\mathbb{L}^d}{1-\mathbb{L}}$

so $\quad [X]-[Z] = [Y]-[E]$

$\Rightarrow \quad [X] = [Y]-[E]+[E]\cdot\dfrac{1-\mathbb{L}}{1-\mathbb{L}^d}$

$= [Y] + [E]\left(\dfrac{1-\mathbb{L}}{1-\mathbb{L}^d} - 1\right)$

Notice: $N_{Z/X}$ disappear in this formula,
only E remains!

K_0 construction v.s. "Motivic Property"

<u>~~Deligne's Mixed Hodge Theory~~</u> on compactly supp. coh. of any alg. v. (over $\mathbb{C}$)

$Z \hookrightarrow X$ closed.

$$(*) \quad \cdots \to H^i_c(X-Z) \to H^i_{\underline{c}}(X) \to H^i_{\underline{c}}(Z) \to H^{i+1}_c(X-Z) \to \cdots$$

$\Rightarrow \chi_c(\cdot) := \sum (-1)^i h^i_c(\cdot, \mathbb{C})$ satisfies

$\chi_c(X) = \chi_c(X-Z) + \chi_c(Z)$ hence

$$\begin{tikzcd} Sch_k \arrow[r, "\chi_c"] \arrow[d] & \mathbb{Z} \\ M \arrow[ur, dashed] & \end{tikzcd}$$

i.e. $\chi_c(X)$ depends only on the class $[X]$. in M.

<u>Thm (Deligne)</u> : $\exists$ MHS $H^{p,q}(H^i_c(X, \mathbb{C}))$.
functorial
compatible with $(*)$.

Cor. If define $\chi^{p,q}_c = \sum (-1)^i H^{p,q}(H^i_c(\cdot, \mathbb{C}))$

take as a V.S. (or Hodge structure)

then

$$\begin{tikzcd} Sch_k \arrow[r, "\chi^{p,q}_c"] \arrow[d] & K_0(Hdg) \\ M \arrow[ur, dashed] & \end{tikzcd}$$

(or just $\mathbb{Z}$)

<u>THEOREM</u>: If X, X' are smooth K-equivalent in the strong sense that

$$\begin{tikzcd} & Y \arrow[dl, dotted, "\varphi"'] \arrow[dr, dotted, "\varphi'"] & \\ X & & X' \end{tikzcd}$$

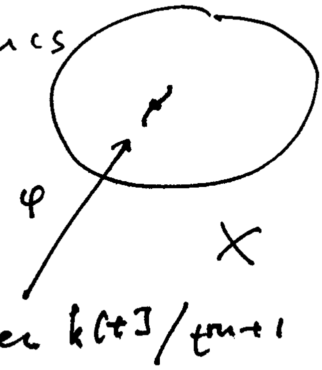
φ, φ' = sequence of blow-ups st. $\varphi^* K_X = \varphi'^* K_{X'}$. Then $[X] = [X']$ in M hence that same Hodge structure.
(not yet canonically isomorphic)

II. Nash's Space of Formal arcs. let $\dim X = d$.

$$\mathcal{L}_m(X) = \mathrm{Mor}_{k\text{-sch}}(\mathrm{Spec}\, k[t]/t^{m+1}, X)$$

i.e. $X(k[t]/t^{m+1})$ truncated arcs

$\mathcal{L}(X) := \varprojlim \mathcal{L}_m(X)$ formal arcs



Natural maps

$\mathcal{L}_{m+1}(X) \xrightarrow{\theta_m} \mathcal{L}_m(X)$

$\pi_m : \mathcal{L}(X) \longrightarrow \mathcal{L}_m(X)$

Fact: both maps are surj if X is smooth.

Nash: Structure of θ_m, π_m if X singular
(~ 1960)

$\mathcal{B} = \mathcal{B}(\mathcal{L}(X))$ — all semi-algebraic subset
(defined by boolean condi on $\underline{\underline{\mathrm{ord}_t}}$ of polynomial functions etc.)
e.g. $\mathrm{ord}_t f(\varphi(t)) \geq \mathrm{ord}_t g(\varphi(t))$

$A \subset \mathcal{L}(X)$. Stable at level k if

- A = union of fibers of π_k
- $\pi_{m+1}(\mathcal{L}(X)) \longrightarrow \pi_m(\mathcal{L}(X))$
  is a piecewise trivial fibration over $\pi_m(A)$
  with fiber $\mathbb{A}_k^d$, $\forall m \geq k$

Prop: Every s.alg set $A = \coprod_{i \in \mathbb{N}} A_i$ (mod $\cap$ FP)
with A_i stable (at some k_i).

Definition of motivic measure

(if A stable (at level k) :

$$\mu_X(A) := [\pi_k(A)] \cdot \mathbb{L}^{-kd} \in M[\mathbb{L}^{-1}]$$

(this is indep. of choice of k since

$$[\pi_{k+1}(A)] = [\pi_k(A)] \cdot \mathbb{L}^d .$$)

$\Rightarrow$ hence $\mu_X : \mathbb{B} \longrightarrow \widehat{M}$.

e.g. $X = \mathbb{A}^d_k = \operatorname{Spec} k[x_1, \cdots, x_d]$

$$\underline{\mathcal{L}_n(X)} = \operatorname{Mor}_k\left(k[t]/t^{n+1}, X\right)$$
$$\cong \operatorname{Mor}\left(k[x_1, \cdots, x_d],\ k[t]/t^{n+1}\right)$$
$$\cong \left(k[t]/t^{n+1}\right)^d \cong \underline{\mathbb{A}_k^{d(n+1)}}$$

trivial product; each $\mathcal{L}_{n+1}(X) \to \mathcal{L}_n(X)$ is trivial bundle, fiber $\cong \mathbb{A}^d$.

e.g. $X = \mathbb{P}^d_k = \operatorname{Proj}(k[x_0, \cdots, x_d])$

$$= \mathbb{A}^d_k \cup \mathbb{A}^{d-1}_k \cup \cdots \cup \mathbb{A}^0_k$$

So $\mathcal{L}_n(X) \xrightarrow[\pi]{} \mathcal{L}_0(X) = X$
is a piecewise trivial fibration, fiber $\cong \mathbb{A}^{nd}_k$
in fact, this is NOT globally trivial since

$k \hookrightarrow k[t]/t^{n+1} \Rightarrow$ $\quad$ $k[t]/t^{n+1}$ points of X

$$\mathcal{L}_n(X) = \mathcal{L}_0(X) \otimes_k \left(k[t]/t^{n+1}\right)$$

(i.e. $X \times_k \underline{\operatorname{Spec}\left(k[t]/t^{n+1}\right)}$ extension by scalar

one point with nontrivial structure sheaf.

locally trivial $\mathbb{A}^n_k$ bundle

$$\begin{tikzcd} X(R) \arrow[r, shift left] \arrow[d] & X \arrow[l, shift left] \arrow[d] \\ \operatorname{Spec} R \arrow[r, shift left] & \operatorname{Spec} k \arrow[l, shift left] \end{tikzcd}$$

as zero section

on the other hand
$\exists!\ k[t]/t^{n+1} \longrightarrow k$
by specialize $t \mapsto 0$,
but need $t^{n+1} = 0$!

Rmk: $\mathcal{L}(X)$ corresponds to the trivial deformation of X.

eg. $\mathcal{L}_1(\mathbb{P}^1) \underset{\pi}{\overset{\sigma}{\rightleftarrows}} \mathcal{L}_0(\mathbb{P}^1)$

$\mathbb{P}^1 \cong \{xy = z^2\} \subset \mathbb{P}^2$

$x(t)\, y(t) = z(t)^2$

$(x_0 + x_1 t)(y_0 + y_1 t) = (z_0 + z_1 t)^2 \mod t^2$

$\underline{x_0 y_0 = z_0^2}$; $\underline{x_1 y_0 + x_0 y_1 = 2 z_0 z_1}$

I. $\qquad$ II.

$\nearrow$ $\mathcal{L}_1(\mathbb{P}^1)$ $\quad$ (or $x_1^2 y_0^2 + 2 x_1 y_0 x_0 y_1 + x_0^2 y_1^2 = 4 z_0^2 z_1^2$.)

if $\exists$ section σ. ie. x_1, y_1, z_1 are determined by x_0, y_0, z_0 up to scalar ($\neq 0$)

then get further eq'n III, IV, V. $\leftarrow$ "σ"

but $[x_0, y_0, z_0, x_1, y_1, z_1] \in \mathbb{P}^5$

five eq^n has nontrivial $\cap$.

$\Rightarrow \sigma$ must has zero. So, π is not trivial bundle

Q: what is this bundle?

Answer: $\mathcal{L}_1(X) \cong T_X$. tangent bundle

in general, Zariski tang. space (if X smooth)

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eg. $\mathcal{L}_1(A_1) \longrightarrow \mathcal{L}_0(A_1) = \{xy = z^2\} \subset \mathbb{C}^3$

eq'n II is completely trivial over $(0,0,0)$. $\quad$ an OPP. (affine cone over $\mathbb{P}^1$)

$\mathcal{L}_2(A_1) \longrightarrow \mathcal{L}_1(A_1) \longrightarrow \mathcal{L}_0(A_1)$

over $(x_0, y_0, z_0) = (0,0,0)$:

$(x_1 t + x_2 t^2)(y_1 t + y_2 t^2) = (z_1 t + z_2 t^2)^2 \mod t^3$

$t^2 (\underline{x_1}\, \underline{y_1} + (\underline{x_1} y_2 + x_2 \underline{y_1}) t) = (\underline{z_1} + z_2 t)^2 t^2 \mod t^3$

$= (\underline{z_1}^2 + 2 z_1 z_2 t + z_2^2 t^2) t^2$

ie. only for those arc st. $x_1 y_1 = z_1^2$ has preimage in $\mathcal{L}_2(A_1)$

and they are all (x_2, y_2, z_2).

Now fix a $(x_1, y_1, z_1)t$:

$$\left(x_1 t + x_2 t^2 + x_3 t^3 \underset{+x_4 t^4}{}\right) \cdot \left(y_1 t + y_2 t^2 + y_3 t^3 \underset{+y_4 t^4}{}\right)$$

$$\equiv (z_1 t + z_2 t^2 + z_3 t^3 \underset{+z_4 t^4}{})^2 \quad \text{mod } t^4$$

i.e. $\underline{x_1 y_1}\, t^2 + (x_1 y_2 + x_2 y_1) t^3 + (x_3 y_1 + x_2 y_2 + x_1 y_3) t^4$

$$\equiv \underline{z_1^2}\, t^2 + 2 z_1 z_2 t^3 \quad (t^4)$$
$+(z_2^2 + 2 z_1 z_3) t^4$

mod t^5

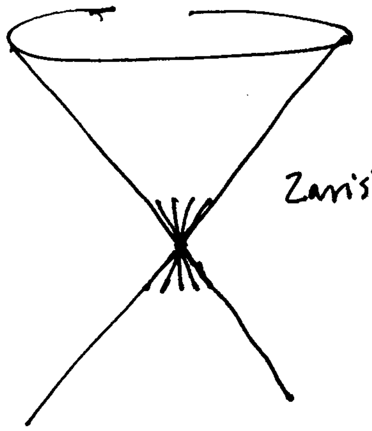
2-dim'l linear subspace

only for those (x_2, y_2, z_2) st. $x_1 y_2 + x_2 y_1 = 2 z_1 z_2$ $\mathcal{H}$

has preimage in $\mathcal{L}_3(A_1)$ and they are all

a 2-dim'l linear subspace

$x^2 + y^2 = t^2$



$\mathcal{L}_1$

$\uparrow$

Zariski tangent space

Still A^3 bundle $\mathcal{L}_3$

Assume here stablize!

$\mathcal{L}_2 = A^3$ bundle $\supset \mathcal{H}$

some hyperplane

θ_1

$\mathcal{L}_1 \supset \{x_1^2 + y_1^2 = z_1^2\} = G$; tangent space

tangent cone at 0

θ_0

$\mathcal{L}_0(X) \supset X_{sing} \amalg X_{smooth}$

$\{0\}$

It is important to notice that $\theta_1 : \mathcal{H} \to G$ is an A^2 bundle

$\pi_2(\mathcal{L}(x)) \xrightarrow[\theta_1]{} \pi_1(\mathcal{L}(x))$

(where $2 = \dim X =: d$)

only over $\pi_1(\mathcal{L}^{(1)}(X))$, in this case, it excludes the case $(x_1, y_1, z_1) = 0$.

$C - \{0\}$ $\quad = \mathcal{L}(x) - \pi_{1,X}^{-1} \mathcal{L}_1(X_{sing})$

Hence that

$X_{sing} = \{x = 0, y = 0, z = 0\}$

$$[\pi_2(\mathcal{L}^{(1)}(X))] = [\pi_1(\mathcal{L}^{(1)}(X))] \cdot [\mathbb{A}^2]$$

so $\mathcal{L}_1(\{0\}) =$ all $\{(x(t), y(t), z(t)\} \equiv 0 \pmod{t^2}$

in general this reads

i.e. $* t^2 + \cdots$

$$[\pi_n \mathcal{L}^{(e)}(X)] = [\pi_{ce} \mathcal{L}^{(e)}(X)] \cdot \mathbb{L}^{d(n-ce)} \quad \text{for } n \geq ce.$$

Kähler differential $\neq$ canonical differential
eg. $0 \in X \subset \mathbb{A}^{n+1}$ ordinary "k-fold" sing.

eg. $X = \{ f(x) = x_0^k + \cdots + x_n^k = 0 \}$.

then

$$\begin{array}{ccc} E & \subset & Y \\ \downarrow & & \downarrow h \\ 0 & \in & X \end{array}$$

single blow up resolve the sing.
with $E \subset \mathbb{P}^n$ with same eq'n
$(x): \; x_0^k + \cdots + x_n^k = 0$.

Expected correspondence
compute the Jacobian of h. $\equiv$ sheaf Ω_X^1

~~Kähler~~ differential (is in general bigger than) regular differential

$\xi : \; R \longrightarrow \Omega^1$

has kernel $3y\,dx - 2x\,dy$ $\leftarrow$ this is not 0 in R.

the only relation in R: $R = k\,dx + k\,dy \big/ 2y\,dy - 3x^2dx$

what is Ω_X^1?

eg. $X = \{ y^2 = x^3 \} = \operatorname{Spec} A$, $A = k[x,y]/(y^2 - x^3)$

Kähler differential Ω_X^1 is given by
df. $f \in A$ subjects to relation $\underline{d(ab) = adb + bda}$
$\underline{d\lambda = 0 \text{ for } \lambda \in k}$

(Notice that $A = k[x] + k[x]y$
$\Rightarrow \xi(3y\,dx - 2x\,dy) = 0$

$$\begin{aligned} 2y\,dy &= 3x^2dx \\ 2yx\,dy &= 3x^3dx \\ &= 3y^2dx \end{aligned}$$

this is useless! $\quad * \quad y(3y\,dx - 2x\,dy) = 0$)

$y_0 y_1 \cdots y_n$

$$\begin{cases} x_0 = y_0 \\ x_1 = y_0 y_1 \\ \vdots \\ x_n = y_0 y_n \end{cases}$$

$x_0^k + \cdots + x_n^k = 0$

$y_0^k (1 + y_1^k + \cdots + y_n^k) = 0$

$$\begin{tikzcd} \mathbb{A}^{n+1} \arrow[d] & \supset & Y \arrow[d, "h"] \\ \mathbb{A}^{n+1} & \supset & X \end{tikzcd}$$

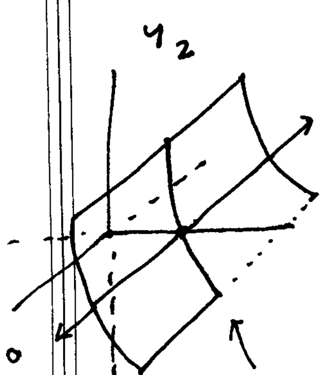
Y: $1 + y_1^k + \cdots + y_n^k = 0$ $\quad y_0$ arbitrary

X: $x_0^k + \cdots + x_n^k = 0 \leftarrow f$

on Y use coor. $y_0, y_1, \ldots, y_{n-1}$

(eg. surface case : $n=2$)

$$\left(\frac{\partial f}{\partial y_0}, \ldots, \frac{\partial f}{\partial y_{n-1}} \right)$$



contracted curve (divisor)

$f(y_0, \ldots, y_{n-1}) = y_0^k (1 + y_1^k + \cdots + y_n^k)$

$$\underline{dx_0} \wedge \cdots \overset{\text{i-omitted}}{\downarrow} \wedge dx_n$$

$= \underline{y_0^{n-1}}\ \underline{dy_0} \wedge \cdots \wedge dy_n$

regular form

$dx_1 \wedge \cdots \wedge dx_n$

$(y_0 dy_1 + y_1 dy_0) \wedge \cdots \wedge (y_0 dy_n + y_n dy_0)$ ← appear at most once

$= y_0^n dy_1 \wedge \cdots \wedge dy_n$

$+ \underline{y_0^{n-1}} \cdot y_i \, dy_0 \wedge \cdots \overset{\text{i-th}}{\wedge} \cdots \wedge dy_n$

$\Omega^1_X = \langle dx_0, \ldots, dx_n \rangle / k(x_0^{k-1} dx_0 + \cdots x_n^{k-1} dx_n)$ ← subject to this

$\Omega_X^{d=n} = \Lambda^{d=n} \Omega_X \quad \sum dx_0 \wedge \cdots \wedge dx_n$ (i-omitted)

$h: Y \to X$ birat'l, $\dim X, Y = d$. smooth

$\Delta_e \subset \mathcal{L}(Y)$ st. $\mathrm{ord}_t J(\varphi) = e$

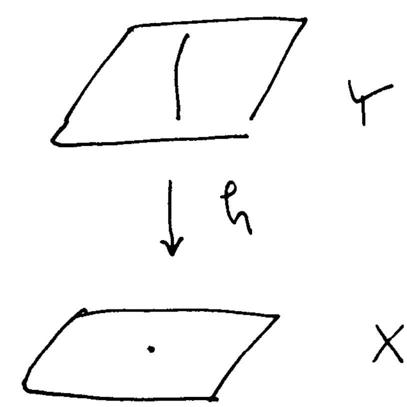
what's the structure of $\Delta_e \longrightarrow \mathcal{L}(X)$?

claim: for $n >> e$

$$\begin{tikzcd} \mathcal{L}(Y) \arrow[r, "\pi_n"] \arrow[d, "h"'] & \mathcal{L}_n(Y) \arrow[d, "h_n"] \\ \mathcal{L}(X) \arrow[r, "\pi_n"'] & \mathcal{L}_n(X) \end{tikzcd}$$

- $\pi_n \Delta_e = \cup$ fibers of h_n
- $h_n |_{\pi_n \Delta_e}$ is a piece-wise trivial $\mathbb{A}^e$ fibration. (onto its image)

eg. blow up a point, for an affine piece.



$$h: \quad \begin{aligned} x_1 &= y_1 \\ x_2 &= y_1 y_2 \\ &\vdots \\ x_d &= y_1 y_d \end{aligned}$$

$J = \text{"}(d-1)E\text{"} = (y_1^{d-1})$

for $\varphi \in \mathcal{L}(Y)$ $\varphi = (y_1(t), \cdots, y_d(t))$

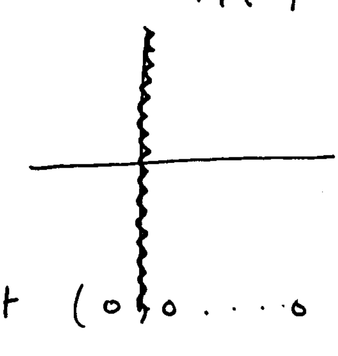
then $\mathrm{ord}_t J(\varphi) = \mathrm{ord}_t [y_1(t)^{d-1}]$

$= (\mathrm{ord}\, y_1(t)) \cdot (d-1) = \underline{\underline{s}}(d-1)$

e can only be of the form $e = s(d-1)$

if $y_1(t) = a_0 + a_1 t + \cdots$ $\qquad x_1(0) \neq 0$.

$\underline{e=0}$. ($\Leftrightarrow a_0 \neq 0$) $\Rightarrow$ isom.



when glue all pieces together
image
should get image only miss point $(0, 0, \ldots, 0)$
ie. all curves of the form $t \cdot v$

This part is tautology!

bec. for every $x(t)$, $y(t)$ is uniquely determined (even for finite level)

hence get a $\mathbb{A}^0$ bundle!

$e = d-1$: i.e. $y_1(t) = a_1 t + \cdots \quad (s=1)$
$a_1 \neq 0$

$x_1(t) = y_1(t)$
$x_2(t) = y_1(t)\, y_2(t)$
$\vdots$
$x_d(t) = y_1(t)\, y_d(t)$

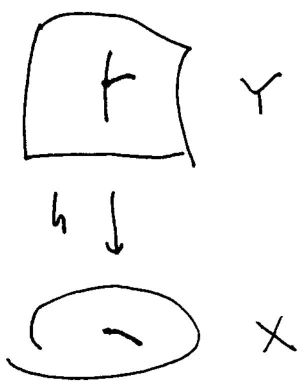
all $x_2(t) \cdots x_d(t)$
st $x_2(0) \underline{\underline{=}} 0, \ldots x_d(0) \underline{\underline{=}} 0$
(i.e. with term $t \ldots$)
can be mapped

image is all $t(u_1, u_2 \ldots u_d)$ but $u_1(0) \neq 0$
gluing together set image $= t(u_1, \ldots, u_d)$
but $u_i(0)$ not $0\ \forall i$

for any such $x(t) = t \cdot v(t)$. $v_i(0) \neq 0$

$y(t)$ is again <u>uniquely solvable</u>

but if look at finite truncation. $n \gg 0$:



for $\mathcal{L}_n(X)$:

$$\tilde{x}(t) := x(t) + \underline{t^{n+1}}\, \tilde{v}$$
$$\exists!\ \tilde{y}(t) = y(t) + \underline{t^{(n-e)+1}}\, \tilde{u}$$

difference t^e

$$\tilde{x}_1(t) = \tilde{y}_1(t) = y_1(t) + t^{n+1-e}\, \tilde{u}_1$$
$$\overset{\shortparallel}{x_1(t) + t^{n+1}\tilde{v}_1} \qquad \text{may set } \tilde{u}_1 = t^e \cdot \tilde{v}_1$$

$$\tilde{x}_2(t) = \tilde{y}_1(t)\, \tilde{y}_2(t) = \left(y_1(t) + t^{n+1}\tilde{v}_1\right)\left(y_2(t) + t^{n+1-e}\tilde{u}_2\right)$$
$$\overset{\shortparallel}{x_2(t) + \underline{t^{n+1}}\tilde{v}_2}$$
$$= y_1(t) y_2(t) + \underline{y_1(t)\, t^{n+1-e}\, \tilde{u}_2}$$
$$+ \underline{t^{n+1}\tilde{v}_1\, y_2(t) + t^{2(n+1)-e}\tilde{v}_1 \tilde{u}_2}$$

Need to solve:

$$\tilde{u}_2\left(y_1(t)\, t^{n+1-e} + t^{2(n+1)-e}\tilde{v}_1\right) = t^{n+1}\tilde{v}_2 - t^{n+1}\tilde{v}_1\, y_2(t)$$

compare order in t: LHS $= (n+1-e+1, \quad)$ must be smaller
$\wedge\!\!\backslash$
RHS $= (n+1)$

hence OK.

Same for $\tilde{x}_i(t) = \tilde{y}_1(t)\, \tilde{y}_i(t)$ *.

In

$$\begin{array}{ccccc} \mathcal{L}(Y) & \longrightarrow & \mathcal{L}_n(Y) & & \varphi \\ \downarrow h & & \downarrow h_n & & \downarrow \\ \mathcal{L}(X) & \longrightarrow & \mathcal{L}_n(X) & & h(\varphi) \end{array}$$

Solve $h(\varphi + \underline{t^{n+1-e} u}) = h(\varphi) + \underline{t^{n+1} v}$

thus $\mathcal{L}(Y) \bmod t^{n+1-e}$ $\qquad$ $\mathcal{L}(X) \bmod t^{n+1}$

$\mathcal{L}_{n-e}(Y)$ $\qquad$ no such map.

implies

$\underline{\pi_n \Delta_e \ni \bar{\varphi}} \in \mathcal{L}_n(Y) \xrightarrow{h_n} \mathcal{L}_n(X)$

the fiber of $h_n(\bar{\varphi})$ is contained in

$$\{ \bar{y} \in \mathcal{L}_n(Y) \mid \bar{y} \equiv \bar{\varphi} \bmod t^{n-e+1} \}$$

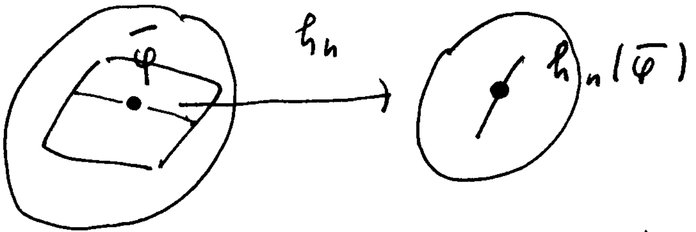
this is contained in $\pi_n \Delta_e$ for $n >> 0$:

because $n-e \geq e$ (so $\bar{y} \equiv \bar{\varphi} \bmod t^{e+1}$ and so $\mathrm{ord}_t J(y) = \mathrm{ord}_t J(\varphi) = e$.)

i.e.
- $\pi_n \Delta_e$ contains all h_n-fibers
- the fibers of $\pi_n \Delta_e \xrightarrow{h_n} \mathcal{L}_n(X)$ are exactly fiber of $\underbrace{\mathcal{L}_n(Y) \longrightarrow \mathcal{L}_{n-e}(Y)}$

hence an $\mathbb{A}^{de}$ fibration.

in fact, only an $\mathbb{A}^e$-hypersurface in it.



What is the set: $\{ \bar{y} \in \mathcal{L}_n(Y) \mid \bar{y} \equiv \bar{\varphi} \bmod t^{n-e+1} \}$

only in charts, eg $Y \cong \mathbb{A}^d$ can write in this form.

this means $\bar{y}$, which projects into same element of $\bar{\varphi}$ in $\mathcal{L}_{n-e}(Y)$