# Lie groups and Lie algebras

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2022 Fall

## 1 Introduction, 9/5

Sample Problems:

1) Given two matrices $A$, $B \in \mathrm{M}_{n\times n}(\mathbb{C})$, $\operatorname{tr}[A,B]=0$, where $[A,B]=AB-BA$ is the Lie bracket. Conversely, if $\operatorname{tr} C=0$, can we find $A$, $B$ such that $C=[A,B]$?

2) We know that $e^Ae^B=e^{A+B}$ when $[A,B]=0$, where

$$e^A:=\sum_{k=0}^{\infty}\frac{A^k}{k!}.$$

If $[A,B]\neq 0$, what should be the RHS? (Baker–Campbell–Hausdorff formula)

Dynkin's formula: for $X_i \in \mathrm{M}_{n\times n}(\mathbb{C})$, define

$$[x_n,\dots,x_1]=[x_n,[x_{n-1},\cdots,x_2,x_1]]$$

recursively. More generally, define

$$\left[x_n^{(i_n)},\dots,x_1^{(i_1)}\right]=\big[\underbrace{x_n,\dots,x_n}_{i_n},\dots,\underbrace{x_1,\dots,x_1}_{i_1}\big].$$

Then we have $e^Xe^Y=e^Z$, where

$$Z=\sum_{n,I,J}\frac{(-1)^{n-1}}{n}\frac{1}{i_1+j_1+\cdots+i_n+j_n}\frac{\left[x_1^{(i_1)},y_1^{(j_1)},\cdots,x_n^{(i_n)},y_n^{(j_n)}\right]}{i_1!j_1!\cdots i_n!j_n!}$$

3) Consider the PDE: $\triangle u+e^u=0$ on $U=B_0(1)\subseteq\mathbb{R}^2$. Liouville: The solution can be explicitly written down! (integrable system).

More generally, consider $u_1$, …, $u_n:=U\to\mathbb{R}$ such that

$$\triangle u_i+\sum_{j=1}^{n}a_{ij}e^{u_j}=0.$$

Can the solution be written down explicitly (locally)? Toda: Yes, if $A = (a_{ij})$ is the Cartan matrix of a simple Lie algebra.

Let $V$ be a vector space over a field $F$. For $s$, $t \in \operatorname{End}(V)$, we define

$$[s, t] = st - ts.$$

We have the Jacobi identity:

$$[s, [t, u]] + [t, [u, s]] + [u, [s, t]] = 0.$$

**Definition 1.1.** A Lie algebra $L$ is a vector space over $F$ with a bilinear map

$$[-, -] : L \times L \to L$$

such that $[x, x] = 0$ for each $x \in L$ and $[-, -]$ satisfies the Jacobi identity.

A **Lie algebra homomorphism** $\varphi : L \to L'$ is a linear map over $F$ satisfies

$$\varphi([x, y]) = [\varphi(x), \varphi(y)].$$

A subspace $K \subseteq L$ is a subalgebra of the Lie algebra $L$ if for each $x$, $y \in K$, $[x, y] \in K$. A subspace $K \subseteq L$ is an ideal of $L$, denoted by $K \trianglelefteq L$, if $[x, y] \in K$ for each $x \in K$ and $y \in L$.

If $K$ is an ideal of $L$, we can define the quotient Lie algebra $L/K$ with the natural Lie bracket $[\overline{x}, \overline{y}] = \overline{[x, y]}$. For a Lie algebra homomorphism $\varphi : L \to L'$, $\ker \varphi$ is an ideal, and there is a Lie algebra isomorphism $L/\ker \varphi \cong \operatorname{Im} \varphi \subseteq L'$.

Classical Lie algebra:

For a vector space $V$, we define $\mathfrak{gl}(V) = (\operatorname{End}(V), [-, -])$, where $[x, y] = xy - yx$. If $V = F^n$, we write $\mathfrak{gl}(V) = \mathfrak{gl}(n, F) = \mathrm{M}_{n\times n}(F)$.

There are 4 special types of classical subalgebra of $\mathfrak{gl}(V)$:

- $A_\ell$: special linear Lie algebra. $\dim V = \ell + 1$, $A_\ell = \mathfrak{sl}(V) = \{x \in \mathfrak{gl}(V) \mid \operatorname{tr} x = 0\}$.
- $B_\ell$: orthogonal Lie algebra. $\dim V = 2\ell + 1$, $B_\ell = \{x \in \mathfrak{gl}(V) \mid x^\mathsf{T} A + Ax = 0\}$, where $A$ is the bilinear form
$$\begin{pmatrix} 1 & & \\ & & I_\ell \\ & I_\ell & \end{pmatrix}$$

- $C_\ell$: symplectic Lie algebra. $\dim V = 2\ell$, $C_\ell = \{x \in \mathfrak{gl}(V) \mid x^\mathsf{T} A + Ax = 0\}$, where $A$ is the bilinear form
$$\begin{pmatrix} & I_\ell \\ -I_\ell & \end{pmatrix}$$
- $D_\ell$: orthogonal Lie algebra. $\dim V = 2\ell$, $D_\ell = \{x \in \mathfrak{gl}(V) \mid x^\mathsf{T} A + Ax = 0\}$, where $A$ is the bilinear form
$$\begin{pmatrix} & I_\ell \\ I_\ell & \end{pmatrix}$$

Note that for $x$, $y$ satisfying $x^\mathsf{T} A + Ax = y^\mathsf{T} A + Ay = 0$, we have

$$[x, y]^\mathsf{T} A + A[x, y] = 0.$$

**Remark 1.2.** Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a orthogonal transformation. Then $\langle Ta, Tb \rangle = \langle a, b \rangle$. An infinitesimal orthogonal transformation then satisfies

$$\langle xa, b \rangle + \langle a, xb \rangle = 0,$$

which is equivalent to $x^\mathsf{T} + x = 0$

A **representation**, or a **module**, of a Lie algebra is a Lie homomorphism

$$\varphi : L \longrightarrow \mathfrak{gl}(V).$$

Can you find one? Yes, the adjoint representation

$$\begin{aligned} L &\xrightarrow{\quad \text{ad} \quad} \mathfrak{gl}(L) \\ x &\longmapsto \operatorname{ad} x = [y \mapsto [x, y]]. \end{aligned}$$

**Definition 1.3.** The **center** of a Lie algebra $L$ is

$$Z(L) := \ker \operatorname{ad} = \{y \in L \mid [x, y] = 0,\ \forall x \in L\}.$$

There is an embedding $L/Z(L) \hookrightarrow \mathfrak{gl}(L)$.

**Definition 1.4.** For a Lie algebra $L$, define $L^{(i)} = \left[L^{(i-1)}, L^{(i-1)}\right]$ recursively, where $L^0 = L$. The sequence

$$L = L^{(0)} \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \cdots$$

is called the **derived series** of $L$.

We say $L$ is **commutative** (or **abelian**) (resp. **solvable**) if $L^{(1)} = 0$ (resp. $L^{(n)} = 0$ for some positive integer $n$).

Note that $L^{(1)}$ is an ideal of $L$ and $L/L^{(1)}$ is commutative.

**Definition 1.5.** For a Lie algebra $L$, define $L^i = [L, L^{i-1}]$ recursively, where $L^1 = L$.

We say $L$ is **nilpotent** if $L^n = 0$ for some positive integer $n$.

## 2 Three giants, 9/7

From now on, we will assume that the Lie algebras are finite dimensional.

Let

$$\begin{aligned}\mathfrak{t}(n, F) &= \{x \in \mathfrak{gl}(n) \mid \ x \text{ is upper triangular}\},\\ \mathfrak{n}(n, F) &= \{x \in \mathfrak{gl}(n) \mid \ x \text{ is strictly upper triangular}\},\\ \mathfrak{d}(n, F) &= \{x \in \mathfrak{gl}(n) \mid \ x \text{ is diagonal}\}\end{aligned}$$

Then $\mathfrak{t}(n, F)$ is solvable, $\mathfrak{n}(n, F)$ is nilpotent and $\mathfrak{d}(n, F)$ is commutative.

We say a Lie algebra $L$ is **ad-nilpotent** if $\operatorname{ad} x$ is nilpotent for each $L$.

**Theorem 2.1** (Engel)**.** An ad-nilpotent algebra is nilpotent.

**Theorem 2.2** (M)**.** Let $L \subseteq \mathfrak{gl}(V)$ be a Lie subalgebra. If $a$ is nilpotent for each $a \in L$, then there exists a (simultaneous) 0-eigenvector of $L$.

*Proof.* Induction on $\dim L$ for all $V$. The base case $\dim L = 1$ is trivial.

If $\dim L > 1$, take any $K \subsetneqq L$ subalgebra. Consider the adjoint representation $\operatorname{ad} : K \to \mathfrak{gl}(L)$. Then $\operatorname{ad} x$ is nilpotent for all $x \in K$ (also on $\mathfrak{gl}(L/K)$). Indeed,

$$x^n = 0 \quad \Longrightarrow \quad (\operatorname{ad} x)^{2n-1} = (x_L - x_R)^{2n-1} = 0.$$

The induction hypothesis tells us that there exists a zero eigenvector $\overline{x} = x + K$ of "K" (under ad), i.e., $[y,x] = (\operatorname{ad} y)x \in K$ for each $y \in K$, or equivalently, $K$ is a proper ideal of $N_L(K)$.

Pick $K$ to be a maximal proper Lie subalgebra of $L$, we see that $N_L(K) = L$, i.e., $K \trianglelefteq L$. Note that $\dim(L/K) = 1$ (otherwise $K$ is not maximal). Say $L = K + Fz$.

Let $W = \{v \in V \mid Kv = 0\}$, which is nonzero by induction. Then $LW \subseteq W$:

$$y(xw) = x(yw) - [x,y]w = 0$$

for $x \in L$, $y \in K$ and $w \in W$. Hence, $z$ is a nilpotent element that acts on $W$. So there exists a nonzero element $v \in W$ such that $zv = 0$. Thus, $Lv = 0$. ■

*Proof of (2.1).* Let $L$ be an ad-nilpotent Lie algebra. Apply (2.2) to the embedding $\operatorname{ad} L \subseteq \mathfrak{gl}(L)$. There exists a nonzero element $x \in L$ such that $[L,x] = 0$. Hence $Z(L) \neq 0$.

The $\dim(L/Z(L)) < \dim L$ and is also adjoint nilpotent. By induction on dimension, it remains to show that $L/Z(L)$ is nilpotent implies $L$ is nilpotent, which follows from the observation:

$$L^{(n)} \subseteq Z(L) \quad\Longrightarrow\quad L^{(n+1)} = 0. \qquad ■$$

**Corollary 2.3.** Under the setting in (2.2), there exists a flag

$$V = V_0 \supset V_1 \supset \cdots \supset V_n = 0$$

such that $LV_i \subseteq V_{i+1}$, i.e., there exists a basis of $V$ such that $L \subseteq \mathfrak{n}(n,F)$.

*Proof.* Induction on dimension. Pick $v \in V$ such that $Lv = 0$ then consider the action of $L$ on $W = V/Fv$. ■

From now on, we assume that $F$ is algebraically closed and $\operatorname{char} F = 0$.

**Theorem 2.4** (Lie)**.** If $L \subseteq \mathfrak{gl}(V)$ is a solvable Lie subalgebra, then there exists a common eigenvector of $L$.

*Proof.* This is clearly true when $\dim L = 0$ or 1. Induction on $\dim L$.

Consider the quotient

$$L \longrightarrow L/[L,L].$$

Since $L/[L,L]$ is abelian, any subspace of it is an ideal. Take $\overline{K} \trianglelefteq L/[L,L]$ with codimension 1 (note that $L/[L,L]$ is nontrivial since $L$ is solvable) and consider its preimage $K \trianglelefteq L$. Since $K$ is also solvable, the subspace

$$W = \{w \in V \mid xw = \lambda(x)w, \forall x \in K\} \subseteq V$$

is nonzero. Let us fix this $\lambda$ as a function on $K$.

**Claim** (Dynkin)**.** The subspace $W$ is fixed by $L$.

*Proof of Claim.* Let $x \in L$ and $w \in W$. Then for each $y \in K$,

$$y(xw) = x(yw) - [x,y]w = \lambda(y)xw - \lambda([x,y])w.$$

So our goal $xw \in W$ is equivalent to $\lambda([x,y]) = 0$.

Consider

$$W_i = \langle w, xw, x^2w, \ldots, x^{i-1}w\rangle \subseteq V.$$

Let $r$ be the smallest integer such that $W_r = W_{r+1}$. Then $W_{r+j} = W_r$ for all positive integer $j$. We claim that $yx^jw \equiv \lambda(y)x^jw \pmod{W_j}$:

Induction on $j$. The base case $j = 0$ is true. For $j > 0$,

$$\begin{aligned} yx^jw &= xyx^{j-1}w - [x,y]x^{j-1}w \\ &= x(\lambda(y)x^{j-1}w + w') - \lambda([x,y])x^{j-1}w, \end{aligned}$$

where $w' \in W_{j-1}$.

Hence, $y \in K$ acts on $W_r$ has

$$\operatorname{tr}_{W_r} y = r\lambda(y).$$

This shows that for $[x,y] \in K$,

$$r\lambda([x,y]) = \operatorname{tr}_{W_r}[x,y] = 0,$$

which implies $\lambda([x,y]) = 0$ if $\operatorname{char} F = 0$. □

Say $L = K + Fz$, then we can find a nonzero element $v_0 \in W$ such that $zv_0 = \lambda v_0$, this $v_0$ is expected! ■

**Corollary 2.5.** Under the setting in (2.4), $L$ stabilizes some flag in $V$, i.e., there exists a basis of $V$ such that $L \subseteq \mathfrak{t}(n, F)$.

*Proof.* Using the theorem and induction on $\dim V$. ■

**Corollary 2.6.** If $L$ is a solvable Lie algebra, then there exists a chain of ideals

$$0 = L_0 \subset L_1 \subset \cdots \subset L_n = L$$

such that $\dim L_i = i$.

*Proof.* Consider

$$\phi = \mathrm{ad} : L \to \mathrm{gl}(L).$$

Since $\phi(L)$ is solvable, a flag is simply a chain of ideals. ■

**Corollary 2.7.** If $L$ is solvable, then $\mathrm{ad}_L\, x$ is nilpotent for $x \in [L, L]$. In particular, $[L, L]$ is nilpotent (by (2.1)).

*Proof.* Since $\mathrm{ad}\, L \subseteq \mathfrak{t}(n, F)$, we have $\mathrm{ad}[L, L] = [\mathrm{ad}\, L, \mathrm{ad}\, L] \subseteq \mathfrak{n}(n, F)$. ■

**Theorem 2.8** (Cartan's criterion)**.** Suppose $L \subseteq \mathfrak{gl}(V)$ is a Lie subalgebra such that

$$\mathrm{tr}(xy) = 0, \quad \forall x \in [L, L],\ y \in L.$$

Then $L$ is solvable.

*Proof.* It is enough to prove $\mathrm{ad}_{[L,L]}\, x$ is nilpotent for all $x \in [L, L]$. (This implies that $[L, L]$ is nilpotent by (2.1), which gives us the solvability of $L$. )

Let

$$M = \{z \in \mathfrak{gl}(V) \mid [z, L] \subseteq [L, L]\} \supseteq L.$$

Then for all $z \in M$, $x \in [L, L]$, we have $\mathrm{tr}(xz) = 0$: assume that $x = [u, v]$, then

$$\mathrm{tr}(xz) = \mathrm{tr}(uvz - vuz) = \mathrm{tr}(uvz - uzv) = \mathrm{tr}(u[v, z]) = 0$$

by the assumption.

Now, let $x = x_s + x_n$ be the Jordan decomposition, where $x_s$ is the semi-simple part and $x_n$ is the nilpotent part. Recall that $x_s$, $x_n$ are uniquely determined and there exists $p(T)$, $q[T] \in F[T]$ with $p(0) = q(0) = 0$ such that $x_s = p(x)$, $x_n = q(x)$.

Write

$$[x_s]_{\mathcal{B}} = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_m \end{pmatrix}$$

with $a_i \in F \supseteq \mathbb{Q}$. Let $E = \sum \mathbb{Q}\, a_i \subseteq F$. We want $E = 0$. In fact, we will show $\operatorname{Hom}(E, \mathbb{Q}) = 0$.

Let $f \in \operatorname{Hom}(E, \mathbb{Q})$ and consider

$$y = \begin{pmatrix} f(a_1) & & \\ & \ddots & \\ & & f(a_m) \end{pmatrix} \in \mathfrak{gl}(V).$$

It is easy to get

$$\operatorname{ad} x_s(e_{ij}) = (a_i - a_j) \cdot e_{ij} \quad \text{and} \quad \operatorname{ad} y(e_{ij}) = (f(a_i) - f(a_j)) \cdot e_{ij}. \tag{$\Upsilon$}$$

Find $r(T) \in F[T]$ such that

$$r(a_i - a_j) = f(a_i) - f(a_j), \quad \forall\, i, j.$$

We see from ($\Upsilon$) that

$$\operatorname{ad} y = r(\operatorname{ad} s) = (r \circ p)(\operatorname{ad} x).$$

Since $(\operatorname{ad} x)L \subseteq [L, L]$ and $(r \circ p)(0) = 0$, we must have $(\operatorname{ad} y)L \subseteq [L, L]$, i.e., $y \in M$. Then

$$0 = \operatorname{tr}(xy) = \sum a_i f(a_i) \quad \xRightarrow{f} \quad \sum f(a_i)^2 = 0 \quad \xRightarrow{f(a_i) \in \mathbb{Q}} \quad f \equiv 0. \qquad \blacksquare$$

# 3 Simple Lie algebra and Weyl's theorem, 9/12

**Definition 3.1.** A Lie algebra $L$ is **simple** if the only Lie ideals of $L$ are $0$ and $L$ also $L$ is not abelian.

A Lie algebra $L$ is **semi-simple** if $\operatorname{Rad} L$, the maximal solvable ideal in $L$, is $0$, i.e., $L$ has no (nonzero) abelian ideal. (If $I \trianglelefteq L$ is solvable with $I^{(n-1)} \neq 0$ and $I^{(n)} = 0$, then $I^{(n-1)}$ is abelian.)

**Definition 3.2.** The **Killing form** of $L$ is

$$\begin{aligned} \kappa = \kappa_L : \quad L \times L &\longrightarrow F \\ (x, y) &\longmapsto \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y). \end{aligned}$$

This is a symmetric bilinear form on $L$.

- $\kappa$ is "associative" (anti-symmetry), i.e.,

$$\begin{tikzcd}
\kappa([x,y],z) \arrow[r, equal] \arrow[d, phantom, "\shortparallel"] & \kappa(x,[y,z]) \arrow[d, phantom, "\shortparallel"] \\
-\kappa(\operatorname{ad} y(x),z) & \kappa(x,\operatorname{ad} y(z)).
\end{tikzcd}$$

The "null space" $\operatorname{rad}\kappa = \{x \in L \mid \kappa(x,y) = 0,\ \forall y \in L\}$ is an ideal of $L$. Indeed,

$$\kappa([x,z],y) = \kappa(x,[z,y]) = 0$$

for every $x \in \operatorname{rad}\kappa$ and $y$, $z \in L$.

**Fact.** If $I$ is an Lie ideal of $L$, then $\kappa_I$, the Killing form of $I$, is equal to $\kappa_L|_{I\times I}$.

This is easy by completing a basis from $I$ to $L$ via $L/I$.

**Theorem 3.3.** The followings are equivalent:

1. $L$ is semi-simple;
2. $\kappa_L$ is non-degenerate;
3. $L = \bigoplus I_i$ as Lie algebra, where each $I_i$ is a simple ideal of $L$.

*Proof.* 1. $\Rightarrow$ 2. : Let $S = \operatorname{rad}\kappa$. Then $\operatorname{tr}(\operatorname{ad} x \operatorname{ad} y) = 0$ for $x \in S$ and $y \in [S,S] = 0$. By Cartan's criterion, $\operatorname{ad}_L S$ is solvable. Since $\operatorname{ad} : L \to \mathfrak{gl}(L)$ is an embedding (otherwise the center $Z(L)$ is nontrivial, which is an abelian ideal), $S$ is solvable, which implies $S \subseteq \operatorname{Rad} L = 0$.

2. $\Rightarrow$ 1. : It is enough to show that every abelian ideal $I$ of $L$ lies in $S = \operatorname{rad}\kappa$. Let $x \in I$ and $y \in L$. Then

$$(\operatorname{ad} x \operatorname{ad} y)^2(L) \subseteq \operatorname{ad} x \operatorname{ad} y(I) \subseteq \operatorname{ad} x(I) \subseteq [I,I] = 0.$$

This implies $\operatorname{tr}(\operatorname{ad} x \operatorname{ad} y) = 0$. Since this is true for all $x$ and $y$, $I \subseteq S$.

1.2. $\Rightarrow$ 3. : Let $I$ be any Lie ideal of $L$. Then $I^\perp$, the orthogonal complement of $I$ with respect to $\kappa$, is an ideal of $L$ by the associativity of $\kappa$. Let $J = I \cap I^\perp$. Our goal is to show that $J = 0$ (this gives us the decomposition $L = I \oplus I^\perp$).

Since $\kappa_J = \kappa|_{J\times J}$, for each $x$, $y \in J$ we have $\kappa_J(x,y) = 0$. By Cartan's criterion, $J$ is solvable, and hence equal to 0.

Now, for an ideal $K \trianglelefteq I$, we have $K \trianglelefteq L$ since

$$[L, K] = [I \oplus I^{\perp}, K] = [I, K] \subseteq K.$$

(Note that $[I^{\perp}, K] \subseteq [I^{\perp}, I] \subseteq J = 0$.) This gives us the desired decomposition by induction on the dimension of $L$.

The uniqueness of decomposition: Let $I \trianglelefteq L$ be a simple ideal. Then $[I, L] \trianglelefteq I$ and is nonzero since $Z(L) = 0$. So

$$I = [I, L] = \bigoplus [I, I_i].$$

Then $I = [I, I_i] \subseteq I_i$ for some $i$, which shows that $I = I_i$ by the simpleness of $I_i$.

3. $\Rightarrow$ 1. : If $L$ is simple, then $\operatorname{Rad} L = 0$ or $L$. The latter case implies $[L, L] \subsetneqq L$, so $[L, L] = 0$, i.e., $L$ is abelian, which is a contradiction. Hence, $L$ is semi-simple.

Also, we know that direct sum of semi-simple Lie algebras is semi-simple. ■

**Corollary 3.4.** Let $L$ be a semi-simple Lie algebra. Then $L = [L, L]$.

Recall: $\operatorname{ad} L \trianglelefteq \operatorname{Der} L = \{\delta \in \mathfrak{gl}(L) \mid \delta[x, y] = [\delta x, y] + [x, \delta y]\}$. This comes from the Jacobi identity and the formula $[\delta, \operatorname{ad} x] = \operatorname{ad}(\delta x)$.

**Theorem 3.5.** Let $L$ be a semi-simple Lie algebra. Then $\operatorname{ad} L = \operatorname{Der} L$.

Recall that an **$L$-module**, or a **representation** of $L$, is a Lie homomorphism

$$\varphi : L \longrightarrow \mathfrak{gl}(V),$$

where $V$ is a (finite dimensional) vector space over $F$.

A representation $\varphi$ is **irreducible** if the only sub $L$-modules are 0 and $V$.

For a $L$-module $V$, we define the Lie action on $V^* = \operatorname{Hom}(V, F)$ by

$$(x \cdot f)(v) = -f(x \cdot v), \quad \forall f \in V^*.$$

For two $L$-modules $V$ and $W$, we define the Lie action on $V \otimes W$ by the Leibniz rule

$$x \cdot (v \otimes w) = (x \cdot v) \otimes w + v \otimes (x \cdot w),$$

and define the Lie action on $\operatorname{Hom}(V, W)$ by

$$(x \cdot f)(v) = x \cdot f(v) - f(x \cdot v).$$

**Theorem 3.6** (Weyl)**.** Let $L$ be a semi-simple Lie algebra and $\varphi : L \to \mathfrak{gl}(V)$ be a representation. Then $\varphi$ is completely irreducible, i.e., $\varphi$ is a direct sum of irreducible representations.

We represent Serre's proof here.

**Fact.** $\varphi(L) \subseteq \mathfrak{sl}(V)$ and hence $= 0$ on 1-dimensional $L$-module: since $L = [L, L]$ and $\mathfrak{sl}(V) = [\mathfrak{gl}(V), \mathfrak{gl}(V)]$.

May assume $\varphi$ is faithful.

**Definition 3.7** (Casimir element)**.** Let $\beta : L \times L \to F$ be a non-degenerate symmetric bilinear associative form. For a basis $x_1, \ldots, x_n$ of $L$, there exists a basis $y_1, \ldots, y_n$ of $L$ such that $\beta(x_i, y^j) = \delta_i^j$. For each $x \in L$, write

$$[x, x_i] = \sum a_i^j x_j, \quad [x, y^j] = \sum b_i^j y^i,$$

then the associativity of $\beta$ gives us $a_i^j = -b_i^j$. We define the **Casimir element** of $\beta$ to be

$$c_\varphi(\beta) = \sum \varphi(x_i)\varphi(y^i) \in \mathfrak{gl}(V).$$

We see that

$$[\varphi(x), c_\varphi(\beta)] = \sum \big(\varphi(a_i^j x_j)\varphi(y^i) + \varphi(x_i)\varphi(b_i^j y^i)\big) = 0,$$

i.e., it is $\varphi(L)$-linear.

For $\beta(x, y) = \operatorname{tr}(\varphi(x)\varphi(y))$, we get the Casimir element of $\varphi$: $c_\varphi = c_\varphi(\beta)$, with

$$\operatorname{tr} c_\varphi = \sum \beta(x_i, y^i) = \dim L \neq 0.$$

If $\varphi : L \to \mathfrak{gl}(V)$ is irreducible, then Schur's lemma implies that

$$c_\varphi = \frac{\dim L}{\dim V} \cdot \operatorname{id}_V.$$

To prove (3.6), let us consider the special case first: suppose that there exists a $L$-submodule $W \subset V$ of codimension 1.

$$0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0$$

The space $V/W \cong F$ has a trivial action by $L$. Now, we induction on $\dim W$. If $W$ is irreducible, then $c_\varphi|_W$ is a nonzero scalar, but $c_\varphi \equiv 0$ on $F$, i.e., the kernel of $c_\varphi : V \to W$ is 1-dimensional and its intersection with $W$ is 0. Thus, $c_\varphi$ gives us the desired splitting map.

If $W$ is not irreducible, then there exists a nonzero proper $L$-submodule $W'$ of $W$ and we get the exact sequence

$$0 \longrightarrow W/W' \longrightarrow V/W' \longrightarrow F \longrightarrow 0.$$

By induction, $V/W' = W/W' \oplus \overline{W}/W'$ for some $\overline{W}$ and we have the exact sequence

$$0 \longrightarrow W' \longrightarrow \overline{W} \longrightarrow F \longrightarrow 0.$$

Induction hypothesis tells us that $\overline{W} = W' \oplus X$ for some $X$. Hence, $V = W \oplus X$ since $W \cap X = 0$.

For the general case, let $W$ be a nonzero proper $L$-submodule of $V$.

$$0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0.$$

Define

$$\mathscr{V} = \{f \in \operatorname{Hom}(V, W) \mid f|_W = a\,\mathrm{id}_W, \text{ for some } a \text{ in } F\}$$

and $\mathscr{W}$ its codimension 1 subspace corresponds to $a = 0$. Then for $x \in L$, $f \in \mathscr{V}$, and $w \in W$ we have

$$(x \cdot f)(w) = x \cdot f(w) - f(x \cdot w) = x(aw) - a(xw) = 0.$$

So there is a exact sequence of $L$-modules:

$$0 \longrightarrow \mathscr{W} \longrightarrow \mathscr{V} \longrightarrow F \longrightarrow 0.$$

The special case tells us that $\mathscr{V} = \mathscr{W} \oplus \mathscr{U}$ for some $\mathscr{U}$. Let $\mathscr{U}$ be spanned by $f$ such that $f|_W = 1|_W$. Again, $L$ acts on $\mathscr{U}$ trivially, so

$$0 = (x \cdot f)(v) = x \cdot f(v) - f(x \cdot v),$$

i.e., $f$ is an $L$-homomorphism. Hence, $V = W \oplus \ker f$.

# 4 $\mathfrak{sl}(2, F)$-representation, 9/14

The Lie algebra $\mathfrak{sl}(2, F)$ is spanned by 3-elements:

$$x = \begin{pmatrix} & 1 \\ & \end{pmatrix}, \quad y = \begin{pmatrix} & \\ 1 & \end{pmatrix}, \quad h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

Note that $h$ is a semi-simple matrix. It is easy to see that

$$[h, x] = 2x, \quad [h, y] = 2y, \quad [x, y] = h.$$

Let $V$ be an $\mathfrak{sl}(2, F)$-module. Then $h$ acts on $V$ semi-simply, which gives us the decomposition $V = \bigoplus_\lambda V_\lambda$, called the weight decomposition, where

$$V_\lambda = \{v \in V \mid h \cdot v = \lambda v\}.$$

For $v \in V_\lambda$, we see that

$$h \cdot (y \cdot v) = y \cdot (h \cdot v) + -2y \cdot v = (\lambda - 2)(y \cdot v),$$

i.e., $y \cdot v \in V_{\lambda-2}$. Similarly, $x \cdot v \in V_{\lambda+2}$.

Consider $v \in V_\lambda$ such that $x \cdot v = 0$ and the subspace

$$V_v := \left\langle v, yv, \ldots, y^m v \neq 0, y^{m+1} v = 0 \right\rangle \subseteq V.$$

To show that $V_v$ is irreducible, it remains to show that $x$ acts on $V_v$.

**Lemma 4.1.** Let $v = v_0$, $v_i = y^i v_0 / i!$. Then for each $i \geq 1$,

$$x \cdot v_i = (\lambda - i + 1) v_{i-1}.$$

*Proof.* By induction (as before). ■

Taking $i = m + 1$, we see that

$$0 = x \cdot v_{m+1} = (\lambda - m) v_m.$$

Hence,

**Corollary 4.2.** The eigenvalue $\lambda$ of $v$ is equal to $m$.

Denote by $V(m)$ the space

$$V_m \oplus V_{m-2} \oplus \cdots \oplus V_{-m},$$

where each $V_j$ is a 1-dimensional subspace. Then each irreducible representation of $\mathfrak{sl}(2, F)$ is of the form $V(m)$, where $m$ is a non-negative integer.

Let $L$ be a semi-simple Lie algebra such that $\operatorname{ad} : L \to \mathfrak{gl}(L)$ is an embedding.

**Definition 4.3.** A subalgebra $T$ of $L$ is a **toral** subalgebra if all its elements are semi-simple.

**Fact I.** $T$ is abelian: for $x \in T$, take a $\lambda$-eigenvector $y \in T$ of $\operatorname{ad}_T x(y)$, i.e., $\operatorname{ad}_T x(y) = \lambda y$. Suppose that $\lambda \neq 0$. Note that $y$ is a 0-eigenvector of $\operatorname{ad}_T y$. Write $x$ as a linear combination of eigenvectors of $\operatorname{ad}_T y$. Then $\operatorname{ad}_T y(x) = -\lambda y$ gives us a contradiction ($\operatorname{ad}_T y(y) = 0$).

Fix such a $T$, call it $H$. Then $\operatorname{ad}_L H$ is simultaneously diagonalizable (since $H$ is abelian). Hence,

$$L = \bigoplus_{\alpha \in \Phi} L_\alpha \oplus L_0,$$

where $\alpha \in H^\vee := \operatorname{Hom}(H, F)$

$$L_\alpha = \{x \in L \mid \operatorname{ad} h(x) = \alpha(h)x, \ \forall h \in H\}.$$

This is called the Cartan decomposition of $L$, elements in $\Phi$ are called roots of $L$.

**Fact II.** For all $\alpha$, $\beta \in H^\vee$, $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$: for any $h \in H$, $x \in L_\alpha$ and $y \in L_\beta$,

$$\operatorname{ad} h[x, y] = [\operatorname{ad} h(x), y] + [x, \operatorname{ad} h(y)] = (\alpha + \beta)(h)[x, y].$$

Hence, if $x \in L_\alpha$ for some $\alpha \neq 0$, then $\operatorname{ad} x$ is nilpotent (since $\Phi$ is a finite set).

**Fact III.** $L_\alpha \perp L_\beta$ if $\alpha + \beta \neq 0$ with respect to the Killing form $\kappa$: for any $h \in H$, $x \in L_\alpha$ and $y \in L_\beta$,

$$0 = \kappa([h, x], y) + \kappa(x, [h, y]) = (\alpha + \beta)(h)\kappa(x, y).$$

Since $\alpha + \beta \neq 0$, we take $h \in H$ such that $(\alpha + \beta)(h) \neq 0$, then $\kappa(x, y) = 0$.

In particular,

$$L_0 \perp \bigoplus_{\alpha\in\Phi} L_\alpha.$$

If $z \in L_0 \cap \operatorname{rad} \kappa|_{L_0}$, then $z \in \operatorname{rad} \kappa = 0$. Hence, $\kappa|_{L_0}$ is nondegenerate.

**Proposition 4.4.** If $H$ is a maximal toral, then $L_0 = C_L(H) = H$.

*Proof.* Reading. ■

So $\kappa|_H$ is nondegenerate and induces the isomorphism

$$\begin{aligned} H^\vee &\longrightarrow H \\ \varphi &\longmapsto t_\varphi, \end{aligned}$$

where $t_\varphi \in H$ is the unique element such that $\kappa(t_\varphi, -) = \varphi$.

## 5 Root system, 9/19

**Proposition 5.1.** Let

$$L = \bigoplus_{\alpha\in\Phi} L_\alpha \oplus H$$

be a Cartan decomposition of a semi-simple Lie algebra $L$. Then

(a) $\Phi$ spans $H^\vee$;

(b) $\alpha \in \Phi$ implies $-\alpha \in \Phi$;

(c) for $x \in L_\alpha$ and $y \in L_{-\alpha}$, $[x, y] = \kappa(x, y)t_\alpha$;

(d) $\alpha \in \Phi$ implies $[L_\alpha, L_{-\alpha}] = F \cdot t_\alpha$ is 1-dimensional;

(e) $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$;

(f) for each non-zero $x_\alpha \in L_\alpha$, there exists $y_\alpha \in L_{-\alpha}$ such that there is an isomorphism

$$\begin{aligned} \langle x_\alpha, y_\alpha, h_\alpha = [x_\alpha, y_\alpha]\rangle &\xrightarrow{\sim} \mathfrak{sl}(2, F) = \langle x, y, h\rangle \\ x_\alpha, y_\alpha, h_\alpha &\longmapsto x, y, h. \end{aligned}$$

(g) $h_{-\alpha} = -h_\alpha$.

*Proof.* (a) If not, dually, there exists a non-zero $h \in H$ such that for each $\alpha \in \Phi$, $\alpha(h) = 0$. Then $[h, L_\alpha] = 0$, which implies $h \in Z(L)$, a contradiction.

(b) If $\alpha \notin \Phi$, then $\alpha + \beta \neq 0$ for each $\beta \in \Phi$. Then $L_\alpha \perp L$, contradicting the nondegeneracy of $\kappa$.

(c) For each $h \in H$,

$$\kappa(h, [x, y]) = \kappa([h, x], y) = \alpha(h)\kappa(x, y) = \kappa(t_\alpha \kappa(x, y), h).$$

(d) As in (b), if $x \in L_\alpha$ is a non-zero element, with $[x, L_{-\alpha}] = 0$, then $\kappa(x, L_{-\alpha}) = 0$. Hence, $\kappa(x, L) = 0$, a contradiction.

(e) If $\alpha(t_\alpha) = 0$, then $[t_\alpha, x] = 0 = [t_\alpha, y]$ for any $x \in L_\alpha$ and any $y \in L_{-\alpha}$. From (d), we can fix $x$, $y$ such that $[x, y] = t_\alpha$. Then $S := \langle x, y, t_\alpha \rangle$ is solvable and $S \cong \operatorname{ad}_L S \hookrightarrow \mathfrak{gl}(L)$. It follows that $\operatorname{ad}_L[S, S]$ is nilpotent. This tells us that $\operatorname{ad}_L t_\alpha$ is both semi-simple and nilpotent, which is 0. Hence, $t_\alpha \in Z(L) = 0$, a contradiction.

(f) Find $y_\alpha$ such that $\kappa(x_\alpha, y_\alpha) = \frac{2}{\kappa(t_\alpha, t_\alpha)} \neq 0$ and set $h_\alpha = \frac{2}{\kappa(t_\alpha, t_\alpha)} t_\alpha$. Then

$$\begin{aligned}
&[x_\alpha, y_\alpha] = \kappa(x_\alpha, y_\alpha) t_\alpha = h_\alpha, \\
&[h_\alpha, x_\alpha] = \frac{2}{\kappa(t_\alpha, t_\alpha)}[t_\alpha, x_\alpha] = \frac{2}{\alpha(t_\alpha)}\alpha(t_\alpha) x_\alpha = 2x_\alpha, \\
&[h_\alpha, y_\alpha] = \frac{2}{\kappa(t_\alpha, t_\alpha)}[t_\alpha, y_\alpha] = \frac{2}{\alpha(t_\alpha)}(-\alpha)(t_\alpha) y_\alpha = -2y_\alpha.
\end{aligned}$$

(g) By $t_{-\alpha} = -t_\alpha$ and $\kappa(t_\alpha, t_\alpha) = \kappa(-t_\alpha, -t_\alpha)$. ■

For $\alpha \in \Phi$, let $M = M_\alpha := H \oplus \bigoplus_{c \in F^\times} L_{c\alpha}$. Then $S_\alpha = \langle x_\alpha, y_\alpha, h_\alpha \rangle \cong \mathfrak{sl}(2, F)$ acts on $M$ by adjoint representation. $M$ has weights (for $h_\alpha$) $c\alpha(h_\alpha) \in \mathbb{Z}$. Since $\alpha(h_\alpha) = \frac{2}{\kappa(t_\alpha, t_\alpha)}\alpha(t_\alpha) = 2$, we see that $c \in \frac{1}{2}\mathbb{Z}$. Note that $M$ contains $S_\alpha$ as an irreducible $S_\alpha$-submodule. The weight 0 part of $M$ is

$$H = \ker \alpha \oplus F \cdot h_\alpha.$$

Hence, $V(0) \subset M$ occurs $\dim H - 1$ times, $V(2) = S_\alpha \subset M$, and there is no other even weights. This shows that $2\alpha \notin \Phi$ and $\frac{1}{2}\alpha \notin \Phi$ neither. Hence, 1 is not a weight of $\alpha \in M$ and $M = \ker \alpha \oplus S_\alpha = H + S_\alpha$, which implies that $\dim L_\alpha = 1$. Also, $S_\alpha = L_\alpha \oplus L_{-\alpha} \oplus [L_\alpha, L_{-\alpha}]$ is unique.

Next, consider the action of $S_\alpha$ on $K_\beta := \sum_{i \in \mathbb{Z}} L_{\beta + i\alpha}$, where $\beta \neq \pm\alpha$. Each 1-dimensional space $L_{\beta + i\alpha}$ has weight $\beta(h_\alpha) + 2i$. Hence, $K_\beta$ is irreducible. Let $q$ and $r$ be the largest integers such that $\beta + q\alpha$ and $\beta - r\alpha$ are roots. Then

$$\beta(h_\alpha) + 2q = -(\beta(h_\alpha) - 2r) \quad \Longrightarrow \quad 2 \cdot \frac{\kappa(t_\beta, t_\alpha)}{\kappa(t_\alpha, t_\alpha)} = \beta(h_\alpha) = r - q \in \mathbb{Z}.$$

On $H^\vee$, put an inner product $(\lambda, \mu) := \kappa(t_\lambda, t_\mu)$ for $\lambda$, $\mu \in H^\vee$. For any basis $\alpha_1, \ldots,$ $\alpha_\ell \in \Phi$ of $H^\vee$, we have $\Phi \subset E_{\mathbb{Q}} := \bigoplus \mathbb{Q}\,\alpha_i$ (by the integrality of $\beta(h_\alpha)$) and

$$(\lambda, \mu) = \kappa(t_\lambda, t_\mu) = \sum_{\alpha \in \Phi} \alpha(t_\lambda)\alpha(t_\mu)$$

is positive definite (on $E_{\mathbb{Q}}$).

**Theorem 5.2** (Root system)**.** For the root system $\Phi$,

(R1) $\Phi$ spans $E$, and $|\Phi| < \infty$;

(R2) if $\alpha \in \Phi$, then $c\alpha \in \Phi$ if and only if $c = \pm 1$;

(R3) for $\alpha$, $\beta \in \Phi$, the reflection $\beta - \frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ of $\beta$ with respect to $\alpha^\perp$ lies in $\Phi$;

(R4) for $\alpha$, $\beta \in \Phi$, $\langle \beta, \alpha \rangle := 2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

Now, we study the abstract root system $\Phi \subset (E, (-,-))$, i.e., $\Phi$ satisfies (R1) ̃(R4).

**Lemma 5.3.** For $\sigma \in \mathrm{GL}(E)$ with $\sigma(\Phi) = \Phi$, $\sigma(\alpha) = -\alpha$ for some $\alpha \in \Phi$, and $\sigma = \mathrm{id}$ on some hyperplane, we have $\sigma = \sigma_\alpha$, the reflection $\beta \mapsto \beta - \langle \beta, \alpha \rangle \alpha$.

*Proof.* Define $\tau = \sigma \circ \sigma_\alpha$. Then $\tau(\Phi) = \Phi$, $\tau(\alpha) = \alpha$, and $\tau = \mathrm{id}$ on $E/\mathbb{Q}\,\alpha$. So all eigenvalues of $\tau$ is 1. The minimal polynomial $P$ of $\tau$ satisfies $P \mid (T-1)^{\ell = \dim E}$. Choose $K \gg 1$ such that $\tau^K|_\Phi = \mathrm{id}$, then $P \mid T^K - 1$. Hence, $P = T - 1$. ■

**Definition 5.4.** Let $\mathscr{W} \subset \mathrm{GL}(E)$ be the subgroup generated by $\sigma_\alpha$, $\alpha \in \Phi$. $\mathscr{W}$ is called the **Weyl group** of $\Phi$, and is a subgroup of $S_{|\Phi|}$.

**Lemma 5.5.** Let $\sigma \in \mathrm{GL}(E)$ with $\sigma(\Phi) = \Phi$. Then $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$ for each $\alpha \in \Phi$ and $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle$.

*Proof.* $\sigma\sigma_\alpha\sigma^{-1}(\Phi) = \Phi$ fixes $\sigma(P_\alpha)$ ($P_\alpha$ is the hyperplane fixed by $\sigma_\alpha$) pointwisely and maps $\sigma(\alpha)$ to $-\sigma(\alpha)$. Applying the previous lemma, we see that $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$.

Now,

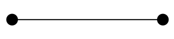
$$\sigma(\beta) - \langle \sigma(\beta), \sigma(\alpha) \rangle \sigma(\alpha) = \sigma_{\sigma(\alpha)}(\sigma(\beta)) = \sigma(\sigma_\alpha(\beta)) = \sigma(\beta - \langle \beta, \alpha \rangle \alpha). \qquad \blacksquare$$

**Corollary 5.6.** If $(\Phi, E) \cong (\Phi', E')$, then $\mathscr{W} \cong \mathscr{W}'$. In particular, $\mathscr{W} \subseteq \operatorname{Aut} \Phi$.

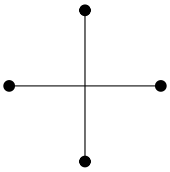
**Definition 5.7.** The dual root system $\Phi^\vee = \left\{ \alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)} \mid \alpha \in \Phi \right\}$ is a root system with the same $\mathscr{W}$.

**Example 5.8.** Some root systems:

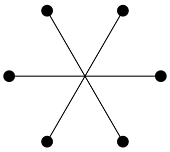
$A_1$:



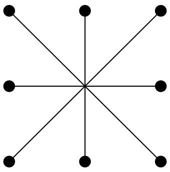
$A_1 \times A_1$:



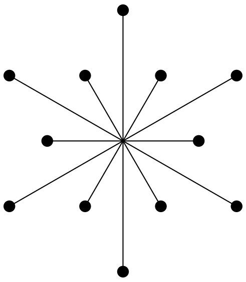
$A_2$:



$B_2$:



$G_2$:



Since

$$\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha))} = \frac{2|\beta| \cos\theta}{|\alpha|},$$

where $\theta$ is the angle between $\alpha$ and $\beta$, $\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 4\cos^2\theta \in \mathbb{Z}$. We get the table ($\alpha \neq \pm\beta$ and WLOG let $|\beta| \geq |\alpha|$):

| $\langle \alpha, \beta \rangle$ | $\langle \beta, \alpha \rangle$ | $\theta$ | $\lvert\beta\rvert^2/\lvert\alpha\rvert^2$ |
|---|---|---|---|
| 0 | 0 | 90° | ? |
| 1 | 1 | 60° | 1 |
| -1 | -1 | 120° | 1 |
| 1 | 2 | 45° | 2 |
| -1 | -2 | 135° | 2 |
| 1 | 3 | 30° | 3 |
| -1 | -3 | 150° | 3 |

**Lemma 5.9.** For $\alpha$, $\beta \in \Phi$, we have

$$(\alpha, \beta) > 0 \implies \alpha - \beta \in \Phi.$$

Similarly,

$$(\alpha, \beta) < 0 \implies \alpha + \beta \in \Phi.$$

*Proof.* Suppose that $(\alpha, \beta) > 0$. From the table, $\langle \alpha, \beta \rangle = 1$ or $\langle \alpha, \beta \rangle = 1$. The former case together with (R3) gives us $\sigma_\beta(\alpha) = \alpha - \langle \alpha, \beta \rangle \beta = \alpha - \beta \in \Phi$. Similarly, the latter case gives us $\beta - \alpha \in \Phi$, which implies $\alpha - \beta \in \Phi$ by (R2). ■

**Corollary 5.10.** For $\beta \neq \pm\alpha$, all roots $\beta + i\alpha$, $i \in \mathbb{Z}$ is unbroken of length $\leq 4$.

*Proof.* If $\beta + p\alpha$, $\beta + s\alpha \in \Phi$ with $p < s$ and $\beta + (p+1)\alpha$, $\beta + (s-1)\alpha \notin \Phi$. The lemma implies $(\alpha, \beta + p\alpha) \geq 0$ and $(\alpha, \beta + s\alpha) \leq 0$. Then

$$(s-p)(\alpha, \alpha) = (\alpha, \beta + s\alpha) - (\alpha, \beta + p\alpha) \leq 0,$$

a contradiction.

The length it at most 4: if $q$ and $r$ are the largest integers such that $\beta + q\alpha$, $\beta - r\alpha \in \Phi$, then $q + r = \langle \beta + p\alpha, \alpha \rangle < 4$. ■

# 6 Weyl group, 9/21

**Definition 6.1.** We call $\Delta \subseteq \Phi$ a **base** if

(B1) $\Delta$ is a basis of $E$;

(B2) for $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha \in \Phi$, either all $k_\alpha \in \mathbb{Z}_{\geq 0}$ or all $k_\alpha \in \mathbb{Z}_{\leq 0}$.

**Fact.** For distinct $\alpha$, $\beta \in \Delta$, we have $(\alpha, \beta) \leq 0$, and $\alpha - \beta \notin \Phi$: if $(\alpha, \beta) > 0$, then $\alpha - \beta \in \Phi$ by (5.9), which contradicts (B2).

**Theorem 6.2.** Every root system has a base. In fact,

(1) let $\gamma \in E \setminus \bigcup_{\alpha \in \Phi} P_\alpha$, where $P_\alpha$ is the hyperplane fixed by $\sigma_\alpha$. Then

$$\Delta(\gamma) := \{\,\text{indecomposable roots in } \Phi^+(\gamma)\,\}$$

is a base, where $\Phi^+(\gamma) = \{\alpha \in \Phi \mid (\alpha, \gamma) > 0\}$ (a root $\alpha$ is said to be **indecomposable** if $\alpha$ cannot be written as $\alpha_1 + \alpha_2$ for some $\alpha_1$, $\alpha_2 \in \Phi^+(\gamma)$). Elements in $\Delta(\gamma)$ is called a **simple root** relative to $\Delta(\gamma)$.

(2) Any base come from such a way.

*Proof.* Since $\Delta(\gamma)$ spans $\Phi^+(\gamma)$ in $\mathbb{Z}_{\geq 0}$, hence spans $E$. If $\alpha$, $\beta \in \Delta$ are distinct, then $(\alpha, \beta) \leq 0$, otherwise

$$\begin{aligned} \alpha - \beta \in \Phi^+(\gamma) \quad &\Longrightarrow \quad \alpha = \beta + (\alpha - \beta), \\ \beta - \alpha \in \Phi^+(\gamma) \quad &\Longrightarrow \quad \beta = \alpha + (\beta - \alpha). \end{aligned}$$

Hence, $\Delta(\gamma)$ is a linearly independent set: suppose that $\varepsilon = \sum s_\alpha \alpha = \sum t_\beta \beta$ with $s_\alpha$, $t_\beta > 0$. Then

$$0 \leq (\varepsilon, \varepsilon) = \sum_{\alpha, \beta} s_\alpha t_\beta (\alpha, \beta) \leq 0$$

tells us that $\varepsilon = 0$.

(2) is left in Exercise 7. ■

**Definition 6.3.** The set $E \setminus \bigcup_{\alpha \in \Phi} P_\alpha$ is a union of (connected) open cones, each open cone is called a **Weyl chamber**.

Every element in a Weyl chamber defines same base. Conversely, every base determines a Weyl chamber.

**Lemma 6.4.**

(a) For $\alpha \in \Phi^+ \setminus \Delta$, there exists $\beta \in \Delta$ such that $\alpha - \beta \in \Phi^+$. Hence, we can write $\alpha = \sum_{i=1}^{k} \alpha_i$, where $\alpha_i \in \Delta$, such that $\sum_{i=1}^{j} \in \Phi^+$ for all $j \leq k$.

(b) For $\alpha \in \Delta$, $\sigma_\alpha$ permutes $\Phi^+ \setminus \{\alpha\}$. In particular, $\sigma(\delta) = \delta - \alpha$ for $\delta = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta$.

(c) (Cancellation lemma) Let $\sigma_i = \sigma_{\alpha_i}$. If

$$\sigma_1 \cdots \sigma_{t-1} \sigma_t(\alpha_t) \succ 0,$$

then there exists $s < t$ such that $\sigma_1 \cdots \sigma_t = \sigma_1 \cdots \sigma_{s-1} \sigma_{s+1} \cdots \sigma_{t-1}$. Here $\alpha \succ \beta$ if $\alpha - \beta \in \Phi^+$.

*Proof.* (a) Suppose that $(\alpha, \beta) \leq 0$ for each $\beta \in \Delta$, then $\Delta \cup \{\alpha\}$ is a linearly independent set (cf. Proof of (6.2)), a contradiction. So there exists $\beta \in \Delta$ such that $(\alpha, \beta) > 0$, and hence $\alpha - \beta \in \Phi^+$ (Note that $\alpha - \beta \in \Phi^- \implies \beta = \alpha + (\beta - \alpha)$).

(b) For $\beta \in \Phi^+ \setminus \{\alpha\}$, $\beta = \sum_{\gamma \in \Delta} k_\gamma \gamma$ with $k_\gamma \geq 0$ for all $\gamma$ and $k_{\gamma_0} > 0$ for some $\gamma_0 \neq \alpha$. The element

$$\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$$

has the same $k_{\gamma_0}$, so $\sigma_\alpha(\beta) \in \Phi^+ \setminus \{\alpha\}$.

(c) Let

$$\beta_i = \sigma_{i+1} \cdots \sigma_{t-1}(\alpha_t), \quad i = 0, \ldots, t-2.$$

Then $\beta_{t-1} = \alpha_t \succ 0$, $\beta_0 \prec 0$. So there exists smallest $s$ such that $\beta_s \succ 0$. Since $\beta_{s-1} \prec 0$, we must have $\beta_s = \alpha_s$. Therefore

$$\sigma_s = (\sigma_{s+1} \cdots \sigma_{t-1}) \sigma_t (\sigma_{t-1} \cdots \sigma_{s+1}),$$

i.e., $\sigma_1 \cdots \sigma_{s-1} \sigma_s \sigma_{s+1} \cdots \sigma_t = \sigma_1 \cdots \sigma_{s-1} \sigma_{s+1} \cdots \sigma_{t-1}$. ■

**Theorem 6.5.** The group $\mathscr{W}$ acts on $\{\text{ base of } \Phi\}$ simply and transitively, and $\mathscr{W}$ is generated by $\sigma_\alpha$, $\alpha \in \Delta$, for any base $\Delta$.

*Proof.* Let $\mathscr{W}' \subseteq \mathscr{W}$ generated by $\sigma_\alpha$, $\alpha \in \Delta$. If $\gamma$ is regular, choose $\sigma \in \mathscr{W}'$ with $(\sigma(\gamma), \delta)$ largest. Then

$$(\sigma(\gamma), \delta) \geq (\sigma_\alpha \cdot \sigma(\gamma), \delta) = (\sigma(\gamma), \sigma_\alpha(\delta)) = (\sigma(\gamma), \delta) - (\sigma(\gamma), \alpha),$$

i.e., $(\sigma(\gamma), \alpha) \geq 0$. Also, $(\sigma(\gamma), \alpha) \neq 0$, otherwise $\gamma \perp \sigma^{-1}\alpha$, a contradiction. Hence, $\sigma(\gamma)$ lies in the Weyl chamber $\mathscr{C}(\Delta)$ corresponds to $\Delta$ and $\sigma \colon \mathscr{C}(\gamma) \to \mathscr{C}(\Delta)$.

Any $\alpha \in \Phi$ lies in some base: take any $\gamma \in P_\alpha \setminus \bigcup_{\beta \neq \pm\alpha} P_\beta$. Let $\gamma'$ "close to" $\gamma$ such that $(\gamma', \alpha) = \varepsilon > 0$, $|(\gamma', \beta)| > \varepsilon$. Then $\alpha \in \Delta(\gamma')$.

In particular, there exists $\sigma \in \mathscr{W}'$ such that $\beta = \sigma(\alpha) \in \Delta$. Then $\sigma_\beta = \sigma_{\sigma(\alpha)} = \sigma\sigma_\alpha\sigma^{-1}$ tells us that $\sigma_\alpha = \sigma^{-1}\sigma_\beta\sigma \in \mathscr{W}'$. Hence, $\mathscr{W}' = \mathscr{W}$.

It remains to show that the action $\mathscr{W}$ on $\{ \text{base of } \Phi \}$ is simple. If $\sigma \neq \mathrm{id}$ with $\sigma(\Delta) = \Delta$, write $\sigma = \sigma_1 \cdots \sigma_t$ (minimal length). Then $\sigma(\alpha_t) < 0$ by (6.4, c), a contradiction. ■

**Definition 6.6.** For $\sigma \in \mathscr{W}$, let $\ell(\sigma)$ be the minimal length of the expression $\sigma = \sigma_1 \cdots \sigma_t$ (relative to a base $\Delta$). For a root $\alpha = \sum_{\beta \in \Delta} k_\beta \beta \in \Phi$, we define the height of $\alpha$ to be $\mathrm{ht}(\alpha) = \sum_{\beta \in \Delta} k_\beta \in \mathbb{Z}$.

A root system $\Phi$ is called **irreducible** if $\Phi = \Phi_1 \sqcup \Phi_2$ with $\Phi_1 \perp \Phi_2$ (this is equivalent to $\Delta = \Delta_1 \sqcup \Delta_2$ for some base $\Delta$). Otherwise, $\Phi$ is called reducible. For example, $A_1 \times A_1$ is reducible.

**Lemma 6.7.** Let $\Phi$ be an irreducible root system. Then

(a) there exists a unique element $\beta \in \Phi^+$ maximum with respect to $\succ$;

(b) the action $\mathscr{W}$ on $E$ is irreducible;

(c) there are at most 2 lengths "$|\alpha|$" $\forall \alpha \in \Phi$ (by key table), and $|\alpha| = |\beta| \implies \beta = w(\alpha)$ for some $w \in \mathscr{W}$.

(d) The unique maximal element $\beta$ is the longer one.

# 7 Classification of root systems, 9/26

Let $\Phi \subseteq E$ be a root system, $\mathscr{W}$ be its Weyl group, $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ be a base.

**Proposition 7.1.** The Cartan matrix $(\langle \alpha_i, \alpha_j \rangle)_{i,j=1}^{\ell} \in \mathrm{M}_\ell(\mathbb{Z})$ determines $\Phi$ up to an isomorphism.

*Proof.* For a vector space isomorphism $\phi : E \to E'$, where $\phi(\alpha_i) = \alpha_i'$, the diagram

$$\begin{tikzcd}
E \arrow[r, "\phi"] \arrow[d, "\sigma_\alpha"] & E' \arrow[d, "\sigma_{\alpha'}"] \\
E \arrow[r, "\phi"] & E'
\end{tikzcd}$$

commutes when $(\langle \alpha_i, \alpha_j \rangle)_{i,j=1}^{\ell} = (\langle \phi(\alpha_i), \phi(\alpha_j) \rangle)_{i,j=1}^{\ell}$. Indeed,

$$\sigma_{\phi(\alpha)}(\phi(\beta)) = \phi(\beta) - \langle \phi(\beta), \phi(\alpha) \rangle \phi(\alpha) = \phi(\beta - \langle \beta, \alpha \rangle \alpha).$$

Hence, $\phi \mathscr{W} \phi^{-1} = \mathscr{W}'$ by (6.5).

For each $\beta \in \Phi$, $\beta = \sigma(\alpha)$ for some $\sigma \in \mathscr{W}$, so $\phi(\beta) = (\phi \sigma \phi^{-1}) \phi(\alpha) \in \mathscr{W}' \Delta' = \Phi'$. ■

**Definition 7.2.** The **Coxeter graph** $\Gamma = \Gamma_\Phi$ of $\Phi$ is a weighted graph $(V, E)$ with $\ell$ vertices $V = \{\alpha_1, \ldots, \alpha_\ell\}$ and edges

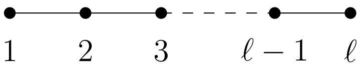
$$E = \left\{ (\overline{\alpha_i \alpha_j}, \langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle \neq 0) \right\}.$$

The **Dynkin diagram** of $\Phi$ is the directed weighted graph $\Gamma$ with $\overline{\alpha_i \alpha_j}$ replaced by $\overrightarrow{\alpha_i \alpha_j}$ if $|\alpha_i| > |\alpha_j|$.

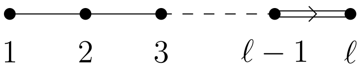
**Fact.** There is a one-to-one correspondence between the irreducible components of $\Phi$ and the connected components of $\Gamma_\Phi$.

**Theorem 7.3.** If $\Phi$ is irreducible, then the Dynkin diagram $\Gamma_\Phi$ is isomorphic to one of followings:

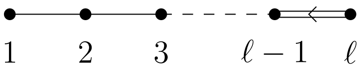
$A_\ell$:



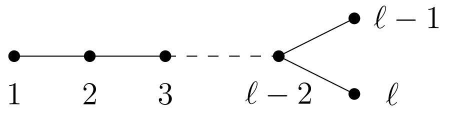
$B_\ell$:



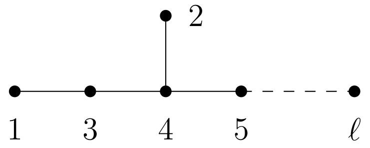
$C_\ell$:



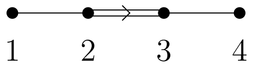
$D_\ell$:



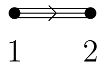
$E_\ell$ ($\ell = 6$, 7, 8):



$F_4$:



$G_2$:



*Proof.* Let $\hat{\alpha}_i = \alpha_i/|\alpha_i|$. Then

$$2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \cdot 2\frac{(\alpha_j, \alpha_i)}{(\alpha_j, \alpha_j)} = 4(\hat{\alpha}_i, \hat{\alpha}_j)^2.$$

Hence, we call a set of unit vectors $A = \{\varepsilon_1, \ldots, \varepsilon_n\}$ admissible if $4(\varepsilon_i, \varepsilon_j)^2 \in \{0, 1, 2, 3\}$ for all $i \neq j$.

(1) The admissible property is preserved under removing a vertex.

(2) The number of edges is at most $\#A - 1$. Let $n = \#A$ and $\varepsilon = \sum \varepsilon_i$. Then

$$0 \leq (\varepsilon, \varepsilon) = n + 2\sum_{i<j} (\varepsilon_i, \varepsilon_j).$$

Since for an edge $(i, j)$, we have $2(\varepsilon_i, \varepsilon_j) \leq -1$. The number of the edges is at most $n - 1$.

(3) There are no cycles in $\Gamma$. Take any cycle $\Gamma' \subseteq \Gamma$. Then the correspond $A' \subseteq A$ is admissible, but it has $\#A'$ edges, a contradiction.

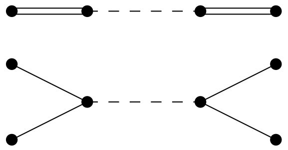
(4) At any $\varepsilon \in A$, the number of edges that connects with $\varepsilon$ is at most 3 (counted with multiplicity). Suppose $\eta_1$, …, $\eta_k \in A$ are connected to $\varepsilon$, then $(\eta_i, \eta_j) = \delta_{ij}$ by (3). Find a unit vector $\eta_0 \in \langle \varepsilon, \eta_1, \ldots, \eta_k \rangle$ that is perpendicular to $\langle \eta_1, \ldots, \eta_k \rangle$. Then

$$\varepsilon = \sum_{i=0}^{k} (\varepsilon, \eta_i)\eta_i \quad \Longrightarrow \quad 1 = (\varepsilon, \varepsilon) = \sum_{i=0}^{k} 4(\varepsilon, \eta_i)^2 < 4(\varepsilon, \varepsilon) - 4(\varepsilon, \eta_0)^2 < 4.$$

(5) The only case with a weight 3 edge is $G_2$ itself.

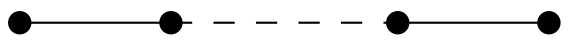
(6) Shrinking a simple chain to a point is OK.

(7) Hence, there is no subgraphs of the form

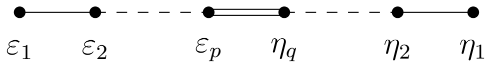


(8) $\Gamma$ belongs to 4 types:

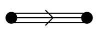
(i)



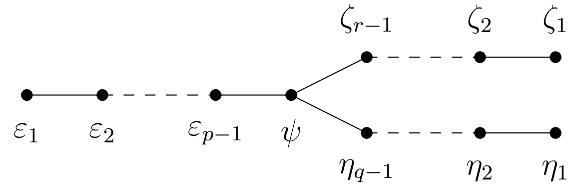
(ii)



(iii)



(iv)



(i) and (iii) corresponds to $A_{n-1}$ and $G_2$, respectively.

(9) For (ii), consider $\varepsilon = \sum i\varepsilon_i$, $\eta = \sum j\eta_j$. Since $2(\varepsilon_i, \varepsilon_{i+1}) = -1 = 2(\eta_j, \eta_{j+1})$, we get

$$(\varepsilon, \varepsilon) = \sum_{i=1}^{p} i^2 - \sum_{i=1}^{p} i(i+1) = p^2 - \frac{p(p-1)}{2} = \frac{p(p+1)}{2}.$$

Similarly, $(\eta, \eta) = \frac{q(q+1)}{2}$. By definition and Cauchy-Schwarz inequality,

$$(\varepsilon, \eta)^2 = \frac{p^2q^2}{2} < \frac{p(p+1)}{2} \cdot \frac{q(q+1)}{2} \qquad \Longrightarrow \qquad (p-1)(q-1) < 2$$

If one of $p$ or $q$ is 1, then $\Gamma$ is isomorphic to $B_\ell$ or $C_\ell$. Otherwise, $p = q = 2$, in this case we get $F_4$.

(10) For (iv), consider $\varepsilon = \sum i\varepsilon_i$, $\eta = \sum j\eta_j$, $\zeta = \sum k\zeta_k$. As in (4), let $\theta_1$, $\theta_2$, $\theta_3$ be the angles between $\psi$ and $\varepsilon$, $\eta$, $\zeta$, respectively. Then $\sum \cos^2\theta_\ell < 1$. As in (9),

$$\cos^2\theta_1 = \frac{(\varepsilon,\psi)^2}{(\varepsilon,\varepsilon)(\psi,\psi)} = (p-1)^2 \cdot \frac{1}{4} \cdot \frac{2}{p(p-1)} = \frac{1}{2}\left(1 - \frac{1}{p}\right).$$

Hence,

$$\frac{1}{2}\left(3 - \frac{1}{p} - \frac{1}{q} - \frac{1}{q}\right) < 1,$$

i.e., $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$. Say $r = 2$, then $q = 2$ gives us $D_n$, while $q = 3$ gives us $E_{p+3}$ ($p = 3$, 4, 5). ■

**Remark 7.4.** The automorphism group $\operatorname{Aut}\Phi$ is isomorphic to $\boldsymbol{\gamma} \rtimes \mathscr{W}$, where $\boldsymbol{\gamma} = \{\sigma \in \operatorname{Aut}\Phi \mid \sigma(\Delta) = \Delta\}$, which can be related to $\operatorname{Aut}\Gamma_\Phi$.

**Definition 7.5.** Given a root system $\Phi \subset E$, we define the **weight lattice** to be

$$\Lambda = \{\lambda \in E \mid \langle \lambda, \alpha \rangle \in \mathbb{Z},\ \forall \alpha \in \Phi\} \supseteq \Phi.$$

It is clear that we only need to check the condition $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ for $\alpha \in \Delta$. Given $\Delta = (\alpha_1, \ldots, \alpha_\ell)$ (an ordered base), we get $\lambda_i$ such that $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$, called the fundamental weights. Then $\Lambda$ is a lattice generated by $\lambda_1, \ldots, \lambda_\ell$. Hence,

$$\alpha_i = \sum_k \langle \alpha_i, \alpha_k \rangle \lambda_k.$$

Let $\Lambda_r$ be the lattice generated by $\Phi$. Then $\Lambda_r \subseteq \Lambda$ and $|\Lambda/\Lambda_r| = \det C$, where $C = (\langle \alpha_i, \alpha_j \rangle)$ is the Cartan matrix.

**Examples.** For $A_2$,

$$C_{A_2} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \implies \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

For $G_2$,

$$C_{G_2} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \implies \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

Note that $\mathscr{W}(\Lambda) = \Lambda$: $\sigma_i\lambda_j = \lambda_j - \delta_{i,j}\alpha_i \in \Lambda$. So any weight $\lambda$ can be conjugate to a dominant weight, i.e., it lies in the dominant set

$$\Lambda^+ = \{\lambda \in \Lambda \mid (\lambda, \alpha) \geq 0\} = \overline{\mathscr{C}(\Delta)} \cap \Lambda.$$

The strictly dominant set is defined to be

$$\{\lambda \in \Lambda \mid (\lambda, \alpha) > 0\} = \mathscr{C}(\Delta) \cap \Lambda.$$

Although $\lambda \succ \mu$ with $\mu \in \Lambda^+$ does not imply $\lambda \in \Lambda^+$, but $\lambda \in \Lambda^+$ implies that there are only finitely many $\mu \in \Lambda^+$ with $\lambda \succ \mu$.

**Example.** The vector $\delta = \frac{1}{2} \sum_{\alpha \succ 0} \alpha = \sum_{j=1}^{\ell} \lambda_j$ is a strictly positive weight.

**Lemma 7.6.** Let $\mu \in \Lambda^+$ and $\nu \in \mathscr{W}(\mu)$. Then $|\nu + \delta| \le |\mu + \delta|$, and the equality holds if and only if $\nu = \mu$.

# 8 Final step I, 10/3

Recall: For a semi-simple Lie algebra $L$, we choose a maximal toral subalgebra $H$, which induces a root space decomposition $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$. Note that $H$ is self-normalizing (in $L$), i.e., $N_L(H) = H$.

In fact, any 2 choices of maximal torals $H_1$, $H_2$ are conjugate by some automorphism. This gives us the classification of semi-simple Lie algebra as $A \sim G$.

Let $V$ be a finite dimensional vector space over $F$ and $A : V \to V$ be a linear map. Consider its characteristic polynomial $f_A(T) = \prod (T - \lambda_i)^{m_i} = \prod p_i(T)$. We get the decomposition $V = \bigoplus V_i$, where $V_i = \ker p_i(A)$.

Take $V = L$, the action $\operatorname{ad} x : L \to L$ gives us the decomposition $L = \bigoplus_{a \in F} L_a(\operatorname{ad} x)$, where $L_a(\operatorname{ad} x) = \bigcup_n \ker(\operatorname{ad} x - a)^n$.

**Fact.** $[L_a(\operatorname{ad} x), L_b(\operatorname{ad} x)] \subseteq L_{a+b}(\operatorname{ad} x)$:

$$\begin{aligned}
&(\operatorname{ad} x - a - b)[y, z] = [(\operatorname{ad} x - a)y, z] + [y, (\operatorname{ad} x - b), z] \\
\implies\ &(\operatorname{ad} x - a - b)^m [y, z] = \sum_{i=0}^{m} \binom{m}{i} \left[(\operatorname{ad} x - a)^i y, (\operatorname{ad} x - b)^{m-i} z\right] = 0
\end{aligned}$$

for $y \in L_a$ and $z \in L_b$ with $m \gg 0$.

This tells us that $L_0(\operatorname{ad} x)$ is a Lie subalgebra, called an **Engel subalgebra**, and $L_{a \neq 0}(\operatorname{ad} x)$ is ad-nilpotent.

**Lemma 8.1.** Let $K$ be a Lie subalgebra of $L$ that contains $L_0(\operatorname{ad} x)$. Then $K$ is self-normalizing (in $L$), i.e., $N_L(K) = K$.

*Proof.* Consider the action $\operatorname{ad} x : N_L(K)/K \to N_L(K)/K$. All the eigenvalues of the action is nonzero. Note that $x \in K$, so $[N_L(K), x] \subseteq K$, which means that the action is 0. ■

**Lemma 8.2.** Let $K$ be a Lie subalgebra of $L$, and let $L_0(\operatorname{ad} z)$ be minimal among all such $z \in K$. If moreover, it contains $K$, then it is totally minimal.

*Proof.* Fix an arbitrary $x \in K$ and consider the pencil $\{\operatorname{ad}(z+cx) \mid c \in F\}$. Since $x \in K$, these elements all stabilize $K_0 = L_0(\operatorname{ad} z)$, hence stabilize $L/K_0$ as well.

The characteristic polynomial $f_c(T) = f(T,c)g(T,c)$, where $f(T,c)$ is the characteristic polynomial of $\operatorname{ad}(z+cx)|_{K_0}$ and $g(T,c)$ is the characteristic polynomial of $\operatorname{ad}(z+cx)|_{L/K_0}$. Write

$$\begin{aligned} f(T,c) &= T^r + f_1(c)T^{r-1} + \cdots + f_r(c) \\ g(T,c) &= T^{n-r} + g_1(c)T^{n-r-1} + \cdots + g_r(c). \end{aligned}$$

We know that each $f_i$, $g_i$ are polynomials in $c$ of degree at most $i$.

For $c = 0$, the 0-eigenspace of $\operatorname{ad} z$ lies in $K_0$, so $g_{n-r}(0) \neq 0$. So we can find $c_1, \ldots, c_{r+1} \in F$ such that $g_{n-r}(c_i) \neq 0$ for all $i$. Then 0 is not an eigenvalue of $\operatorname{ad}(z + c_i x)$ on $L/K_0$, and hence $L_0(\operatorname{ad}(z + c_i x)) \subseteq K_0$.

Since $K_0$ is minimal, $K_0 = L_0(\operatorname{ad} z) = L_0(\operatorname{ad}(z + c_i x))$, i.e., $\operatorname{ad}(z + c_i x)$ has only 0-eigenvalue on $K_0$. So $f(T, c_i) = T^r$, i.e., $f_i \equiv 0$. Hence, $L_0(\operatorname{ad}(z + cx)) \supseteq K_0$ for all $c \in F$. Since $x$ is arbitrary, $K_0$ is totally minimal. ■

**Definition 8.3.** A **Cartan subalgebra** (CSA) $H$ of a Lie algebra $L$ is a self-normalizing nilpotent subalgebra.

For example, a maximal toral of a semi-simple Lie algebra is Cartan.

**Theorem 8.4.** Let $H$ be a Lie subalgebra of $L$. Then $H$ is a CSA if and only if $H$ is a minimal Engel subalgebra (hence it exists).

*Proof.* ($\Leftarrow$) $H$ is self-normalizing by (8.1). Also, by (8.2), $H = L_0(\operatorname{ad} z) \subseteq L_0(\operatorname{ad} x)$ for all $x \in H$, i.e., $\operatorname{ad}_H x$ is ad-nilpotent for all $x \in H$. Hence, by Engel's theorem (2.1), $H$ is nilpotent.

($\Rightarrow$) Let $H$ be a CSA. The nilpotency of $H$ implies that $H \subseteq L_0(\operatorname{ad} x)$ for all $x \in H$. We claim the equality holds for some minimal one.

If not, take $L_0(\operatorname{ad} z \in H)$ be a minimal one. By (8.2), $L_0(\operatorname{ad} z) \subseteq L_0(\operatorname{ad} x)$ for all $x \in H$. So the action of $H$ on $L_0(\operatorname{ad} z)/H$ acts as nilpotent endomorphisms. By some ancient theorem, there exists a 0-eigenvector $y + H$, $y \notin H$, such that $[H, y] \subseteq H$. Since $H$ is self-normalizing, $y \in H$, a contradiction. ■

**Corollary 8.5.** Let $L$ be a semi-simple Lie algebra over $F$. Then CSA $\equiv$ maximal toral ($\equiv C_L(s)$ for some semi-simple element $s$).

*Proof.* ($\Leftarrow$) is already done. ($\Rightarrow$) Let $H$ be a CSA. Then $H = L_0(\operatorname{ad} x)$ for some $x \in H$. Write $x = x_s + x_n$, then $H = L_0(\operatorname{ad} x) = L_0(\operatorname{ad} x_s) = C_L(x_s)$. Since $C_L(x_s)$ contains $Fx_s$ and $Fx_s$ is contained in some maximal toral $C$, which is abelian, we have $H \supseteq C$. Since $C$ is a CSA, $H = C$. ■

**Remark 8.6.** Functorialities:

(a) If $\phi : L \to L'$ is surjective, then the image $\phi(H)$ of a CSA $H$ of $L$ is a CSA of $L'$.

(b) Let $H' \subseteq L'$ be a CSA. Then any CSA $H$ of $\phi^{-1}(H')$ is also a CSA of $L$.

**Definition 8.7.** An element $x \in L$ is strongly ad-nilpotent if $x \in L_{a\neq 0}(\operatorname{ad} y)$ for some $y \in L$.

Let $\mathcal{N}(L) = \{\text{strongly ad-nilpotent}\}$, and let

$$\mathcal{E}(L) = \left\langle e^{\operatorname{ad} x} \mid x \in \mathcal{N}(L) \right\rangle \trianglelefteq \operatorname{Aut} L.$$

For a subalgebra $K$ of $L$,

$$\mathcal{E}(L; K) = \left\langle e^{\operatorname{ad}_L x} \mid x \in \mathcal{N}(K) \right\rangle.$$

**Idea.** $\mathcal{E}(L)$ is “better than” $\operatorname{Int} L$.

**Facts.** $K \subseteq L$ implies $\mathcal{N}(K) \subseteq \mathcal{N}(L)$, hence $\mathcal{E}(K) = \mathcal{E}(L; K)|_K$.

For a surjective homomorphism $\phi : L \to L'$, $\phi(\mathcal{N}(L)) = \mathcal{N}(L')$. Moreover, for each $\sigma' \in \mathcal{E}(L')$, there exists $\sigma \in \mathcal{E}(L)$ such that the diagram

$$\begin{tikzcd} L \arrow[r, "\phi"] \arrow[d, "\sigma"] & L' \arrow[d, "\sigma'"] \\ L \arrow[r, "\phi"] & L' \end{tikzcd}$$

commutes: say $\sigma' = e^{\operatorname{ad}_{L'} x'}$, where $x' = \phi(x)$ for some $x \in \mathcal{N}(L)$. Then for each $z \in L$,

$$\begin{aligned} \left(\phi \circ e^{\operatorname{ad}_L x}\right)(z) &= \phi\left(z + [x, z] + \frac{1}{2}[x, [x, z]] + \cdots\right) \\ &= \phi(z) + [x', \phi(z)] + \frac{1}{2}[x', [x', \phi(z)]] + \cdots \\ &= \left(e^{\operatorname{ad}_{L'} x'} \circ \phi\right)(z). \end{aligned}$$

**Theorem 8.8.** Let $L$ be a solvable Lie algebra. Then any two CSA's $H_1$, $H_2$ are conjugated under $\mathcal{E}(L)$.

*Proof.* Introduction on $\dim L$. If $\dim L = 1$ or $L$ is nilpotent, CSA $= L$, done!

If $L$ is not nilpotent, take $A \trianglelefteq L$ to be an abelian ideal with smallest dimension.

Let $\phi : L \to L' = L/A$ be the quotient map. Then the images $H_1' = \phi(H_1)$, $H_2' = \phi(H_2)$ are CSA's of $L'$. By induction hypothesis, there exists $\sigma' \in \mathcal{E}(L')$ such that $\sigma'(H_1') = H_2'$. Take $\sigma \in \mathcal{E}(L)$ such that $\sigma' \circ \phi = \phi \circ \sigma$. Then $\sigma$ maps $K_1 = \phi^{-1}(H_1)$ to $K_2 = \phi^{-1}(H_2)$ and $\sigma(H_1)$, $H_2$ are CSA's of $K_2$.

If $K_2 \neq L$, then by the induction hypothesis there exists $\tau' = \tau|_K \in \mathcal{E}(K_2) = \mathcal{E}(L; K_2)|_{K_2}$ such that $H_2 = \tau'(\sigma(H_1)) = (\tau\sigma)(H_1)$, as desired.

Otherwise $L = K_2 = \sigma(K_1) = K_1$, and hence $L = H_2 + A = H_1 + A$. Write $H_2 = L_0(\operatorname{ad} x)$. Since $A$ is $\operatorname{ad} x$-stable,

$$A = A_0(\operatorname{ad} x) \oplus A_*(\operatorname{ad} x) = A_0 \oplus A_*.$$

Then both $A_0$ and $A_*$ are $L = H_2 + A$ stable. It follows from the minimality of the dimension of $A$ that $A = A_0$ or $A = A_*$.

If $A = A_0$, then $A \subseteq H_2$. Then $L = H_2$ is nilpotent, a contradiction. Hence $A = A_*(\operatorname{ad} x)$. But $L = H_1 + A$ shows that $x = y + z$ for some $y \in H_1$ and $z \in A = A_*(\operatorname{ad} x)$, i.e., $z = [x, z']$ since $\operatorname{ad} x$ is invertible on it.

Since $A$ is abelian, $(\operatorname{ad} z')^2 = 0$. So

$$e^{\operatorname{ad} z'} x = (1 + \operatorname{ad} z')(x) = x - [x, z'] = y.$$

So $H = L_0(\operatorname{ad} y)$ is also a CSA that contains $H_1$, which implies $H = H_1$, i.e., $e^{\operatorname{ad} z'}$ maps $H_2$ to $H_1$. Write $z' = \sum_{a \neq 0} z'_a$, $z'_a \in A_a(\operatorname{ad} x)$, we see that all $z'_a$ commutes. So

$$e^{\operatorname{ad} z'} = \prod e^{\operatorname{ad} z'_a} \in \mathcal{E}(L). \qquad \blacksquare$$

# 9 Final step II, 10/5

**Theorem 9.1.** For a Lie algebra $L$ over an algebraically closed field $F$ with $\operatorname{char} F = 0$, any CSA is conjugate to each other.

The case $F = \mathbb{C}$ is proved by Cartan and Weyl using analysis (differential geometry). For a general field, it is proved by Chevalley and Bourbaki using algebraic geometry. A purely algebraic proof was given by Winter.

We do the case $F = \mathbb{C}$ first. Let $n = \dim L$. For each element $x \in L$, consider the characteristic polynomial

$$f_x(T) := \det(\operatorname{ad} x - T) = (-1)^n T^n + g_1(x) T^{n-1} + \cdots + g_{n-r}(x) T^r,$$

where $r$ is the smallest integer such that the polynomial $g_{n-r}(x) \neq 0$. We define the rank of $L$, denoted by $\operatorname{rank} L$, to be such $r$, and call $x \in L$ regular, or generic, if $g_{n-r}(x) \neq 0$. Then a CSA $H = L_0(\operatorname{ad} x)$ has dimension $k \geq r$.

**Fact.** Regular elements form a Zariski open subset in $L \cong \mathbb{C}^n$, hence it is path connected and dense open.

Given CSA's $H_0 = L_0(\operatorname{ad} x_0)$, $H_1 = L_0(\operatorname{ad} x_1)$, and take any path $x_-$ in the Zariski open subset connecting $x_0$ and $x_1$. Then for any $t \in [0,1]$, $L_0(\operatorname{ad} x_t)$ is a CSA. If we can prove that any point $y$ near $x = x_t$, $L_0(\operatorname{ad} y)$ is conjugate to $L_0(\operatorname{ad} x)$, then the statement holds by applying compact argument.

To do this, apply IFT to

$$\begin{aligned} H \times \mathbb{C}^{n-k} &\xrightarrow{\ \psi\ } L \cong \mathbb{C}^n \\ (h, t) &\longmapsto \textstyle\prod_{i=1}^{n-k} e^{\operatorname{ad}(t_i y_i)} h, \end{aligned}$$

where $y_i$ are the generalized eigenvectors of $\operatorname{ad} x$.

**Exercise.** This is invertible!

**Definition 9.2.** A subalgebra $B \subseteq L$ is **Borel** if it is a maximal solvable subalgebra.

(A) A Borel subalgebra is self-normalizing: if $[x, B] \subseteq B$, then $[B + Fx, B + Fx] \subseteq B$, which implies $B + Fx$ is solvable. By maximality of $B$, $x \in B$.

(B) If $\operatorname{Rad} L \subsetneq L$, then the set of Borel subalgebras in $L$ is 1-1 corresponds to the set of Borel subalgebras in $L/\operatorname{Rad} L$. Indeed, the sum of a solvable subalgebra and the solvable ideal $\operatorname{Rad} L$ is a solvable subalgebra.

(C) For a semi-simple Lie algebra $L$, $H$ a CSA with base $\Delta \subseteq \Phi$,

$$B(\Delta) := H \oplus \bigoplus_{\alpha \in \Phi^+(\Delta)} L_\alpha,$$

called a standard Borel relative to $H$, is Borel. Any standard Borel subalgebra is conjugate to each other via $\mathcal{E}(L)$. Indeed, let $N(\Delta) = \bigoplus_{\alpha \in \Phi^+(\Delta)} L_\alpha$. Then $[B(\Delta), B(\Delta)] = N(\Delta)$, which is nilpotent, so $B(\Delta)$ is solvable. If $B(\Delta)$ is not maximal, say $K \supsetneq B(\Delta)$ is also solvable, then $K \supseteq L_{-\alpha}$ for some $\alpha \in \Phi^+$. Then $K$ contains a semi-simple Lie algebra $S_\alpha$, a contradiction. Now, for a root $\alpha \in \Phi$, the action $\sigma_\alpha$ on $H$ extends to $\tau_\alpha \in \mathcal{E}(L)$: take $x_\alpha \in L_\alpha$, $y_\alpha \in L_{-\alpha}$ that defines $S_\alpha$, and define $\tau_\alpha = e^{\operatorname{ad} x_\alpha} e^{-\operatorname{ad} y_\alpha} e^{\operatorname{ad} x_\alpha}$. Then $\tau_\alpha$ maps $B(\Delta)$ to $B(\sigma_\alpha \Delta)$. Hence, any standard Borel subalgebra is conjugate to each other since the Weyl group $\mathscr{W}$ acts on the bases transitively.

**Theorem 9.3.** All Borel subalgebras (BSA) are $\mathcal{E}(L)$-conjugate. In particular, all CSA's are $\mathcal{E}(L)$-conjugate.

*Proof.* We prove the latter statement first (using the former statement): for CSA's $H$ and $H'$, we can put them in BSA's $B$ and $B'$, respectively. Take any $\sigma \in \mathcal{E}(L)$ such that $\sigma(B) = B'$, then $\sigma(H)$, $H'$ are CSA in $B'$. The statement now reduce to the solvable case.

For the former statement, induction on $\dim L$. The base case $\dim L = 1$ is trivial. Using (B) together with the lifting of $\mathcal{E}(L)$ under $L \to L' = L/\operatorname{Rad} L$, we may assume that $L$ is semi-simple. And it suffices to prove that any Borel subalgebra $B'$ of $L$ is conjugate to a standard Borel subalgebra $B = B(\Delta)$ relative to some CSA $H$.

Next, we induction on $\dim(B \cap B')$ downward. The base case $B \cap B' = B$, which is equivalent to $B = B'$, is trivial. So let $B \supsetneq B \cap B'$.

(1) If $B \cap B' \neq 0$, then

  (i) 1. all nilpotent elements $N'$ in $B \cap B'$ is nonzero. $N'$ is an ideal in $B \cap B'$ (using $[B, B] = N(\Delta)$), but not in $L$. So $K := N_L(N') \subsetneq L$.

   2. $B \cap B' \subsetneq B \cap K$: consider the adjoint action $N'$ on $B/B \cap B' \neq 0$. Then there exists a 0-eigenvector $y + B \cap B'$, but $x \in N'$ implies $[x, y] \in [B, B]$, and thus in $N'$, i.e., $y \in N_B(N') = B \cap K$.

   3. Take BSA's $C$, $C'$ of $K \subsetneq L$ that contains $B \cap K$, $B' \cap K$, respectively.

  By (first) induction hypothesis, there exists $\sigma \in \mathcal{E}(L; K)$ such that $\sigma(C') = C$. By (second) induction hypothesis, there exists $\tau \in \mathcal{E}(L)$ such that $\tau(B_1) = B$, where $B_1$ is some BSA that contains $\sigma(C')$. Then

$$B \cap \tau\sigma(B') \supseteq \tau\sigma(C') \cap \tau\sigma(B') \supseteq \tau\sigma(B' \cap K) \supsetneq \tau\sigma(B \cap B').$$

  By (second) induction hypothesis, $B$ is conjugate to $\tau\sigma(B')$.

  (ii) If $N' = 0$, left for reading.

(2) $B \cap B' = 0$, left for reading. ■

# 10 Existence theorem I, 10/12

**Definition 10.1.** For a vector space $V$ over $F$, we define the tensor algebra

$$T(V) := \oplus_{i=0}^{\infty} T^i(V), \quad T^i(V) = V^{\otimes i}.$$

For a Lie algebra, the **universal enveloping algebra** of $L$ is defined to be

$$\mathfrak{U}(L) := T(L)/J,$$

where $J$ is the 2-sided ideal generated by $x \otimes y - y \otimes x - [x, y]$, $x$, $y \in L$.

The universal enveloping algebra $\mathfrak{U}(L)$ satisfies the following universal property: for a linear map $j : L \to \mathfrak{A}$, where $\mathfrak{A}$ is an associative $F$-algebra, such that $j[x, y] = j(x)j(y) - j(y)j(x)$, $x$, $y \in L$, there exists a linear map $\mathfrak{U}(L) \to \mathfrak{A}$ that completes the diagram:

$$\begin{tikzcd}
L \arrow[r, "j"] \arrow[d] & \mathfrak{A} \\
T(L) \arrow[r, "\pi"] & \mathfrak{U}(L) \arrow[u, "\exists"']
\end{tikzcd}$$

**Definition 10.2.** Let $T_m = T^0 \oplus \cdots \oplus T^m$, $U_m = \pi(T_m)$. We see that $U_i \cdot U_j \subseteq U_{i+j}$, Define $G^m = U_m/U_{m-1}$, $\mathfrak{G} = \bigoplus_{m=0}^{\infty} G^m$.

**Theorem 10.3** (PBW, Poincaré-Birkhoff-Witt)**.** There is an isomorphism $w : S(L) \to \mathfrak{G}$, where $S(L)$ is the symmetric algebra of $L$.

The surjectivity is easy: $T^m \to U_m \to G^m$ is onto, so $\phi : T \to \mathfrak{G}$ is onto. Also, $\phi(I) = 0$, where $I$ is the 2-sided ideal generated by $x \otimes y - y \otimes x$.

This defines a surjection from $w : S(L) \to \mathfrak{G}$. The injectivity is hard (left for reading).

**Corollary 10.4.** (A) For $W \subseteq T^m \to S^m$ satisfying $\pi : W \cong S$, $\pi(W)$ is complement to $U_{m-1}$ in $U_m$.

(B) $i : L \to \mathfrak{U}(L)$ is injective: taking $W = T^1 = L$ $(m = 1)$.

(C) For any ordered basis, $x_1$, …, $x_n$ of $L$. $x_{i(1)} \cdots x_{i(m)}$ with $i(1) \leq \cdots \leq i(m)$ form a basis of $\mathfrak{U}(L)$: Take $W = \langle x_{i(1)} \otimes \cdots \otimes x_{i(m)} \rangle \subseteq T^m$

**Definition 10.5.** Let $X$ be a set. The free Lie algebra generated by $X$ over $F$ is defined to be the Lie subalgebra $\mathbf{X}$ in $T(V)$ generated by $X$, where $V$ is the vector space over $F$ with $X$ as basis.

Let $L$ be a semi-simple Lie algebra, $H$ a CSA of $L$. Let $\Phi$ be the root system induced by $H$, $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ a base of $\Phi$, $A = (c_{ij}) = (\langle \alpha_i, \alpha_j \rangle)$ the Cartan matrix. For each $i$, let $S_{\alpha_i} = \langle x_i, y_i, h_i \rangle$ be the Lie algebra generated by $L_{\alpha_i}$ and $L_{-\alpha_i}$.

**Proposition 10.6** (Serre relations)**.** (S1) $[h_i, h_j] = 0$, (S2) $[x_i, y_j] = \delta_{ij} h_i$,

(S3) $[h_i x_j] = c_{ji} x_j$, $[h_i y_j] = -c_{ji} y_j$,

($\mathrm{S}_{ij}^{+}$) $(\operatorname{ad} x_i)^{-c_{ji}+1} x_j = 0$ $(i \neq j)$,

($\mathrm{S}_{ij}^{-}$) $(\operatorname{ad} y_i)^{-c_{ji}+1} y_j = 0$ $(i \neq j)$.

*Proof.* We only prove ($\mathrm{S}_{ij}^{+}$). Since $\alpha_j - \alpha_i \notin \Phi$, we get the $\alpha_j$-string $\alpha_j$, $\alpha_j + \alpha_i$, …, $\alpha_j + q\alpha_i$. Since $0 - q = c_{ji}$, we get $(\operatorname{ad} x_i)^{-c_{ji}+1} x_j = (\operatorname{ad} x_i)^{q+1} x_j = 0$. ■

**Theorem 10.7** (Serre)**.** These relations are complete (for semi-simple Lie algebra $L$).

*Proof.* **Step 1.** Let $\hat{L}$ be the free Lie algebra generated by $X = \{x_i, y_i, h_i\}_{i=1}^{\ell}$, $\hat{K}$ the 2-sided ideal generated by (S1), (S2), and (S3), $L_0$ the quotient $\hat{L}/\hat{K}$. Then $L_0 = H \oplus X \oplus Y$, where $H$, $X$, and $Y$ are lie subalgebras generated by $\{h_i\}$, $\{x_i\}$, and $\{y_i\}$, respectively, and $H = \oplus F h_i$.

Let $\mathbf{V} = T(F^{\ell})$. Fix a basis $v_1$, …, $v_\ell$ of $F^\ell$ and define the representation $\hat{\phi} : \hat{L} \to \mathfrak{gl}(\mathbf{V})$ by $h_j \cdot 1 = x_j \cdot 1 = x_j \cdot v_j = 0$, $y_j \cdot 1 = v_j$, and

$$\begin{cases} h_j \cdot v_{i_1} \otimes \cdots \otimes v_{i_t} = -(c_{i_1 j} + \cdots + c_{i_t j}) v_{i_1} \otimes \cdots \otimes v_{i_t}, \\ x_j \cdot v_{i_1} \otimes \cdots \otimes v_{i_t} = v_{i_1} \otimes (x_j \cdot v_{i_2} \otimes \cdots \otimes v_{i_t}) - \delta_{i_1 j} \left(\sum_{k=2}^{t} c_{i_k j}\right) v_{i_2} \otimes \cdots \otimes v_{i_t}, \\ y_j \cdot v_{i_1} \otimes \cdots \otimes v_{i_t} = v_j \otimes v_{i_1} \otimes \cdots \otimes v_{i_t}. \end{cases}$$

We check that $\hat{K}_0 := \ker \hat{\phi} \supseteq \hat{K}$, i.e., the $\mathfrak{gl}(\mathbf{V})$ is in fact an $L_0$-module.

(1) $[h_i, h_j] \in \hat{K}_0$: since $h_i$ acts diagonally, $[\hat{\phi}(h_i), \hat{\phi}(h_j)] = 0$,

(2) $[x_i, y_j] - \delta_{ij} h_j \in \hat{K}_0$:

$$\begin{aligned} x_i y_j \cdot v_{i_2} \otimes \cdots \otimes v_{i_t} - y_j x_i \cdot v_{i_2} \otimes \cdots \otimes v_{i_t} &= -\delta_{ji} \left( \sum_{k=2}^{t} c_{i_k j} \right) v_{i_2} \otimes \cdots \otimes v_{i_t} \\ &= \delta_{ij} h_j v_{i_2} \otimes \cdots \otimes v_{i_t}. \end{aligned}$$

(3) $[h_i, y_j] + c_{ji} y_j \in \hat{K}_0$:

$$\begin{aligned}
(h_i y_j - y_j h_i) \cdot 1 &= h_i v_j = -c_{ji} v_j = -c_{ji} y_j \cdot 1, \\
(h_i y_j - y_j h_i) \cdot v_{i_1} \otimes \cdots \otimes v_{i_t} &= h_i \cdot v_j \otimes v_{i_1} \otimes \cdots \otimes v_{i_t} \\
&\quad + (e_{i_1 i} + \cdots + e_{i_t i}) v_j \otimes v_{i_1} \otimes \cdots \otimes v_{i_t} \\
&= e_{ji} y_j v_{i_1} \otimes \cdots \otimes v_{i_t}.
\end{aligned}$$

(4) $[h_i, x_j] - c_{ji} x_j \in \hat{K}_0$:

**Claim.** $h_i \cdot (x_j \cdot v_{i_1} \otimes \cdots \otimes v_{i_t}) = -(c_{i_1 i} + \cdots + c_{i_t i} - c_{ji}) x_j \cdot v_{i_1} \otimes \cdots \otimes v_{i_t}$.

Induction on $t$. The base case $t = 0$ is trivial. For simplicity, let $v = v_{i_2} \otimes \cdots \otimes v_{i_t}$. By induction hypothesis,

$$h_i \cdot (x_j \cdot v) = -(c_{i_2 i} + \cdots + c_{i_t i} - c_{ji}) x_j \cdot v. \tag{II}$$

Since

$$y_{i_1} h_i x_j = (h_i + c_{i_1 i}) y_{i_1} x_j = (h_i + c_{i_1 i})(x_j y_{i_1} - \delta_{ji_1} h_j),$$

we get

$$\begin{aligned}
h_i \cdot (x_j \cdot v_{i_1} \otimes v) &= h_i x_j y_{i_1} \cdot v, \\
&= y_{i_1}(h_i x_j \cdot v) - c_{i_1 i} x_j y_{i_1} \cdot v + \delta_{ji_1}(h_i + c_{i_1 i}) h_j \cdot v \\
&= -(c_{i_2} + \cdots + c_{i_t i} - c_{ji}) y_{i_1} x_j \cdot v - c_{i_1} x_j \cdot v_{i_1} \otimes v \\
&\quad + \delta_{ji_1}(-c_{i_1 i} + c_{i_2 i} + \cdots + c_{i_t i})(c_{i_2 j} + \cdots + c_{i_t j}) v \\
&= -(c_{i_2} + \cdots + c_{i_t i} - c_{ji})(x_j y_{i_1} + \delta_{ji_1} h_j) \cdot v - c_{i_1} x_j \cdot v_{i_1} \otimes v \\
&\quad + \delta_{ji_1}(-c_{i_1 i} + c_{i_2 i} + \cdots + c_{i_t i})(c_{i_2 j} + \cdots + c_{i_t j}) v \\
&= -(c_{i_1 i} + \cdots + c_{i_t i} - c_{ji}) x_j \cdot v_{i_1} \otimes v \\
&\quad + \delta_{ji_1}(c_{i_2 i} + \cdots + c_{i_t i} - c_{ji})(c_{i_2 j} + \cdots + c_{i_t j}) v \\
&\quad + \delta_{ji_1}(-c_{i_1 i} + c_{i_2 i} + \cdots + c_{i_t i})(c_{i_2 j} + \cdots + c_{i_t j}) v \\
&= -(c_{i_1 i} + \cdots + c_{i_t i} - c_{ji}) x_j \cdot v_{i_1} \otimes v,
\end{aligned}$$

as desired.

Hence, $(h_i x_j - x_j h_i) \cdot 1 = 0$ and

$$(h_i x_j - x_j h_i) \cdot v_{i_1} \otimes \cdots \otimes v_{i_t} = c_{ji} x_j \cdot v_{i_1} \otimes \cdots \otimes v_{i_t}.$$

# 11 Existence theorem II, 10/17

So there is a nontrivial $L_0$-module $\mathfrak{gl}(\mathbf{V})$. Then $L_0 = H + X + Y$, where $H = \sum_i Fh_i$, $X = \langle x_i \rangle$, $Y = \langle y_i \rangle$.

**Exercise.** Prove that $X$ (resp. $Y$) is generated by $\{x_i\}$ (resp. $\{y_i\}$) freely.

- For all $h_i$, $[h_i, H] = 0$, $[h_i, [x_j, x_k]] = (c_{ji} + c_{ki})[x_j, x_k]$, induction get the main calculation:

  $$[h_i, [x_{i_1}, [\dots, [x_{i_{t-1}}, x_{i_t}] \dots]]] = (c_{i_1 i} + \cdots + c_{i_t i})[x_{i_1}, [\dots, [x_{i_{t-1}}, x_{i_t}] \dots]] \in X.$$

  A similar result also holds for $Y$.

- For all $x_i$. $[x_i, H + X] = X$,

  $$\begin{aligned} [x_i, [y_j, y_k]] &= [[x_i, y_j], y_k] + [y_j, [x_i, y_k]] \\ &= \delta_{ij}[h_i, y_k] + \delta_{ik}[y_j, h_i] = -\delta_{ij} c_{ki} y_k + \delta_{ik} c_{ji} y_j \in Y. \end{aligned}$$

  By induction, we get $[x_i, Y] \subseteq Y$.

- For all $y_i$, we get $[y_i, L_0] \subseteq Y$ similarly.

**Claim.** $\phi(h_i)$ are linearly independent and the sum $L_0 = H + X + Y$ is direct.

If $\sum a^i \phi(h_i) = 0$, then for each $j$,

$$0 = \sum a^i \phi(h_i) v_j = -\sum a_i c_{ji} e_j \implies \sum a^i c_{ji} = 0.$$

Since $j$ is arbitrary, $a^i = 0$ for all $i$.

By the calculation above, $L_0 = H + X + Y$ is a decomposition of $L_0$ into eigenspaces of $\operatorname{ad} H$. Indeed, the eigenvalue is $\lambda = \sum_j k_j \alpha_j > 0$ on $X$ ($< 0$ on $Y$), any iterative $[\dots]$ in $X$ of $x_{i_1}, \dots, x_{i_t}$ has eigenvalue $\sum_k c_{i_k}$. Evaluate at $h_i$, this eigenvalue is of the form $\sum m_j c_{ji}$, where $m_j \geq 0$ and $\sum m_j = t$. So $X \cap Y = 0$. (Otherwise, we get $\sum m_j c_{ji} = -\sum n_j c_{ji}$ for some $m_j$, $n_j \geq 0$, then $\sum (m_j + n_j) c_{ji} = 0$. Since $C$ is nondegenerate, this leads to a contradiction.)

**Step 2.** Adding relations $(\mathrm{S}_{ij}^{+})$, $(\mathrm{S}_{ij}^{-})$:

$$\begin{aligned} I &= \left\langle x_{ij} := (\operatorname{ad} x_i)^{-c_{ji}+1} x_j \,\middle|\, i \neq j \right\rangle \trianglelefteq X, \\ J &= \left\langle y_{ij} := (\operatorname{ad} y_i)^{-c_{ji}+1} y_j \,\middle|\, i \neq j \right\rangle \trianglelefteq Y. \end{aligned}$$

Then $J$, and hence $I$, $K = I + J$, is an ideal of $L_0$.

**Lemma 11.1.** $[x_k, y_{ij}] = 0$.

*Proof of Lemma.* If $k \neq i$, then $[x_k, y_i] = 0$ implies that

$$\operatorname{ad} x_k(y_{ij}) = (\operatorname{ad} y_i)^{-c_{ji}+1} \operatorname{ad} x_k(y_j) = 0.$$

If $k = i$, then

$$\operatorname{ad} x_k(\operatorname{ad} y_i)^t y_j = t(c_{ji} - t + 1)(\operatorname{ad} y_i)^{t-1} y_j$$

by induction on $t$. The result now follows by letting $t = -c_{ji} + 1$. $\square$

Now we check that $J \trianglelefteq L_0$: As the calculation above, we have

$$(\operatorname{ad} h_k) y_{ij} = (-c_{jk} + (c_{ji} - 1)c_{ik}) y_{ij}.$$

Together with $\operatorname{ad} h_k(Y) \subseteq Y$, we get $\operatorname{ad} h_k(J) \subseteq J$ by Jacobi's identity. Using the Lemma and the fact $\operatorname{ad} x_k(Y) \subseteq Y + H$, we get $\operatorname{ad} x_k(J) \subseteq J$ (again by Jacobi's identity).

**Step 3.** Hence, $L := L_0/K = H \oplus N^+ \oplus N^-$, where $N^+ := X/I$ and $N^- := Y/J$, and this is the semi-simple Lie algebra we want!

For $\lambda \in H^\vee$, $L_\lambda := \{x \in L \mid [h, x] = \lambda(h)x\}$ as before. We had seen $H = L_{\vec{0}}$, $N^+ = \bigoplus_{\lambda > 0} L_\lambda$, $N^- = \bigoplus_{\lambda < 0} L_\lambda$, and each piece has finite dimension.

The operators $\operatorname{ad} x_i$ and $\operatorname{ad} y_i$ are locally nilpotent, i.e., for each $z \in L$, there exists $k \geq 0$ such that $(\operatorname{ad} x_i)^k z = (\operatorname{ad} y_i)^k z = 0$: let

$$M_i = \{\text{ all such } z\,\}.$$

Then $x_j \in M_i$ by ($\mathrm{S}_{ij}^+$), hence $h_j \in M_i$ by (S3), and hence $y_j \in M_i$ by (S2). Note that $M_i$ is a Lie algebra:

$$(\operatorname{ad} x_i)^n [y, z] = \sum_{j=0}^{n} \binom{n}{j} [(\operatorname{ad} x)^j y, (\operatorname{ad} x)^{n-j} z] = 0$$

by taking $n$ large enough. We get $M_i = L$.

Now, $\tau_i := e^{\operatorname{ad} x_i} e^{-\operatorname{ad} y_i} e^{\operatorname{ad} x_i} \in \operatorname{Aut} L$ is well-defined. In fact, if $\sigma_i \lambda = \mu$, where $\sigma_i = \sigma_{\alpha_i}$ is the reflection, then $\tau_i = \sigma_i$ on $L_\lambda \oplus L_\mu$ as a reflection. So $\dim L_\lambda = \dim L_\mu$. This result also holds for $\sigma\lambda = \mu$, where $\sigma \in \mathscr{W}$.

It is clear that $\dim L_{\alpha_i} = 1$ by the main calculation and $L_{k\alpha_i} = 0$ if $k \neq -1$, 0, 1 (since $[x_i, \ldots, x_i] = 0$). By some exercise before, $L_\lambda \neq 0$ if and only if $\lambda \in \Phi$ or $\lambda = \vec{0}$. In particular, $\dim L = \dim H + |\Phi| < \infty$. $L$ is semi-simple: let $A \trianglelefteq L$ be an abelian ideal, $A = (A \cap H) \oplus \bigoplus_{\alpha \in \Phi}(A \cap L_\alpha)$. We see that $A \cap L_\alpha = 0$ for all $\alpha \in \Phi$ (otherwise $A \supseteq \langle L_\alpha, L_{-\alpha} \rangle$). Hence, $A \subseteq H$ and $[L_\alpha, A] = 0$ for all $\alpha$. So $A \subseteq \bigcap_{\alpha \in \Phi} \ker \alpha = 0$.

Now, $H$ is abelian and self normalizing, so $H$ is a CSA with root system $\Phi$. The proof is complete. ■

For the classical case $A_\ell$, $B_\ell$, $C_\ell$, $D_\ell$, we want to show that they are simple.

**Definition 11.2.** A Lie algebra $L$ is **reductive** if $\operatorname{rad} L = Z(L)$.

If $L$ is reductive, then $L' = L/Z(L)$ is semisimple. So there is a (completely) action of $\operatorname{ad} L = \operatorname{ad} L'$ on $L = M \oplus Z(L)$, where $M \trianglelefteq L$ is an ideal. Then $[L, L] = [M, M] \subseteq M \cong L'$. Hence this inclusion is an identity, so $L = [L, L] \oplus Z(L)$.

**Proposition 11.3.** Let $L \subseteq \mathfrak{gl}(V)$. If the action of $L$ on $V$ is irreducible, then $L$ is reductive and $\dim Z(L) \leq 1$. If moreover $L \subseteq \mathfrak{sl}(V)$, then $L$ is semi-simple.

*Proof.* Let $S = \operatorname{rad} L$, and let $v$ be a common eigenvector $v$ (exists by (2.4)). Then $s \cdot v = \lambda(s)v$ for all $s \in S$ for some $\lambda$. For $x \in L$, we have

$$s \cdot (x \cdot v) = x \cdot (s \cdot v) + [s, x] \cdot v = \lambda(s)x \cdot v + \lambda([s, x])v.$$

Since $L \cdot v = V$, all matrices of $S$ is upper diagonal in some basis with diagonal entries $\lambda(s)$.

Since $\operatorname{tr}[S, L] \equiv 0$, $\lambda|_{[S,L]} = 0$, so the calculation above shows that the action of $S$ on $V$ is just scalar. So $S = Z(L)$ and $\dim S \leq 1$. Also, if $L \subseteq \mathrm{sl}(V)$, then $S = 0$. ■

**Example 11.4.** $L = A_\ell$, $B_\ell$, $C_\ell$, $D_\ell$ are semi-simple: it suffices to check that the actions of $B_\ell$, $C_\ell$, $D_\ell$ on $V$ are irreducible.

Let $W \subseteq V$ be an $L$-invariant subspace. Then $W$ is invariant under $\langle \mathrm{id}, L, +, \circ \rangle \subseteq \operatorname{End} V$. For $L = B_\ell$, $C_\ell$, $D_\ell$, we get all $\operatorname{End} V$.

In fact, $L = A_\ell$, $B_\ell$, $C_\ell$, $D_\ell$ are simple with $H \cong C_L(H)$.

# 12 Representation theory of semi-simple Lie algebra, 10/19

In this section, we fix a Lie algebra $L$, its CSA $H$, root system $\Phi$, base $\Delta$, and Weyl group $\mathscr{W}$.

**Facts.** Let $V$ be a $L$-module. Then $H$ acts on $V$ diagonally and for each $\lambda \in H^\vee$, $V_\lambda$ is defined. It is easy to see that

(a) $L_\alpha : V_\lambda \to V_{\lambda+\alpha}$;

(b) $V' := \sum V_\lambda$ is direct (⚠: $V'$ could be 0);

(c) if $\dim V < \infty$, then $V = V'$.

**Definition 12.1.** Suppose a maximal vector $v^+$ exists, i.e., $v^+ \in V$ and $L_\alpha v^+ = 0$ for all $\alpha > 0$. (For example, when $\dim L$ is finite, then Lie's theorem tells us that there exists a common eigenvector $v^+$ of $B = B(\Delta)$.) We may further assume that $v^+ \in V_\lambda$ for some $\lambda$. We call $\lambda$ a highest weight and call $v^+$ a highest weight vector.

If $V = \mathfrak{U}(L) \cdot v^+$, then $V$ is called a **standard cyclic** (or **irreducible**) $L$-module.

**Notation.** Let $\Phi^+ = \{\beta_1, \dots, \beta_n\}$. Then PBW theorem tells us that $\mathfrak{U}(L)$ has a basis $\{z_{i_1}^{k_1} \dots z_{i_t}^{k_t} \mid i_1 < \dots < i_t\}$, where $\{z_i\} = \{h_\bullet, x_\bullet, y_\bullet\}$ and the order is given by

$$y_{\beta_1} < \dots < y_{\beta_m} < h_1 < \dots < h_\ell < x_{\beta_1} < \dots < x_{\beta_m}.$$

**Proposition 12.2.** Suppose $V$ is cyclic.

(i) Then $V$ is spanned by $y_{\beta_1}^{i_1} \cdots y_{\beta_m}^{i_m} v^+$ ($i_j \geq 0$), hence $V = \bigoplus_{\lambda \in H^\vee} V_\lambda$. $V$ has weights of the form $\mu = \lambda - \sum_{i=1}^{\ell} k_i \alpha_i$, $k_i \geq 0$. Each $V_\mu$ has finite dimension, and $\dim V_\lambda = 1$.

(ii) Every $L$-submodule $W$ of $V$ is the direct sum of its weight spaces. Hence

- $V$ is indecomposable with unique maximal proper submodule and unique irreducible quotient module.
- In particular, if there is a surjective map $V \to V'$, then $V'$ is also standard cyclic of weight $\lambda$.

*Proof.* (i) Consider the (vector space) decomposition $L = N^- \oplus B$. We have $\mathfrak{U}(L) = \mathfrak{U}(N^-) \otimes \mathfrak{U}(B)$ (as vector space). Then $V = \mathfrak{U}(N^-) \cdot v^+$. The last assertion follows from the fact that the solutions of $\sum i_j \beta_j = \sum k_i \alpha_i$ is finite for each fixed $\{k_i\}$.

(ii) Let $w = \sum_{i=1}^n v_i \in W$ with $v_i \in V_{\mu_i}$. We claim that $v_i \in W$ for each $i$. If not, then there exists a $w$ with smallest $n \geq 2$ such that $v_i \notin W$ for all $i$. Find $h \in H$ such that $\mu_1(h) \neq \mu_2(h)$. Then

$$hh \cdot w = \sum \mu_i(h) v_i \implies 0 \neq w' := (h - \mu_1(h)) \cdot w = \sum_{i=2}^n (\mu_i(h) - \mu_1(h)) v_i,$$

a contradiction.

Now, if $V = W_1 \oplus W_2$, then $V_\lambda \nsubseteq W_i$. This implies $W_1 \oplus W_2 \subsetneq V$ a contradiction. This shows that $\sum_{W \subsetneq V} W \subsetneq V$ is the unique maximal proper submodule. ■

**Theorem 12.3.** For each $\lambda \in H^\vee$, there exists a unique (up to isomorphism) irreducible standard cyclic $L$-module of highest weight $\lambda$ (may be infinite dimensional).

*Proof.* If $V$ is an irreducible module, then the maximal vector $v^+$ is unique up to scalar. Indeed, for $w \in L_\mu$, $\mathfrak{U}(L) \cdot w \subseteq \mathfrak{U}(L) \cdot v^+$ and the equality holds if and only if $\lambda = \mu$.

Given irreducible modules $V = \mathfrak{U}(L) \cdot v^+$ and $W = \mathfrak{U}(L) \cdot w^+$. Let $X = V \oplus W$. Then $(v^+, w^+) \in X_\lambda$ is a highest vector. Let $Y = \mathfrak{U}(L) \cdot (v^+, w^+) \subseteq X$ and consider the projections $p$ and $q$ to $V$ and $W$, respectively. We see that $p(Y) = V$ and $q(Y) = W$. Since $V$ and $W$ are irreducible quotient modules of $Y$, they are isomorphic. This proves the uniqueness.

We prove the existence via induced module technique. Notice that $V = \mathfrak{U}(L) \cdot v^+$ has a 1-dimensional $B$-submodule $V_\lambda$. Thus, we define $D_\lambda = F v^+$ as $B$-module via

$$\left(h + \sum x_\alpha\right) \cdot v^+ := h \cdot v^+ = \lambda(h) v^+.$$

Then $D$ is also a $\mathfrak{U}(B)$-module. Define $Z(\lambda) = \mathfrak{U}(L) \otimes_{\mathfrak{U}(B)} D_\lambda$, which is a left $\mathfrak{U}(L)$-module. The vector $1 \otimes v^+ \in Z(\lambda)$ is nonzero by PBW theorem.

Since $\mathfrak{U}(L) = \mathfrak{U}(N^-) \otimes_F \mathfrak{U}(B)$, we get $Z(\lambda) = \mathfrak{U}(N^-) \otimes F(1 \otimes v^+)$. Now take $Y(\lambda) \subsetneq Z(\lambda)$ be the unique maximal proper submodule. We define $V(\lambda) = Z(\lambda)/Y(\lambda)$, which is the desired module. ■

# 13 Existence theorem, 10/24

**Definition 13.1.** An element $\lambda \in H^\vee$ is **integral** (resp. **dominant**, $(\lambda \in \Lambda)$) if $\lambda(h_i) \in \mathbb{Z}$ (resp. $\lambda(h_i) \in \mathbb{N}$) for all $i$.

**Theorem 13.2.** There exists a one-to-one correspondence between $\lambda \in \Lambda^+$ and finite dimensional irreducible $L$-modules $V(\lambda)$. Also, the set $\Pi(\lambda)$ of weights of $V(\lambda)$ is permuted by $\mathscr{W}$.

*Proof.* Similar as in Serre's theorem. Let $m_i = \lambda(h_i) \in \mathbb{Z}_{\geq 0}$, $\phi : L \to \mathfrak{gl}(V)$ the representation, and $v^+ \in V(\lambda)$ the highest weight vector.

**Lemma 13.3.** In $\mathfrak{U}(L)$, we have

(a) $[x_j, y_i^{k+1}] = 0$, $j \neq i$;

(b) $[h_j, y_i^{k+1}] = -(k+1)\alpha(h_j)y_i^{k+1}$;

(c) $[x_i, y_i]^{k+1} = -(k+1)y_i(k - h_i)$

*Proof of (13.3).* (a). Since $[R_{y_i}, L_{y_i}] = 0$, we have

$$[x_j, y_i^{k+1}] = (R_{y_i}^{k+1} - L_{y_i}^{k+1})x_j = (R_{y_i}^k + \cdots + L_{y_i}^k)(R_{y_i} - L_{y_i})x_j = (R_{y_i}^k + \cdots + L_{y_i}^k)[x_j, y_i] = 0.$$

(b) Induction on $k$. The case $k = 0$ follows from the definition. For $k > 0$, we have

$$\begin{aligned}[h_j, y_i^{k+1}] &= (h_j y_i^k - y_i^k h_j)y_i + y_i^k(h_j y_i - y_i h_j) \\ &= -k\alpha(h_j)y_i^{k+1} - y_i^k \alpha(h_j) y_i = -(k+1)\alpha(h_j)y_i^{k+1}.\end{aligned}$$

(c) Induction on $k$. The case $k = 0$ again follows from the definition. For $k > 0$, we have

$$\begin{aligned}[x_i, y_i]^{k+1} &= [x_i, y_i]^k y_i + y_i^k [x_i, y_i] \\ &= -k y_i^{k+1}(k - 1 - h_i)y_i + y_i^k h_i = -(k+1)y_i(k - h_i).\end{aligned}$$

□

Now, for each $i$, $y_i^{m_i+1} \cdot v^+ = 0$: Let $w = y_i^{m_i+1} v^+$. Then $x_j \cdot v^+ = 0$ implies that $x_j \cdot w = 0$ for all $j \neq i$ (by (a)). By (c),

$$x_i \cdot w = y_i^{m_i+1} x_i \cdot v^+ - (m_i + 1) y_i^{m_i}(m_i - h_i)v^+ = 0.$$

If $w \neq 0$, then it is a highest vector whose weight is not equal to $\lambda$, a contradiction.

Hence, $V$ contains a finite dimensional $S_i := S_{\alpha_i}$-module $\langle v^+, y_i \cdot v^+, \ldots, y_i^{m_i} \cdot v^+ \rangle$. Note that this is $S_i$-stable since it is $y_i$-stable, $h_i$-stable by (b), and $x_i$-stable by (c).

For any fixed $i$, let $V' := V_i'$ be the sum of all finite dimensional $S_i$-submodule in $V$. Then $V' = V$: say $W$ is a finite dimensional $S_i$-submodule. Then $x_\alpha \cdot W$, $\alpha \in \Phi$ is still a finite dimensional $S_i$-module. Hence, $V'$ is stable under $S_{\alpha_i}$. Since $V' \neq 0$, $V' = V$.

So any $v \in V$ lies in a finite (sum of) finite $S_i$-module. Therefore $\phi(x_i)$ and $\phi(y_i)$ are locally nilpotent, and hence $s_i := e^{\phi(x_i)} e^{-\phi(y_i)} e^{\phi(x_i)} \in \operatorname{Aut}(V)$ and $s_i V_\mu = V_{\sigma_i \mu}$. This tells us that $\mathscr{W}$ maps $\Pi(\lambda)$ to itself and $\Pi(\lambda)$ is finite. Indeed, for each $\mu \in \Pi(\lambda)$, there exists $w \in \mathscr{W}$ such that $w\mu \in \Lambda^+$. Then $w\mu \prec \lambda$ and thus

$$|\Pi(\lambda)| \leq |\mathscr{W}| \cdot |\{\nu \in \Lambda^+ \mid \nu \prec \lambda\}| < \infty.$$

Since each weight space $V_\mu$ is finite dimensional, $V$ is finite dimensional. ■

**Definition 13.4** (weight string). For $\mu \in \Lambda$ and $\alpha \in \Phi$, the $\alpha$-string through $\mu$ is the set

$$\{\mu + i\alpha \in \Pi(\lambda) \mid i \in \mathbb{Z}\} \subseteq \Pi(\lambda).$$

$S_\alpha$ acts on $\bigoplus V_{\mu+i\alpha}$, so it must be connected, i.e.,

$$\{\mu + i\alpha\} = \{\mu - r\alpha, \ldots, \mu + q\alpha\}.$$

As before, $r - q = \langle \mu, \alpha \rangle$ and $\sigma_\alpha$ reverse it.

**Corollary 13.5.** Let $\mu \in \Lambda$. Then $\mu \in \Pi(\lambda)$ if and only if $w\mu \prec \lambda$ for all $w \in \mathscr{W}$.

*Proof.* $\Pi(\lambda)$ is saturated, i.e., $\mu \in \Pi(\lambda)$ and $\alpha \in \Phi$ implies $\mu - i\alpha \in \Pi(\lambda)$ for all $i$ between 0 and $\langle \mu, \alpha \rangle$.

Choose $w\lambda \in \Lambda^+$, then we may obtain $w\mu$ from $\lambda$ by saturated roots. ■

Main questions on representation theory: In terms of Euclidean system, what's $\deg \lambda := \dim V(\lambda)$? What's $m_\lambda(\mu) := \dim V(\lambda)_\mu$? What's the irreducible decomposition of $V(\lambda_1) \otimes V(\lambda_2)$?

**Definition 13.6.** Let $\{k^i\} \subseteq H$ be the dual basis of $\{h_i\}$ (with respect to the killing form). For each $\alpha \in \Phi$, let $z_\alpha = \frac{(\alpha,\alpha)}{2} y_\alpha$ so that $[x_\alpha, z^\alpha] = t_\alpha = ((\alpha,\alpha)/2)h_\alpha$. We define the universal Casimir element $c_L := \sum_{i=1}^{\ell} h_i k^i + \sum_{\alpha \in \Phi} x_\alpha z^\alpha \in \mathfrak{U}(L)$.

Let $\phi : L \to \mathfrak{gl}(V)$ be a nontrivial representation. For $L$ simple, we get the ordinary Casimir element $c_\phi = a \cdot \phi(c_L)$ for some $a \in F$. Indeed, $\phi(x,y) := \operatorname{tr}(\phi(x)\phi(y))$ is nondegenerate and associative, and hence proportional to $\kappa(x,y)$ by Schur's lemma.

For $L = L_1 \oplus \cdots \oplus L_t$ semi-simple, $c_L = c_1 + \cdots + c_t$, $\phi(c_L)$ is not necessary proportional to $c_\phi$, but it commutes with $c_\phi$. So if $\phi$ is irreducible, $\phi(c_L)$ is scalar.

**Proposition 13.7** (traces on weight spaces)**.** Let $V = V(\lambda)$ for some $\lambda \in \Lambda^+$ with representation $\phi : L \to \mathfrak{gl}(V)$. Then for each $\mu \in \Pi(\lambda)$,

$$\operatorname{tr}(\phi(x_\alpha)\phi(z_\alpha); V_\mu) = \sum_{i=0}^{\infty} m_\lambda(\mu + i\alpha) \cdot (\mu + i\alpha, \alpha).$$

*Proof.* For $\alpha$ fixed, an irreducible $S_\alpha$-module $V(m)$ of highest weight $m$ has a basis $\{v_0, \ldots, v_m\}$, where $v_0 \in V_m$, $v_i = y^i \cdot v_0/i!$. Now we scale $v_i$: let $w_i = ((\alpha,\alpha)/2)^i i! \cdot v_i = z_0^i \cdot v_0$. Then

$$\begin{aligned} t_\alpha \cdot w_i &= (m - 2i)\frac{(\alpha,\alpha)}{2} \cdot w_i, \\ z_\alpha \cdot w_i &= w_{i+1}, \\ x_\alpha \cdot w_i &= i(m - i - 1)\frac{(\alpha,\alpha)}{2} \cdot w_{i-1}. \end{aligned}$$

Hence

$$\operatorname{tr}(\phi(x_\alpha)\phi(z_\alpha); V(m)) = \sum_i (i+1)(m-i)\frac{(\alpha,\alpha)}{2}.$$

Let $\mu \in \Pi(\lambda)$ with $\mu + \alpha \notin \Pi(\lambda)$. We get the $\alpha$-string through $\mu$: $\mu - m\alpha$, $\ldots$, $\mu$, where $m = \langle \mu, \alpha \rangle$. For $i$ between 0 and $\lfloor m/2 \rfloor$.

Consider the $S_\alpha$-module $W = V_{\mu - m\alpha} \oplus \cdots \oplus V_\mu$. Write $W = \bigoplus_{i=0}^{\lfloor m/2 \rfloor} V(m-2i)^{n_i}$. Let $0 \le k \le m/2$, $0 \le i \le k$. We see that

$$\phi(x_\alpha)\phi(z_\alpha)w_{k-i} = (k - i + 1)(m - 1 - k)\frac{(\alpha,\alpha)}{2} \cdot w_{k-i}.$$

Using the relation $\sum_{i=0}^{j} n_i = m_\lambda(\mu - j\alpha)$, we get

$$\begin{aligned}\operatorname{tr}(\phi(x_\alpha)\phi(z_\alpha); V_{\mu-k\alpha}) &= \sum_{i=0}^{k} n_i(k-i+1)(m-i-k)\frac{(\alpha,\alpha)}{2} \\ &= \sum_{i=0}^{k} m_\lambda(\mu - i\alpha)(m-2i)\frac{(\alpha,\alpha)}{2} \\ &= \sum_{i=0}^{k} m_\lambda(\mu - i\alpha)\cdot(\mu - i\alpha, \alpha).\end{aligned}$$

Reflection by $\sigma_\alpha$, we get the case $m/2 < k \le m$:

$$\begin{aligned}\operatorname{tr}(\phi(x_\alpha)\phi(z_\alpha); V_{\mu-k\alpha}) &= \sum_{i=0}^{m-k-1} m_\lambda(\mu - i\alpha)\cdot(\mu - i\alpha, \alpha) \\ &= \sum_{i=0}^{k} m_\lambda(\mu - i\alpha)\cdot(\mu - i\alpha, \alpha)\end{aligned}$$

by $(\mu - i\alpha, \alpha) = -(\mu - (m-i)\alpha, \alpha)$. This completes the proof. ∎

**Proposition 13.8** (Freudenthal's formula). The number $m(\mu) := m_\lambda(\mu)$ is given recursively by

$$\big((\lambda+\delta, \lambda+\delta) - (\mu+\delta, \mu+\delta)\big)\cdot m(\mu) = 2\sum_{\alpha\succ 0}\sum_{i=1}^{\infty} m(\mu + i\alpha)\cdot(\mu + i\alpha, \alpha).$$

*Proof.* Since $V$ is irreducible, $\operatorname{tr}(\phi(c_L); V_\mu) = c\cdot m(\mu)$, where $c$ is independent of $\mu$. By the definition of $c_L$,

$$\begin{aligned}\operatorname{tr}(\phi(c_L); V_\mu) &= \sum_{i=1}^{\ell}\phi(h_i)\phi(k^i) + \sum_{\alpha\in\Phi}\sum_{i=0}^{\infty} m(\mu + i\alpha)\cdot(\mu + i\alpha, \alpha) \\ &= m(\mu)\cdot(\mu,\mu) + \sum_{\alpha\in\Phi}\sum_{i=1}^{\infty} m(\mu + i\alpha)\cdot(\mu + i\alpha, \alpha),\end{aligned}$$

where the $i = 0$ term is cancelled for $\alpha$, $-\alpha$.

**Claim.** For each $\alpha \in \Phi$ and $\mu \in \Lambda$,

$$\sum_{i=-\infty}^{\infty} m(\mu + i\alpha)\cdot(\mu + i\alpha, \alpha) = 0.$$

Indeed, let $\mu - r\alpha$, …, $\mu + q\alpha$ be the $\alpha$-string through $\mu$. Since $\frac{q-r}{2} = -\frac{(\mu,\alpha)}{(\alpha,\alpha)}$ and

$m(\mu - (r-j)\alpha) = m(\mu + (q-j)\alpha)$,

$$\begin{aligned}\sum_{i=-\infty}^{\infty} m(\mu+i\alpha)\cdot(\mu+i\alpha,\alpha) &= \sum_{i<\frac{q-r}{2}} m(\mu+i\alpha)\cdot(\mu+i\alpha,\alpha)\\ &\quad+\sum_{i>\frac{q-r}{2}} m(\mu+i\alpha)\cdot(\mu+i\alpha,\alpha)\\ &=0.\end{aligned}$$

By the claim,

$$\begin{aligned}c\cdot m(\mu) &= (\mu,\mu)m(\mu)+\sum_{\alpha\succ 0}(\mu,\alpha)\cdot m(\mu)+2\sum_{\alpha\succ 0}\sum_{i=1}^{\infty} m(\mu+i\alpha)\cdot(\mu+i\alpha,\alpha)\\ &=(\mu,\mu+2\delta)\cdot m(\mu)+2\sum_{\alpha\succ 0}\sum_{i=1}^{\infty} m(\mu+i\alpha)\cdot(\mu+i\alpha,\alpha).\end{aligned}$$

For $\mu=\lambda$, we get $c=(\lambda,\lambda+2\delta)$. So the statement now follows from the identity

$$(\lambda+2\delta,\lambda)-(\mu+2\delta,\mu)=(\lambda+\delta,\lambda+\delta)-(\mu+\delta,\mu+\delta).$$

Also, $w\mu\prec\lambda$ for all $w\in\mathscr{W}$ implies that $(\mu+\delta,\mu+\delta)<(\lambda+\delta,\lambda+\delta)$. ■

# 14 Character theory, 10/26

Let $\lambda\in\Lambda^+$ be a weight, and let $V(\lambda)=\bigoplus_{\mu\in\Pi(\lambda)}V(\lambda)_\mu^{\oplus m_\lambda(\mu)}$ be the corresponding irreducible module. We define its formal character to be

$$\mathrm{ch}_\lambda=\mathrm{ch}_{V(\lambda)}=\sum_{\mu\in\Pi(\lambda)} m_\lambda(\mu)e(\mu)\in Z[\Lambda],$$

where $\{e(\mu)\}$ is a free basis of the group ring.

For a finite dimensional module $V\in\mathrm{Rep}\,L$, we define $\mathrm{ch}_V$ similarly. Then $\mathrm{ch}_{V\oplus V'}=\mathrm{ch}_V+\mathrm{ch}_{V'}$, and $\mathrm{ch}_{V\otimes V'}=\mathrm{ch}_V\cdot\mathrm{ch}_{V'}$. Hence, there is a homomorphism $\mathrm{ch}:\mathrm{Rep}\,L\to\mathbb{Z}[\Lambda]$.

Under the correspondence

$$\mathbb{Z}[\Lambda]\quad\longleftrightarrow\quad\mathbb{Z}^{\oplus\Lambda}=\{f:\Lambda\to\mathbb{Z}\mid f\text{ has finite support}\},$$

$e(\mu)$ corresponds to $e_\mu$ (or $\varepsilon_\mu$), where

$$e_\mu(\lambda)=\begin{cases}1, & \text{if }\lambda=\mu,\\ 0, & \text{if }\lambda\neq\mu.\end{cases}$$

**Definition 14.1.** (a) The **Kostant function** $p(\lambda)$ is the number of ways to write $\lambda = \sum_{\alpha \prec 0} k_\alpha \alpha$ with $k_\alpha \geq 0$.

(b) The **Weyl function** $q = \prod_{\alpha \succ 0}(e_{\alpha/2} - e_{-\alpha/2})$, where we view $e_{\alpha/2} = e(\alpha/2)$, $e_{-\alpha/2} = e(-\alpha/2) \in \mathbb{Z}[\Lambda/2]$, and

$$q = \sum_{\sigma \in \mathscr{W}} (-1)^{|\sigma|} e_{\sigma\delta} \in Z[\Lambda]$$

since $\delta = \frac{1}{2}\sum_{\alpha \succ 0} \alpha \in \Lambda$.

**Theorem 14.2** (Kostant)**.** For $\lambda \in \Lambda^+$,

$$m_\lambda(\mu) = \sum_{\sigma \in \mathscr{W}} (-1)^{|\sigma|} p(\mu + \delta - \sigma(\lambda + \delta)).$$

**Theorem 14.3** (Weyl character formula)**.** For $\lambda \in \Lambda^+$,

$$q \cdot \mathrm{ch}_\lambda = \sum_{\sigma \in \mathscr{W}} (-1)^{|\sigma|} e_{\sigma(\lambda+\delta)}.$$

**Corollary 14.4.** The degree of $\lambda$, i.e., $\dim V(\lambda)$, is equal to

$$\frac{\prod_{\alpha \succ 0}(\lambda + \delta, \alpha)}{\prod_{\alpha \succ 0}(\delta, \alpha)}.$$

**Theorem 14.5** (Steinberg)**.** For $\lambda'$, $\lambda'' \in \Lambda^+$, if we write $V(\lambda') \otimes V(\lambda'') = \bigoplus_{\lambda \in \Lambda^+} V(\lambda)^{\oplus d_\lambda}$, then

$$d_\lambda = \sum_{\sigma, \tau \in \mathscr{W}} (-1)^{|\sigma| + |\tau|} p(\lambda + 2\delta - \sigma(\lambda' + \delta) - \tau(\lambda'' + \delta)).$$

**Theorem 14.6** (Weyl)**.** Let $G$ be a compact Lie group. Then a two $G$-representations $(V, \rho)$, $(V', \rho')$ are isomorphic if and only if $\chi_\rho$

Harish-Chandra proved this result for semi-simple Lie algebras.

For a $L$-module $V$, let $P(V) = S(V^*)$. For example, $P(H)$ is spanned by pure powers $\lambda^k$ (exercise). For an element $f$, we define its symmetrization $\operatorname{Sym} f = \sum_{\sigma \in \mathscr{W}} f^\sigma$, where $f^\sigma(x) = \sigma \cdot f(x) = f(\sigma^{-1}x)$. Then $P(V)^{\mathscr{W}}$ is spanned by $\operatorname{Sym} \lambda^k$'s.

Let $G = \operatorname{Int} L = \langle e^{\operatorname{ad} x} \mid x \text{ nilpotent} \rangle$ acts on $P(V)$ in the obvious way. We get $P(V)^G$, the $G$-invariant polynomial functions.

**Theorem 14.7** (Chevalley). The map

$$\theta \colon P(L)^G \longrightarrow P(H)^{\mathscr{W}}$$

is surjective, where $\theta(f) = f|_H$.

**Definition 14.8.** For $\lambda \in H^\vee$, the character $\chi_\lambda : Z = Z(\mathfrak{U}(L)) \to F$ is defined by mapping $z \in Z$ to $z \cdot v^+/v^+$. Note that $z \cdot v^+ = a \cdot v^+$ for some $a$ since $h \cdot z \cdot v^+ = z \cdot h \cdot v^+ = z \cdot \lambda(h)v^+$ and $x_\alpha \cdot z \cdot v^+ = z \cdot x_\alpha \cdot v^+ = 0$.

**Proposition 14.9** (Linkage). For $\lambda$, $\mu \in H^\vee$, we say $\mu$ is equivalent to $\lambda$, denoted by $\mu \sim \lambda$, if $\lambda + \delta = w(\mu + \delta)$ for some $w \in \mathscr{W}$. Then $\lambda \sim \mu$ implies $\chi_\lambda = \chi_\mu$.

*Proof.* We have, by PBW bases, that

$$Z(\lambda) = \mathfrak{U}(L)/I(\lambda),$$

where $I(\lambda) = \mathfrak{U}(L)\langle x_\alpha, h_\alpha - \lambda(h_\alpha) \cdot 1\rangle$.

If $m := \langle \lambda, \alpha\rangle \geq 0$, $\overline{y}_\alpha^{m+1}$ is still a maximal vector, and is not 0 if $\lambda(\alpha_j) < 0$ for some $j$. For

$$\begin{aligned} \mu &= \sigma_\alpha(\lambda + \delta) - \delta \\ &= (\lambda - \langle \lambda, \alpha\rangle\alpha) - (\delta - (\delta - \alpha)) \\ &= \lambda - (m+1)\alpha, \end{aligned}$$

$Z(\lambda)$ contains image of $Z(\mu)$. This implies that $\chi_\lambda = \chi_\mu$.

If $m < 0$, then

$$\langle \mu, \alpha\rangle = \langle \lambda, \alpha\rangle - 2(\langle \lambda, \alpha\rangle + 1) = -\langle \lambda, \alpha\rangle - 2.$$

$m = -1$ is equivalent to $\mu = \lambda$, and $m \leq -2$ implies that $\langle \mu, \alpha\rangle \geq 0$, which reduce to the case $m \geq 0$. ■

# 15 The proof of Harish-Chandra's theorem and Kostant/Weyl formulas, 10/31

**Theorem 15.1** (Harish-Chandra). For $\lambda$, $\mu \in H^\vee$. If $\chi_\lambda = \chi_\mu$, then $\lambda \sim \mu$.

*Proof.* Let $\xi : \mathfrak{U}(L) \to \mathfrak{U}(H)$ via PBW bases. Let $v^+$ be a maximal vector of $V(\lambda)$. Then

$$\prod_{\alpha \succ 0} y_\alpha^{i_\alpha} \prod_i h_i^{k_i} \prod_{\alpha \succ 0} x_\alpha^{j_\alpha} v^+ = 0$$

if there exists $j_\alpha > 0$, or maps to lower weight vector if there exists $i_\alpha > 0$. Hence, the only bases contribute $\chi_\lambda(z)$ are from $\mathfrak{U}(H)$, i.e., $\chi_\lambda(z) = \lambda(\xi(z))$ for $z \in Z$. Here, we extend $\lambda \in H^\vee$ to $\lambda : \mathfrak{U}(H) \to F$.

Consider the Lie algebra homomorphism

$$\begin{tikzcd} H \arrow[r] \arrow[dr, hook, "i"'] & \mathfrak{U}(H) \\ & \mathfrak{U}(H). \arrow[u, "\eta", "h_i \mapsto h_i - 1"'] \end{tikzcd}$$

Let

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[rrr, bend right, "\psi"'] & \mathfrak{U}(L) \arrow[r, "\xi"] & \mathfrak{U}(H) \arrow[r, "\eta"] & \mathfrak{U}(H). \end{tikzcd}$$

Since $\delta = \frac{1}{2} \sum_{\alpha \succ 0} \alpha = \sum \lambda_i$,

$$(\lambda + \delta)(h_i - 1) = \lambda(h_i) + 1 - (\lambda + \delta) \cdot 1 = \lambda(h_i).$$

So

$$(\lambda + \delta)(\psi(z)) = \lambda(\xi(z))) = \chi_\lambda(z).$$

If $\lambda \in \Lambda$, all $\mathscr{W}$-conjugates of $\mu = \lambda + \delta$ are equal at $\psi(z)$, so $\mathscr{W}$ fixes $\psi(z)$ for each $z \in Z$. Hence, there is a homomorphism $\psi : Z \to S(H)^{\mathscr{W}}$. Thus, if $\lambda \sim \mu$, then $\chi_\lambda = \chi_\mu$ for all $\lambda$, $\mu \in H^\vee$.

Conversely, let $\chi_\lambda = \chi_\mu$. Then $\lambda + \delta = \mu + \delta$ on $\psi(Z) \subseteq S(H)^{\mathscr{W}}$. If $\psi(Z) = S(H)^{\mathscr{W}}$, then

$$\lambda + \delta = w(\mu + \delta)$$

for some $w \in \mathscr{W}$ and done!

Let $G = \operatorname{Int} L$. Recall that $S(L) \cong \mathfrak{U}(L)$ only as $G$-module (not algebra). So we have a diagram via the isomorphism $H^\vee \xrightarrow{\sim} H$ induced by the killing form:

$$\begin{tikzcd} \mathfrak{U}(L)^G \arrow[r] \arrow[d, leftrightarrow] & S(H)^{\mathscr{W}} \arrow[d, leftrightarrow] \\ P(L)^G \arrow[r] & P(H)^{\mathscr{W}}, \end{tikzcd}$$

where $P(-)$ is the polynomial function functor.

**Lemma 15.2.** The center $Z = Z(\mathfrak{U}(L))$ is equal to $\mathfrak{U}(L)^G$.

*Proof of Lemma.* Let $z \in Z$. We see that $e^{\operatorname{ad} x} z = z$ and hence $\sigma(z) = z$ for each $\sigma \in G$. Conversely, let $x \in \mathfrak{U}(L)^G$ and let $n = \operatorname{ad} x_\alpha$ with $n^t \neq 0$, $n^{t+1} = 0$. Take distinct numbers $a_1, \dots, a_{t+1} \in F$. Then

$$e^{a_i n} = 1 + a_i n + \cdots + \frac{a_i^t}{t!} n^t \in G,$$

and

$$n = b_1 e^{a_1 n} + \cdots + b_{t+1} e^{a_{t+1} n}$$

for some $b_i$'s. So

$$(\operatorname{ad} x_\alpha)(x) = \left( \sum_{i=1}^{t+1} b_i \right) x$$

and $\sum b_i = 0$ since $n$ is nilpotent. Hence, $[x_\alpha, x] = 0$. Since $\alpha$ is arbitrary, $x \in Z$. $\square$

$\blacksquare$

To apply it, let $\mathfrak{X}$ be the space of functions $f : H^\vee \to F$ supported on region of the form $\lambda = \sum_{\alpha \succ 0} \mathbb{Z}_{\geq 0} \alpha$.

Let $\theta(\lambda) = \{\mu \in H^\vee \mid \mu \prec \lambda, \mu \sim \lambda\}$.

**Main example.** $\operatorname{ch}_{Z(\lambda)} \in \mathfrak{X}$. We compute $\operatorname{ch}_\lambda = \operatorname{ch}_{V(\lambda)}$ via $\operatorname{ch}_{Z(\mu)}$'s within $\mathfrak{X}$. By Harish-Chandra's theorem, an easy induction shows that $Z(\lambda)$ has a composition series with factor of the form $V(\mu)$, $\mu \in \theta(\lambda)$. Reversing it! By triangular system, we write

$$\operatorname{ch}_{V(\lambda)} = \sum_{\mu \in \theta(\lambda)} c(\mu) \operatorname{ch}_{Z(\mu)},$$

where $c(\mu) \in \mathbb{Z}$ and $c(\lambda) = 1$. For $\lambda \in \Lambda^+$, $\sigma(\operatorname{ch}_\lambda) = \operatorname{ch}_\lambda$ for each $\sigma \in \mathscr{W}$. We have

$$\sigma(q * \operatorname{ch}_\lambda) = \sigma(q) * \sigma(\operatorname{ch}_\lambda) = (-1)^{|\sigma|} q * \operatorname{ch}_\lambda.$$

Also,

- $\operatorname{ch}_{Z(\lambda)}(\mu) = P(\mu - \lambda) = (P * e_\lambda)(\mu)$;
- $q * p * e_{-\delta} = e_\delta * \prod_{\alpha \succ 0} (e_0 - e_{-\alpha}) * p * e_{-\delta}$
  $= \prod_{\alpha \succ 0} (e_0 - e_{-\alpha}) \prod_{\alpha \succ 0} (e_0 + e_{-\alpha} + e_{-2\alpha} + \cdots) = e_0$.

Hence, $q * \mathrm{ch}_{Z(\lambda)} = e_{\lambda+\delta}$, and thus

$$q * \mathrm{ch}_{V(\lambda)} = \sum_{\mu \in \theta(\lambda)} c(\mu) e_{\mu+\delta}.$$

Since $\mathscr{W}$ acts on $\{\mu + \delta \mid \mu \in \theta(\lambda)\}$ transitively, $c(\mu) = (-1)^{|\sigma|}$, where $\sigma(\mu + \delta) = \lambda + \delta$. So we get

$$q * \mathrm{ch}_{\lambda} = \sum_{\sigma \in \mathscr{W}} (-1)^{|\sigma|} e_{\sigma(\lambda+\delta)}.$$

**Definition 15.3.** (a) A **Lie group** $G$ is a ($C^\infty$) manifold such that its group law

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, h) &\longmapsto gh^{-1} \end{aligned}$$

is $C^\infty$.

(b) $f : G \to H$ is a Lie group homomorphism if it is a group homomorphism and $C^\infty$.

(c) If $f$ is an immersion, i.e., the tangent map $df_a \colon T_a G \to T_{f(a)} H$ is injective, we call $G \hookrightarrow H$ an (immersed) Lie subgroup.

If $f(G) \subseteq H$ is closed, then $\mathsf{Top}(G)$ is diffeomorphic to $\mathsf{Top}(H)|_{f(G)}$.

**Main example.** $\mathrm{GL}(n, F) \subseteq \mathrm{M}_{n\times n}(F) \cong F^{n^2}$. Since $y^{-1} = \operatorname{adj} y / \det y$, $y^{-1}$ is a rational function in $y_i^j$'s, which is $C^\infty$ outside $\det^{-1}(0)$. Hence, $\mathrm{GL}(n, F)$ is a Lie group (in fact an algebraic group).

For the quaternion numbers $\mathbb{H}$, we define

$$\begin{aligned} \mathrm{M}_{n\times n}(\mathbb{H}) &= \{g : \mathbb{H}^n \to \mathbb{H}^n \text{ (right) linear over } \mathbb{H}\}, \\ \mathrm{GL}(n, \mathbb{H}) &= \{g \in \mathrm{M}_{n\times n}(\mathbb{H}) \text{ invertible}\}. \end{aligned}$$

If we write $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$:

$$a + bi + cj + dk = (a + bi) + j(c - di),$$

then we can view $\mathrm{GL}(n, \mathbb{H})$ as a subgroup of $\mathrm{GL}(2n, \mathbb{C})$: since

$$(u + jv) \cdot j = j\overline{u} - \overline{v} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} =: J \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix},$$

$g \in \mathrm{M}_{n\times n}(\mathbb{H})$ if and only if

$$g \in \mathrm{GL}(2n, \mathbb{C})_{\mathbb{H}} := \{Y \in \mathrm{M}_{n\times n}(\mathbb{C}) \mid YJ = J\overline{Y}\} = \left\{ Y = \begin{pmatrix} A & -\overline{B} \\ B & -\overline{A} \end{pmatrix} \right\}.$$

**Compact Lie groups.**

$$\mathrm{O}(n) = \{g \in \mathrm{GL}(n,\mathbb{R}) \mid g^{\mathsf{T}} g = \mathrm{id}\} \supseteq \mathrm{SO}(n) = \{g \in \mathrm{O}(n) \mid \det g = 1\},$$
$$\mathrm{U}(n) = \{g \in \mathrm{GL}(n,\mathbb{C}) \mid g^* g = \mathrm{id}\} \supseteq \mathrm{SU}(n) = \{g \in \mathrm{U}(n) \mid \det g = 1\},$$

where $g^* = \overline{g}^{\mathsf{T}}$. Since $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are defined by polynomials, we can define $\mathrm{O}(n,F)$ and $\mathrm{SO}(n,F)$ over every field $F$.

The **symplectic group** is defined by

$$\mathrm{Sp}(n) = \{g \in \mathrm{M}_{n\times n}(\mathrm{H}) \mid g^* g = \mathrm{id}\} \subseteq \mathrm{GL}(n,\mathbb{H}),$$

where $\overline{a+bi+cj+dk} = a - bi - cj - dk$, i.e., $g \in \mathrm{Sp}(n)$ preserves the inner product $(z,w) = \sum \overline{z}_i w_i$. Under the identification $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$, we have

$$\mathrm{Sp}(n) = \mathrm{SU}(2n) \cap \mathrm{M}_{2n\times 2n}(\mathbb{C})_{\mathbb{H}} = \mathrm{SU}(2n) \cap \mathrm{Sp}_{2n},$$

where

$$\mathrm{Sp}_{2n} := \{g \in \mathrm{GL}(n,\mathbb{C}) \mid g^{\mathsf{T}} J g = J\}.$$

(Note that under the condition $g^* g = 1$, $gJ = J\overline{g}$ if and only if $g^{\mathsf{T}} J g = J$.)

By definition, $\mathrm{Sp}(1) = \mathrm{SU}(2) \cong S^3$, where $\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}$ is mapped to $(a,b) \in \mathbb{C}^2 \cong \mathbb{R}^4$. In fact, there is a 2-1 cover from $\mathrm{Sp}(1)$ to $\mathrm{SO}(3)$. Moreover, since $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2\mathbb{Z}$ for all $n \geq 3$, there exists a simply connected double cover $\mathrm{Spin}_n(\mathbb{R}) \to \mathrm{SO}(n)$ called the spin group. When $n = 3$, $\mathrm{Spin}_3(\mathbb{R})$ is just $\mathrm{Sp}(1)$.

**Definition 15.4.** The **Clifford algebra** on $V = (\mathbb{R}^n, (-,-))$ is

$$\mathrm{Cl}_n(\mathbb{R}) = \mathrm{Cl}(V) := T(V)/\langle x \otimes x + (x,x)\rangle,$$

i.e., $xy + yx = -2(x,y)$.

**Examples.** $\mathrm{Cl}_0(\mathbb{R}) \cong \mathbb{R}$, $\mathrm{Cl}_1(\mathbb{R}) \cong \mathbb{C}$, $\mathrm{Cl}_2(\mathbb{R}) \cong \mathbb{H}$.

Let $e_1$, …, $e_n$ be a basis of $V$. Then $\mathrm{Cl}(V)$ has basis $\{e_{i_1} \cdots e_{i_k} \mid i_1 < \cdots < i_k\}$. As a vector space, $\mathrm{Cl}(V)$ is isomorphic to $\bigwedge V$.

**Definition 15.5.** Clifford module structure on $\bigwedge V$: for $x \in V$, $c(x) = \epsilon(x) - \iota(x) = (x\wedge) - (x \lrcorner)$. Here,

$$x \lrcorner (y_1 \wedge \cdots \wedge y_k) = \sum_{i=1}^{k} (-1)^{i-1} (x, y_i) y_1 \wedge \cdots \wedge \widehat{y_i} \wedge \cdots y_k.$$

By checking on standard basis, we can show that $c(x)^2 = -(x,x)$.

**Definition 15.6.** We define the homomorphisms

$$\begin{aligned} \Phi\colon \quad \mathrm{Cl}(V) &\longrightarrow \mathrm{End}(\textstyle\bigwedge V) \\ x_1 \cdots x_k &\longmapsto c(x_1) \cdots c(x_k) \end{aligned}$$

and

$$\begin{aligned} \Psi\colon \quad \mathrm{Cl}(V) &\longrightarrow \textstyle\bigwedge V \\ v &\longmapsto v \cdot 1. \end{aligned}$$

Now, we construct $\mathrm{Spin}_n(\mathbb{R})$:

**Facts.** $\mathrm{Sp}(n)$ for $n \geq 1$ and $\mathrm{SU}(n)$ for $n \geq 2$ are simply connected. $\pi_1(\mathrm{SO}(2)) = \mathbb{Z}$, $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$. Indeed, for a Lie group $G$ and its Lie subgroup $H$, we can consider the homogeneous space (coset space) $G/H$. There is a fiber bundle

$$\begin{tikzcd} H \arrow[r] & G \arrow[d, "\pi"] \\ & G/H, \end{tikzcd}$$

so hence an induced long exact sequence

$$\cdots \longrightarrow \pi_k(H) \longrightarrow \pi_k(G) \longrightarrow \pi_k(G/H) \longrightarrow \pi_{k-1}(H) \longrightarrow \cdots .$$

For the case $G = \mathrm{SO}(n)$ and $G/H = S^{n-1}$, $H \cong \mathrm{Stab}(x) \cong \mathrm{SO}(n-1)$ for all $x \in G/H$. Thus, the statement $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2\mathbb{Z}$ for $n \geq 4$ is equivalent to $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2\mathbb{Z}$.

To show that $\mathrm{SO}(3) \cong S^3/\{\pm 1\}$, we note that $\mathrm{SO}(3) = \mathrm{O}(\mathrm{Im}\,\mathbb{H})^\circ$. So the adjoint map

$$\mathrm{Ad}\colon \mathrm{Sp}(1) \longrightarrow \mathrm{SO}(3),$$

where

$$\mathrm{Ad}(g)(u) = gug^{-1} = gu\overline{g},$$

is well-defined. For $\{i, j, k\}$ is an orthogonal basis of $\mathrm{Im}\,\mathbb{H}$. By checking on this basis, $\mathrm{Ad}(\cos\theta + v\sin\theta)$ is equal to the rotation $R_{2\theta}$ in $i$-$j$ plane. We see that Ad is surjective and $\ker \mathrm{Ad} = \{\pm 1\}$. Hence, $\mathrm{Spin}_3(\mathbb{R}) = \mathrm{SU}(2) = \mathrm{Sp}(1) = S^3$.

**Definition 15.7.** Write $\mathrm{Cl}(V) = \mathrm{Cl}(V)^+ \oplus \mathrm{Cl}(V)^-$ (under the identification $\bigwedge V = (\bigwedge V)^+ \oplus (\bigwedge V)^-$). There is a main involution $\alpha$ defined by

$$\alpha(x_1 \cdots x_k) = x_1 \cdots x_k.$$

It is easy to see that $\alpha$ is a homomorphism. The conjugation on $\mathrm{Cl}(V)$ is defined to be

$$(x_1 \cdots x_k)^* = \alpha(x_k \cdots x_1).$$

The spin group and the pin group are now defined to be

$$\mathrm{Spin}(V) = \{g \in \mathrm{Cl}(V)^+ \mid gg^* = \mathrm{id},\ gVg^* = V\}$$
$$\mathrm{Pin}(V) = \{g \in \mathrm{Cl}(V) \mid gg^* = \mathrm{id},\ gVg^* = V\}.$$

These groups lie in $\mathrm{Cl}(V)^\times$, and hence are Lie subgroups.

**Theorem 15.8.** There are exact sequences

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Pin}_n(\mathbb{C}) \xrightarrow{\rho} \mathrm{O}(n) \longrightarrow 1,$$
$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}_n(\mathbb{C}) \xrightarrow{\rho} \mathrm{SO}(n) \longrightarrow 1,$$

where $\rho(g)(v) = \alpha(g)vg^*$. Moreover, $\mathrm{Pin}_n(\mathbb{R})$ has 2 connected components and $\mathrm{Pin}_n(\mathbb{R}) = \mathrm{Spin}_n(\mathbb{R})^\circ$.

*Proof.* For $\mathrm{Pin}_n(\mathbb{R})$,

$$|\rho(g)x|^2 = -\alpha(g)xg^*(\alpha(g)xg^*)^* = \alpha(g)xg^*g^{**}x^*\alpha(g)^* = \alpha(g)|x|^2\alpha(g)^*$$

- $\rho$ surjects reflections: $r_x := \rho(x)$.
- $\ker\rho = \{\pm 1\}$: it suffices to show $\ker\rho \subseteq \mathbb{R}$. Let $g \in \ker\rho$, so that $\alpha(g)x = xg$ for all $x \in V$. Write $g = e_1a + b$, where $b$ has no $e_1$ in its products. Take $x = e_1$, we get

  $$-e_1\alpha(a)e_1 + \alpha(b)e_1 = -a + e_1b.$$

  Since $-e_1\alpha(a)e_1 = a$, $\alpha(b)e_1 = e_1b$, we get $a = 0$. By symmetry, there is no $e_i$ component in $g$ for each $i$. Hence, $g \in \mathbb{R}$.

So

$$\mathrm{Pin}_n(\mathbb{R}) = \{x_1 \cdots x_k \mid |x_i| = 1,\ k \le 2n\}$$

and

$$\mathrm{Spin}_n(\mathbb{R}) = \{x_1 \cdots x_k \mid |x_i| = 1,\ k \text{ even}\}.$$

Finally, $\mathrm{Spin}_n(\mathbb{R})$ is connected (for $n \ge 2$):

$$\gamma(t) = \cos t + e_1e_2\sin t = e_1(-e_1\cos t + e_2\sin t) \in \mathrm{Spin}_n(\mathbb{R})$$

connects $\ker\rho = \{\pm 1\}$. Also, $\mathrm{Pin}_n(\mathbb{R}) = x\,\mathrm{Spin}_n(\mathbb{R}) \sqcup \mathrm{Spin}_n(\mathbb{R})$ for any $x \in S^{n-1}$. ∎

# 16 Integration, 11/7

**Proposition 16.1.** Let $G$ be a connected Lie group. Then $G = \bigcup_{n\geq 1} U^n$, where $U$ is any neighborhood of the identity $e \in G$. In particular, $G$ is second countable.

*Proof.* Let $V = U \cap U^{-1}$, which is open, $H = \bigcup_{n\geq 1} V^n \subseteq G$. For each $g \in G$, $gH$ is also open. Hence, $G = \bigsqcup_{\alpha\in G/H} g_\alpha H$. Since $G$ is connected, $G = eH = H$. ■

**Proposition 16.2.** Let $H$ be a discrete normal subgroup of a connected Lie group $G$. Then $H$ lies in the center of $G$.

*Proof.* For $h \in H$, consider the set $C_h = \{ghg^{-1} \mid g \in G\} \subseteq H$. Since $G$ is connected, $C_h$ is connected. Since $H$ is discrete, $C_h = \{h\}$, which implies $h \in Z(G)$. ■

**Theorem 16.3.** Let $G$ be a connected Lie group. The universal cover $\widetilde{G}$ of $G$ is a Lie group, such that the canonical map $\pi : \widetilde{G} \to G$ is a group homomorphism. In particular, $K := \ker \pi$ is a normal discrete subgroup of $G$, hence abelian.

*Proof.* We only need to define the Lie group structure on $\widetilde{G}$. Fix $\widetilde{e} \in \pi^{-1}(e)$. Consider

$$\begin{aligned} M : \widetilde{G} \times \widetilde{G} &\xrightarrow{\quad s \quad} G \\ (\widetilde{g}, \widetilde{h}) &\longrightarrow \pi(\widetilde{g})\pi(\widetilde{h})^{-1}. \end{aligned}$$

There exists a unique map $\widetilde{s}$: $M \to \widetilde{G}$ such that $\pi \circ \widetilde{s} = s$. This $\widetilde{s}$ defines the group structure on $\widetilde{G}$ (and that $\pi$ is a group homomorphism). ■

**Example.** Let $G$ be a Lie group. Then $\pi_k(G)$ is abelian for each $k \geq 1$, $\pi_0(G) \cong G/G^\circ$, where $G^\circ$ is the connected component of $G$. The composition law in $\pi_k$ is equal to the group law in $G$.

Indeed, let $\phi_1$, $\phi_2 : (I^k, \partial I^k) \to (G, e)$ be 2 continuous maps. Then

$$\phi_1 * \phi_2 \sim (\phi_1 * \phi_0) * (\phi_0 * \phi_2) = \phi_1 \cdot \phi_2,$$

where the $\cdot$ is the group law in $G$.

To show that $\pi_k$ is abelian for $k \geq 2$, simply note that

$$\begin{array}{|c|c|}\hline \phantom{\mathrm{id}} & \phantom{\mathrm{id}} \\ \phi_1 & \phi_2 \\ \phantom{\mathrm{id}} & \phantom{\mathrm{id}} \\ \hline \end{array} \sim \begin{array}{|c|c|}\hline \phi_1 & \mathrm{id} \\ \hline \mathrm{id} & \phi_2 \\ \hline \end{array} \sim \begin{array}{|c|c|}\hline \mathrm{id} & \phi_1 \\ \hline \phi_2 & \mathrm{id} \\ \hline \end{array} \sim \begin{array}{|c|c|}\hline \phantom{\mathrm{id}} & \phantom{\mathrm{id}} \\ \phi_2 & \phi_1 \\ \phantom{\mathrm{id}} & \phantom{\mathrm{id}} \\ \hline \end{array}.$$

**Fact.** The tangent bundle $TG$ is trivial, i.e., $TG \cong_{C^\infty} G \times T_eG$, for example, via left invariant vector fields. For $v \in T_eG$, let $\widetilde{v}(g) = \ell_{g*}v$, where $\ell_g$ is the left translation, while $r_g$ is the right translation. $\widetilde{v}$ is a left invariant vector fields by its value at $T_eG$. Using this construction, we can also define left invariant metric $\langle -, - \rangle$, left invariant volume form, denoted by $\omega_g = dg$, unique up to scalar. If $G$ is compact, we can choose a unique $dg$ such that

$$\int_G dg = 1.$$

**Theorem 16.4.** If $G$ is compact, then $dg$ is also right invariant and inversion invariant.

*Proof.* Since $dg$ is left invariant,

$$\ell_g^*(r_h^* dg) = r_h^* \ell_g^* dg = r_h^* dg$$

is also left invariant, and hence there exists $c(h) \in \mathbb{R}^\times$ such that $r_h^* dg = c(h)^{-1}\, dg$. Then $c\colon G \to \mathbb{R}^\times$ is a homomorphism. Since $G$ is compact, $\operatorname{Im} c \subseteq \{\pm 1\}$. Note that $c(h) = -1$ if and only if $r_h$ is orientation reversing.

Now,

$$\int_G f(gh)\, dg = \int_G f(gh)\, d(gh) \cdot c(h) = \int_G f(g)\, dg. \qquad \blacksquare$$

**Theorem 16.5** (Fubini)**.** Let $G$ be a compact Lie group, $H \subseteq G$ a closed subgroup. If $\ell_h^* = \mathrm{id}$ on $\bigwedge^{\mathrm{top}}(G/H)_{\overline{e}}$, then $G/H$ has a unique left invariant volume form $\omega_{G/H} = d(gH) = d\overline{g}$ such that

$$\int_{G/H} F\, d\overline{g} = \int_G (F \circ \pi)\, dg,$$

where $\pi\colon G \to G/H$ is the quotient map. Moreover,

$$\int_G f(g)\, dg = \int_{G/H} \int_H f(gh)\, dh\, d(gH).$$

# 17 Representation of Lie groups, 11/9

A group representation $(\pi, V)$ of $G$ is a (continuous) homomorphism $\pi : G \to \mathrm{GL}(V)$, where $G$ is a Lie group and $V$ is a finite dimensional vector space over $\mathbb{C}$. For two representations $(\pi, V)$, $(\pi', V')$, the set of morphisms between them are

$$\mathrm{Hom}_G(V, V') = \{T\colon V \to V' \mid T \circ \pi(g) = \pi'(g) \circ T,\, \forall g \in G\}.$$

**Examples.**

1) Standard representation: If $G$ is a subgroup of $\mathrm{GL}(n, F)$, $F = \mathbb{R}$, $\mathbb{C}$, then the inclusion $G \hookrightarrow \mathrm{GL}(n, F)$ is a representation, where $V = \mathbb{C}^n$. Also, $G$ acts on functions on $V$ by $(g \cdot f)(v) = f(g^{-1}v)$.

2) Let $V_m(\mathbb{R}^n) = \mathbb{R}[x_1, \ldots, x_n]_m$, the space of homogeneous degree $m$ polynomials. We see that $\dim V_m(\mathbb{R}^n) = \binom{n+m-1}{m}$. Let $G = \mathrm{O}(n) \subseteq \mathrm{GL}(n, \mathbb{R})$. Then elements in $G$ commutes with the Laplacian $\triangle = \sum \partial_i^2$, i.e.,

$$\triangle(g \cdot f) = g(\triangle f).$$

   Hence, $G$ acts on the harmonic polynomials $\mathscr{H}_m(\mathbb{R}^n) = \{f \in V_m(\mathbb{R}^n) \mid \triangle f = 0\}$.

3) Consider the action of $G = \mathrm{SU}(2)$ on $V_n(\mathbb{C}^2) = \mathbb{C}[z_1, z_2]_2$. This is an irreducible representation. In fact,

$$g \cdot f = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \cdot z_1^k z_2^{n-k} = z_1^k z_2^{n-k} \circ \begin{pmatrix} \overline{a} & \overline{b} \\ -b & a \end{pmatrix} = (\overline{a}z_1 + \overline{b}z_2)^k(-bz_1 + az_2)^{n-k}$$

   and it is easy to see that every nonzero element in $V_n(\mathbb{C}^2)$ generates $V_n(\mathbb{C}^2)$ under $G$.

   Alternatively, consider $V_n' = \mathrm{Hol}_0(\mathbb{C})_{\leq n} = \{a_0 + a_1 z + \cdots a_n z^n\}$, which is isomorphic to $V_n(\mathbb{C}^2)$ as an vector space via Möbius transformation. Hence, the action of $G$ on $V_n'$ is

$$(g \cdot f)(z) = (-bz + a)^n f\left(\frac{\overline{a}z + \overline{b}}{-bz + a}\right).$$

   Since holomorphic functions on $\mathbb{C}$ corresponds to harmonic functions on $\mathbb{R}^2$, we know that $\mathscr{H}_m(\mathbb{R}^2) = 2$.

4) Consider the 2-1 cover $G = \operatorname{Spin}_n(\mathbb{R}) \to \mathrm{SO}(n)$. A genuine representation is a representation not from $\mathrm{SO}(n)$. Let $V = (\mathbb{R}^n, (-,-)) \otimes \mathbb{C}$, where $(z, w) = \sum z_i w_i$. Let $m = \lfloor \frac{n}{2} \rfloor$. We can write $V = W \oplus W'$ if $n = 2m$ and $V = W \oplus W' \oplus \mathbb{C}e_n$ if $n = 2m+1$, where

$$W = \{(z_1, \dots, z_m, iz_1, \dots, iz_m)\}, \quad W' = \{(z_1, \dots, z_m, -iz_1, \dots, -iz_m)\}.$$

**Theorem 17.1.** Let $S = \bigwedge^\bullet(W)$ be the spinor. Then

$$\mathrm{Cl}(V) \cong \begin{cases} \operatorname{End} S, & \text{if } n = 2m, \\ \operatorname{End} S \oplus \operatorname{End} S, & \text{if } n = 2m+1 \end{cases}$$

as an algebra. Since $\operatorname{Spin}(\mathbb{R})$ is a subset of $\mathrm{Cl}(V)$, we get a faithful representation of $\operatorname{Spin}_n(\mathbb{R})$.

*Proof.* For $n$ even, define $\varphi : V \to \operatorname{End} S$ by $\varphi(z) = \alpha\epsilon(w) - \beta\iota(w')$, where $z = w = w'$ with $w \in W$, $w' \in W'$ and $\alpha$, $\beta$ are two numbers such that $\alpha\beta = 2$. We see that

$$\varphi(z)^2 = -2(\epsilon(w)\iota(w') + \iota(w')\epsilon(w)) = -2(w, w') = -(z, z),$$

and hence $\varphi$ defines a map $\mathrm{Cl}(V) \to \operatorname{End} S$. Note that $\dim \mathrm{Cl}(V) = \dim \operatorname{End} S$. Hence, to show that it is an isomorphism, it suffices to show that it is surjective.

Take a basis $\{w_i\}$ of $W$ and a basis $\{w_i'\}$ of $W'$ such that $(w_i, w_j') = \delta_{ij}$. Note that $w_{i_1} \cdots w_{i_k} w_{i_1}' \cdots w_{i_k}'$ maps $\bigwedge^p W$ to 0 if $p < k$, onto $w_{i_1} \wedge \cdots \wedge w_{i_k}$ if $p = k$, and an induction shows that it is surjective it $p > k$.

For $n$ odd, write $z = w + w' + \zeta e_n$ and define

$$\varphi^\pm(z) = \alpha\epsilon(w) - \beta\iota(w') \pm (-1)^p i\zeta$$

on $\bigwedge^p W$. Again, these defines maps $\varphi^\pm\colon \mathrm{Cl}(V) \to \operatorname{End}(S)$ and these maps are surjective. ■

**Theorem 17.2.** As an algebra,

$$\mathrm{Cl}(V) \cong \begin{cases} \operatorname{End} S^+ \oplus \operatorname{End} S^-, & \text{if } n = 2m, \\ \operatorname{End} S, & \text{if } n = 2m+1. \end{cases}$$

*Proof.* For $n$ even, $\varphi$ preserves $S^{\pm}$ on $\mathrm{Cl}^+(V)$. So $\varphi : \mathrm{Cl}^+(V) \hookrightarrow \operatorname{End} S^+ \oplus \operatorname{End} S^-$. Since they have same dimensions, $\varphi$ is an isomorphism.

For $n$ odd, the definition of $\varphi^{\pm}$ mixes degree. So $\varphi^{\pm}$ does not preserve $S^{\pm}$. But take one piece $\varphi^+$ and dimension count, we still get an isomorphism. ∎

**Example.** For $n = 3$, $m = 1$, $\mathrm{Spin}_3(\mathbb{R}) = \mathrm{SU}(2) = S^3$. $S = \bigwedge W = \mathbb{C}^2$ and there is a map $\mathrm{Spin}_3(\mathbb{R}) \to \operatorname{End} S = M_{2\times 2}(\mathbb{C})$.

Since $-1 \in \mathrm{Spin}_n(\mathbb{R}) \subseteq \mathrm{Cl}^+(V)$ maps to $1 \in \mathrm{SO}(n)$, and $-1$ is nontrivial on $S$, $S$ is a genuine module.

## 18 Representation of Lie groups II, 11/21

Let $G$ acts on finite dimensional $\mathbb{C}$-vector spaces $V$, $W$. There is a natural action on $V \otimes_{\mathbb{C}} W$ by Leibniz rule:

$$g \cdot (v \otimes w) = gv \otimes w + v \otimes gw.$$

Let $\rho \colon G \to \mathrm{GL}(V)$ be the representation, $\mathcal{B} = \{v_1, \dots, v_n\}$ a basis of $V$. Write $M_g = [\rho(g)]_{\mathcal{B}}^{\mathcal{B}}$. Then $(M_g)_i^j = v^j(gv_i)$, where $\mathcal{B}^\vee = \{v^i\} \subseteq V^\vee$ is the dual basis of $\mathcal{B}$. Hence,

$$((M_g^\vee)^\mathsf{T})_i^j = v_i(gv^j) = v^j(g^{-1}v_i) = (M_{g^{-1}})_i^j = (M_g^{-1})_i^j,$$

i.e., $M_g^\vee = (M_g^{-1})^\mathsf{T}$.

For $\overline{V}$, the same abelian group as $V$ but with different $G$-module structure: $z \odot v = \overline{z} \cdot v$, where $\odot$, $\cdot$ denote the multiplications on $\overline{V}$, $V$, respectively. Then there is a representation $\overline{\rho} \colon G \to \overline{V}$.

For $G$ compact, there exists a $G$-invariant inner product $(-,-)$ on $V$ by taking

$$(v, w) = \int_G \langle gv, gw \rangle \, dg,$$

where $\langle -, - \rangle$ is any inner product on $V$. We may choose $v_i$ to be an orthonormal (unitary) basis. Then $\rho$ maps $G$ into $U(n) \subseteq \mathrm{GL}(n) \cong \mathrm{GL}(V)$. Hence, $\rho(g)^{-1} = \overline{\rho(g)}^\mathsf{T}$ and as $G$-modules, $V^\vee \cong \overline{V}$. Also, we get Weyl's completely reducibility theorem: for a $G$-submodule $W \subseteq V$, we see that $W^\perp \subseteq V$ is also a $G$-module. We say that a $G$-module $V$ is irreducible if every $G$-submodule of $V$ is either $\{0\}$ or $V$.

**Theorem 18.1** (Schur's Lemma)**.** Let $V$, $W$ be irreducible finite dimensional $G$-modules. Then

$$\operatorname{Hom}_G(V, W) = \begin{cases} \mathbb{C}, & \text{if } V \cong W, \\ 0, & \text{else.} \end{cases}$$

*Proof.* For a nonzero $G$-homomorphism $T \in \operatorname{Hom}_G(V, W)$, $\ker T = 0$ and $\operatorname{Im} T = W$. So $V \cong W$ as $G$-modules. Fix a $G$-isomorphism $T_0 \colon V \to W$. For any $T \colon V \to W$, since $\det(TT_0^{-1} - \lambda I) \neq 0$, we get $TT_0^{-1} = \lambda I$ for some $\lambda$. ■

**Corollary 18.2.** Let $G$ be a compact Lie group. Then a finite dimensional $G$-module $V$ is irreducible if and only if $\operatorname{Hom}_G(V, V) \cong \mathbb{C}$. In this case, the $G$-invariant inner product $(-, -)$ is unique up to scalar.

*Proof.* If $V$ is not irreducible, say $V = V_1 \oplus V_2$ with $V_1$, $V_2 \neq 0$, then

$$\dim \operatorname{Hom}_G(V, V) \geq \dim \operatorname{Hom}_G(V_1, V_1) + \dim \operatorname{Hom}_G(V_2, V_2) \geq 2.$$

Given two $G$-invariant inner products $(-, -)_1$, $(-, -)_2$. These give us two isomorphisms

$$T_i \in \operatorname{Hom}(\overline{V}, V^\vee) \cong \mathbb{C}$$

by sending $v \in \overline{V}$ to $(-, v)_i$, $i = 1$, 2. Then $T_1 = cT_2$ for some $c \neq 0$. ■

**Corollary 18.3.** Let $V_1$, $V_2$ be irreducible $G$-submodules of $(V, (-, -))$, where $(-, -)$ is a $G$-invariant inner product. If $V_1$ and $V_2$ are non-isomorphic, then $V_1 \perp V_2$.

*Proof.* If not, then $W = \{v \in V_1 \mid v_1 \perp v_2\}$ is a proper submodule of $V_1$, which is 0 by the irreducibility of $V_1$. Hence, $(-, -) \colon V_1 \otimes V_2 \to \mathbb{C}$ is a nondegenerate pairing, and thus $\overline{V}_1 \cong V_2^\vee \cong \overline{V}_2$. ■

Let $\widehat{G}$ be the set of equivalence elements of irreducible (unitary) representation $(\pi, E_\pi)$'s. For a finite dimensional $G$-module $V$, let $V_{[\pi]}$ be the $\pi$-isotropic component, i.e., the largest subspace of $V$ which is isomorphic to $E_\pi^{m_\pi}$ for some $m_\pi \geq 0$.

**Theorem 18.4.** There is an isomorphism $\iota_\pi \colon \operatorname{Hom}_G(E_\pi, V) \otimes E_\pi \to V_{[\pi]}$ by sending

$T \otimes v$ to $Tv$. Hence

$$\bigoplus_{\pi \in \widehat{G}} \mathrm{Hom}_G(E_\pi, V) \otimes E_\pi \xrightarrow{\sim} \bigoplus_{\pi \in \widehat{G}} V_{[\pi]} = V,$$

called the canonical decomposition of $V$.

*Proof.* Let $T \in \mathrm{Hom}_G(E_\pi, V)$ be a nonzero element. Then $\ker T = 0$ and therefore $E_\pi \cong T(E_\pi)$. By the definition of $V_{[\pi]}$, $T(E_\pi) \subseteq V_{[\pi]}$. Since $\iota_\pi$ is a $G$-morphism, onto, so we only have to check that it is injective.

Since

$$\dim \mathrm{Hom}_G(E_\pi, V) = \dim \mathrm{Hom}_G(E_\pi, V_{[\pi]}) = m_\pi$$

by Schur's lemma, $\dim \mathrm{LHS} = m_\pi \cdot \dim E_\pi = \dim V_{[\pi]}$.

Finally, $V = \sum_{[\pi] \in \widehat{G}} V_{[\pi]} = \bigoplus_{[\pi] \in \widehat{G}} V_{[\pi]}$. ■

**Examples.**

(1) The action of SU(2) on $V_n(\mathbb{C}^2)$ is irreducible.

(2) The action of SO($n$) on $\mathscr{H}_m(\mathbb{R}^n)$ is irreducible for $n \geq 3$. For $n = 2$, only O(2) irreducible.

**Fact 1.** Under the algebra isomorphism

$$\begin{aligned} V(\mathbb{R}^n) &\longrightarrow D(\mathbb{R}^n) \\ x_i &\longmapsto \partial_{x_i}, \end{aligned}$$

where $D(\mathbb{R}^n)$ is the space of differential operator with constant coefficient, define $(p, q) = \overline{\partial_q} p$, which is a hermitian inner product on $V_m(\mathbb{R}^n)$. There is an orthonormal basis $x_1^{k_1} \cdots x_n^{k_n}$ with $\sum k_i = m$. Also,

$$\mathscr{H}_m(\mathbb{R}^n) = \left(|x|^2 V_{m-2}(\mathbb{R}^n)\right)^\perp.$$

Indeed,

$$(p, |x|^2 q) = \overline{\partial_{|x|^2 q}} p = \overline{\partial_q} \triangle p = (\triangle p, q).$$

As a consequence,

$$V_m(\mathbb{R}^n) = \mathscr{H}_m(\mathbb{R}^n) \oplus^\perp |x|^2 V_{m-2}(\mathbb{R}^n) = \mathscr{H}_m(\mathbb{R}^n) \oplus \mathscr{H}_{m-2}(\mathbb{R}^n) \oplus \cdots$$

as $\mathrm{O}(n)$-modules.

**Fact 2.** Under $\mathrm{O}(n-1) \hookrightarrow \mathrm{O}(n)$, $g \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & g \end{smallmatrix}\right)$,

$$\mathscr{H}_m(\mathbb{R}^n)|_{\mathrm{O}(n-1)} = \mathscr{H}_m(\mathbb{R}^{n-1}) \oplus \mathscr{H}_{m-1}(\mathbb{R}^{n-1}) \oplus \mathscr{H}_{m-2}(\mathbb{R}^{n-1}) \oplus \cdots .$$

Write $V_m(\mathbb{R}^n) \ni p = \sum x_1^k p_k$, where $p_k \in V_{m-k}(\mathbb{R}^{n-1})$. Then $V_m(\mathbb{R}^n) \cong \bigoplus V_{m-k}(\mathbb{R}^{n-1})$ as $\mathrm{O}(n-1)$-modules. So

$$\begin{aligned} V_m(\mathbb{R}^n)|_{\mathrm{O}(n-1)} &\cong \mathscr{H}(\mathbb{R}^n)|_{\mathrm{O}(n-1)} \oplus V_{m-2}(\mathbb{R}^n)|_{\mathrm{O}(n-1)} \\ &\cong \mathscr{H}(\mathbb{R}^n)|_{\mathrm{O}(n-1)} \oplus \bigoplus V_{m-2-k}(\mathbb{R}^{n-1}). \end{aligned}$$

On the other hand,

$$V_m(\mathbb{R}^n)|_{\mathrm{O}(n-1)} \cong \bigoplus V_{m-k}(\mathbb{R}^{n-1}) \oplus \bigoplus V_{m-2-k}(\mathbb{R}^{n-1}).$$

So it suffices to show the "cancellation": if $G$ is a compact Lie group and $V \oplus U \cong W \oplus U$, then $V \cong W$. This is true by the canonical decomposition.

Now, we show that $\mathscr{H}_m(\mathbb{R}^n)$ is an irreducible $\mathrm{SO}(n)$-module. If $f \in \mathscr{H}_m(\mathbb{R}^n)$ is $\mathrm{SO}(n)$-invariant, then $f = c|x|^m$ and $\triangle f = 0$. which implies that $m = 0$ or $c = 0$. It follows from Fact 2 that $\mathscr{H}_m(\mathbb{R}^n)|_{\mathrm{SO}(n-1)}$ has a unique $\mathrm{SO}(n-1)$-invariant function, up to scalar.

**Claim.** For an $\mathrm{SO}(n)$-invariant finite dimensional subspace $V$ of $C^0(S^{n-1})$, there exists a (nonzero) $\mathrm{SO}(n-1)$-invariant function $f \in V$.

Indeed, there exists $f \in V$ such that $f(1, 0, \dots, 0) \neq 0$ (otherwise $V = 0$). Let

$$\tilde{f}(s) = \int_{\mathrm{SO}(n-1)} f(gs)\, dg,$$

$\{f_i\}$ a basis of $V$. Since $gf = \sum c^i(g) f_i$ for some functions $c^i \colon G \to \mathbb{C}$, $\tilde{f} = \sum \left( \int_{\mathrm{SO}(n-1)} c^i(g)\, dg \right) f_i \in V$. So $\tilde{f}$ is the desired function since $\tilde{f}(1, 0, \dots, 0) \neq 0$.

Now, if $\mathscr{H}_m(\mathbb{R}^n) = V_1 \oplus V_2$ with $V_i$ being $\mathrm{SO}(n)$-invariant, $V_i|_{S^{n-1}}$ contains a nonzero $\mathrm{SO}(n-1)$-invariant function $f_i$, $i = 1$, 2, which contradicts the uniqueness of such functions (up to scalar).

(3) For $n$ even, the action of $\mathrm{Spin}_n(\mathbb{R})$ on $S^\pm$ is irreducible. For $n$ odd, the action of $\mathrm{Spin}_n(\mathbb{R})$ on $S$ is irreducible.

# 19 Character theory, 11/23

Let $G$ be a compact Lie group. Then there is a $G$-invariant metric on $G$ and hence a $G$-invariant volume form (Haar measure) $dg$. We normalize the form so that

$$|G| = \int_G dg = 1.$$

Let $\rho\colon G \to \mathrm{GL}(V)$, $\rho'\colon G \to \mathrm{GL}(V')$ be representations, where $V$, $V'$ are finite dimensional $\mathbb{C}$-vector spaces. Consider $\rho''\colon G \to \mathrm{GL}(\mathrm{Hom}(V, V'))$, $\rho''(g)(e) = \rho'(g) \circ e \circ \rho(g^{-1})$.

**Lemma 19.1** (Symmetrization)**.** For a homomorphism $e : V \to V'$, the element $\eta(e) = \int_G \rho''(g)(e)\, dg$ lies in $\mathrm{Hom}_G(V, V')$.

*Proof.* By defintion

$$\begin{aligned}\rho'(h)\eta(e) &= \int_G \rho'(hg)e\rho(g^{-1})\, dg = \int_G \rho'(g)e\rho(h^{-1}g)^{-1}\, d(h^{-1}g)\\ &= \int_G \rho'(g)e\rho(g)^{-1}\, dg\, \rho(h) = \eta(e)\rho(h).\end{aligned}$$ ■

**Corollary 19.2.** If $\rho$, $\rho'$ are irreducible, then

(i) $\rho \not\cong \rho'$ implies $\eta(e) = 0$ for all $e \in \mathrm{Hom}(V, V')$;

(ii) $\rho \cong \rho'$ implies $\eta(e) \cong cI_V$ under an identification $V \cong V'$.

**Theorem 19.3** (Schur's orthogonality relations)**.** Let $(\rho, V)$, $(\rho', V')$ be irreducible representations. Write $\rho(g) = (T^i_j(g))$, $\rho'(g) = ({T'}^k_\ell(g))$ in some basis $\mathcal{B} \subset V$, $\mathcal{B}' \subset V'$. Then

$$\int_G T^i_j(g){T'}^k_\ell(g^{-1})\, dg = \begin{cases} 0, & \text{if } \rho \not\cong \rho', \\ \frac{|G|}{\dim V}\delta^i_\ell \delta^k_j, & \text{if } \rho = \rho',\ \mathcal{B} = \mathcal{B}'. \end{cases}$$

*Proof.* Let $e = e^k_j$ be the elementary matrix. Then the integral

$$\int_G T^i_j(g){T'}^k_\ell(g^{-1})\, dg = \int_G \rho'(g^{-1})e^k_j\rho(g)\, dg = (\eta(e^k_j))^i_\ell.$$

When $\rho \not\cong \rho'$, this is 0. For the case $\rho = \rho'$, $(\eta(e^k_j))^i_\ell = c^k_j \cdot \delta^i_\ell$ for some $c^k_j$. So

$$c^k_j = \frac{1}{\dim V}\int_G \sum_{i=\ell} (T^i_j(g)T^k_\ell(g)^{-1})\, dg = \frac{1}{\dim V}\int_G T^i_j(g)T^k_i(g)^{-1}\, dg = |G| \cdot \delta^k_j.$$ ■

Now we set $\chi_\rho = \chi_V := \operatorname{tr} \circ \rho \colon G \to \mathbb{C}$, called the character of $(\rho, V)$. Then $\chi_\rho \in C^\infty(G)$ and $\chi_\rho(e) = \dim V$.

Let $\mathbb{C}$ be the trivial representation, i.e., $G \to \{\mathrm{id}\} \subset \mathrm{GL}(\mathbb{C})$. Then $\chi_{\mathbb{C}} \equiv 1$.

$\chi$ defines a map from $\operatorname{Rep} G$ to $C^\infty(G)$. We see that $\chi_{V \oplus V'} = \chi_V + \chi_{V'}$ and $\chi_{V \otimes V'} = \chi_V \cdot \chi_{V'}$. Since $\chi_V(hgh^{-1}) = \chi_V(g)$, $\chi_V$ is a class function. Also,

$$\chi_{V^\vee}(g) = \chi_{\overline{V}}(g) = \overline{\chi_V(g)} = \chi_V(g^{-1})$$

by taking a unitary basis.

**Theorem 19.4.** Let $V$, $W$ be finite dimensional $G$-representations over $\mathbb{C}$.

(1) $\langle \chi_V, \chi_W \rangle := \int_G \chi_V(g) \overline{\chi_W(g)}\, dg = \dim \operatorname{Hom}_G(V, W)$.

(2) $V \cong W$ if and only if $\chi_V = \chi_W$.

*Proof.* Choose a unitary bases of $V$, $W$, etc.. If $V$, $W$ are irreducible, we get $\overline{T}'(g) = T'^{\mathsf{T}}(g^{-1})$. So

$$\langle \chi_V, \chi_W \rangle = \begin{cases} 0, & \text{if } V \ncong W, \\ \frac{1}{\dim V} \delta^i_\ell \delta^k_j \delta^j_i \delta^\ell_k = 1, & \text{if } V \cong W. \end{cases}$$

In general, write $V = \bigoplus E_\pi^{m_\pi}$, $W = \bigoplus E_\pi^{m'_\pi}$. Then $\chi_V = \sum m_\pi \chi_\pi$, $\chi_W = \sum m'_\pi \chi_\pi$. So

$$\langle \chi_V, \chi_W \rangle = \operatorname{Hom}(V, W).$$

Since $\{m_\pi\}$ (resp. $\{m'_\pi\}$) determines the isomorphic type of $V$ (resp. $W$) and

$$m_\pi = \langle \chi_\pi, \chi_V \rangle, \quad m'_\pi = \langle \chi_\pi, \chi_W \rangle,$$

we get (2). ■

**Corollary 19.5.** Let $V^G$ be the $G$-invariant vectors in $V$. Then

$$\int_G \chi_V(g)\, dg = \langle \chi_V, \chi_{\mathbb{C}} \rangle = \dim V^G$$

since $V^G = \operatorname{Hom}_G(\mathbb{C}, V)$. Also, $V$ is irreducible if and only if $\|\chi_V\| = 1$.

**Theorem 19.6.** For compact Lie groups $G_1$, $G_2$, a finite dimensional representation $W$ of $G_1 \times G_2$ is irreducible if and only if $W \cong V_1 \otimes V_2$, where $V_i$ is a irreducible $G_i$-representation, $i = 1, 2$.

*Proof.* Let $V_i$ be a irreducible $G_i$-representation, $i = 1, 2$. The invariant measure on $G_1 \times G_2$ is given by $dg_1 \wedge dg_2$. So

$$\chi_{V_1 \otimes V_2}(g_1 g_2) = \chi_{V_1}(g_1) \cdot \chi_{V_2}(g_2)$$

implies that $\|\chi_{V_1 \otimes V_2}\| = \|\chi_{V_1}\| \cdot \|\chi_{V_2}\| = 1$.

Conversely, let $W$ be an irreducible $G_1 \times G_2$-representation. Write

$$W = \bigoplus_{[\pi] \in \widehat{G}_2} \mathrm{Hom}_{G_2}(E_\pi, W) \otimes E_\pi$$

as $G_2$-modules. The equation above is in fact a $G_1 \times G_2$-morphism, since $\mathrm{Hom}_{G_2}(E_\pi, W)$ has a natural $G_1$ action. Since $W$ is irreducible, $W = \mathrm{Hom}_{G_2}(E_\pi, W) \otimes E_\pi$ for some $\pi$. ■

Be more concern with your character than your representation!

## 20 Peter-Weyl theorem, 11/28

Let $G$ be a compact Lie group. Then $C(G)$ is a Banach space with respect to

$$\|f\|_{C(G)} = \sup_{g \in G} |f(g)|;$$

$L^2(G)$ is a Hilbert space with respect to

$$\langle f_1, f_2 \rangle = \int_G f_1 \overline{f}_2 \, dg, \quad \|f\|_{L^2(G)} = \left( \int_G |f|^2 \, dg \right)^{1/2}.$$

Since $G$ is compact, $C(G)$ is dense in $L^2(G)$. There are two natural action of $G$ on $C(G)$, $L^2(G)$:

$$\begin{aligned} \ell : \quad G \times C(G) &\longrightarrow C(G) \\ (g, f) &\longmapsto \ell_g f = [h \mapsto f(g^{-1}h)], \\ r : \quad G \times C(G) &\longrightarrow C(G) \\ (g, f) &\longmapsto r_g f = [h \mapsto f(hg)]. \end{aligned}$$

The action of $G$ on $C(G)$ is continuous: for each $h \in G$, since $f_1$ is uniformly continuous,

$$\begin{aligned} |\ell_{g_1} f_1(h) - \ell_{g_2} f_2(h)| &= |f_1(g_1^{-1}h) - f_2(g_2^{-1}h)| \\ &\le |f_1(g_1^{-1}h) - f_1(g_2^{-1}h)| + |f_1(g_2^{-1}h) - f_2(g_2^{-1}h)| \to 0 \end{aligned}$$

as $(g_1, f_1)$ tends to $(g_2, f_2)$. The action of $G$ on $L^2(G)$ is also continuous:

$$\begin{aligned}
\|\ell_{g_1} f_1 - \ell_{g_2} f_2\|_{L^2(G)} &= \|f_1 - \ell_{g_1^{-1} g_2} f_2\|_{L^2(G)} \\
&\leq \|f_1 - f_2\|_{L^2(G)} + \|f_2 - \ell_{g_1^{-1} g_2} f_2\|_{L^2(G)} + \|\ell_{g_1} f_2 - \ell_{g_2} f_2\|_{L^2(G)} \\
&\leq \|\ell_{g_1} f_2 - \ell_{g_1} f\|_{L^2(G)} + \|\ell_{g_1} f - \ell_{g_2} f\|_{L^2(G)} + \|\ell_{g_2} f - \ell_{g_2} f_2\|_{L^2(G)} \\
&\leq \|f_2 - f\|_{L^2(G)} + \|\ell_{g_1} f - \ell_{g_2} f\|_{L^2(G)} + \|f_2 - f\|_{L^2(G)} \\
&\leq \|\ell_{g_1} f - \ell_{g_2} f\|,
\end{aligned}$$

where $f \in C(G)$ is an element such that $f \to f_2$ in $L^2$-norm.

**Definition 20.1.** Let $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ be a family of Hilbert spaces with inner product $\langle -, - \rangle_\alpha$ on $V_\alpha$. We define

$$\widehat{\bigoplus_{\alpha \in \mathcal{A}}} V_\alpha = \left\{ (v_\alpha) \;\middle|\; v_\alpha \in V_\alpha,\ \sum_{\alpha \in \mathcal{A}} \|v_\alpha\|_\alpha^2 < \infty \right\}$$

and

$$\langle (v_\alpha), (v'_\alpha) \rangle = \sum_\alpha \langle v_\alpha, v'_\alpha \rangle_\alpha.$$

Then $\bigoplus_\alpha V_\alpha$ is dense in $\widehat{\bigoplus}_\alpha V_\alpha$ and $V_\alpha \perp V_\beta$ for all $\alpha \neq \beta$.

Let $T$ be a bounded self-adjoint on operator on $V$. The spectral projection of $T$ is the family $\{P_\Omega = \chi_\Omega(T)\}$ where $\chi_\Omega$ is the indicator function of the Borel measurable set $\Omega$ such that

(1) $P_\Omega$ is an orthogonal projection;

(2) $P_\varnothing = 0$, $P_{(-a,a)} = \mathrm{id}$ for some $a > 0$;

(3) If $\Omega = \bigsqcup_{i=1}^\infty \Omega_i$, then $\lim_{N \to \infty} \sum_{i=1}^N P_{\Omega_i} = P_\Omega$.

(The spectrum of $T$ is the set

$$\{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is not invertible}\},$$

and $P_\lambda = \chi_\lambda(T)$.)

For each $v \in V$, $\lambda \mapsto \langle v, P_\lambda v \rangle$ is a measure. Since $T$ is self-adjoint,

$$\langle v, Tv \rangle = \int_{\mathbb{R}} \lambda \, d(\langle v, \mathbb{P}_\lambda v \rangle).$$

**Fact.** There is a one-to-one correspondence

$$\begin{aligned}\{\,\text{projection valued measures}\,\} &\longrightarrow \{\,\text{bounded self-adjoint operators}\,\}\\ \{P_\Omega\} &\longmapsto \langle v, Tw\rangle = \textstyle\int_{\mathbb{R}} \lambda\, d(\langle v, P_\lambda w\rangle).\end{aligned}$$

**Lemma 20.2** (Schur's lemma for Hilbert spaces)**.** If $V$ is irreducible, then $\operatorname{Hom}_G(V, V) = \mathbb{C} \cdot \mathrm{id}$.

*Proof.* For a $G$-operator $T$, write

$$T = \frac{T + T^*}{2} - i\frac{T - T^*}{2i}.$$

Since $T$ is a $G$-operator, then $T^*$ is also a $G$-operator. So we may assume that $T$ is self-adjoint. For each $g \in G$, $g \circ T = T \circ g$ implies that $g \circ P_\Omega = P_\Omega \circ g$, so $\ker g$ and $\operatorname{Im} g$ are $G$-submodules. Hence, $P_\Omega = \mathrm{id}$ or $0$.

Now, $P_{(-a,a)} = \mathrm{id}$ for some $a > 0$. So there exists $\lambda$ such that $P_\lambda = \mathrm{id}$. Hence, $T = \lambda \cdot \mathrm{id}$. ■

**Theorem 20.3.** Let $V$ be a Hilbert space and $\rho\colon G \to \mathrm{GL}(V)$ an irreducible representation. Then there exists finite dimensional irreducible $G$-submodules $V_\alpha \subseteq V$ such that $V = \widehat{\bigoplus}_\alpha V_\alpha$.

This shows that every irreducible unitary representation of $G$ are all finite dimensional, and the set of $G$-finite vectors (i.e., $v \in V$ such that $\dim\langle Gv\rangle < \infty$) is dense in $V$.

**Fact.** Let $(\rho, V)$ be a unitary representation of $G$ on $V$. Then there exists a nonzero $G$-subspace of $V$ with $\dim W < \infty$.

*Proof.* Let $T_0$ be a nonzero finite rank projection (self-adjoint, positive, compact) in $\operatorname{Hom}(V, V)$,

$$T = \int_G \rho(g) \circ T_0 \circ \rho(g)^{-1}\, dg.$$

Then $T$ is $G$-invariant. Since $T_0$ is positive,

$$\langle Tv, v\rangle = \int_G (T \circ \rho(g)^{-1}(v), \rho(g)^{-1}v)\, dg$$

shows that $T$ is positive. Since $T_0$ is self-adjoint, $T$ is self-adjoint. If $T$ is compact, self-adjoint, then there exists $\lambda \in \mathbb{C}$ such that $\dim \ker(T - \lambda I) < \infty$ and we know that $\ker(T - \lambda I)$ is a $G$-submodule. ■

Now, consider

$$\mathcal{S} = \{\{V_\alpha \mid \alpha \in \mathcal{A},\ \dim V_\alpha < \infty,\ V_\alpha \perp V_\beta \text{ for } \alpha \neq \beta\}\}.$$

By Zorn's lemma, there exists a maximal element $\{V_\alpha \mid \alpha \in \mathcal{A}\}$ in $\mathcal{S}$.

**Claim.** $\widehat{\bigoplus}_{\alpha \in \mathcal{A}} V_\alpha = V$.

If not, the orthogonal complement of $\widehat{\bigoplus}_{\alpha \in \mathcal{A}} V_\alpha$ is closed and $G$-invariant. So it contains a finite dimensional subspace $V_\gamma$, a contradiction.

Consider the $\pi$-isotypic component $V_{[\pi]}$ of $V$. $\mathrm{Hom}_G(E_\pi, V)$ forms a Hilbert space: $\langle T_1, T_2 \rangle_{\mathrm{Hom}} \,\mathrm{id} = T_2^* \circ T_1$. For $x_1$, $x_2 \in E_\pi$,

$$\langle T_1 x, T_2 x_2 \rangle_V = \langle T_2^* T_1 x_1, x_2 \rangle_{E_\pi} = \langle \langle T_1, T_2 \rangle_{\mathrm{Hom}} x_1, x_2 \rangle = \langle T_1, T_2 \rangle_{\mathrm{Hom}} \langle x_1, x_2 \rangle_{E_\pi}.$$

**Definition 20.4.** For $V_1$, $V_2$, we define $V_1 \mathbin{\widehat{\otimes}} V_2$ to be the completion of $V_1 \otimes V_2$ with respect to

$$\langle v_1 \otimes v_2, v_1' \otimes v_2' \rangle = \langle v_1, v_1' \rangle \langle v_2, v_2' \rangle.$$

Hence,

$$V = \widehat{\bigoplus_{[\pi] \in \widehat{G}}} V_{[\pi]} = \widehat{\bigoplus_{[\pi] \in \widehat{G}}} \mathrm{Hom}_G(E_\pi, V) \mathbin{\widehat{\otimes}} E_\pi.$$

# 21 Peter-Weyl theorem II, 11/30

**Theorem 21.1.** As $G \times G$-modules,

$$L^2(G) \cong \widehat{\bigoplus_{[\pi] \in \widehat{G}}} E_\pi^\vee \otimes E_\pi.$$

*Proof.* Recall that

$$L^2(G) = \widehat{\bigoplus_{[\pi] \in \widehat{G}}} L^2(G)_{[\pi]} = \widehat{\bigoplus_{[\pi] \in \widehat{G}}} \mathrm{Hom}_G(E_\pi, L^2(G)) \mathbin{\widehat{\otimes}} E_\pi.$$

Consider $C(G)_{G\text{-fin}} \subseteq C(G) \subseteq L^2(G)$, where $C(G)_{G\text{-fin}}$ contains the elements that has finite dimensional $G$-orbit.

**Lemma 21.2.** We have

(1) $\operatorname{Hom}_G(E_\pi, C(G)_{G\text{-fin}}) \cong E_\pi^\vee$, and

(2) $C(G)_{G\text{-fin}} \cong \bigoplus_{[\pi]\in\widehat{G}} E_\pi^\vee \otimes E_\pi$.

*Proof of Lemma.* We see that $C(G)_{G\text{-fin}}$ is equal to $\mathrm{MC}(G)$, the set of functions of the form

$$f_{u,v}^V \colon g \mapsto \langle gu, v\rangle,$$

where $V$ is a finite dimensional unitary representation of $G$. Indeed, via the left action $\ell\colon G \to \mathrm{GL}(C(G))$,

$$(\ell_g f_{u,v}^V)(h) = f_{u,v}^V(g^{-1}h) = \langle g^{-1}hu, v\rangle = \langle hu, gv\rangle = f_{u,gv}^V(h).$$

So

$$\langle \ell_g f_{u,v}^V \mid g \in G\rangle \subseteq \langle f_{u,v'}^V \mid v' \in V\rangle \in \mathrm{Ob}(\mathsf{Vect}_{\text{fin}}),$$

and hence $f_{u,v}^V \in C(G)_{G\text{-fin}}$. Conversely, if $f \in C(G)_{G\text{-fin}}$, say $\dim V < \infty$ and $f \in V$. Consider $\overline{V} = \{\overline{f} \mid f \in V\}$ with action $g \cdot \overline{f} = \overline{g \cdot f}$. Then $\overline{V}$ is a $G$-submodule of $C(G)$ and $\overline{V}$ has an induced norm from $L^2(G)$. Now, for each $\overline{f} \in \overline{V}$, $\overline{f}(e) \in \mathbb{C}$, so there is exist an $\overline{f}_0 \in \overline{V}$ such that $\overline{f}(e) = \langle \overline{f}, \overline{f}_0\rangle$ for all $\overline{f} \in \overline{V}$. Hence,

$$\overline{f}(g) = \ell_{g^{-1}}\overline{f}(e) = \langle \ell_{g^{-1}}\overline{f}, \overline{f}_0\rangle = \langle \overline{f}, \ell_g\overline{f}_0\rangle$$

implies that

$$f_{\overline{f}_0,\overline{f}}^{\overline{V}}(g) = \langle g\overline{f}_0, \overline{f}\rangle = \overline{\overline{f}(g)} = f(g),$$

i.e., $f \in \mathrm{MC}(G)$.

From the proof above, we also see that $C(G)_{G\text{-fin}}$ with respect to $\ell$ is equal to $C(G)_{G\text{-fin}}$ with respect to $r$. Indeed, for $f \in C(G)_{G\text{-fin}}$ with respect to $r$, there exists $V \in C(G)$ with $\dim V < \infty$ and $f \in V$. Similarly, there exists $f_0 \in V$ such that $f(e) = (f, f_0)$ for all $f \in V$. So $f(g) = r_g f(e) = \langle r_g f, f_0\rangle$ implies that $f = f_{f,f_0}^V \in \mathrm{MC}(G)$.

Now,

$$C(G)_{G\text{-fin}} = \bigoplus_{\pi\in\widehat{G}} \operatorname{Hom}_G(E_\pi, C(G)_{G\text{-fin}}) \otimes E_\pi$$

as left $G$-modules. In fact, $C(G)_{G\text{-fin}}$ is a $G \times G$-module by

$$((g_1, g_2)f)(h) = (r_{g_1}\ell_{g_2}f)(h) = f(g_2^{-1}hg_1).$$

The second $G$-action on $\operatorname{Hom}_G(E_\pi, C(G)_{G\text{-fin}}) \otimes E_\pi$ is trivial on the second component and is defined by

$$(gT)(x) = r_g(T(x))$$

on the first component $(\ell_{g'}(Tg)(x) = \ell_{g'}r_gT(x) = r_gT(\ell_{g'}x) = (Tg)(\ell_{g'}x))$.

Recall that $E_\pi^\vee$ is a (left) $G$-module: for $\lambda \in E_\pi^\vee$, $(\lambda g)(x) = \lambda(g^{-1}x)$.

Consider

$$\begin{tikzcd}
\operatorname{Hom}_G(E_\pi, C(G)_{G\text{-fin}}) \arrow[r, shift left, "\varphi"] & E_\pi^* \arrow[l, shift left, "\psi"]
\end{tikzcd}$$

$$\begin{tikzcd}
T \arrow[r, mapsto] & \lambda_T\colon x \mapsto (Tx)(e) \\
T_\lambda\colon x \mapsto [h \mapsto \lambda(h^{-1}x)] & \lambda \arrow[l, mapsto]
\end{tikzcd}$$

We see that $\varphi$ is a $G$-morphism:

$$\begin{aligned}(\lambda_T g)(x) &= \lambda_T(g^{-1}x) = T(g^{-1}x)(e) = (\ell_{g^{-1}}(Tx))(e) \\ &= (Tx)(g) = ((Tx)g)(e) = ((Tg)(x))(e) = \lambda_{Tg}(x).\end{aligned}$$

$T_\lambda \in$ LHS:

$$\ell_g(T_\lambda(x))(h) = T_\lambda(x)(g^{-1}h) = \lambda(h^{-1}gx) = (T_\lambda(gx))(h),$$

so $\ell_g(T_\lambda(x)) = T_\lambda(gx)$. Similarly, $\psi$ is a $G$-morphism.

It is easy to check that $\varphi \circ \psi = \mathrm{id}$ and $\psi \circ \varphi = \mathrm{id}$: $\lambda_{T_\lambda}(x) = (T_\lambda(x))(e) = \lambda(x)$,

$$(T_{\lambda_T}(x))(h) = \lambda_T(h^{-1}x) = (T(h^{-1}x))(e) = (\ell_{h^{-1}}(T(x)))(e) = T(x)(h).$$

This proves (1). For (2), consider

$$\begin{tikzcd}
\displaystyle\bigoplus_{[\pi] \in \widehat{G}} E_\pi^\vee \otimes E_\pi \arrow[r] & C(G)_{G\text{-fin}} \\
\lambda \otimes v \arrow[r, mapsto] & f_{\lambda \otimes v}\colon g \mapsto \lambda(g^{-1}v).
\end{tikzcd}$$

This is a $G \times G$-morphism.

First, we check that $\varphi$ is surjective. Since $\mathrm{MC}(G) = C(G)_{G\text{-fin}}$ is generated by $f^{E_\pi}_{v_i^\pi, v_j^\pi}$, where $\{v_i^\pi\}$ is a basis of $E_\pi$, it suffices to show that $f^{E_\pi}_{v_i^\pi, v_j^\pi}$ lies in the image. Pick $\lambda = \langle -, u \rangle \in E_\pi^\vee$. Then

$$f_{\lambda \otimes v}(g) = \lambda(g^{-1}v) = \langle g^{-1}v, u \rangle = \langle v, gu \rangle = f^{E_\pi}_{u,v}(g),$$

as desired.

Suppose that $\varphi$ is not injective, say $0 \neq \sum \lambda_i \otimes v_i \in \ker \varphi$. We may assume that $\sum \lambda_i \otimes v_i \in \sum_{j=1}^{N} E_{\pi_j}^{\vee} \otimes E_{\pi_j}$ for some $\pi_j \in \widehat{G}$. Then $\langle \sum \lambda_{ji} \otimes v_{ji} \rangle_{G \times G} \subseteq E_{\pi_j}^{\vee} \otimes E_{\pi_j}$. But for $0 \neq \lambda \otimes v \in E_{\pi_i}^{\vee} \otimes E_{\pi_i}$, there exists $h$ such that $f_{\lambda \otimes v}(h) \neq 0$, a contradiction. $\square$

We claim that $C(G)_{G\text{-fin}}$ is dense in $C(G)$ and thus in $L^2(G)$. By Stone-Weierstrass theorem, we only need to show that $C(G)_{G\text{-fin}}$ separates points, i.e., for each $g_0 \in G$, there exists $f \in C(G)_{G\text{-fin}}$ such that $f(g_0) \neq f(e)$.

Choose $e \in U \subseteq G$ such that $U \cap g_0 U = \varnothing$. Let $\chi_U$ be the characteristic function of $U$. Then $\ell_{g_0} \chi_U = \chi_{g_0 U}$ implies that $\langle \ell_{g_0} \chi_U, \chi_U \rangle = 0$. Since $\langle \chi_U, \chi_U \rangle > 0$, $\ell_{g_0} \neq \mathrm{id}_{L^2(G)}$. Also, $L^2(G) = \widehat{\bigoplus} V_\alpha$ implies that there exists $V_{\alpha_0}$ and $x \in V_{\alpha_0}$ such that $\ell_{g_0} x \neq x$. So there exists $y \in V_{\alpha_0}$ such that $\langle \ell_{g_0} x, y \rangle \neq \langle x, y \rangle$. Pick $f = f_{x,y}^{V_{\alpha_0}}$. We get $f(g_0) \neq f(e)$, as desired.

Let

$$\iota \colon \widehat{\bigoplus_{[\pi] \in \widehat{G}}} \mathrm{Hom}_G(E_\pi, L^2(G)) \mathbin{\widehat{\otimes}} E_\pi \xrightarrow{\sim} L^2(G).$$

We need to show that the inclusion $\kappa \colon E_\pi^{\vee} \to \mathrm{Hom}_G(E_\pi, L^2(G))$ is an isomorphism.

If not, $\mathrm{Im}\, \kappa \subsetneqq \mathrm{Hom}_G(E_\pi, L^2(G))$. Since $\iota$ is an isomorphism and $\dim E_\pi^{\vee} < \infty$, so the inclusion

$$\iota(\kappa(E_\pi^{\vee}) \otimes E_\pi) \subsetneqq \iota(\mathrm{Hom}_G(E_\pi, L^2(G)) \otimes E_\pi$$

is closed. Pick $f \neq 0$ lies in the orthogonal complement of the LHS in the RHS. Then

$$f \in \left( \bigoplus_{[\pi'] \in \widehat{G}} \iota(\kappa(E_{\pi'}^{\vee}) \otimes E_{\pi'}) \right)^{\perp} = (C(G)_{G\text{-fin}})^{\perp},$$

a contradiction. $\blacksquare$

# 22 Applications of Peter-Weyl theorem, 12/5

Let $G$ be a compact Lie group. Then there is a decomposition (21.1)

$$L^2(G) = \widehat{\bigoplus_{[\pi] \in \widehat{G}}} E_\pi^{\vee} \otimes E_\pi = \widehat{\bigoplus_{[\pi] \in \widehat{G}}} \mathrm{End}\, E_\pi.$$

For $f \in L^2(G)$, what is the corresponding element in $\operatorname{End} E_\pi$? For $G = S^1$, this is Fourier series (note that $\widehat{S^1} \cong \mathbb{Z}$). What is the algebra structure in the RHS corresponds to the algebra structure (via convolution) in the LHS?

1. Let $f_{ij}^{E_\pi}$ be the matrix coefficient of $E_\pi$. Then

$$\left\{ \sqrt{\dim E_\pi} f_{ij}^{E_\pi} \;\middle|\; [\pi] \in \widehat{G} \right\}$$

   is an orthonormal basis of $L^2(G)$.

2. There exists a finite dimensional faithful representation $\rho\colon G \hookrightarrow \mathrm{GL}(V)$, and hence $G$ is isomorphic to a subgroup of $\mathrm{U}(N)$ ($N = \dim V$).

   If $\dim G > 0$, take $e \neq g_1 \in G^\circ$. Then there exists a representation $(\rho_1, V_1)$ such that $\pi_1(g_1) \neq I_V$ (by P-W). Then $G_1 := \ker \pi_1$ is a closed subgroup of $G$ (and hence a compact submanifold) that contains $g_1$. Since $G_1$ cannot contain a neighborhood of $e$, $\dim G_1 < \dim G$. If $\dim G_1 > 0$, then continue this process to get $(\rho_i, V_i)_{i=1}^N$. Then $\dim \ker(\rho_1 \oplus \cdots \oplus \rho_N) = 0$, so $\ker(\rho_1 \oplus \cdots \oplus \rho_N) = \{h_j\}_{j=1}^M$ is a finite group. For each $i = 1, \ldots, M$, choose $\rho_{N+i}(h_i) \neq \mathrm{id}$. Then $\rho_1 \oplus \cdots \oplus \rho_{N+M}$ is the desired representation.

3. Let $\underline{\chi}$ be the set of irreducible characters $\chi_\pi$, $\pi \in \widehat{G}$.

   (3.1) $\langle \underline{\chi} \rangle = C_{\mathrm{cl}}(G)_{G\text{-fin}}$, the set of $G$-finite class functions.

   Indeed, there is an isomorphism

$$C_{\mathrm{cl}}(G)_{G\text{-fin}} \cong \bigoplus_{[\pi] \in \widehat{G}} (\operatorname{End} E_\pi)_{\mathrm{cl}}.$$

   For $f \in C(G)$, $f \in C(G)_{\mathrm{cl}}$ if and only if the diagonal action $g \cdot f = f$, where $g \cdot f(h) := f(g^{-1} h g)$, i.e., $f$ corresponds to $\{T_\pi \in \operatorname{End}_G E_\pi\}_{[\pi] \in \widehat{G}}$. By Schur's lemma, $T_\pi = \lambda_\pi(g) I_{E_\pi}$.

   Note that $I_{E_\pi} = \sum_i \langle -, e_i \rangle \otimes e_i \in E_\pi^\vee \otimes E_\pi$ maps to

$$g \mapsto \sum_i \langle g^{-1} e_i, e_i \rangle = \sum_i \langle e_i, g e_i \rangle = \sum \overline{\langle g e_i, e_i \rangle},$$

   i.e., $\overline{\chi}_\pi$.

   (3.2) $\langle \underline{\chi} \rangle$ is dense in $C_{\mathrm{cl}}(G)$.

Indeed, for $f \in C(G)$ and for each $\varepsilon > 0$, there exists $\varphi \in C(G)_{G\text{-fin}}$ such that the sup norm $\|f - \varphi\|_0 < \varepsilon$. Let

$$\widetilde{\varphi}(h) = \int_G \pi(g^{-1}hg)\, dg \in C_{\mathrm{cl}}(G),$$

then

$$\|f - \widetilde{\varphi}\|_0 \le \sup_{h \in G} \int_G |f(g^{-1}hg) - \varphi(g^{-1}hg)|\, dg \le \|f - \varphi\|_0 < \varepsilon.$$

Now, $\widetilde{\varphi}$ is $G$-finite: write

$$\varphi(h) = \sum_i \langle hx_i, y_i\rangle,$$

where $x_i$, $y_i \in E_{\pi_i}$ and the sum is finite. Then

$$\begin{aligned}\widetilde{\varphi}(h) &= \sum_i \int_G \langle g^{-1}hgx_i, y_i\rangle\, dg \\ &= \sum_i \left\langle \int_G g^{-1}hg\, dg \cdot x_i, y_i \right\rangle \\ &= \sum_i \frac{\chi_i}{\dim E_{\pi_i}} \langle x_i, y_i\rangle,\end{aligned}$$

where $\chi_i = \chi_{\pi_i} = \operatorname{tr} \pi_i$. Here, we use the fact that

$$\int_G \pi(g^{-1}hg)\, dg \in \operatorname{End}_G E_\pi = \mathbb{C} \cdot \mathrm{id}$$

and that

$$\operatorname{tr}\left(\int_G \pi(g^{-1}hg)\, dg\right) = \int_G \operatorname{tr} \pi(g^{-1}hg)\, dg = \int_G \operatorname{tr} \pi(h)\, dg = \chi(h).$$

(3.3) $\underline{\chi}$ is an orthonormal basis of $L^2_{\mathrm{cl}}(G)$, i.e., for $f \in L^2_{\mathrm{cl}}(G)$,

$$f = \sum_{[\pi] \in \widehat{G}} \langle f, \chi_\pi\rangle \chi_\pi.$$

Indeed, choose $\varphi \in C(G)_{G\text{-fin}}$ such that $\|f - \varphi\|_2 < \varepsilon$ by P-W theorem. As above, $\widetilde{\varphi} \in \langle \underline{\chi}\rangle$. Also,

$$\begin{aligned}\|f - \widetilde{\varphi}\|_2 &= \left(\int_G |f(h) - \widetilde{\varphi}(h)|^2\, dh\right)^{1/2} \\ &= \left(\int_G \left|\int_G f(g^{-1}hg) - \varphi(g^{-1}hg)\, dg\right|^2 dh\right)^{1/2} \\ &\le \int_G \left(\int_G \left|f(g^{-1}hg) - \varphi(g^{-1}hg)\right|^2 dh\right)^{1/2} dg = \|f - \varphi\|_2 < \varepsilon.\end{aligned}$$

4. As a corollary, we have $\mathbb{N} \cong \widehat{\mathrm{SU}(2)}$ by mapping $n \in \mathbb{N}$ to $V_n(\mathbb{C}^2)$.

The isomorphism

$$L^2(G) \cong \widehat{\bigoplus_{[\pi]\in\widehat{G}}} \operatorname{End} E_\pi$$

can be extended to an unitary/algebra isomorphism. The inner product on $L^2(G)$ is the natural one, and the product structure on $L^2(G)$ is the convolution:

$$(f_1 * f_2)(g) = \int_G f_1(gh^{-1}) f_2(h)\, dh.$$

The inner product on the RHS is the Hilbert-Schmidt inner product:

$$\langle (T_\pi), (S_\pi)\rangle = \sum \operatorname{tr}(S_\pi^* \circ T_\pi).$$

The product structure on $L^2(G)$ is the operator product structure:

$$(T_\pi)\cdot(S_\pi) = \left(\frac{T_\pi \circ S_\pi}{\sqrt{\dim E_\pi}}\right).$$

On one component $[\pi] \in \widehat{G}$, let $\pi\colon L^2(G) \to \operatorname{End} E_\pi$ be

$$\pi(f)\cdot v := \int_G f(g)\cdot gv\, dg.$$

Then in fact

(1) $\pi(f_1 * f_2) = \pi(f_1)\circ\pi(f_2)$, and

(2) $\pi(f)^* = \pi(\tilde{f})$, where $\tilde{f}(g) = \overline{f(g^{-1})}$.

Indeed, this follows from

$$\begin{aligned}\pi(f_1 * f_2)\cdot v &= \int_G\int_G f_1(gh^{-1}) f_2(h) g\cdot v\, dhdg \\ &= \int_G f_1(g)\left(g\cdot\int_G f_2(h)hv\right) dhdg = \pi(f_1)\circ\pi(f_2)\cdot v,\end{aligned}$$

and

$$\langle \pi(f_1)v, w\rangle = \int_G f(g)\langle gv, w\rangle\, dg = \int_G \langle v, \overline{f(g)}g^{-1}w\rangle\, dg = \langle v, \pi(\tilde{f})\cdot w\rangle.$$

**Definition 22.1.** The operator valued Fourier transform is

$$L^2(G) \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \operatorname{Op}(\widehat{G}),$$

where $\mathrm{Op}(\widehat{G})$ is just $\widehat{\bigoplus}\,\mathrm{End}\,E_\pi$ with the inner product structure and the product structure,

$$\mathcal{F}f := \left(\sqrt{\dim E_\pi}\cdot\pi(f)\right)_{\pi\in\widehat{G}},$$
$$\mathcal{G}(T_\pi) := \sum_\pi \sqrt{\dim E_\pi}\cdot\mathrm{tr}(T_\pi\circ\pi(g^{-1})).$$

**Theorem 22.2** (Plancherel)**.** The maps $\mathcal{F}$ and $\mathcal{G}$ are unitary, algebra, $G\times G$-isomorphisms and inverse to each other.

**Corollary 22.3.** We have

(1) $\|f\|^2 = \sum \dim E_\pi\cdot\|\pi(f)\|^2$;

(2) $\mathcal{G}I_{E_\pi} = \sqrt{\dim E_\pi}\cdot\chi_{\overline{E}_\pi}$;

(3) $f = \sum \dim E_\pi\cdot f*\chi_\pi$;

(4) $\langle f_1, f_2\rangle = \sum \dim E_\pi\cdot\mathrm{tr}\,\pi(\tilde{f}_2 * f_1)$.

**Definition 22.4.** For $f\in L^2(G)$, its scalar valued Fourier transform is

$$\widehat{f}(\pi) := \mathrm{tr}\,\pi(f) = \sum_i\langle\pi(f)v_i, v_i\rangle = \int_G f(g)\sum_i\langle gv_i, v_i\rangle\,dg = \langle f, \chi_{\overline{E}_\pi}\rangle$$

**Corollary 22.5.** There is an isomorphism

$$\begin{tikzcd} L^2_{\mathrm{cl}}(G) \arrow[r, "\widehat{\ }"] & \ell^2(\widehat{G}) \\ f \arrow[r, mapsto] & \widehat{f}. \end{tikzcd}$$

# 23 Lie algebras coming from Lie groups, 12/7

Let $G$ be a Lie group. Then the Lie algebra of $G$, denoted by $\mathrm{Lie}\,G$ or $\mathfrak{g}$, is the left invariant vector field on $G$ under Lie bracket:

$$[X,Y]f = XYf - YXf.$$

If $X = a^i\frac{\partial}{\partial x^i}$ and $Y = b^j\frac{\partial}{\partial x^j}$, then

$$[X,Y] = XY - YX = a^i\frac{\partial b^j}{\partial x^i}\frac{\partial}{\partial x^j} - b^i\frac{\partial a^i}{\partial x^j}\frac{\partial}{\partial x^i}.$$

Since $X$, $Y$ are left invariant, $[X, Y]$ is also left invariant.

**Fact.** $\mathfrak{gl}(n, \mathbb{R}) = \mathrm{GL}(n, \mathbb{R})$, i.e., $[\widetilde{A}, \widetilde{B}]_e = AB - BA$, where $\widetilde{A}$ (resp. $\widetilde{B}$) is the left invariant vector field determined by $A \in T_e \mathrm{GL}(n, \mathbb{R})$ (resp. $B$). Indeed, let $h$ be a curve on $G = \mathrm{GL}(n, \mathbb{R})$ such that $h'(0) = A$. Then $(gh(t))' = gh'(t)$. So in particular $\ell_{g*}A = gA$. Write $A = \left(a^i_j \frac{\partial}{\partial x^i_j}\right)$, $g = (x^i_j(g))$. Notice that

$$\frac{\partial}{\partial x^i_j}(x^k_m b^m_\ell) = \delta^k_i \delta^j_m b^m_\ell = \delta^k_i b^j_\ell.$$

So

$$\begin{aligned}[\widetilde{A}, \widetilde{B}]_e &= a^i_j \frac{\partial}{\partial x^i_j}(gB)^k_\ell \frac{\partial}{\partial x^k_\ell} - b^i_j \frac{\partial}{\partial x^i_j}(gA)^k_\ell \frac{\partial}{\partial x^k_\ell}\bigg|_{g=e} \\ &= (AB - BA)^i_\ell \frac{\partial}{\partial x^i_\ell}\bigg|_e .\end{aligned}$$

Consider the (unique) curve $\gamma$ with $\gamma(0) = e$, $\gamma'(0) = X \in T_eG$, $\gamma'(t) = \widetilde{X}_{\gamma(t)}$. If $G \subseteq \mathrm{GL}(n, \mathbb{C})$, then in fact $\gamma(t) = e^{tX}$:

$$\gamma'(t) = e^{tX}X = \gamma(t)X = \widetilde{X}_{\gamma(t)}.$$

This says that $\widetilde{X}$ determines an one parameter group of diffeomorphism on $G$ by right translations.

**Fact.** The exponential map $\exp\colon X \mapsto \gamma(1) = e^X$ is complete, i.e., $\gamma(t)$ is defined for all $t \in \mathbb{R}$ and is a diffeomorphism.

*Proof.* Notw that $\frac{d}{dt}e^{tX}\big|_{t=0} = X$ implies $(d\exp)_0)_0 = \mathrm{id}$. The result then follows from the inverse function theorem. ■

Caution: $\exp \mathfrak{g}$ generate a neighborhood of $G$, hence generate $G^\circ$. But it may not be onto. True if $G$ is compact!

**Example 23.1.** $\mathfrak{sl}(n, F)$: $\det e^{tX} = e^{t \operatorname{tr} X}$. So $\det e^{tX} = 1$ for all $t$ if and only if $\operatorname{tr} X = 0$.

$\mathfrak{su}(n) = \mathfrak{u}(n) \cap \mathfrak{sl}(n, \mathbb{C})$: $e^{tX}(e^{tX})^* = e^{tX}e^{tX^*} = 1$ for all $t$ if and only if $X^* = -X$.

Note that $\dim_{\mathbb{R}} \mathfrak{sl}(n, \mathbb{R}) = \dim_{\mathbb{R}} \mathfrak{su}(n) = n^2 - 1$. In fact, $\mathfrak{sl}(n, \mathbb{C}) \cong \mathfrak{su}(n) \otimes_{\mathbb{R}} \mathbb{C}$.

$\mathfrak{so}(n) = \mathfrak{o}(n)$: we have $X^\mathsf{T} = -X$, and note that this implies $\operatorname{tr} X = 0$ automatically.

$\mathfrak{sp}(n)$: reading.

**Proposition 23.2.** Let $\varphi\colon H \to G$ be a Lie group homomorphism, i.e., a $C^\infty$ group homomorphism. Then $d\varphi\colon \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra homomorphism, the diagram

$$\begin{tikzcd}
\mathfrak{h} \arrow[r, "d\varphi"] \arrow[d, "\exp"] & \mathfrak{g} \arrow[d, "\exp"] \\
H \arrow[r, "\varphi"] & G
\end{tikzcd}$$

commutes, and if $H$ is connected, then $d\colon \operatorname{Hom}(H, G) \to \operatorname{Hom}(\mathfrak{h}, \mathfrak{g})$ is injective.

*Proof.* $d\varphi([X, Y]) = [d\varphi(X), d\varphi(Y)]$ follows from the $C^\infty$ structure. Since $\varphi(gg') = \varphi(g)\varphi(g')$, $\varphi \circ \ell_g = \ell_{\varphi(g)} \circ \varphi$. By chain rule,

$$d\varphi \circ d\ell_g = d\ell_{\varphi(g)} \circ d\varphi,$$

i.e., left invariant vector field are compatible with $d\varphi$, hence also integral curve. This implies that the diagram commutes by the construction of exp. Then the injectivity of $d$ follows from the commutative diagram. ■

Consider the inner automorphism $I_g = \ell_g r_{g^{-1}}$. The adjoint representation is

$$\begin{tikzcd}
G \arrow[r, "\mathrm{Ad}"] & \operatorname{Aut}\mathfrak{g} \\
g \arrow[r, mapsto] & dI_g,
\end{tikzcd}$$

this is a Lie group homomorphism. If $Z(G)$ is trivial, then $G \hookrightarrow \mathrm{GL}(\mathfrak{g})$, and hence $G$ is a matrix group. We define

$$\mathrm{ad} = d\,\mathrm{Ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}).$$

**Fact.** Explicit formulas for matrix groups. They are all as expected.

$$\begin{aligned}
\mathrm{Ad}(g)(X) &= (ge^{tX}g^{-1})'(0) = gXg^{-1} \\
\mathrm{ad}(X)Y &= (e^{tX}Ye^{-tX})'(0) = XY - YX = [X, Y].
\end{aligned}$$

Also, $\mathrm{Ad}\, e^X = e^{\mathrm{ad}\, X}$.

**Theorem 23.3.** There is a one to one correspondence between subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ and connected Lie subgroup $H$ of $G$.

*Proof.* Fix a basis $\{X_i\}$ of $\mathfrak{h}$. We get a distribution $\mathscr{H}_g = \langle \widetilde{X}_{ig} \rangle$ for each $g \in G$. Let $\mathscr{H} = \bigsqcup_{g \in G} \mathscr{H}_g$. We show that this distribution is integrable:

$$\left[f^i \widetilde{X}_i, g^j \widetilde{X}_j\right] = f^i g^j [\widetilde{X}_i, \widetilde{X}_j] + f^i (\widetilde{X}_i g^j) \widetilde{X}_j - g^j (\widetilde{X}_j f^i) \widetilde{X}_i \in \mathscr{H}_g.$$

Take $H$ to be the maximal integral submanifold that contains $e$. It is easy to check that $H$ is indeed a subgroup. ■

**Corollary 23.4.** If $H$ is simply connected, $G$ is connected, then there exists natural bijection between $\operatorname{Hom}(H, G)$ and $\operatorname{Hom}(\mathfrak{h}, \mathfrak{g})$.

*Proof.* Let $\rho\colon H \to G$. Then the graph $\Gamma_\rho \subseteq H \times G$ is a group and $\Gamma_\rho \to H$ is a bijection. Then it can be reduced to the previous case. ■

# 24 Exponential map, 12/12

Consider $G \subseteq \mathrm{GL}(n, \mathbb{C})$. Then $[X, Y] = 0$ if and only if $e^{tX}e^{sY} = e^{tX+sY}$ for all $t$, $s \in \mathbb{R}$. Indeed, if the latter condition holds, then

$$e^{tX}e^{sY} = e^{sY}e^{tX}.$$

Applying $\partial_s\partial_t|_{s=t=0}$ on the both sides we get $XY = YX$. Hence,

**Corollary 24.1.** If $A \subseteq G$ is connected, then $A$ is abelian if and only if $\mathfrak{a} := \operatorname{Lie} A$ is abelian.

**Definition 24.2.** A ($k$-)**torus** is a Lie group $T^k := (S^1)^k = \mathbb{R}^k/\mathbb{Z}^k$.

**Proposition 24.3.** A compact abelian Lie group $G$ is isomorphic to $T^k \times F$ for some $k$, where $F$ is a finite abelian group.

*Proof.* Consider the exponential map $\exp\colon \mathfrak{g} \to G^\circ$, which is a group homomorphism, and hence surjective. Since exp is locally diffeomorphic near 0, its kernel $\ker \exp$ is discrete, and thus is isomorphic to $\mathbb{Z}^{\dim \mathfrak{g}}$ (since $\mathfrak{g}/\ker \exp \cong G^0$).

Now, $G/G^\circ$ is a finite abelian group $F \cong \prod \mathbb{Z}/n_i\mathbb{Z}$. Let $g_i \in G$ with $\overline{g}_i = 1 + n_i\mathbb{Z} \in \mathbb{Z}/n_i\mathbb{Z}$. Then $g_i^{n_i} \in G^\circ$ implies that there exists an $x_i$ such that $e^{n_i x_i} = g_i^{n_i}$. Let $h_i = g_i e^{-x_i} \in g_i G^\circ$. Then $h_i^{n_i} = e$ and

$$\begin{tikzcd}
G^\circ \times \prod \mathbb{Z}/n_i\mathbb{Z} \arrow[r] & G \\
(g, (\overline{m_i})_i) \arrow[r, mapsto] & g \prod h_i^{m_i}
\end{tikzcd}$$

is the desired isomorphism. ■

**Definition 24.4.** A **maximal torus** of a compact Lie group $G$ is a maximal connected abelian group. A **Cartan subalgebra** of $\mathfrak{g} = \operatorname{Lie} G$ is a maximal abelian subalgebra.

**Corollary 24.5.** Let $T$ be a connected subgroup of a compact Lie group $G$. Then $T$ is a maximal torus of $G$ if and only if $\mathfrak{t} := \operatorname{Lie} T$ is Cartan. In particular, $\mathfrak{t}$ (and hence $T$) always exists!

**Example 24.6.** (1) Let

$$T = \{\operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})\} \subseteq \mathrm{U}(n)$$
$$\mathfrak{t} = \{\operatorname{diag}(i\theta_1, \dots, i\theta_n)\} \subseteq \mathfrak{u}(n).$$

Then $T$ is a maximal torus of $\mathrm{U}(n)$, $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{u}(n)$. A similar results holds for $\mathrm{SU}(n)$ and $\mathfrak{su}(n)$ with additional condition $\sum \theta_i = 0$.

(2)

$$T = \left\{\operatorname{diag}\left(\begin{pmatrix}\cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i\end{pmatrix}\right)\right\} \subseteq \mathrm{SO}(2n),$$
$$\mathfrak{t} = \left\{\operatorname{diag}\left(\begin{pmatrix}0 & -\theta_i \\ \theta_i & 0\end{pmatrix}\right)\right\} \subseteq \mathfrak{so}(2n).$$

(3)

$$T = \left\{\operatorname{diag}\left(\begin{pmatrix}\cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i\end{pmatrix}, 1\right)\right\} \subseteq \mathrm{SO}(2n+1),$$
$$\mathfrak{t} = \left\{\operatorname{diag}\left(\begin{pmatrix}0 & -\theta_i \\ \theta_i & 0\end{pmatrix}, 0\right)\right\} \subseteq \mathfrak{so}(2n+1).$$

**Theorem 24.7.** Let $G$ be a compact Lie group, $\mathfrak{t}$ a Cartan subalgebra. Then for each $X \in \mathfrak{g}$, there exists $g \in G$ such that $\operatorname{Ad}(g)X \in \mathfrak{t}$.

*Proof.* Any finite dimensional representation $(\rho, V)$ has a $G$-invariant inner product, in particular for $(\operatorname{Ad}, \mathfrak{g})$, we call it $\langle -, - \rangle$.

**Lemma 24.8.** Let $\mathfrak{t} = \mathfrak{z}(y)$ for some regular element $Y \in \mathfrak{g}$.

So we want to find $g \in G$ such that $[\mathrm{Ad}(g)X, Y] = 0$, i.e.,

$$\langle [\mathrm{Ad}(g)X, Y], Z\rangle = -\langle Y, [\mathrm{Ad}(g), Z]\rangle = 0$$

for all $Z \in \mathfrak{g}$. Let $g_0$ achieves the maximal of the $C^\infty$ function

$$f(g) = \langle Y, \mathrm{Ad}(g)X\rangle.$$

Then $t \mapsto \langle Y, \mathrm{Ad}(e^{tZ})\,\mathrm{Ad}(g_0)X\rangle$, $t \in \mathbb{R}$, has maximum at $t = 0$ for each $Z \in \mathfrak{g}$. Hence,

$$0 = \frac{d}{dt}\bigg|_{t=0} \langle Y, \mathrm{Ad}(e^{tZ})\,\mathrm{Ad}(g_0)X\rangle = \langle Y, \mathrm{ad}(Z)\,\mathrm{Ad}(g_0)X\rangle = -\langle Y, [\mathrm{Ad}(g_0)X, Z]\rangle. \qquad \blacksquare$$

**Corollary 24.9.** (a) $\mathrm{Ad}(G)$ acts transitively on the set of Cartan subalgbras.

(b) $G$ acts transitively on maximal tori of $G$ by conjugation.

*Proof.* For (a), let $\mathfrak{t}_1 = \mathfrak{z}(X)$, and let $g \in G$ such that $\mathrm{Ad}(g)X \in \mathfrak{t}_2$. Then

$$\mathrm{Ad}(g)\mathfrak{t}_1 = \{\mathrm{Ad}(g)Y \mid [Y, X] = 0\}.$$

Write $Y' = \mathrm{Ad}(g)Y$. Then

$$[\mathrm{Ad}(g)^{-1}Y', X] = 0 \implies [Y', \mathrm{Ad}(g)X] = 0.$$

So $\mathfrak{t}_2 \subseteq \mathfrak{z}(\mathrm{Ad}(g)X)$. By the maximality of $\mathfrak{t}_2$, $\mathrm{Ad}(g)\mathfrak{t}_1 = \mathfrak{t}_2$.

For (b), let $T_i = \exp \mathfrak{t}_i$. Then

$$gT_1g^{-1} = g\exp(\mathfrak{t}_1)g^{-1} = \exp(\mathrm{Ad}(g)\mathfrak{t}_1) = \exp(\mathfrak{t}_2) = T_2. \qquad \blacksquare$$

Recall that if $G$ is connected, then $\mathrm{Ad}(g) = \mathrm{id}$ if and only if $g \in Z(G)$.

**Theorem 24.10.** Let $G$ be a compact connected Lie group. Then $\exp \mathfrak{g} = G$ and for each $g_0 \in G$, there exists $g \in G$ such that $gg_0g^{-1} \in T$.

*Proof.* Indeed, $g_0$ lies in some maximal torus $T'$, and $gT'g^{-1} = T$ for some $g \in G$. $\blacksquare$

**Theorem 24.11.** Let $G \subseteq \mathrm{GL}(n, \mathbb{C})$, $\gamma\colon \mathbb{R} \to \mathfrak{g}$ a $C^\infty$ curve. Then

$$\frac{d}{dt}\gamma(t) = \left(\frac{e^{\mathrm{ad}\,\gamma(t)} - 1}{\mathrm{ad}\,\gamma(t)}\right)\gamma'(t) \cdot e^{\gamma(t)} = e^{\gamma(t)} \cdot \left(\frac{1 - e^{-\mathrm{ad}\,\gamma(t)}}{\mathrm{ad}\,\gamma(t)}\right)\gamma'(t).$$

Note that $(e^z - 1)/z$ and $(1 - e^{-z})/z$ are invertible power series in $z$.

*Proof.* Consider the $C^\infty$ function $\varphi(s,t) = e^{-s\gamma(t)}\frac{\partial}{\partial t}e^{s\gamma(t)}$. Then $\varphi(0,t) = 0$ and

$$\frac{\partial}{\partial s}\varphi(s,t) = -e^{-s\gamma}\gamma\frac{\partial}{\partial t}e^{s\gamma} + e^{-s\gamma}\frac{\partial}{\partial t}(\gamma e^{s\gamma}) = \operatorname{Ad}(e^{-s\gamma})\gamma' = e^{-s\operatorname{ad}\gamma}\gamma'.$$

So

$$\begin{aligned} e^{-\gamma(t)}\frac{\partial}{\partial t}e^{\gamma(t)} = \varphi(1,t) &= \int_0^1 \frac{\partial}{\partial s}\varphi(s,t)\,ds = \int_0^1 e^{-s\operatorname{ad}\gamma}\gamma'\,ds \\ &= \left(\int_0^1 \sum_n \frac{(-s)^n}{n!}(\operatorname{ad}\gamma)^n\right)\gamma' = \frac{1-e^{-\operatorname{ad}\gamma}}{\operatorname{ad}\gamma}\gamma'. \end{aligned}$$ ■

**Corollary 24.12.** The tangent map $(d\exp)_X$ is nonsingular if and only if

$$\operatorname{Spec}(\operatorname{ad}X) \subseteq (\mathbb{C}\setminus 2\pi i\mathbb{Z}) \cup \{0\}.$$

*Proof.* Simply take $\gamma(t) = X + tY$ with $(\operatorname{ad}X)Y = \lambda Y$. Then

$$\left(\frac{1-e^{-\operatorname{ad}X}}{\operatorname{ad}X}\right)Y = \begin{cases} \dfrac{1-e^{-\lambda}}{\lambda}Y, & \text{if } \lambda \neq 0, \\ Y, & \text{if } \lambda = 0. \end{cases}$$ ■

**Theorem 24.13** (Dynkin's formula)**.** For any $X$, $Y \in \mathfrak{gl}(n)$, we have $e^Xe^Y = e^Z$, where

$$Z = \sum_{i_k+j_k\geq 1} \frac{(-1)^{n+1}}{n}\frac{1}{(i_1+j_1)\cdots(i_k+j_k)}\cdot\frac{[X^{(i_1)}Y^{(j_1)}\dots X^{(i_k)}Y^{(j_k)}]}{i_1!j_1!\cdots i_k!j_k!}.$$

*Proof.* There exists a unique $C^\infty$ function $Z(t)$ such that $e^{Z(t)} = e^{tX}e^{tY}$ near $t = 0$. Then

$$\left(\frac{e^{\operatorname{ad}Z}-1}{\operatorname{ad}Z}\right)Z'\cdot e^Z = Xe^Z + e^ZY.$$

Hence,

$$\begin{aligned} Z' &= \left(\frac{\operatorname{ad}Z}{e^{\operatorname{ad}Z}-1}\right)(X + \operatorname{Ad}(e^Z)Y) \\ &= \left(\frac{\operatorname{ad}Z}{e^{\operatorname{ad}Z}-1}\right)(X + \operatorname{Ad}(e^{tX})Y) = \left(\frac{\operatorname{ad}Z}{e^{\operatorname{ad}Z}-1}\right)(X + e^{t\operatorname{ad}X}Y). \end{aligned}$$

Note that

$$\operatorname{ad}Z = \log(1 + (e^{\operatorname{ad}Z}-1)) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n}(e^{\operatorname{ad}Z}-1)^n.$$

So

$$\left(\frac{\operatorname{ad}Z}{e^{t\operatorname{ad}Z}-1}\right)(X + e^{t\operatorname{ad}X}Y) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n}(e^{t\operatorname{ad}X}e^{t\operatorname{ad}Y}-1)^{n-1}(X + e^{t\operatorname{ad}X}Y).$$

The result now follows by an easy calculation. ■

**Corollary 24.14.** Let $N \subseteq \mathrm{GL}(n, \mathbb{C})$ be a connected subgroup such that $\mathfrak{n} := \operatorname{Lie} N$ is contained in the set of strict upper triangular matrices. Then $N = \exp \mathfrak{n}$.

*Proof.* Consider the equation $e^X e^Y = e^Z$ near 0 (so that exp is one-to-one). The matrix coefficients of $Z$ are polynomial in $X = (x^i_j)$, $Y = (y^i_j)$ by Dynkin's formula. So the equality holds everywhere. Hence, $(\exp \mathfrak{n})^2 \subseteq \exp \mathfrak{n}$. Since $\exp \mathfrak{n}$ generated $N$, $\exp \mathfrak{n} = N$. ■

**Theorem 24.15.** Let $G$ be a compact Lie group. Then $\mathfrak{g}$ is reductive.

*Proof.* Let $\langle -, - \rangle$ be a Ad-invariant inner product on $\mathfrak{g}$. Then $\mathfrak{a} \subseteq \mathfrak{g}$ implies $\mathfrak{a}^\perp \subseteq \mathfrak{g}$. Hence,

$$\mathfrak{g} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k \oplus \mathfrak{z}_1 \oplus \cdots \mathfrak{z}_k,$$

where $\dim \mathfrak{s}_i \geq 2$ and $\dim \mathfrak{z}_i = 1$. It is easy to check that $[\mathfrak{s}_i, \mathfrak{s}_j] = 0$ if $i \neq j$ and $Z(\mathfrak{g}) = \bigoplus \mathfrak{z}_j$. ■

**Theorem 24.16** (Structure of compact Lie group).

(a) Let $G'$ be the normal subgroup generated by commutators $[g, h] = ghg^{-1}h^{-1}$. If $G$ is compact connected, then $G'$ is connected, closed in $G$ and $\operatorname{Lie} G' = [\mathfrak{g}, \mathfrak{g}]$.

(b) $G = G' \times Z(G)^\circ / F$, where $F = G' \cap Z(G)^\circ$ is a finite abelian group.

(c) For $\mathfrak{g}' = \bigoplus \mathfrak{s}_i$, $S_i = \exp(\mathfrak{s}_i) \trianglelefteq G'$ is connect, closed, with only proper closed normal subgroup being finite central in $G$.

# 25 Reduce Lie group representations to Lie algebra representations, 12/14

Let $G$ be a Lie subgroup of $\mathrm{GL}(n, \mathbb{C})$, $\rho\colon G \to \mathrm{GL}(V)$ a finite dimensional representation. Then $\rho(e^X) = e^{d\rho(X)}$, so $d\rho$ determines $\rho|_{G^\circ}$. Also, $\rho$ determines $d\rho$. Hence, for $G$ connected, $W \subseteq V$ is $\rho(G)$-invariant if and only if $W$ is $d\rho(\mathfrak{g})$-invariant. For $G$ compact connected, $V$ is irreducible if and only if $V$ is irreducible as a $\mathfrak{g}_\mathbb{C}$-representation, where $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$.

**Observation.** We can put $\mathfrak{g} \subseteq \mathfrak{u}(n) \subseteq \mathfrak{gl}(n, \mathbb{C}) = \mathfrak{u}(n) \oplus i\mathfrak{u}(n) = \mathfrak{u}(n)_\mathbb{C}$. So there is a natural inclusion $\mathfrak{g}_{\mathbb{C}} \to \mathfrak{u}(n)_{\mathbb{C}}$.

Note that elements in $\mathfrak{u}(n)$ are skew-Hermitian, while elements in $i\mathfrak{u}(n)$ are Hermitian. So elements in $\mathfrak{u}(n) \cup i\mathfrak{u}(n)$ are normal.

**Example 25.1.**

$$\begin{aligned} \mathfrak{su}(n)_{\mathbb{C}} &= \mathfrak{sl}(n, \mathbb{C}) \\ \mathfrak{so}(n)_{\mathbb{C}} &= \{X^{\mathsf{T}} = -X\}, \\ \mathfrak{sp}(n)_{\mathbb{C}} &= (\mathfrak{u}(2n) \cap \mathfrak{sp}(n, \mathbb{C}))_{\mathbb{C}} = \mathfrak{sp}(n, \mathbb{C}). \end{aligned}$$

We see that $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$ are real compact Lie groups, while $\mathrm{SL}(n)$, $\mathrm{Sp}(n)$ are non-compact.

**Theorem 25.2.** For any semisimple Lie algebra $L$ over $\mathbb{C}$, there exists a compact real form, i.e., there exists a real compact Lie group $G$ such that $L \cong \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$.

Let $G$ be a compact Lie group that acts on $V$ by $\rho$, $\langle -, - \rangle$ a $G$-invariant inner product on $\mathbb{C}$, $\mathfrak{t} \subseteq \mathfrak{g}$ a Cartan subalgebra. Then $\mathfrak{t}_{\mathbb{C}}$ acts on $V$ as a family of commuting normal operators, and hence simultaneously diagonalizable. So the Cartan subalgebra defined here is same as the Cartan subalgebra defined in the theory of Lie algebra.

Now, fix a maximal torus $T \subseteq G$, $\mathfrak{t} = \operatorname{Lie} T$. For a $G$-module $(\rho, V)$, consider the weight space decomposition

$$V = \bigoplus_{\alpha \in \Phi(V)} V_\alpha, \quad H \cdot v = d\rho(H) \cdot v = \alpha(H) \cdot v, \quad \forall H \in \mathfrak{t}_{\mathbb{C}},\ v \in V_\alpha.$$

Take $(\rho, V) = (\mathrm{Ad}, \mathfrak{g}_{\mathbb{C}})$. Then we have the root decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Phi(\mathfrak{g}_{\mathbb{C}})^\times} \mathfrak{g}_\alpha.$$

Then $\Phi(\mathfrak{g}_{\mathbb{C}})^\times$ could be decomposed into the positive part $\Phi^+$ and the negative part $\Phi^-$.

**Example 25.3.** Let $G = \mathrm{SU}(n)$,

$$\mathfrak{t} = \left\{ \operatorname{diag}(i\theta_1, \dots, i\theta_n) \,\middle|\, \sum \theta_i = 0 \right\}, \quad \mathfrak{t}_{\mathbb{C}} = \left\{ \operatorname{diag}(z_1, \dots, z_n) \,\middle|\, \sum z_i = 0 \right\}.$$

Then $\Phi = \{\pm(\varepsilon_i - \varepsilon_j) \mid i < j\}$, where $\varepsilon_i(\operatorname{diag}(z_j)) = z_i$. This is indeed $A_{n-1}$.

As in the Lie algebra representation, an element $v \in V_{\lambda_0}$ is a highest weight vector if $X \cdot v = 0$ for all $X \in \mathfrak{n}^+$. New feature: analytically integral weight,

$$A = A(T) = \{\lambda \in (i\mathfrak{t})^\vee \mid \lambda(H) \in 2\pi i\mathbb{Z},\ \forall e^H = \mathrm{id}\}.$$

We see that $A$ is isomorphic to the character group $\chi(T) = \mathrm{Hom}(T, \mathbb{C}^\times)$ of $T$ by $\xi_\lambda(e^H) = e^{\lambda(H)}$.

**Theorem 25.4.** Let $G$ be a connected compact Lie group, $V$ a finite dimensional irreducible representation. Then there exists a unique highest weight $\lambda_0$ which is dominant, integral, and analytically integral.

**Definition 25.5.** An element $g \in G$ is regular if $Z_G(g)^\circ$ is a maximal torus. The set of regular elements in $G$ is denoted by $G^{\mathrm{reg}}$, and is open dense in $G$.

For $t \in T$, define $d(t) = \prod_{\alpha\in\Phi}(1 - \xi_{-\alpha}(t))$, which is nonzero if and only if $t$ is regular.

**Theorem 25.6** (Weyl integral formula)**.** For $f \in C(G)$,

$$\int_G f(g)\,dg = \frac{1}{|W(G)|}\int_T d(t)\int_{G/T} f(gtg^{-1})\,d(gT)dt,$$

where $W(G) = N_G(T)/T$, which is in fact isomorphic to the Weyl group of $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{t}_\mathbb{C}$.

*Proof.* Consider

$$\psi\colon G/T \times T^{\mathrm{reg}} \longrightarrow G^{\mathrm{reg}}$$

by multiplication. This map is surjective, and is a $|W(G)|$ to 1 local diffeomorphism. Now use

$$\psi^*\omega_G = d(t)\pi_1^*\omega_{G/T} \wedge \pi_2^*\omega_T. \qquad \blacksquare$$

**Theorem 25.7.** Let $V = V(\lambda)$ be the representation with highest weight $\lambda$. For $g \in G^{\mathrm{reg}}$, $g$ is conjugate to $e^H \in T$ for some $H \in \mathfrak{t}$, then

$$\chi_\lambda(g) = \Theta_\lambda(g) := \frac{\sum_{w\in W(G)} \det w \cdot e^{w(\lambda+\Phi)(H)}}{\prod_{\alpha\in\Phi^+}(e^{\alpha(H)/2} - e^{-\alpha(H)/2})},$$

where $\Phi = \frac{1}{2}\sum_{\alpha\succ 0}\alpha$.

## 26 Borel-Weil theorem, 12/19

**Definition 26.1.** Let $G$ be a compact connected Lie group, $T$ a maximal torus of $G$. Then we can embed $G$ into $\mathrm{U}(n) \subseteq \mathrm{GL}(n, \mathbb{C})$. Fix $\Phi^+(\mathfrak{g}_\mathbb{C})$, we get a Borel subalgebra $\mathfrak{b} = \mathfrak{t}_\mathbb{C} \oplus \mathfrak{n}^+$ and $\mathfrak{g}_\mathbb{C} = \mathfrak{n}^- \oplus \mathfrak{t}_\mathbb{C} \oplus \mathfrak{n}^+$. Let $N$, $B$, $A$, $G_\mathbb{C}$ be the connected Lie subgroup in $\mathrm{GL}(n, \mathbb{C})$ correspond to $\mathfrak{n}^+$, $\mathfrak{b}$, $\mathfrak{a} = i\mathfrak{t}$, $\mathfrak{g}_\mathbb{C} \subseteq \mathfrak{gl}(n, \mathbb{C}) = \mathfrak{u}(n)_\mathbb{C}$.

The Cartan involution $\theta$ (an abstact version of complex conjugation) is defined to be $\theta(x \otimes z) = x \otimes \overline{z}$. Hence, $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$ is the eigenspace decomposition of $\theta$ (with eigenvalue $1$, $-1$, respectively). Since $\mathfrak{g} \subseteq \mathfrak{u}(n)$, $\theta Z = -Z^*$:

$$Z = X + iY \quad \Longrightarrow \quad -Z^* = -X^* + iY^* = X - iY.$$

**Proposition 26.2.** Let $\alpha \in \Phi(\mathfrak{g}_\mathbb{C})$ be a root. Then $\alpha$ is purely imaginary on $\mathfrak{t}$, equivalently, $\alpha$ is real on $\mathfrak{a}$. In particular, $\theta \mathfrak{g}_\alpha = \mathfrak{g}_{-\alpha}$.

*Proof.* The first statement follows from the facts that $\alpha$ skew-hermitian on $\mathfrak{t}$ and hermitian on $i\mathfrak{t}$. For $H \in \mathfrak{t}$, $Z = X + iY \in \mathfrak{g}_\alpha$,

$$\alpha(H)(X + iY) = [H, X] + i[H, Y]$$

implies that $\alpha(H)X = i[H, Y]$, $\alpha(H)Y = [H, X]$. Hence,

$$\mathrm{ad}(H)(\theta Z) = [H, X] - i[H, Y] = -\alpha(H)(X - iY) = (-\alpha)(H)(\theta Z). \qquad \blacksquare$$

**Remark 26.3.** For $\mathbb{G}$ compact, $\mathfrak{g}$ semisimple, the Killing form $B(X, Y) = \mathrm{tr}(\mathrm{ad}\, X \,\mathrm{ad}\, Y)$ is negative definite on $\mathfrak{g}$ since

$$B(X, X) = \sum_{\alpha \in \Phi} \alpha(X)^2 < 0.$$

So we prefer to consider $\alpha \in \mathfrak{a}^*$, so that $\alpha(H) = B(H, u_\alpha)$ for some $u_\alpha \in \mathfrak{a}$, and get

$$h_\alpha = \frac{2}{B(u_\alpha, u_\alpha)} \cdot u_\alpha, \quad \alpha(h_\alpha) = 2.$$

These give us the standard $\mathfrak{sl}(2, \mathbb{C})$ triple: take $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha = -\theta e_\alpha$, then $[e_\alpha, f_\alpha] \parallel h_\alpha$ and we may assume that $[e_\alpha, f_\alpha] = h_\alpha$.

Let $M$ be a $C^\infty$ manifold, $\mathbf{V} \to M$ a complex vector bundle of rank $n$. Assume that a Lie group $G$ acts on $\mathbf{V}$ fiberwisely, i.e., $g \cdot \mathbf{V}_x \subseteq \mathbf{V}_{g(x)}$ for some $g(x) \in M$. We say that $V$ is a **homogeneous vector bundle** if $\mathbf{V}_x \xrightarrow{g} \mathbf{V}_{g(x)}$ is a linear isomorphism. Then $G$ acts on $M$ and on $\Gamma(M, \mathbf{V})$ by $(g \cdot s)(x) = g \cdot s(g^{-1}x)$.

Let $H \subseteq G$ be a closed subgroup, $V$ a finite dimensional representation of $H$. Then

$$\mathbf{V} = G \times_H V := {}^{G \times V}\!\big/\!{\sim} \longrightarrow M := G/H$$

is a homogeneous vector bundle, where $(gh, v) \sim (g, hv)$ and $g' \cdot [(g, v)] = [(g'g, v)]$.

**Proposition 26.4.** There is a 1-1 correspondence between homogeneous vector bundles over $G/H$ and finite dimensional representations of $H$.

*Proof.* Indeed, $\mathbf{V}_{eH}$ is a representation of $H$. ■

**Definition 26.5.** Let $H$ be a closed subgroup of $G$, $\rho\colon H \to \mathrm{GL}(V)$ a representation. The **induced representation** $\mathrm{Ind}_H^G(\rho) = \mathrm{Ind}_H^G(V)$ of $\rho$ (or $V$) is

$$\{f\colon G \to V \mid f(gh) = h^{-1} \cdot f(g)\}$$

with action $(g' \cdot f)(g) = f((g')^{-1}g)$.

**Proposition 26.6.** There is a natural $G$-isomorphism

$$\Gamma(G/H, G \times_H V) \xrightarrow{\sim} \mathrm{Ind}_H^G(V).$$

*Proof.* Identify $(G_H \times V)_{eH} \cong V\colon (h, v) \mapsto h^{-1}v$. For $s \in \Gamma(G/H, G \times_H V)$, it corresponds to $f_s(g) = g^{-1}s(gH)$. For $f \in \mathrm{Ind}_H^G(V)$, it corresponds to $s_f(gH) = (g, f(g))$. ■

**Theorem 26.7** (Frobenius reciprocity)**.** Let $H$ be a closed subgroup of $G$, $V$ an $H$-module, $W$ a $G$-module. Then

$$\mathrm{Hom}_G(W, \mathrm{Ind}_H^G(V)) \cong \mathrm{Hom}_H(W|_H, V)$$

as $\mathbb{C}$-vector spaces.

*Proof.* Reading. ■

**Lemma 26.8.** The exponential maps $\exp\colon \mathfrak{n}^+ \to N$, $\mathfrak{a} = i\mathfrak{t} \to A$ are bijections, $N$, $B$, $A$ are closed subgroups of $G_{\mathbb{C}}$, and

$$\begin{aligned} T \times \mathfrak{a} \times \mathfrak{n}^+ &\longrightarrow B \\ (t, iH, X) &\longmapsto te^{iH}e^X \end{aligned}$$

is a diffeomorphism.

*Proof.* This follows from Dynkin's formula. ■

**Theorem 26.9.** We have $G/T \cong G_{\mathbb{C}}/B$, hence it is a complex (homogeneous) manifold.

*Proof.* Since $\mathfrak{g} = \{X + \theta X \mid X \in \mathfrak{g}_{\mathbb{C}}\}$, $\mathfrak{g}/\mathfrak{t}$ and $\mathfrak{g}_{\mathbb{C}}/\mathfrak{b}$ both are spanned by the image of $X_\alpha + \theta X_\alpha$, where $X_\alpha \in g_\alpha$, $\alpha \in \Phi^+$. So $p\colon G \to G_{\mathbb{C}}/B$ has $dp$ surjective at $e \in G$. Then $\operatorname{Im} p$ contains a neighborhood of $eB$ and hence open and closed. Thus, $p$ is surjective.

We claim that $G \cap B = T$. First of all, $\mathfrak{g} \cap \mathfrak{b} = \mathfrak{t}$ is known. Let $g \in G \cap B$. Then $\operatorname{Ad}(g)$ preserves $\mathfrak{t} = \mathfrak{g} \cap \mathfrak{b}$, hence $T$, i.e., $g \in N_G(T)$. Let $w$ be the image of $g$ in the Weyl group. Then $g \in B$ implies that $w$ preserves $\Delta^\perp$, hence preserves the fundamental Weyl chamber. Thus, $w = I$ and $g = T$. ■

**Definition 26.10.** For $\lambda \in A(T)$, let $\mathbb{C}_\lambda$ be the $T$-module corresponds to the character $\xi_\lambda\colon T \to \mathbb{C}^\times$, and $L_\lambda = G \times_T \mathbb{C}_\lambda$ the homogeneous line bundle over $G/T$. We extend $\xi_\lambda$ to $\xi_\lambda^{\mathbb{C}}\colon B \to \mathbb{C}^\times$ by

$$\xi_\lambda^{\mathbb{C}}(te^{iH}e^X) = \xi_\lambda(t)e^{i\lambda(H)},$$

and still denote the corresponding $B$-module by $\mathbb{C}_\lambda$. Let $L_\lambda^{\mathbb{C}} = G_{\mathbb{C}} \times_B \mathbb{C}_\lambda$ be the homogeneous (holomorphic) line bundle over $G_{\mathbb{C}}/B$.

**Lemma 26.11.** We have

$$\operatorname{Ind}_T^G(\xi_\lambda) \cong \Gamma(G/T, L_\lambda) \cong \Gamma(G_{\mathbb{C}}/B, L_\lambda^{\mathbb{C}}) \cong \operatorname{Ind}_B^{G_{\mathbb{C}}}(\xi_\lambda^{\mathbb{C}})$$

as $C^\infty$-sections.

Since $L_\lambda^{\mathbb{C}}$ is holomorphic over $G_{\mathbb{C}}/B$, we have $\Gamma_{\mathrm{hol}}(G/T, L_\lambda) := \Gamma_{\mathrm{hol}}(G_{\mathbb{C}}/B, L_\lambda^{\mathbb{C}})$.

**Theorem 26.12** (Borel-Weil)**.** The space

$$\Gamma_{\text{hol}}(G/T, L_\lambda) = \begin{cases} V(-\lambda), & \text{if } -\lambda \text{ is dominant}, \\ 0, & \text{else.} \end{cases}$$

*Proof.* Use

$$C^\infty(G)_{G\text{-fin}} = \bigoplus_{\substack{\gamma \in A(T) \\ \text{dominant}}} V(\gamma)^\vee \otimes V(\gamma)$$

to read out holomorphic property in this decomposition. ■

**Theorem 26.13** (Bott-Borel-Weil)**.** Let $\lambda \in A(T)$, $\delta = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$. If $\lambda + \delta$ lies in a Weyl chamber wall, then

$$H^p(G/T = G_{\mathbb{C}}/B, L_\lambda^\vee) = 0, \quad p > 0.$$

Otherwise, let $w \in W(\Phi^+)$ such that $w * \lambda = w(\lambda + \delta) - \delta$ is dominant, and $\ell(w)$ be the length of $w$, which is equal to the number of $\alpha \in \Phi^+$ such that $B(\lambda + \delta, \alpha) < 0$. Then

$$H^p(G/T, L_\lambda^\vee) = \begin{cases} V(w * \lambda), & \text{if } p = \ell(w), \\ 0, & \text{else.} \end{cases}$$