代數幾何簡介

1999 年台灣大學理學院暑期課程 (對象: 數學系與物理系高年級大學部學生)

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摘要

這個短期課程這個短期課程的主要目的是希望能介紹給同學一些具體可算的例子, 用以說明代數幾何學理最根本的一些思想. 代數幾何過去被認為是艱困的純數學領域. 但是在近 20 年來它與其他科學甚至科技的關聯漸漸被開發了出來. 可以預見的未來, 代數幾何將會是解決許多問題的過程中有用的工具. 因此我們期望利用這個課程讓更多的同學能建立對代數幾何的興趣.

主要的內容將會包括:

Lecture I. 橢圓函數與三次曲線

這個主題的目標在於建立古典富變函數與代數多項式的關係. 我們希望能解釋為什麼橢圓周長的積分式是無法被初等函數所表達的.

Lecture II. Riemann—Roch 定理及其應用

我將會對代數曲線的 Riemann—Roch 問題作較深入的探討. 時間許可之下, 我們將簡介 Riemann—Roch 問題與 Hilbert 多項式在一般維度下的關係.

Lecture III. Blow-Up 技巧與三次曲面的幾何

我們將詳細介紹 blow up 的意義與計算方法, 並用以化解奇異點. 我們利用它來證明三次多項式所定義的曲面上恰有 27 條直線, 並且將概述這與理論物理上 quanum correction 的關係. 最後我們用它來化解著名的 ADE 奇異點.

INTRODUCTION TO ALGEBRAIC GEOMETRY

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Lecture I. — Elliptic Functions and Cubic Curves

1. Origin from Algebra. We lerned since high school days that if X, Y and Z are the length of three sides of a right triangle with Z the long one, then (Pythogarus Theorem)

$$X^2 + Y^2 = Z^2. \tag{1.1}$$

We also learned how to obtain all right triangles with integer side length, that is, integer solutions of (1.1). The answer is

$$X = u^2 - v^2, \quad Y = 2uv, \quad Z = u^2 + v^2, \tag{1.2}$$

for all integers u and v.

While it is trivial to check (1.2) gives some solutions of (1.1), it is not completely obvious why all solutions are given in this way. One way to see this is the following: first of all, the problem is equivalent to find all rational solutions of

$$x^2 + y^2 = 1, \tag{1.3}$$

which is simply the equation of a circle in the plane. Let P be the point $(-1, 0)$ and $Q = (x, y)$ be another point on the circle. Let $t = y/(x+1)$ be the slope of the line PQ. The idea is, since each slope $t \in (-\infty, \infty)$ corresponds exactly to one point Q, one should be able to represent (x, y) in terms of t. To proceed, substitute y by $t(x+1)$ into (1.3). After simplification one get

$$(t^2 + 1)x^2 + t^2 x + (t^2 - 1) = 0. \tag{1.4}$$

That is, $((t^2+1)x + (t^2-1))(x+1) = 0$. The root $x = -1$ corresponds to the original choosen point P and the other root gives the point Q in terms of t:

$$x = \frac{t^2 - 1}{t^2 + 1}, \qquad y = \frac{2t}{t^2 + 1}, \tag{1.5}$$

Which gives all rational solutions of (1.3) if t run through all rational numbers. Write $t = u/v$ as a ratio of two coprime integers, one easily recover (1.2).

Exercise 1. A conic is a plane curve defined by a ploynomial equation $f(x, y) = 0$ with $\deg f \leq 2$. Show that any conic can parametrized by rational functions in the sense that there are rational functions $x(t)$ and $y(t)$ such that $f(x(t), y(t)) = 0$.

A similar equation had been considered by Fermat 300 years ago:

$$X^n + Y^n = Z^n, \quad n \geq 3. \tag{1.6}$$

Fermat claimed there are no nontrivial integer solutions of it. However, "The Proof" has only recently been obtained by Andre Wiles which uses almost all essential machinery in number theory developed during the past 300 years! Instead of dealing with Fermat's question, we may consider a (much) simpler problem, the rational parametrization problem in the sense discussed above: can one find three polynomials $X(t)$, $Y(t)$ and $Z(t)$ such that

$$X(t)^n + Y(t)^n = Z(t)^n, \quad n \geq 3? \tag{1.7}$$

Like Fermat's expectation, the answer is NO. But this time a purely elementary proof is possible. Later in this lecture we will do this using geometric method. In fact we will deal with arbitrary polynomials, not just of Fermat's type.

2. Origin from Analysis. The next example is about indefinite integral which we learned in Calculus. It is a simple result using change of variables through trigonometric functions that we may "integrate out"

$$\int \frac{dx}{\sqrt{x^2 + ax + b}} \tag{2.1}$$

for any a and b to get a closed formula in terms of "elementary functions", that is, in terms of usual polymomials, $x^\alpha, \alpha \in \mathbf{R}$, trigonometric functions, exponential functions, their inverse functions like $\tan^{-1}$, log etc. and all their composition functions.

However, since Newton and Lebnitz created the theory of Calculus in the 17th century, it is generally understood that while differentiation of elementary functions is still elementary, it is definitely not true for indefinite integrals. The first and perhaps the most famous example studied in mathematical history is the "elliptic integral" by Abel and Jacobi in the 19th century:

$$\int \frac{dx}{\sqrt{x^3 + ax + b}}. \tag{2.2}$$

The polynomial inside the square root could also be of higher degree. Abel-Jacobi theory implies that all these indefinite integrals are not expressible by elementary functions. This will be explained at the end of this lecture.

Exercise 2. Show that the calculation of the circonference of an ellipse can be reduced to the following integral:

$$\int \frac{dx}{\sqrt{(1 - x^2)(1 - kx^2)}}. \tag{2.3}$$

This is the historical reason for the name "elliptic integral".

3. Basic Complex Analysis. Let U be an open subset of the complex plane $\mathbf{C} \cong \mathbf{R}^2$ and $f : U \to \mathbf{C}$ be a C^∞ complex valued function. We wish to study f's such that it is holomorphic (= complex differentiable) in the sense that

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \tag{3.1}$$

is well-defined at every point $z \in U$. Write $f(z) =: f(x, y) = u(x, y) + iv(x, y)$ and consider two special way to make h approaching 0, $h = \Delta x$ and $h = i\Delta y$. Then (3.1) has an unique limit implies that $u_x + iv_x = \frac{1}{i}(u_y + iv_y)$, which is equivalent to the Cauchy-Riemann equation:

$$u_x = v_y, \qquad v_x = -u_y. \tag{3.2}$$

(Geometrically this says that the mapping $(x, y) \mapsto (u, v)$ is a conformal mapping.) As in the multivariable calculus, the line integral $\int_C f(z)dz$ is defined for a path in U. It satisfies some nice properties

Cauchy Theorem. *Let f be a holomorphic function on U. Then $\int_C f(z)dz = 0$ for any contractible closed curve C in U.*

Proof. Write $C = \partial D$, then by Green's formula of line integrals,

$$\begin{aligned} \int_C f(z)dz &= \int_C (u + iv)(dx + idy) = \int_{\partial D} (udx - vdy) + i(vdx + udy) \\ &= \int_D (-v_x - u_y)dA + i(u_x - v_y)dA = 0 \end{aligned} \tag{3.3}$$

by the Cauchy-Riemann equation (3.2). Q.E.D.

Exercise 3. Deduce from Cauchy's theorem the following Cauchy integral formula:

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - a} dz,$$

where a is a point inside C. Use this to prove that f is holomorphic if and only if f has a power series expansion (with nonzero radius of convergence) at every point. (This is the Taylor expansion).

We also need to consider moromorphic functions $f(z)$ on U. These are functions which could be locally written as $f(z) = g(z)/h(z)$ where $g(z)$ and $h(z)$ are holomorphic. By Exercise 3. This is equivalent to say that $f(z)$ has a Laurent series expansion at each point $a \in U$:

$$f(z) = c_k(z - a)^k + c_{k+1}(z - a)^{k+1} + \cdots, \quad k \in \mathbf{Z}. \tag{3.4}$$

The least integer k with $c_k \neq 0$ is called $\mathrm{ord}_a f$, the order of f at a. f is said to have a pole at a of order $|k|$ if $k < 0$. Clearly a meromorphic function is holomorphic if and only if it has no poles. It is a simple consequence of the existence of Taylor (Laurent) expansion that the set of zeros and poles is a discrete subset of U (that is, no limit points).

Now a simple compuattion shows that

$$\begin{aligned}\int_{|z|=r} z^k dz &= \int_0^{2\pi} (r^k e^{ik\theta})(ire^{i\theta} d\theta) = i\int_0^{2\pi} r^{k+1} e^{i(k+1)\theta} d\theta \\ &= \begin{cases} 2\pi i, & \text{if } k = -1 \\ 0, & \text{otherwise.}\end{cases}\end{aligned} \tag{3.5}$$

If we define $\mathrm{Res}_a f$, the residue of f at a, to be the coefficient a_{-1} in the Laurent expansion, then Cauchy's theorem and (3.5) leads to

Residue Theorem. For a meromorphic function f on U and a closed curve $C = \partial D \subset U$ not passing through poles of f,

$$\int_C f(z)dz = \sum_{a\in D} \mathrm{Res}_a f. \tag{3.6}$$

For example, let g be holomorphic and let $f = g'/g$. using Taylor's expansion we see that $\mathrm{Res}_a f = \mathrm{ord}_a g$, hence the so-called "argument principle":

$$\int_C \frac{g'(z)}{g(z)} dz = \sum_{a\in D} \mathrm{ord}_a g. \tag{3.7}$$

4. Tori and Weierstrass' $\mathcal{P}$ function. Let $\Lambda \subset \mathbf{C}$ be a rank 2 free abelian group (lattice). Without loss of generality we may assume that $\Lambda = \mathbf{Z} \oplus \mathbf{Z}\tau$, τ a complex number with $\mathrm{Im}\ \tau > 0$. we define a compact space $X = \mathbf{C}/\Lambda$ by identifying z with $z + \lambda$ for all $\lambda \in \Lambda$. It is topologically a torus with one hole in it. But analytically X has a natural one dimensional complex manifold structure. We would like to study the geometry of X. Let D be the "fundamental domain", that is, the parallelogram with vertexes 0, 1, $1 + \tau$ and τ. It is natural to seek of nonconstant holomorphic functions on X. However, one has

Fact. *There is no nonconstant holomorphic functions on a compact complex manifold.*

Although we have not yet defined the notion of complex manifold, this fact is just a simple consequence of the maximal modulus principle. A special proof in the case of torus $X = \mathbf{C}/\Lambda$ is that such a function f, bounded since X is compact, can be lifted to be a bounded holomorphic function on the whole $\mathbf{C}$. Liouville theorem then says that it must be a constant.

Next we seek of meromorphic function f. Consider the closed curve $C = \partial(a + D)$ for suitable $a \in \mathbf{C}$ such that all zeros and poles of f are not on C. Then $\int_C f(z)dz = 0$ because integrals of opposite sides cancel out ($f(z+1) = f(z) = f(z+\tau)$ and the sides have opposite directions). (3.6) or (3.7) then shows that f can not have only a simple pole. Thus f must have at least 2 poles or f has exactly one pole with order ≥ 2. By lefting f to $\mathbf{C}$ we may also regard f as a meromorphic function on the whole complex plane $\mathbf{C}$ which is doubly periodic with periods 1 and τ. In order to construct f with a single double pole at 0, it is necesarily be of the form (to make it Λ-periodic):

$$\frac{1}{z^2} + \sum_{\lambda \in \Lambda - 0} \frac{1}{(z-\lambda)^2}. \tag{4.1}$$

However, (3.8) does not gives a convergent series. To remedy the convergency, Weierstrass introduced his famous correction terms and get the so-called Weierstrass $\mathcal{P}$ function:

$$\mathcal{P}(z) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda - 0} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right). \tag{4.2}$$

Up to translation ($g(z) := f(z+a)$) and addition or multiplication by a constant, this is the unique function with this property. Following Weierstrass, we will show

Theorem (Weierstrass). *The functions $\mathcal{P}$ and $\mathcal{P}'$ satisfy a cubic polynomial equation*

$$\mathcal{P}'(z)^2 = 4\mathcal{P}(z)^3 - g_2 \mathcal{P}(z) - g_3, \tag{4.3}$$

for certain constant g_2, g_3 depends only on Λ.

Proof. $\mathcal{P}'(z)$ is a meromorphic function on X with an uniqhe pole at 0 of order 3. In fact,

$$\mathcal{P}'(z) = -2 \left(\frac{1}{z^3} + \sum_{\lambda \in \Lambda - 0} \frac{1}{(z-\lambda)^3} \right) = -2 \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3}. \tag{4.4}$$

(This time it is a convergent series.) So it is obvious to see that $\mathcal{P}'(z)^2$ and $4\mathcal{P}(z)^3$ are both meromorphic with same 6 order pole at 0 with the same coefficient. This means that their difference is meromorphic with only one pole at 0 of order ≤ 5. The actual calculation will show that there are unexpected cancellations and their difference has only pole of order ≤ 2 at 0. By adding suitable multiple of $\mathcal{P}$ to cancel out the pole of order 2, the only thing left is a holomorphic function on X, hence a constant. Q.E.D.

Exercise 4. Carry out the details of the above proof. Compute explicitly that

$$g_2(\Lambda) = 60 \sum_{\lambda \in \Lambda} \frac{1}{\lambda^4} \quad \text{and} \quad g_3(\Lambda) = 140 \sum_{\lambda \in \Lambda} \frac{1}{\lambda^6}. \tag{4.5}$$

The meanning of Weierstrass' theorem is that while the torus $X = \mathbf{C}/\Lambda$ is a quite analytic object, it is in fact "equivalent" to the "plane algebraic curve" defined by the polynomial equation $y^2 = 4x^3 - g_2 x - g_3$ with $(x, y) \in \mathbf{C}^2$, and the equivalence is given by $z \mapsto (\mathcal{P}(z), \mathcal{P}'(z))$. However, to make this precise we need to take care several problems. The most serious one is that $(\mathcal{P}(z), \mathcal{P}'(z))$ is not defined at $z = 0$. Another one is that we need to show it is a one-one onto map. Both problems could be solved by introducing the "points at infinity", that is, we should wrok with "projective spaces".

5. Affine Varieties and Projective Varieties. An affine variety $V \subset \mathbf{C}^n$ is by definition the common solutions of a set of polynomial equations

$$V = \{(X_1, \dots, X_n) \in \mathbf{C}^n \,|\, F_i(X_1, \dots, X_n) = 0,\ i \in I\}. \tag{5.1}$$

Obviously large part of mathematics is concerned with the study of polynomial equations. However, in many cases we would like to study such solutions of certain restricted type, like integral solutions or rational solutions etc. In general the problem becomes extremely difficult and only a small part of it could be studied in detail. This is usually regarded as "Number Theory" or "Arithmetic Geometry". Algebraic Geometry, however, treats this problem in a different aspect.

Firstly, one consider solutions in algebraically closed field like complex number $\mathbf{C}$. Then equations like

$$X^2 + Y^2 + Z^2 + 1 = 0 \quad \text{and} \quad X^2 + Y^2 - Z^2 + 1 = 0 \tag{5.2}$$

while define totally different surfaces in $\mathbf{R}^3$, do define isomorphic "complex surfaces" in $\mathbf{C}^3$. For those people wish to study integral solutions, algebraic geometry could be regarded as the first step (rough study) toward his question.

Secondly, many techniques in Topology or Geometry apply perfectly only for compact spaces. Affine varieties, however, is never compact. (How to see this?) Thus in algebraic geometry one wish to study certain compactification instead. A easy way to do this is to consider projective spaces

$$\mathbf{P}^n := (\mathbf{C}^{n+1} - 0)/\sim \quad \text{with} \quad (x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n) \quad \text{for} \quad \lambda \neq 0. \tag{5.3}$$

A point in $\mathbf{P}^n$ is thus an equivalence class in $\mathbf{C}^{n+1}$ which corresponds to a complex line passing through 0. It is usually denoted by $[x] = (x_0 : \cdots : x_n)$. Let U_i, $i = 0, \dots, n$, be the open subset defined by

$$\begin{aligned} U_i &:= \{[x] \in \mathbf{P}^n \,|\, x_i \neq 0\} \xrightarrow{\sim} \mathbf{C}^n, \\ &(x_0 : \cdots : x_n) \mapsto \left(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right) \quad (i\text{-th place omitted}). \end{aligned} \tag{5.4}$$

These $n+1$ affine spaces form an open cover of $\mathbf{P}^n$ and the complement of each U_i is again a projective space of dimension $n-1$, which corresponds to "points at infinity with respect to U_i", that is, $x_i = 0$.

Remark. Continue this process, we see that there is a decomposition of $\mathbf{P}^n$ into affine spaces:

$$\mathbf{P}^n = \mathbf{C}^n \cup \mathbf{C}^{n-1} \cup \cdots \cup \mathbf{C}^0, \tag{5.5}$$

which is very useful in topological considerations.

A projective variety $X \subset \mathbf{P}^n$ is by definition the common solutions of a set of homogeneous polynomial equations

$$X = \{(x_0 : \cdots : x_n) \in \mathbf{P}^n \,|\, f_i(x_0, \dots, x_n) = 0,\ i \in I\}. \tag{5.6}$$

It is easy to see that one must consider homogeneous polynomials in order that $f(x) = 0$ will impliy $f(\lambda x) = 0$ for all λ. Since $\mathbf{P}^n$ is compact (why?) and X is a closed subset of it, we see that all projective varieties are compact spaces.

Let $V \subset \mathbf{C}^n$ be an affine variety as before. By identifying $\mathbf{C}^n$ with U_0, each polynomial $F(X_1, \ldots, X_n)$ of highest degree d then gives rise to a homogeneous polynomial of degree d in $x_0, \ldots, x_n$, namely

$$f(x) := x_0^d F\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right). \tag{5.7}$$

In this way we obtain compactification of affine varieties in centain $\mathbf{P}^n$ by homogenized their defining equations.

Exercise 5. Modify the map constrcuted in section 4 by

$$\begin{aligned} \phi : \mathbf{C}/\Lambda &\to \mathbf{P}^2 \\ z &\mapsto (1 : \mathcal{P}(z) : \mathcal{P}'(z)) \quad \text{if } z \neq 0\,. \\ 0 &\mapsto (0 : 1 : 0) \end{aligned} \tag{5.8}$$

Show that ϕ is an ismorphism (bi-holomorphic map) between $\mathbf{C}/\Lambda$ and the projective variety (cubic curve) defined by $x_2^2 x_0 = x_1^3 - g_2 x_1 x_0^2 - g_3 x_0^3$.

6. Riemann Surfaces and Algebraic Curves. It is easy to imagine and not hard to prove that all compact 2 dimensional real orientable surfaces are distinguished by a single invariant – its genus (number of holes). We define a Riemann surface to be a one dimensional complex manifold, hence is automatically a real 2 dimensional surface. For example, $\mathbf{C}/\Lambda$ has genus one. since any two smooth cubic curves can be connected through smooth cubic curves, we see that they all have genus one. In fact later we will see that they are all ismorphic to some tori.

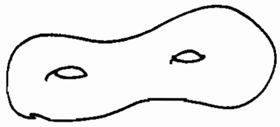
Historically Riemann is the first person to consider abstract spaces (manifolds). He initiated the study of Riemann surfaces and also higher dimensional real manifold theory (Riemannian geometry). We first noticed that a meromorphic function f on a complex manifold X is equivalent to a holomorphic map $\tilde{f} : X \to \mathbf{P}^1$. If $\dim X = 1$, Riemann observed the

Fact. *Existence of such a map $\tilde{f} : X \to \mathbf{P}^1$ implies that X must be an projective algebraic curve.*

Later he also showed the existence by establishing the nowadays so called "Riemann-Roch Theorem". Thus in one dimensional case, compact analytic object (compact Riemann surfaces) is the same as the algebraic object (non-singular complex projective curve). This is in general not true in higher dimensions. We will discuss Riemann-Roch in some detail in next lecture. The proof of the Fact makes use of some elementary field theoretic results, which will not be given here.

7. Genus Formula – Riemann-Hurwicz vs. Adjunction.

Every R.S. has a topological genus g :



$g = 2$.

We will answer the question in § 1 by proving

•a). $X^n + Y^n + Z^n = 0$ is a smooth curve in $\mathbb{P}^2$ with genus $g = \frac{1}{2}(n-1)(n-2)$, hence ≥ 1 if $n \geq 3$.

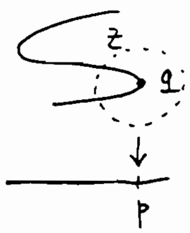
•b). $\nexists$ non constant holomorphic map $\mathbb{P}^1 \longrightarrow X$ if $g(X) \geq 1$.

This will solve (1.7) because the existence of $X(t), Y(t), Z(t)$ is equiv. to a map $\mathbb{P}^1 \dashrightarrow X$. But we have

Exercise 7. Any rational holomorphic map from $\mathbb{P}^1$ – finite points to a compact R.S. X can be extended to $\mathbb{P}^1 \longrightarrow X$.

Both claims follows from Riemann - Hurwicz formula :

Let $\begin{matrix} Y \\ \downarrow f \\ X \end{matrix}$



if locally near q the map f is not 1–1, then there is a number e_q (local degree) st. $f : z \longmapsto z^{e_q}$

Let $\chi(S)$ = Euler number of S := # pts – # lines + # Δ's (for any triangulation of S)

$= 2 - 2g(S)$.

Then one has (let d = deg of f = # $f^{-1}(p)$ for general p) :

$$\chi(Y) + \sum_q (e_q - 1) = d\, \chi(X).$$

- in the special case $X = \mathbb{P}^1$ ($g(X) = 0$), this reads $g(Y) = \frac{1}{2}\sum(e_q - 1) - d + 1$; a) follows from this formula.
- If $Y = \mathbb{P}^1$, it is trivial to see that b) is impossible.

Later in Lecture III, we will also compute a) via "adjunction".

8. Abel-Jacobi Map. (a special case)

In this section we want to show the integral in (2.2) is not given by elementary functions.

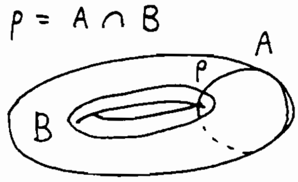
The idea is: Let $X \subset \mathbb{P}^2$ be the curve (affine piece)

$$y^2 = x^3 + ax + b$$

then $\dfrac{dx}{\sqrt{x^3+ax+b}} = \dfrac{dx}{y}$ is a holomorphic 1-form (this will also be discussed later in Lecture II, III)

Here we extend everything to $\mathbb{C}$ (not just in $\mathbb{R}$)

By Riemann-Hurwicz, know $g(X) = 1$, hence topologically there are 2 indep. loops A, B. Let $p = A \cap B$



the integral $\displaystyle\int_p^q \frac{dx}{y} =: F(q)$

is well-defined only if modulo $\displaystyle\underbrace{\int_A \frac{dx}{y}}_{\omega_1}$ and $\displaystyle\underbrace{\int_B \frac{dx}{y}}_{\omega_2}$.

i.e. $F : X \longrightarrow \mathbb{C}/\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$

Abel-Jacobi theorem says that this is an isomorphism. (in fact this case has an easy proof)

this means that F^{-1} is a doubly periodic function.

but the inverse function of an elementary function can have at most one period (like $\sin^{-1}(z)$ etc.)

hence impossible.

For general $\displaystyle\int \frac{dx}{\sqrt{f(x)}}$ with $\deg f = 4$, the curve $X: y^2 = f(x)$ still has $g(X) = 1$, hence the same.

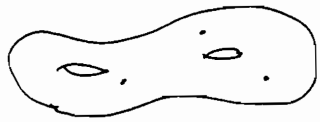
for $\deg f \geq 5$, then $g(X) \geq 2$. One need to use the full strength of Abel-Jacobi's thm. $\square$.

Lecture II. Riemann-Roch Thm and its Applications.

1. Divisors and Linear Systems.

X : cpt R.S. of genus g

Wish to study meromorphic functions on X.



eg. $g = 2$.

Divisors : $D = \sum n_i P_i$, $P_i \in X$, $n_i \in \mathbb{Z}$

Degree : $\deg D := \sum n_i$; $D \geq 0$ (effective) $\overset{\text{def}}{\Longleftrightarrow}$ $\deg D \geq 0$

Principal divisor : f meromorphic function

$$\operatorname{div}(f) \equiv (f) := \sum_{p \in X} (\operatorname{ord}_p f) \cdot p$$

Fact : $\deg (f) = 0$.

ie. # zeros = # of poles (count with multi.)

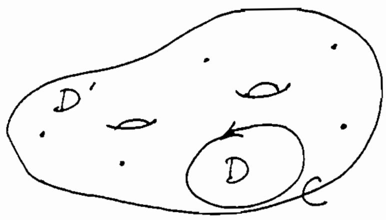
pf: By argument principle

$$\sum_{p \in X} (\operatorname{ord}_p f) \cdot p = \int_C \frac{f'(z)}{f(z)} \, dz$$

view $C = \partial D'$

$$\| \qquad 0$$

view $C = \partial D$



Q.E.D.

Remark: Not true if X is not compact.

Linear System:

$$L(D) := \{ f \text{ mero.} \mid (f) + D \geq 0 \} \cup \{0\}$$

ie. poles of f can only in the given points P_i with $n_i > 0$, and $\operatorname{ord}_{P_i} f \geq -n_i$.

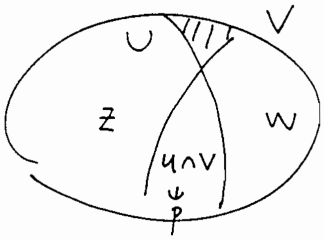
Exercise 1. $L(D)$ is a finite dim. $\mathbb{C}$-vector space and $L(D) \neq \phi \Rightarrow \deg D \geq 0$

- Let $\ell(D) = \dim_{\mathbb{C}} L(D)$. How to compute it ?

What is $L(\operatorname{div}(f))$?

2. Canonical Divisors on X.

meromorphic differential 1-form:



X : R.S.

Let $p \in U$: coor. system z

symbol dz

Subjects to the transformation rule:

for $p \in V$ another coor. sys. w

$$dz = \frac{dz}{dw}\, dw$$

on $U \cap V$: z is a holomorphic fcn of w.

A mero. 1-form is a collection of data

(contravariant 1-tensor)

$\mathcal{U} = \{U\}$ some open cover of X by coor. charts U's

$\alpha = (\alpha_U(z_U)\, dz_U)_U$ st. α_U is mero. on U and

$$\alpha_U(z_U)\, dz_U = \alpha_V(z_V)\, dz_V \qquad \text{on } U \cap V$$

ie. $$\alpha_U = \alpha_V \cdot \frac{dz_V}{dz_U}$$

This is the dual notion of vector fields.

- Given any f mero. df is a mero. 1-form:

  eg. on (U, z). $df := \frac{df}{dz} \cdot dz$

- Definition: Any 2 divisors D_1, D_2 are linearly equiv. $\Leftrightarrow$ $D_1 - D_2 = (f)$ some mero. f.

Exercise 2: Any mero. differential 1-form defines a "canonical divisor" up to linear equivalence.

(Hint: $\alpha = (\alpha_U\, dz_U)$, "$\mathrm{div}(\alpha_U)$" is well-defined on X

Any α, β. α/β is a "global" meromorphic fcn.

Denote the canonical div. class by K.)

3. Riemann – Roch Formula :

X cpt. R.S. $D = \sum n_i p_i$, $\deg D = \sum n_i$

$$\boxed{\ell(D) - \ell(K-D) = \deg D + 1 - g}$$, g = genus of X

The proof will not be given here.

Applications :

- $D = 0 \Rightarrow 1 - \ell(K) = 0 + 1 - g$ ie. $\ell(K) = g$
  there are g linearly independent holomorphic differential 1-forms.
  (for $g=0$, only has mero. 1-form)

- $D = K \Rightarrow \ell(K) - \ell(0) = \deg K + 1 - g$
  so, $\deg K = g - 1 - (1-g) = 2g - 2 = -\chi(X)$
  this recovers the Hopf - Poincaré index thm.

- $g = 0$: let $D = p \in X$.
  $\ell(p) - \ell(K-p) = 1 + 1 - 0 = 2$

But $\ell(K-p) = 0$ since $\deg(K-p) = 2g - 2 - 1 = -3 < 0$

$\Rightarrow \ell(p) = 2$, so

$$L(p) = \underbrace{\mathbb{C}\cdot 1}_{\text{constant}} \oplus \underbrace{\mathbb{C}\cdot f}_{\text{another nontrivial mero. function with only one pole at } p.}$$

$\Rightarrow \tilde{f} : X \longrightarrow \mathbb{P}^1$
is a degree 1 map, hence isomorphism
ie. $X \cong \mathbb{P}^1$.

conclusion :
any R.S. of genus 0 $\cong \mathbb{P}^1$,
$\exists!$ complex structure on S^2, it is usual $\mathbb{P}^1$.

- $g = 1$: Let $D = p \in X$

$$\ell(p) - \underset{\substack{\| \\ 0}}{\ell(K-p)} = 1 + (1-1) = 1$$

since $\deg(K-p) = (\underset{\substack{\| \\ 0}}{2g-2}) - 1 = -1 < 0$

$\ell(p) = 1$: only constant function.

$$\ell(2p) - \underset{\substack{\| \\ 0}}{\ell(K-2p)} = 2 + (1-1) = 2$$

$\deg = -2$

$$L(2p) = \mathbb{C} \cdot \underset{\substack{| \\ \text{const.}}}{1} \oplus \mathbb{C} \cdot x$$

x: mero. function with double pole at p.
(not simple pole, otherwise get $X \cong \mathbb{P}^1$.)

Remark: x is in fact Weierstrass' $\wp$ function.

In general, $\ell(K - kp) = 0$, so

$$\ell(kp) = \deg(kp) + (\underset{\substack{\| \\ 1}}{1-g}) = k$$

$L(3p)$: $1, x, y$ — another mero. function with triple pole at p.

$L(4p)$: $1, x, y, x^2$

$L(5p)$: $1, x, y, x^2, xy$

$L(6p)$: $1, x, y, x^2, xy, \underbrace{x^3, y^2}_{\text{both have 6 order pole at } p.}$

$\ell(6p) = 6 \Rightarrow \exists$ linear relation :

$$a + bx + cy + dx^2 + exy + fx^3 + gy^2 = 0$$

and $f \neq 0$, $g \neq 0$. ie. $X \cong$ cubic plane curve. (elliptic curve)

Weierstrass' normal form :

$$y^2 = 4x^3 - g_2 x - g_3 .$$

Exercise 3. For $g = 1$, show that K is equiv. to 0.

4 Embedding of Riemann Surfaces:

Let $f_0, \cdots, f_n \in L(D)$ be a basis

define $\Phi_D : X \dashrightarrow \mathbb{P}^n$

$$p \longmapsto (f_0(p) : f_1(p) : \cdots : f_n(p))$$

(defined by canceling out common factor)

- Φ_D is well-defined at $p \in X$ if

$$L(D) \supsetneq L(D-p)$$

(eg. say $p \notin D$, then this means $\exists f \in L(D)$ st. $f(p) \neq 0$)

- Φ_D is an embedding if $\forall p, q \in X$ (may $=$)

$$L(D) \supsetneq L(D-p) \supsetneq L(D-p-q)$$

For $p \neq q$: $\Rightarrow$ $\exists f \in L(D)$ st. $f(p) = 0$ but $f(q) \neq 0$

$\exists g \in L(D)$ st. $g(p) \neq 0$ but $g(q) = 0$

so $\Phi_D(p) \neq \Phi_D(q)$ in $\mathbb{P}^n$

(seperate points)

For $p = q$: $L(D) \supsetneq L(D-p) \supsetneq L(D-2p)$ means that functions in $L(D)$ when restricts to nbd. of p generate Taylor expansion to linear order

$$f(z) = a_0 + a_1(z-p) + \cdots .$$

Since $\ell(D-p) \leq \ell(D) \leq \ell(D-p) + 1$:

Φ_D well-defined $\iff$ $\ell(D) = \ell(D-p) + 1 \quad \forall p$

Φ_D embedding $\iff$ $\ell(D) = \ell(D-p-q) + 2 \quad \forall p, q$

Exercise 4. Let D be a divisor on X, use R-R to show (1). $\deg D \geq 2g \Rightarrow \Phi_D$ well-defined

(2). $\deg D \geq 2g+1 \Rightarrow \Phi_D$ embedding .

5. Canonical embedding & Hyperelliptic Curves

X R.S. genus $= g \geq 2$. $\ell(K) = g \geq 2$

Φ_K must be well-defined : $\Phi_K : X \longrightarrow \mathbb{P}^{g-1}$

$$\ell(K-p) - \underbrace{\ell(K-(K-p))}_{\substack{= \ell(p) \\ = 1 \text{ (constant)}}} = \deg(K-p) + (1-g) = 2g-2-1+(1-g) = g-2$$

$\ell(p) = 1$ since $X \not\cong \mathbb{P}^1$ (1 constant)

$\Rightarrow \ell(K-p) = g-1 < \ell(K) = g$.

Is Φ_K an embedding ?

$$\ell(K-p-q) - \underbrace{\ell(K-(K-p-q))}_{= \ell(p+q)} = \deg(K-p-q) + 1-g = g-3$$

$\Rightarrow \ell(K-p-q) = g-3+\ell(p+q)$

since $\ell(p) = 1$, $\ell(p+q) = 1$ or 2.

- $\ell(p+q) = 1 \Rightarrow \ell(K-p-q) = g-2 = \ell(K) - 2$
  $\forall\, p, q \in X$. $\Rightarrow \Phi_K$ is an embedding

- $\ell(p+q) = 2$ for some $p, q \in X$:

  Let $L(p+q) = \mathbb{C}\cdot 1 \oplus \mathbb{C}\cdot f$ — nontrivial mero. fcn with 2 poles at p, q exactly

  $\Rightarrow \tilde{f} : X \longrightarrow \mathbb{P}^1$
  sending p, q to ∞, a degree 2 map.

  X must be of the following form in $\mathbb{P}^2$:

  $(*)$ $y^2 = x^n + a_{n-1}x^{n-1} + \cdots + a_0$

  f = projection to the x-factor.

  this is called " hyper-elliptic " curves.

Exercise 5. Show Equation $(*)$ is singular at ∞ (for $n \geq 4$).

Ex 5+. Find a correct non-singular chart at infinity.

6. Automorphisms of a Riemann Surface :

$g=0$, $X = \mathbb{P}^1$ $\quad \mathrm{Aut}(X) =$ Mobius transformation

$$z \longmapsto \frac{az+b}{cz+d} \text{ on } \mathbb{C} .$$

$g=1$, $X = \mathbb{C}/\Lambda$ $\quad \mathrm{Aut}(X) \supset$ translations

$$z \longmapsto z+\lambda , \quad \lambda \in \Lambda .$$

Theorem : For $g \geq 2$. $\mathrm{Aut}(X)$ is finite.
in fact, $\# \mathrm{Aut}(X) \leq 84(g-1)$.

(Sketch of) pf : First assume X is canonical , ie.

$\Phi_K : X \hookrightarrow \mathbb{P}^{g-1}$ is an embedding

$p \longmapsto (f_0(p) : \cdots : f_{g-1}(p))$

If σ acts on X ; $\sigma : X \xrightarrow{\sim} X$ then

σ acts on the vector space $L(K)$

$\Rightarrow$ σ is an element in $\mathrm{Aut}(\mathbb{P}^{g-1})$

$= PGL(\mathbb{C}^g) = GL(g,\mathbb{C})/\mathbb{C}^\times$

And $\mathrm{Aut}(X) \longrightarrow PGL(\mathbb{C}^g)$ has no kernel.

Since $\chi(X) = 2-2g \leq -2$,

$\nexists$ holomorphic vector fields on X

$\Rightarrow$ $\mathrm{Aut}(X)$ is a discrete group.

This is false -> Now any discrete subgroup in $PGL(\mathbb{C}^g)$
must be finite. * Ex 6+. Fix it using an induced metric!

Exercise 6. Do the hyperelliptic case .

Remark : $\# \mathrm{Aut}(X) \leq 84(g-1)$ can be proved by using Hurwicz formula . and this bound is sharp .

7. Hilbert polynomials :

Notice the following : Let $\deg D = d > 0$

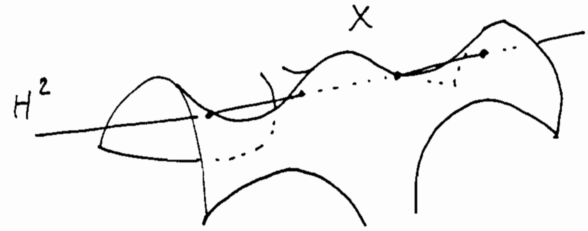
$$\ell(kD) - \ell(K - kD) = \deg(kD) + 1 - g$$

$$\Rightarrow \quad \ell(kD) = (\deg D)\cdot k + 1 - g \text{; when } k \gg 0$$

Consider a general projective variety $X \subseteq \mathbb{P}^N$ with $\dim X = n$.

Degree of X in $\mathbb{P}^N$:

take a "general" linear subspace $H^{N-n} \cong \mathbb{P}^{N-n}$ and count $\#(X \cap H^{N-n})$.



$X \subset \mathbb{P}^3$ $\dim X = 2$

degree $= \#(X \cap H^2) = 4$

Let X be given by the zero set of an ideal

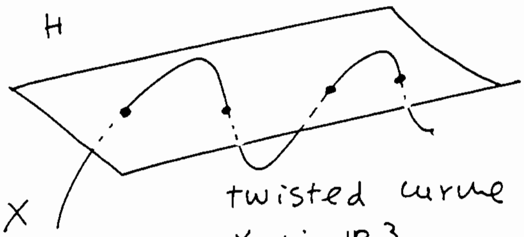
$I = (f_1, \cdots, f_r)$

Consider :

$$R = \mathbb{C}[x_0, \cdots, x_N]/I$$

$$= R_0 \oplus R_1 \oplus R_2 \oplus \cdots$$

R_k = the subspace given by poly of degree k.



twisted curve X in $\mathbb{P}^3$

- Simplest case : $r = 1$. (Hypersurfaces) :

eg. $X \subset \mathbb{P}^2$ given by $f(x,y,z) = 0$, $\deg f = d = \#(X \cap H)$

$$R = \mathbb{C}[x,y,z]/(f) = R_0 \oplus R_1 \oplus \cdots \oplus R_k \oplus \cdots$$

R_k : = all deg k polynomial / $f \cdot$ (all $\deg = k-d$ polynomial)

$$\dim = H^3_k - H^3_{k-d} \qquad \text{(true for } k > d\text{)}$$

$$= C^{3+k-1}_{k} - C^{3+k-d-1}_{k-d} = C^{k+2}_{2} - C^{k+2-d}_{2}$$

$$\dim R_k = C^{k+2}_2 - C^{k+2-d}_2$$

$$= \frac{(k+2)\cdot(k+1)}{2} - \frac{(k+2-d)\cdot(k+1-d)}{2}$$

$$= \frac{1}{2}\left(\cancel{k^2} + 3k + 2 - \cancel{k^2} - (3-2d)k - (2-d)(1-d)\right)$$

$$= \frac{1}{2}\cdot 2dk + \frac{1}{2}\left(2 - (d^2-3d+2)\right)$$

$$= d\cdot k + \frac{1}{2}d(3-d)$$ is a polynomial in k with coeff $= d =$ degree.

By Lecture I. Hurwicz formula : $g = \frac{(d-1)(d-2)}{2}$

$$1-g = \frac{2-(d-1)\cdot(d-2)}{2} = \frac{1}{2}d(3-d)$$

hence for $k \gg 0$:

$$\dim R_k = d\cdot k + \frac{1}{2}d(3-d)$$

$$= \underline{\deg(X\cap H)}\cdot k + (1-g)$$

compare with R-R thm: for $k \gg 0$

$$l(kD) = \underline{(\deg D)}\cdot k + (1-g)$$

By taking divisor $D = H\cap X$, hyperplane section

Exercise 7. In general, Hilbert find that for $k \gg 0$

$$\dim R_k = \frac{\deg(X)}{n!}\cdot k^n + (\deg k \le n-1 \text{ terms})$$

relate to the geometry of X.

Do this for 2 cases:

(1) $X = (\deg = d)$ hypersurface in $\mathbb{P}^{n+1}$ by $f = 0$.

(2) for $\dim X = 2$. write out the whole polynomial explicitly.

This leads to the general Riemann-Roch formula.

S : Noetherian graded ring, eg. $k[x_0, \dots, x_n]$

M : finitely generated graded module / S

eg. $M = I \triangleleft S$ graded ideal, or

$M = S/I$

Fact : $\exists$ filtration $0 = M^0 \subseteq M^1 \subseteq \dots \subseteq M^r = M$

by graded S-mod st.

$$M^i/M^{i-1} \simeq (S/p_i)(l_i)$$

the graded module shifted by l_i $\leftarrow$ $\mathbb{Z}$

$M^\cdot$ may not be unique.

p_i homog. prime.

pf : let Λ = set of sub grad S-mod of M st the filt exists.

$\Lambda \neq \emptyset$ since $\{0\} \in \Lambda$.

let $M' \in \Lambda$ be a maximal element (not unique)

($\exists$ since S Noetherian, M f.g $\Rightarrow$ M Noetherian $\Rightarrow$ AC stops)

if $M' = M$ then done,

otherwise let $M'' = M/M'$

$\mathcal{I} := \{ I_m = \mathrm{Ann}(m) \mid m \in M'' \setminus 0, \text{ homog} \}$

Standard fact : $\exists$ maximal element I_m and in fact I_m is prime p.

Say $ab \in I_m$, $b \notin I_m$

may assume a, b homog.

$0 \neq bm \in M'' \Rightarrow I_m \subseteq I_{bm}$ hence equal

so $abm = 0 \Rightarrow a \in I_{bm} = I_m$ #

if $\deg(m) = l$ then

$N := Sm \subseteq M''$ has $N \simeq (S/p)(-l)$

the inverse image N' of N in M then $\in \Lambda$ too

and $M' \subsetneq N'$ $\times$. ie. must $M \in \Lambda$ #

Hilbert function and polynomial:

M graded over $S = k[x_0, \cdots, x_n]/I$

(in fact, enough to assume $S = k[x_0, \cdots, x_n]$)

$\varphi_M(l) := \dim_k M_l \quad \forall l \in \mathbb{Z}$.

Theorem (Hilbert-Serre): if M f.g. then

$\varphi_M(l) = P_M(l)$ $\forall l \gg 0$ for some poly $P_M \in \mathbb{Q}[z]$.

and $\deg P_M = \dim \underbrace{Z(\mathrm{Ann}\, M)}_{\text{support of } M}$.

pf: $0 \to M' \to M \to M'' \to 0$

exact $\Rightarrow$ $\varphi_M = \varphi_{M'} + \varphi_{M''}$

also $Z(\mathrm{Ann}\, M) = Z(\mathrm{Ann}\, M') \cup Z(\mathrm{Ann}\, M'')$

so it is enough to prove for $M = (S/\mathfrak{p})(\ell)$.

by change $P(z)$ to $P(z+\ell)$, may take $M = S/\mathfrak{p}$

if $\mathfrak{p} = (x_0, \cdots, x_n)$ then $M \cong k$, $\varphi_M(l) = 0$ for $l > 0$

So take $P_M = 0$

in this case

$Z(\mathrm{Ann}\, M) = Z(\mathfrak{p}) = \phi$, define $\dim = -1$.

if $\mathfrak{p} \neq (x_0, \cdots, x_n)$; let $x_i \notin \mathfrak{p}$. (notice $\deg x_i = 1$)

$$0 \to M \xrightarrow{x_i} M \to \underset{\underset{M''}{\parallel}}{M/x_iM} \to 0$$

then $\varphi_{M''}(l) = \varphi_M(l) - \varphi_M(l-1) = (\Delta\varphi_M)(l-1)$

Notice that $Z(\mathrm{Ann}\, M'') = \underset{\underset{\mathfrak{p}}{\parallel}}{Z(\mathrm{Ann}\, M)} \cap \underbrace{H}_{\{x_i = 0\} \text{ hyperplane}}$.

but $H \not\supset Z(\mathfrak{p})$ $\Rightarrow$

$\dim Z(\mathrm{Ann}\, M'') = \dim Z(\mathfrak{p}) - 1$

so induction on $\dim Z(\mathrm{Ann}\, M)$ from the very beginning

get $\varphi_{M''} = P_{M''}$ of $\deg = \dim Z(\mathrm{Ann}\, M'')$

$\Rightarrow$ $\varphi_M = $ some P_M of $\deg = \cdots + 1$ #.

Now for $X \subset \mathbb{P}^n$ proj. var of dim r

the homog. coor. ring $S(X) = S/I$ or some $\mathfrak{p}$

let Hilbert poly of X

$$P_X := P_{S(X)}.$$

the simplest invariants:

$$P_X(\ell) = c_\ell \binom{r}{\ell} + c_{\ell-1} \binom{r}{\ell-1} + \cdots \qquad c_i \in \mathbb{Z}$$

$$= c_\ell \cdot \frac{r^\ell}{\ell!} + \cdots$$

- $\dim X = \deg P_X$
- degree of $X \subset \mathbb{P}^n := c_\ell = \ell! \cdot$ leading coefficient
- how about the lower degree part $\underline{\underline{???}}$

Applications of "degree":

$Y \subset \mathbb{P}^n$, $\dim Y = r$

H hyp. surface $\not\supseteq Y$

$Y \cap H = Z_1, \ldots, Z_s \quad \dim Z_i = r-1$

with homog. prime P_i.

$\exists$ multiplicity $i(Y,H;Z_j) := \mu_{P_j}(S/\underbrace{(I_Y + I_H)}_{M})$

Theorem (Bezout):

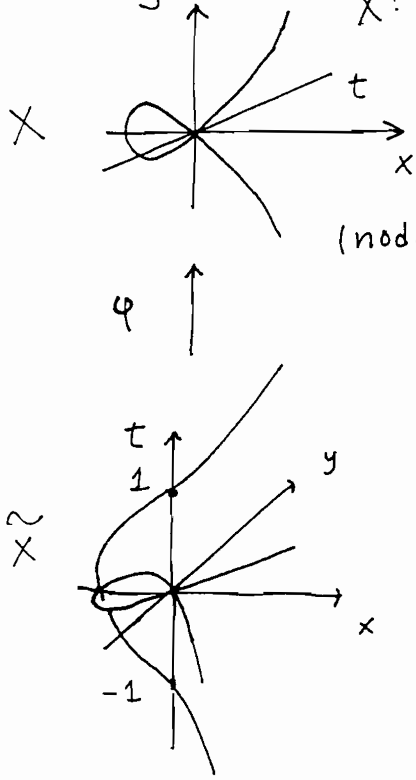
$$\deg Y \cdot \deg H = \sum_{j=1}^{s} i(Y,H;Z_j) \deg Z_j.$$

P_j is a minimal prime of M.

$\mu_p(M)$ is the # of times appear in M.

Lecture III. Blow-up techniques and the Geometry of cubic Surfaces.

1. Resolving curve singularities by slopes:



(node)

$X: \; y^2 = x^2(x+1)$ in $\mathbb{C}^2$

let $t = y/x$ = slope

then in $\mathbb{C}^3 \ni (x,y,t)$

using slopes as an extra coordinate, then

$\tilde{X}: \; (tx)^2 = x^2(x+1)$

$\Rightarrow \; t^2 = x+1$

together with $y = tx$

$\tilde{X}$ = smooth curve cut out by 2 smooth surfaces

$$\begin{cases} y - tx = f(x,y,t) = 0 \\ t^2 - x - 1 = g(x,y,t) = 0 \end{cases}$$

In general may need more blow-ups to get a resolution of singularities.

How do we know $\tilde{X}$ is smooth?

use implicit function theorem:

$$\begin{vmatrix} f_x & f_y & f_t \\ g_x & g_y & g_t \end{vmatrix} = \begin{vmatrix} -t & 1 & -x \\ -1 & 0, & 2t \end{vmatrix}$$ has rank $= 2$

since $\begin{vmatrix} -t & 1 \\ -1 & 0 \end{vmatrix} = 1 \neq 0$.

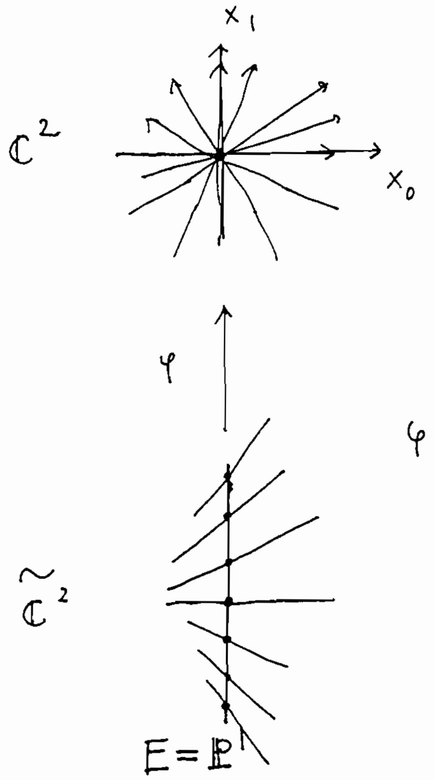
Exercise 1: Resolving curve singularities

(a) $y^2 = x^3$

(b) $y^2 = x^4$

(c) $y^2 = x^5$.

2. Blow up a point in $\mathbb{C}^2$:



exceptional curve (divisor)

disadvantage of slopes:

a curve may have slope $= \infty$

$\Rightarrow$ Need to consider all directions

i.e. projective space $P(\mathbb{C}^2) = \mathbb{P}^1$

$\tilde{\mathbb{C}}^2 \subset \mathbb{C}^2 \times \mathbb{P}^1 \ni (x_0, x_1; Y_0 : Y_1)$

$$\begin{array}{l} \tilde{\mathbb{C}}^2 \\ \varphi \downarrow \\ \mathbb{C}^2 \end{array} \qquad \tilde{\mathbb{C}}^2 = \{ x_0 Y_1 = x_1 Y_0 \}$$

a smooth surface in $\mathbb{C}^2 \times \mathbb{P}^1$

$\tilde{\mathbb{C}}^2 = U_0 \cup U_1$ open cover

$U_i = \{ Y_i \neq 0 \} \cong \mathbb{C}^2$

- $E = \varphi^{-1}(0)$ because $0 = \{ x_0 = 0,\ x_1 = 0 \}$ hence no restriction on $Y_0, Y_1 \rightsquigarrow \mathbb{P}^1$

- outside E and 0, φ is an isomorphism:

  If $x_0 \neq 0$ say, then

  $x_0 Y_1 = x_1 Y_0 \Rightarrow (Y_0 : Y_1) = (x_0 : x_1)$ uniquely determined.

- the restriction of φ to U_0

$$\begin{array}{l} U_0 \cong \mathbb{C}^2 = \{ Y_0 \neq 0 \} \Rightarrow \dfrac{Y_1}{Y_0} = \dfrac{x_1}{x_0} = \text{slope} := y_1 \\ \varphi \downarrow \\ \mathbb{C}^2 \end{array}$$

has coordinates y_0, y_1 with

$\varphi :\ x_0 = y_0;\ x_1 = y_0 y_1$.

Exercise 2. Write down the generalization of blow up a point $0 \in \mathbb{C}^3$. How about $\mathbb{C}^n$?

$\varphi^* C = C + E$

$\varphi^* C\ E$

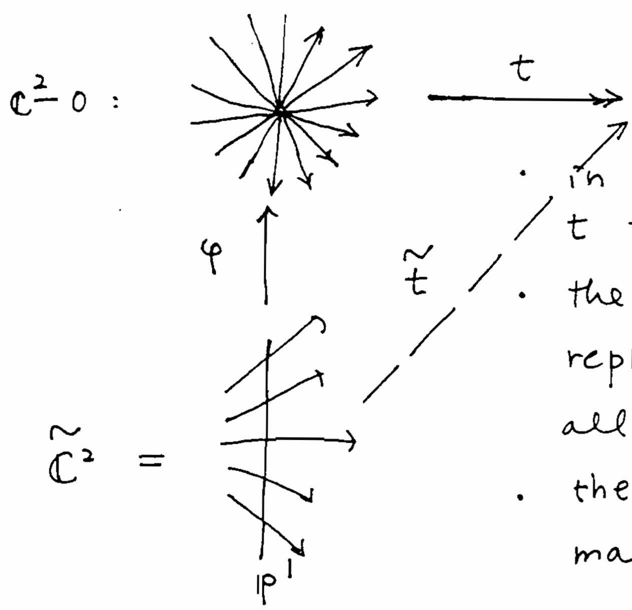
3. Tautological line bundle on $\mathbb{P}^1$.

Recall $\mathbb{P}^1 = (\mathbb{C}^2 - 0)/\sim$

$=$ the space of all directions in $\mathbb{C}^2$

So there is a map $\mathbb{C}^2 - 0 \twoheadrightarrow \mathbb{P}^1$

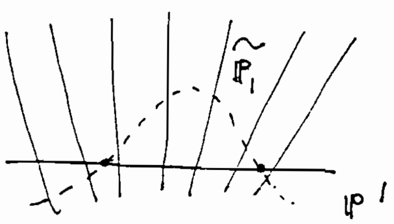
(direction map, slope map)



- in order to extend the map t to "0" (which is impossible)
- the only possibility is to replace 0 by the space of all directions $= E \cong \mathbb{P}^1$
- then $\tilde{t}$ is the identity map on $E \cong \mathbb{P}^1$.

the map $\tilde{t} : \tilde{\mathbb{C}}^2 \twoheadrightarrow \mathbb{P}^1$ is a line bundle.
(tautological)

E^2: self-intersection number of $E \cong \mathbb{P}^1$ inside $\tilde{\mathbb{C}}^2$



$:=$ # of intersections $\begin{cases} \mathbb{P}^1 \text{ (the zero section) and} \\ \tilde{\mathbb{P}}^1 \text{ (any top. section)} \end{cases}$
count with sign (± 1)

$=$ deg of div. of a meromorphic section.

Exercise 3. Show that the constant $X_0 = 1$ defines a meromorphic section of the line bundle with one pole hence conclude $E^2 = -1$. (Hint: consider U_0, U_1).

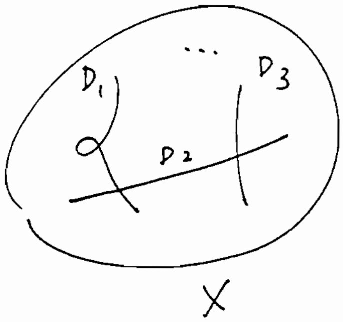
4. Intersection Theory of surfaces

(2-dim'l cpx projective varieties)

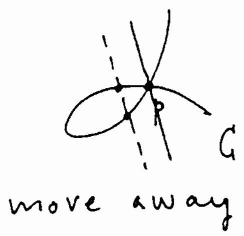
X alg. surface.

Divisors $D = \sum n_i D_i$, D_i irreducible curve in X, $n_i \in \mathbb{Z}$.

- each irreducible curve $C \subset X$ is a "hypersurface" which is locally the zero set $\mathrm{div}(f)$ of a holo. fcn f.



- multiplicity of C at p:



$\mathrm{mult}_p\, C = 2$.

- Pull back of divisors after a blow up.

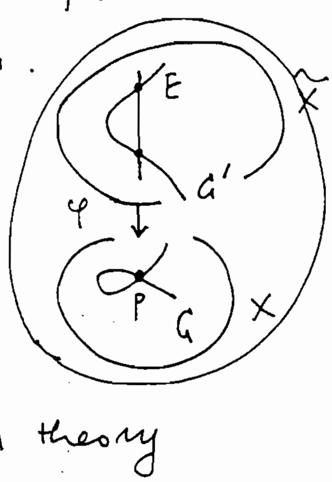
$$\begin{array}{ccc} \tilde{X} & \supset & E \cong \mathbb{P}^1 \\ \varphi \downarrow & & \downarrow \\ X & \ni & p \in C \end{array}$$

then $\varphi^* C = C' + mE$

$C' =$ proper transform of C

$= \overline{\varphi^{-1}(C - p)} \subset X$

$m = \mathrm{mult}_p\, C$.



- fact: intersection number "$A.B$" satisfies

$$(\varphi^* C).D = C.\varphi(D).$$

this gives $(\varphi^* C).E = 0$ and

$$(\varphi^* C_1).(\varphi^* C_2) = C_1.C_2.$$

Exercise 4. Use the above intersection theory to deduce that $E^2 = -1$.

Also $C'^2 = C^2 - m^2$, $m = \mathrm{mult}_p\, C$.

5. Canonical Divisors and Adjunction Formula.

f mero. fcn $\Rightarrow$ $(f) = (f)_0 - (f)_\infty$ principal div.

α mero. n-form $\Rightarrow$ (α) canonical divisor K_X

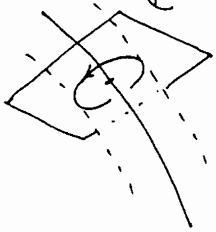
Let $D \subset X$ smooth divisor
(i.e. hypersurface)

How to compute K_D in terms of K_X ?

eg. locally $D = \{f = 0\}$.

By Cauchy's integral formula applied to a transveral direction

$\mathbb{C}$



$$(n-1) \text{ form on } D = \frac{1}{2\pi i} \oint \frac{dx_1 \cdots dx_n}{f}$$

$D = \{f=0\}$

$$\doteq \frac{1}{2\pi i} \oint_{|f|=\varepsilon} \frac{df}{f} \cdot \frac{dx_2 \cdots dx_n}{\partial f / \partial x_1}$$

$$= \frac{dx_2 \cdots dx_n}{\partial f / \partial x_1}$$

Taylor expansion $\Rightarrow$ Adjunction formula

$$\boxed{K_D = (K_X + D)\big|_D}$$

In case X a surface, D a curve. get

$$2g(D) - 2 = \deg K_D = (K_X + D) \cdot D$$

Exercise 5. Let $C \subset \mathbb{P}^2$ be a smooth curve of degree d, show that

$$g(C) = \frac{(d-1)\cdot(d-2)}{2}.$$

6. Quadric Surfaces in $\mathbb{P}^3$.

Let $X \subset \mathbb{P}^3$ be a smooth hypersurface.
$X = (f)$. $\deg f = d$.

- $d=1$: $f(x_0, x_1, x_2, x_3) = a x_0 + b x_1 + c x_2 + d x_3$
  $f = 0$ is simply a hyperplane $\cong \mathbb{P}^2$.

- $d=2$: $f(x_0, x_1, x_2, x_3) = \sum a_{ij} x_i x_j$
  a quadratic form.

linear algebra $\Rightarrow$ over $\mathbb{C}$, $\exists$ new coor. system
st. $f = y_0^2 + y_1^2 + y_2^2 + y_3^2$
Further change of coor. $/\mathbb{C}$ get $f=0 \Leftrightarrow$

$$(y_0 + i y_1)(y_0 - i y_1) = (y_2 + i y_3)(y_2 - i y_3)$$

$$z_0 z_3 = z_1 z_2$$

claim: A quadrtic surface is $\cong \mathbb{P}^1 \times \mathbb{P}^1$.

pf: consider the map (Veronesse map)

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 & \longrightarrow & \mathbb{P}^3 \\ (u:v)\times(s:t) & \longmapsto & (\underset{\overset{\shortparallel}{z_0}}{us} : \underset{\overset{\shortparallel}{z_1}}{ut} : \underset{\overset{\shortparallel}{z_2}}{vs} : \underset{\overset{\shortparallel}{z_3}}{vt}) \end{array}$$

clearly the image is Th $z_0 z_3 = z_1 z_2$. Q.E.D.

So only surfaces of degree ≥ 3 have nontrivial geometry.

Exercise 6. Show that one may get $\mathbb{P}^1 \times \mathbb{P}^1$ by blow up 2 points $p, q \in \mathbb{P}^2$, then blow down the proper transform of the line $\overline{pq}$. Here one need to use Castelnuovo's thm that any curve $\cong \mathbb{P}^1$ with self-intersection number $= -1$ can be blow down.

7. linear systems of cubic curves.

consider the space of all cubic polynomials of $\mathbb{P}^2$. i.e. with 3 variables. Denote this space by $L(3H)$, easy to see $\ell(3H) := \dim L(3H)$

$$= H_3^3 = C_3^5 = C_2^5 = 10.$$

(= number of monomial, deg 3)

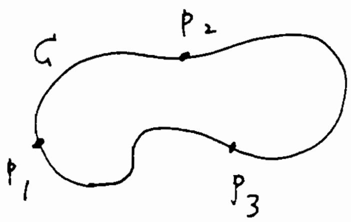
Let $p_1, \cdots, p_r$ be r points in $\mathbb{P}^2$ (general points)

$|3H - p_1 - \cdots - p_r| :=$ all cubic curves $\ni p_i \ \forall i$

Notice that

$|3H| \cong \mathbb{P}(L(3H)) = \mathbb{P}^9$

which is the map



$\Phi_{3H} : \mathbb{P}^2 \longrightarrow \mathbb{P}^9$

Now $|3H - p_1 \cdots - p_r|$ corr. to a subspace of $L(3H)$ of $\dim = 9 - r$

Let's call it $\mathcal{L}$. then the map

$$\Phi_{\mathcal{L}} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^{9-r}$$

is in general not defined on $p_1, \cdots, p_r$

- Theorem: For $r = 0, 1, \cdots, 6$, By blowing up $p_1, \cdots, p_r$ get

$$\begin{tikzcd} \mathbb{P}^2 \arrow[r, dashed] & \mathbb{P}^{9-r} \\ & X \arrow[ul] \arrow[u, hook, "i"'] \end{tikzcd}$$

X is a surface embeded in $\mathbb{P}^{9-r}$ of $\deg X = 9 - r$.

For $r = 6$, get X a cubic surface in $\mathbb{P}^3$.

Exercise 7. Assume that in the thm. already know $X \hookrightarrow \mathbb{P}^{9-r}$ is embeded, explain the degree statement : $\deg X = 9 - r$.

8. 27 lines on a cubic Surface in $\mathbb{P}^3$.

$$\begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[dr] & \\ L \subset \mathbb{P}^2 \arrow[rr, dashed] & & \mathbb{P}^3 \supset \mathcal{H} \end{tikzcd}$$

X cubic surface: by adjunction, Let $\mathcal{H}$ = hyp. plane in $\mathbb{P}^3$.

$$K_X = (K_{\mathbb{P}^3} + X)|_X$$
$$= (-4\mathcal{H} + 3\mathcal{H})|_X = -\mathcal{H}|_X = -H$$

a line $\ell \subset \mathbb{P}^3$ is inside X

$\Leftrightarrow$ ℓ is a $\mathbb{P}^1$ in X and $H.\ell = 1$

$\Leftrightarrow$ ℓ is a $\mathbb{P}^1$ in X and $K.\ell = -1$

Adjunction again:

$$-2 = 2g(\ell) - 2 = (K+\ell).\ell = -1 + \ell^2$$

i.e. ℓ is a (-1) curve. $(\ell^2 = -1)$.

Let $E_1, \cdots, E_6$ be the φ-exceptional curves

Let $\tilde{L} = \varphi^* L$ = pull back of (a line $\mathbb{P}^2$) in X

If $\ell \neq E_i$, may write $\ell = m\tilde{L} - \sum m_i E_i$, $m_i \in \mathbb{Z}$, $m > 0$

$$\begin{cases} \ell.H = 1 \\ \ell.E_i = 0 \text{ or } 1 \end{cases} \Rightarrow \begin{cases} 3m - \sum m_i = 1 \\ m_i = 0 \text{ or } 1 \end{cases}$$

(notice that $H.\tilde{L} = 3$
$E_i.\tilde{L} = 0$.)

$\Rightarrow$ $m = 1$, 2 of $m_i = 1$ $\Rightarrow$ $\ell \in |\tilde{L} - E_i - E_j|$

or $m = 2$, 5 of $m_i = 1$ $\quad$ $\ell \in |2\tilde{L} - E_{i_1} \cdots - E_{i_5}|$

Push ℓ down to $\mathbb{P}^2$, this means

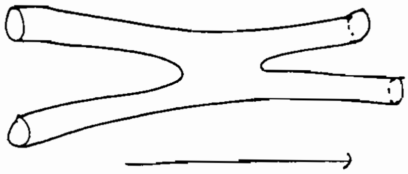
$\ell \longleftrightarrow$	E_i,	$\overline{\langle P_i, P_j \rangle}$,	conics through 5 of P_i
#	6	15	6

Total $= 6 + 15 + 6 = 27$.

Remark: Similar methods works for $X \hookrightarrow \mathbb{P}^d$, deg d case.

9. Quantum Correction by holomorphic Curves.

- String Theory : on space X with Kähler form ω.



$\subset X$

interaction of strings

- Topological correlation functions :

$$\langle \mathcal{O}_1, \cdots, \mathcal{O}_k \rangle := \int_{\varphi : \Sigma \to X} \text{"}D\varphi\text{"}\ \mathcal{O}_1 \cdots \mathcal{O}_k\ e^{-2\pi S[\varphi]}$$

Σ : all R.S.

$$S[\varphi] = \text{action} = \int_\Sigma \|d\varphi\|^2 d\mu$$

$$= \int_\Sigma \|\bar\partial\varphi\|^2 d\mu + \int_\Sigma \varphi^*\omega$$

topological quantity

depends only on homotopy class $[\Sigma, X]$.

$\Rightarrow \langle \mathcal{O}_1, \cdots, \mathcal{O}_k \rangle$

$$= e^{-2\pi\int_\Sigma \varphi^*\omega} \int \text{"}D\varphi\text{"}\ \mathcal{O}_1 \cdots \mathcal{O}_k \cdot e^{-2\pi t\int_\Sigma \|\bar\partial\varphi\|^2 d\mu}$$

by rescaling metric may get factor t, but the whole theory is indep. of t (assume topological)

Now take $t \to \infty$:

RHS $= 0$ unless $\bar\partial\varphi = 0$.

ie. φ is a holomorphic map : $\Sigma \to X$.

$$\Rightarrow \langle \mathcal{O}_1 \cdots \mathcal{O}_k \rangle = \sum_{\substack{\text{homotopy}\\ \text{classes}}} e^{-2\pi\int_\Sigma \varphi^*\omega} \int_{\mathcal{M}} D\varphi \cdot \mathcal{O}_1 \cdots \mathcal{O}_k$$

finite dimensional moduli space of holomorphic maps in a fixed homotopy class.

$\Rightarrow$ Physics vs Geometry in 1990 ~.

<u>End</u>.

ODP: A^1-sing $\underbrace{x_0^2+x_1^2+x_2^2=0}_{f(x_0,x_1,x_2)}$ in $\mathbb{C}^3$

$$\begin{array}{ccccc} Y & \subset & \tilde{\mathbb{C}}^3 & \subset & \mathbb{C}^3\times\mathbb{P}^2 \\ \downarrow\varphi & & \downarrow\varphi & & \\ X & \subset & \mathbb{C}^3 & & \end{array}$$

open cover U_i of $\tilde{\mathbb{C}}^3$, $i=0,1,2$
get open cover on Y
$Y_i := U_i \cap Y$

$\tilde{\mathbb{C}}^3 = (x,Y)$ st: $\dfrac{x_0}{Y_0} = \dfrac{x_1}{Y_1} = \dfrac{x_2}{Y_2}$

$U_0 = (Y_0 \neq 0) \ni$
i.e. trivial line bundle over $V_0 = \mathbb{P}^2 \cap U_0 \cong \mathbb{C}^3$.

$$x_1 = x_0 \cdot \frac{Y_1}{Y_0}$$
$$x_2 = x_0 \cdot \frac{Y_2}{Y_0}$$

for coordinates on U_0, may use

$$\underbrace{y_0 := x_0}_{\text{coordinate on the line bundle}} \;;\quad \underbrace{y_1 := \frac{Y_1}{Y_0} \;;\quad y_2 := \frac{Y_2}{Y_0}}_{\text{simply coor on } \mathbb{P}^2\cap U_0 = V_0}$$

And the blow-up morphism φ is given by

$$\begin{cases} x_0 = y_0 \\ x_1 = y_0 y_1 \\ x_2 = y_0 y_2 \end{cases}$$

exc. div $(0)\times\mathbb{P}^2 = E$ corr. to $x_0=x_1=x_2=0$, i.e. $\underline{y_0=0}$.

Similarly, on U_1: $\begin{cases} x_0 = y_1 y_0 \\ x_1 = y_1 \\ x_2 = y_1 y_2 \end{cases}$ on U_2: $\begin{cases} x_0 = y_2 y_0 \\ x_1 = y_2 y_1 \\ x_2 = y_2 \end{cases}$

Notice the diff meaning of (y_0, y_1, y_2) in U_i.

Blow-up ODP:

on U_0:
$$\begin{aligned} f(x) = f(\varphi(y)) &= f(y_0, y_0y_1, y_0y_2) \\ &= y_0^2 + y_0^2y_1^2 + y_0^2y_2^2 \\ &= \underbrace{y_0^2}_{\text{gives } 2E}(1+y_1^2+y_2^2) = y_0^2\, g(y) \end{aligned}$$

$Y_0 = Y\cap U_0$ is non-singular. Since $Dg = (0, 2y_1, 2y_2)$
no common zero of g and Dg.
By sym. Y_1, Y_2 also non-singular.
Exceptional curve $\ell = Y\cap E$ = conic in $\mathbb{P}^2 \cong \mathbb{P}^1$ ✱

A little generalization : A_n - sing , $n \geq 1$.

$$f(x) = x_0^2 + x_1^2 + x_2^{n+1} = 0 \qquad \text{let } n \geq 2$$

U_0 : $f(y_0, y_0 y_1, y_0 y_2) = y_0^2 + y_0^2 y_1^2 + y_0^{n+1} y_2^{n+1}$

$$= y_0^2 (1 + y_1^2 + y_0^{n-1} y_2^{n+1}) = y_0^2\, g(y)$$

$$Dg = ((n-1) y_0^{n-2} y_2^{n+1},\ 2y_1,\ (n+1) y_0^{n-1} y_2^{n})$$

still no common zero ($y_1 = 0 = y_0 y_2$!)

so Y_0 is smooth.

U_1 : $f(y_1 y_0, y_1, y_1 y_2)$ similar to U_0, $\exists$ Y_1 smooth.

U_2 : $f(y_2 y_0, y_2 y_1, y_2) = y_2^2 y_0^2 + y_2^2 y_1^2 + y_2^{n+1}$

$$= y_2^2 (\underline{y_0^2 + y_1^2 + y_2^{n-1}}) = \underline{y_2^2}\ \underline{g(y)}$$

$y_2 = 0$ gives exc. div $E \simeq \mathbb{P}^2$

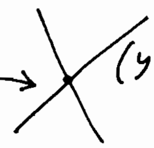
when $n=2$ get smooth

$n \geq 3$ get A_{n-2} sing.

exceptional curves (for $n \geq 2$) :

$$E \cap Y_2 = (y_0^2 + y_1^2 = 0)$$

singular pt (for $n \geq 3$) is still $(0,0,0)$



$(y_0 + y_1 i)(y_0 - y_1 i) = 0$

2 components

So blow-up does not nec. have irred. exc. div.

<u>Conclusion</u>: resolving A_n - sing will need $[\frac{n}{2}]$ blow-ups and introduce a chain of $\mathbb{P}^1$'s of length $= n$

$e =$  .

$$\begin{tikzcd} \tilde{X} \arrow[d, "\varphi"] & \supset & e \arrow[d] \\ X & \ni & 0 \end{tikzcd}$$

Exercise : what is $\operatorname{div}(\varphi^* x_i)$? for $i = 0, 1, 2$.

Another viewpoint:

let $\mu_k = \{ z^k = 1, z \in \mathbb{C} \} \simeq \mathbb{Z}/k\mathbb{Z} = \langle \zeta \rangle$ — k-th roots of 1.

acts on $\mathbb{C}^2$: $\zeta(x, y) = (\zeta x, \zeta^{-1} y)$

let $X = \mathbb{C}^2/\mu_k$, this is again an affine var.

with coor ring $R(X) = (\mathbb{C}[x,y])^{\mu_k} = \mathbb{C}[\underset{u}{\underset{\|}{x^k}}, \underset{w}{\underset{\|}{xy}}, \underset{v}{\underset{\|}{y^k}}]$

$$\cong \mathbb{C}[u, v, w]/(uv - w^k)$$

ie. $X \hookrightarrow \mathbb{C}^3$ as the hyp. surface $uv - w^k = 0$

change coor: get A_{k-1} - sing ✱.

In general, let $G \subset SL(2,\mathbb{C})$, $|G| < \infty$

then $X = \mathbb{C}^2/G \hookrightarrow \mathbb{C}^3$ by a single equation

$$A_n : \quad x_0^2 + x_1^2 + x_2^{n+1} = 0 \qquad n \geq 1$$

$$D_n : \quad x_0^2 + x_1^2 x_2 + x_2^{n-1} = 0 \qquad n \geq 4 \qquad \left(\text{notice } D_3 \cong A_3\right)$$

$$\left\{\begin{array}{ll} E_6 : & x_0^2 + x_1^3 + x_2^4 = 0 \\ E_7 : & x_0^2 + x_1^3 + x_1 x_2^3 = 0 \qquad n = 6, 7, 8. \\ E_8 : & x_0^2 + x_1^3 + x_2^5 = 0 \end{array}\right.$$

These are called: ADE, Du Val, Kleinian, RDP (rational double pts), crepant (canonical)

<u>Theorem</u> (Crepant resolution of surface singularities)

(1) We have the following blowing-up sequence:

$$A_1 \xrightarrow[1]{} sm, \quad A_2 \xrightarrow[+]{} sm, \quad A_n \xrightarrow[+]{} A_{n-2} \quad (n \geq 3)$$

$$D_4 \xrightarrow[1]{} A_1 + A_1 + A_1, \quad D_n \xrightarrow[1]{} A_1 + D_{n-2} \quad (n \geq 5)$$

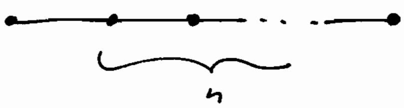
$$E_6 \xrightarrow[1]{} A_5, \quad E_7 \xrightarrow[1]{} D_6, \quad E_8 \xrightarrow[1]{} E_7.$$

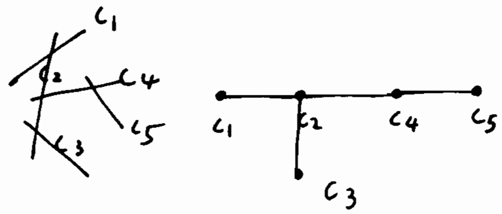
(2) These are precisely all $\underbrace{\text{surface}}_{\text{normal Gorenstein}}$ singularities st.

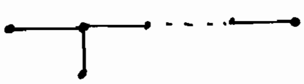
$\exists$ crepant resolution $\tilde{X} \xrightarrow[\varphi]{} X$ st. $K_{\tilde{X}} = \varphi^* K_X$.

Corollary - Exercise: Define the Dynkin diagram of the exc. curves: each curve $\longmapsto \bullet$

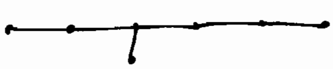
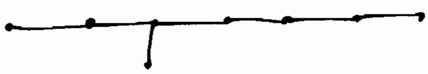
Then $\qquad C_1 \cap C_2 \neq \emptyset \longmapsto$ $\underset{C_1}{\bullet}$—$\underset{C_2}{\bullet}$

A_n :


eg.


D_n :


E_6 :


E_7 :

, E_8 :


<u>example</u> : $D_n \longrightarrow A_1 + D_{n-2}$ $(n \geq 5)$:

$$f(x) = x_0^2 + x_1^2 x_2 + x_2^{n-1}$$

U_0 :
$$f(\varphi(y)) = f(y_0, y_0 y_1, y_0 y_2)$$
$$= y_0^2 + y_0^2 y_1^2 y_0 y_2 + y_0^{n-1} y_2^{n-1}$$
$$= y_0^2 (1 + y_0 y_1^2 y_2 + y_0^{n-3} y_2^{n-1}) = y_0^2 g(y)$$

$$Dg = \left(y_1^2 y_2 + (n-3) y_0^{n-4} y_2^{n-1},\ 2 y_0 y_1 y_2,\ y_0 y_1^2 + (n-1) y_0^{n-3} y_2^{n-2} \right)$$

if $Dg(y) = 0$ then $y_0 y_1 y_2 = 0$, if $g(y) = 0$ too then $y_0 y_2 \neq 0$ hence $y_1 = 0$, but plug in get $Dg \neq 0$. So Y_0 is non-singular.

U_1 :
$$f(\varphi(y)) = f(y_1 y_0, y_1, y_1 y_2)$$
$$= y_1^2 y_0^2 + y_1^2 y_1 y_2 + y_1^{n-1} y_2^{n-1}$$
$$= y_1^2 (\underline{y_0^2} + \underline{y_1 y_2} + y_1^{n-3} y_2^{n-1}) = y_1^2 g(y)$$

No need to do this : ($Dg = (2y_0,\ y_2 + (n-3) y_1^{n-4} y_2^{n-1},\ y_1 + (n-1) y_1^{n-3} y_2^{n-2})$
$(0,0,0)$ is clearly the (only) singular pt.)

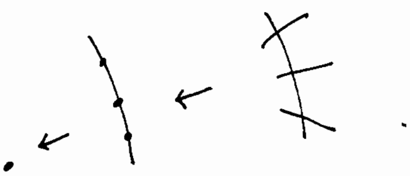
up to <u>analytic change of coor</u> of (y_1, y_2)
or in the completion $\mathbb{C}[[y_1, y_2]]$
this is an A_1-sing.

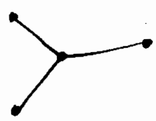
U_2 :
$$f(\varphi(y)) = f(y_2 y_0, y_2 y_1, y_2)$$
$$= y_2^2 y_0^2 + y_2^2 y_1^2 y_2 + y_2^{n-1}$$
$$= y_2^2 (y_0^2 + y_1^2 y_2 + y_2^{n-3}) = y_2^2 g(y)$$

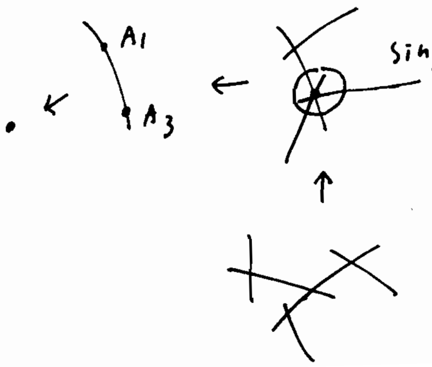
clearly we get an D_{n-2}-sing. #

~~Dynkin~~ <u>diagram</u>:

$D_4 \to A_1 + A_1 + A_1$; $D_5 \longrightarrow A_1 + A_3$



get 



need to locate the singular point

get 

step device corr. to a_ν, ψ^ν :

$$\tilde{z}^\alpha = z^\alpha + \zeta^\alpha \cdot \sqrt{a_\nu} \frac{\cos \lambda \psi^\nu}{\lambda} + \eta^\alpha \cdot \sqrt{a_\nu} \frac{\sin \lambda \psi^\nu}{\lambda}$$

$$\frac{\partial \tilde{z}^\alpha}{\partial x_i} = \frac{\partial z^\alpha}{\partial x_i} + \frac{\partial \zeta^\alpha}{\partial x_i} \cdot \sqrt{a_\nu} \frac{\cos \lambda \psi^\nu}{\lambda} + \zeta^\alpha$$

Example : For any single blowing up of (rational) Gorenstein double point.

$$\begin{array}{ccc} \tilde{X} & \subset & \tilde{\mathbb{C}}^3 \\ \varphi \downarrow & & \downarrow \varphi \\ X & \subset & \mathbb{C}^3 \end{array} \Rightarrow K_{\tilde{X}} = \left(K_{\tilde{\mathbb{C}}^3} + \tilde{X}\right)\Big|_{\tilde{X}}$$

So eventually for ADE we construct the crepant blowup.

$$= \left(\varphi^* K_{\mathbb{C}^3} + \underline{\underline{2E + \tilde{X}}}\right)\Big|_{\tilde{X}}$$

$$= \varphi^*\left(K_{\mathbb{C}^3} + X\right)\Big|_X \qquad \underline{\underline{2E + \tilde{X}}} = \varphi^* X$$

Conversely:

$$= \varphi^* K_X .$$ this is why called double point

For $\varphi : \tilde{X} \longrightarrow X$, $\tilde{X}$ sm. , X Gorenstein, normal surface.

with exceptional curves E_i

then $E_i \cdot K_{\tilde{X}} = E_i \cdot \varphi^* K_X = \varphi(E_i) \cdot K_X = pt. \, K_X = 0$

By Hodge index theorem, $E_i^2 < 0$

hence $2g(E_i) - 2 = (K_{\tilde{X}} + E_i) \cdot E_i = E_i^2 < 0$

$\Longleftrightarrow g(E_i) = 0$ & E_i smooth $\mathbb{P}^1$, with $E_i^2 = -2$.

In fact the quadratic form

$$q(x) := \left(\sum x_i E_i\right)^2 < 0 \quad \text{negative definite}$$

Using some number theoretic classification, the Dynkin diagram are exactly given by ADE ※.