

Index of Elliptic Operator and Heat Kernel Approach. p. 1/7

$E \quad F$
 $\downarrow \quad \downarrow$
 M
 C^+

$P: C(E) \rightarrow C(F)$ diff op. elliptic

$P^*: C(E) \leftarrow C(F)$ adjoint op.

$$\text{index } P := \dim \ker P - \dim \text{coker } P \in \mathbb{Z}$$

finite by compactness

$$\dim \ker P^*$$

bec. $\text{Im } P = (\ker P^*)^\perp$ *

$$(eg \quad C: \langle P s, t \rangle_F = \langle s, P^* t \rangle_E)$$

$$= \dim \ker P^* P - \dim \ker P P^*$$

here, $P^* P: C(F) \rightarrow C(F)$ is self adjoint, elliptic.

also $P P^*$

* Basic Notion of Functional Analysis:

$$\text{If } P f = g$$

$$\text{then } \langle P f, h \rangle = \langle g, h \rangle$$

$$\langle f, P^* h \rangle$$

so for $h \in \ker P^*$ then $\langle g, h \rangle = 0$

$$\text{i.e. } g \in (\ker P^*)^\perp$$

conversely, if $g \in (\ker P^*)^\perp$

then the eqn: (eqn for weak solutions)

$$\langle f, P^* h \rangle = \langle g, h \rangle$$

is solved via Riesz repr:

$\psi: \text{Im } P^* \rightarrow \mathbb{C}$ linear functional via

$$\psi(P^* h) := \langle g, h \rangle$$

why bounded?

$$\text{well-defined: } P^* h_1 = P^* h_2$$

$$\Rightarrow \psi(P^* h_1) - \psi(P^* h_2) = \langle g, h_1 - h_2 \rangle = 0$$

Hahn-Banach

$\Rightarrow \psi$ extends to $\bar{\psi}: C(F) \rightarrow \mathbb{C}$

and $\bar{\psi}(u) = \langle f, u \rangle$ for some $f \in C(F)$

hence $\langle g, h \rangle = \bar{\psi}(P^* h) = \langle f, P^* h \rangle$, when complete

Heat ξq^u : for self adjoint op L

hw: C^0 (hint: P. 2/7
using Garding
ineq.)

$$(*) \quad \frac{\partial}{\partial t} f = LF$$

$$\text{Heat kernel: } H(x, y, t) = \sum e^{-\lambda_i t} \varphi_i(x) \otimes \varphi_i^*(y)$$

$$\text{Heat operator: } e^{-tL} \cdot g = \int_M H(x, y, t) \cdot g(y) \varphi_i(y)$$

$$\frac{\partial}{\partial t} (e^{-tL} \cdot g) = \int_M \frac{\partial}{\partial t} H(x, y, t) \cdot g(y)$$

$$= \int_M L_x H(x, y, t) \cdot g(y)$$

$$= L \left(\int_M H(x, y, t) \cdot g(y) \right) = L (e^{-tL} \cdot g)$$

$$\text{and } \lim_{t \rightarrow 0} e^{-tL} \cdot g = \lim_{t \rightarrow 0} \int_M H(x, y, t) \cdot g(y)$$

$$= g(x)$$

goes to δ function

So $e^{-tL} g$ is the solution of (*) with integrable initial condition g .

Trace of heat operator / heat kernel:

$$\text{tr } e^{-tL} = \sum e^{-\lambda_i t} = \int_M \text{tr } H(x, x, t)$$

$$\dim \ker L + \sum_{\lambda_i \neq 0} e^{-\lambda_i t}$$

If $P^* P \varphi_i = -\lambda_i \varphi_i$ then

$$P P^* (P \varphi_i) = -\lambda_i (P \varphi_i) \quad (\text{notice that } P \varphi_i \neq 0)$$

this gives a bijection of λ_i -eigenspace of $P^* P$ and $P P^*$.

So get McKean-Singer Formula

$$\text{Index } P := \dim \ker P - \dim \text{Coker } P$$

$$= \dim \ker P^* P - \dim \ker P P^*$$

$$= \text{tr } e^{-t P^* P} - \text{tr } e^{-t P P^*}$$

$$= \int_M (\text{tr } H_{P^* P} - \text{tr } H_{P P^*})$$

this is the supertrace

Generalized Harmonic Oscillator on $V = \mathbb{R}^N$

P. 3/7

R skew sym. $n \times n$ \rightarrow left in \mathcal{A}
 F $N \times N$ a comm. algebra

$$H := - \sum_i \nabla_i^2 + F$$

$$= - \sum_i \left(\partial_i + \frac{1}{4} R_{ij} x_j \right)^2 + F$$

acts on $\Gamma(V, \mathcal{A} \otimes \text{End}(\mathbb{C}^N))$

then the heat kernel is

$$h(x, t) = \frac{1}{(4\pi t)^{N/2}} \det^{1/2} \left(\frac{tR/2}{\sinh(tR/2)} \right) \cdot e^{-\frac{1}{4t} \left\langle x \left| \frac{tR}{2} \coth \frac{tR}{2} \right| x \right\rangle} \cdot e^{-tF}$$

ie $\left(\frac{\partial}{\partial t} + H \right) h = 0$ and $\lim_{t \rightarrow 0} h = \delta(x)$.

Starting Point: Gaussian integral

$$A = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} : A^2 = 2\pi \int_0^{\infty} e^{-r^2} r dr$$

$$= 2\pi \left(-\frac{1}{2} e^{-r^2} \right) \Big|_0^{\infty} = \pi$$

Scaling:

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{4t}} d\left(\frac{x}{\sqrt{4t}}\right) = \sqrt{\pi}$$

ie. $\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4t}} dx = 1$.

$$\lim_{t \rightarrow 0} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} = \delta_0(x)$$

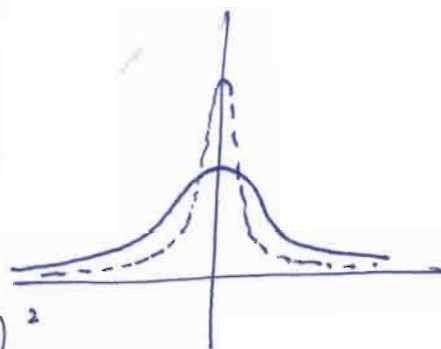
let $p(x, y, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}}$

then $\lim_{t \rightarrow 0} p(x, y, t) = \delta_y(x) = \delta(x-y)$

It satisfies the heat eqⁿ:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) p = 0$$

p is called the "heat kernel" of $\frac{\partial^2}{\partial x^2}$



Same computation works for n -dim. p. 4/7₂

$$p = (4\pi t)^{-n/2} \cdot e^{-\frac{|x-y|^2}{4t}}$$

then $\frac{\partial}{\partial t} p - \Delta p = 0$.

Computation:

$$p = (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}}$$

$$p_t = -\frac{1}{2} (4\pi t)^{-\frac{3}{2}} \cdot 4\pi e^{-\frac{(x-y)^2}{4t}} + p \cdot + \frac{1}{4} \frac{(x-y)^2}{t^2} \rightarrow -p \cdot \frac{1}{2t}$$

$$(p_x = -p \cdot \frac{x-y}{2t})$$

$$p_{xx} = p \cdot \frac{(x-y)^2}{4t^2} - p \cdot \frac{1}{2t}$$

ie. $\boxed{p_t - p_{xx} = 0}$

define $e^{-t\Delta} g(x) := \int_{\mathbb{R}} p(x,y,t) g(y) dy$

then $(\frac{\partial}{\partial t} - \Delta)(e^{-t\Delta} g) = 0$

and $(\lim_{t \rightarrow 0} e^{-t\Delta} g)(x) = g(x) \quad (*)$

require only that $g \in L^1$ for $e^{-t\Delta} g$

well-defined, but at least C^0 to get $(*)$

?!

Harmonic Oscillator: Mehler's formula p. 5/7

$$\text{let } H = -\frac{d^2}{dx^2} + x^2$$

$$\text{heat kernel } P_t(x, y) \text{ st. } \left(\frac{\partial}{\partial t} + H_x \right) P = 0$$

Guess Method :

$$\text{let } P = e^{A \frac{x^2}{2} + Bxy + A \frac{y^2}{2} + C}$$

A, B, C are functions of t only. notice P is sym. in x and y (bec. H is a self-adjoint op.)

$$\left(\frac{\partial}{\partial t} + H_x \right) P = \left[A' \frac{x^2}{2} + B'xy + A' \frac{y^2}{2} + C' - (Ax + By)^2 - A + x^2 \right] P = 0$$
$$\begin{cases} P_x = (Ax + By) P \\ P_{xx} = (Ax + By)^2 P + Ap \end{cases} \quad \begin{matrix} \uparrow \\ \text{want} \end{matrix}$$

$$\Rightarrow \begin{cases} \frac{A'}{2} - A^2 + 1 = 0 & (x^2) \\ B' - 2AB = 0 & (xy) \\ \frac{A'}{2} - B^2 = 0 & (y^2) \\ C' - A = 0 & (\text{const}) \end{cases}$$

Solve ODE: $A' = 2(A^2 - 1) \Rightarrow A(t) = -\text{coth}(2t + c)$

see back side

$$B^2 = \frac{A'}{2} = \text{csch}^2(2t + c)$$

$$\Rightarrow \underline{B(t) = \text{csch}(2t + c)} \quad (\pm \text{ sign is contained in } c)$$

$$\underline{C(t) = \int A(t) dt = -\frac{1}{2} \log \sinh(2t + c) + D}$$

$$\begin{cases} \int \frac{\cosh}{\sinh} dt = \log \sinh(t) \\ \left(\frac{\cosh}{\sinh} \right)' = \frac{\sinh^2 - \cosh^2}{\sinh^2} = -\text{csch}^2 \end{cases}$$

for initial condition: put $y=0$.

then $t \rightarrow 0$. should go to " $\frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ "

$$A(t) = -\coth(2t+c) \sim -\frac{1}{2t} \Rightarrow c=0.$$

$$\text{and } e^{c(t)} = (\sinh 2t)^{-1/2} e^D \sim (4\pi t)^{-1/2}$$

$$\Rightarrow D = \log[(2\pi)^{-1/2}]$$

So the heat kernel is given by: Mehler's formula

$$p = p(x, y, t) = (2\pi \sinh 2t)^{-1/2} e^{-\frac{1}{2} \coth 2t (x^2 + y^2) - \text{csch } 2t \cdot xy}$$

change of variable. (let $y=0$) $t \mapsto \frac{tr}{4}$; $x \mapsto \sqrt{\frac{tr}{4}} x$.

$$(*) \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + \frac{tr^2}{16} x^2 + f \right) p_t(x) = 0$$

has solution:

$$\frac{1}{\sqrt{4\pi t}} \cdot \left[\frac{tr/2}{\sinh(tr/2)} \right]^{1/2} e^{-\left[\frac{tr}{2} \coth\left(\frac{tr}{2}\right) \right] \cdot \frac{x^2}{4t} - tf}$$

proof for $H = -\sum \nabla_i^2 + F = -\sum \left(\partial_i + \frac{1}{4} R_{ij} x_j \right)^2 + F$:

$$p_t(x, R, F) := \frac{1}{\sqrt{4\pi t}^n} \det \left(\frac{tr/2}{\sinh tr/2} \right)^{1/2} e^{-\frac{tr}{4t} \langle x | \frac{tr}{2} \coth \frac{tr}{2} | x \rangle} \cdot e^{-tF}$$

Need to show that $\frac{\partial p_t}{\partial t} = -H p_t$ (algebraic formula!)

both side are analytic functions of R_{ij} .

\Rightarrow may assume that $R_{ij} \in \mathbb{R}$

why not simply diagonalize it in \mathbb{C} ?

May compute this in any cov. system of \mathbb{R}^n

say. the system st. $R = \begin{bmatrix} 0 & -r & 0 \\ 0 & 0 & 0 \\ 0 & 0 & s \end{bmatrix}$ block form

The problem is reduced to the 2-dim case.

! But how about F rules.
 ok. they are of diff rules.

In this case, get

$$H = -(\partial_1^2 + \partial_2^2) - \frac{1}{2} R_{ij} \partial_i(x_j \cdot) - \left(\frac{1}{4} r\right)^2 (x_1^2 + x_2^2) + F$$

" since $i \neq j$

$$- \frac{1}{2} r (x_2 \partial_1 - x_1 \partial_2)$$

$$P_t(x_1, x_2) = \frac{1}{4\pi t} \cdot \frac{tr/2}{\sin tr/2} \cdot e^{-\frac{tr}{2} \cot(\frac{tr}{2}) \cdot \frac{\|x\|^2}{4t}} \cdot e^{-tF}$$

Reason: using the trick of $\sqrt{-1}$ number (see back side lemma)

only here has nontrivial matrix part.

replace r by ir , get the 2-dim'l analogue of (*) except that get one more term

$$- \frac{1}{2} r (x_2 \partial_1 - x_1 \partial_2) - \text{infinitesimal rotation}$$

but this term acts on $\|x\|^2 = 0$

this proves that $\frac{\partial}{\partial t} P_t = -H P_t \quad \square$

Main Idea for Local Index Formula:

$$M-S: \text{index } P = \text{tr } e^{-t P^* P} - \text{tr } e^{-t P P^*}$$

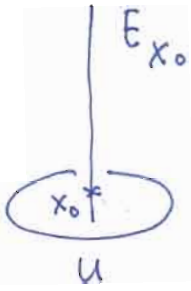
$$= \int_X \text{tr } P \Big|_{\text{for } P^* P} - \text{tr } P \Big|_{\text{for } P P^*}$$

for "generalized Laplace"

has "Weitzenböck type formula":

$$\Delta = \text{tr } \nabla^2 + F$$

some term comes from curvature of \mathcal{g} and of certain bundle E .



for a pt $x_0 \in X$. pick a nbd U .
 trivialize all bundles $(T_x|_U, E|_U)$
knowing the precise value $F(x_0)$ will
 determine the heat kernel at x_0 .

This can be done by rescaling!