Index of Elliptic Operator and Heat Kernel Approach.

$$\begin{array}{ccc} E & & F \\ & \searrow\ \swarrow & \\ & M & \\ & \text{cpt} & \end{array}$$

$P : \Gamma(E) \to \Gamma(F)$ diff op. Elliptic

$P^* : \Gamma(E) \leftarrow \Gamma(F)$ adjoint op.

$\operatorname{index} P := \dim \ker P - \dim \operatorname{coker} P \in \mathbb{Z}$

finite by compactness

$\operatorname{coker} P = \dim \ker P^*$ bec. $\underline{\operatorname{Im} P = (\ker P^*)^\perp}$ *

(eg τ : $\langle Ps, \tau\rangle_F = \langle s, P^*\tau\rangle_E$)
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 0$

$= \dim \ker \underline{P^*P} - \dim \ker \underline{PP^*}$

Here, $P^*P : \Gamma(E) \to \Gamma(E)$ is self adjoint, elliptic.
Also PP^*.

* Basic Notion of Functional Analysis :

If $Pf = g$

then $\langle Pf, h\rangle = \langle g, h\rangle$

$\qquad\quad = \langle f, P^*h\rangle$

so for $h \in \ker P^*$ then $\langle g, h\rangle = 0$

ie. $\underline{g \in (\ker P^*)^\perp}$

Conversely, If $g \in (\ker P^*)^\perp$

then the eqn : (eq'n for weak solutions.)

$\langle f, P^*h\rangle = \langle g, h\rangle$

is solved via Riesz repr :

$\varphi : \operatorname{Im} P^* \to \mathbb{C}$ linear functional via

$\varphi(P^*h) := \langle g, h\rangle$ why bounded?

well-defined : $P^*h_1 = P^*h_2$

$\Rightarrow \varphi(P^*h_1) - \varphi(P^*h_2) = \langle g, h_1 - h_2\rangle = 0$.

Hahn-Banach $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad h_1 - h_2 \in \ker P^*$

$\Rightarrow \varphi$ extends to $\bar\varphi : \Gamma(E) \to \mathbb{C}$

and $\bar\varphi(u) = \langle f, u\rangle$ for some $f \in$ "$\Gamma(E)$"

hence $\langle g, h\rangle = \bar\varphi(P^*h) = \langle f, P^*h\rangle$, when complete.

Heat Eq'n: for self adjoint op L

HW: C^∞ (hint: P. 2/7 using Garding ineq.)

(*) $\frac{\partial}{\partial t} f = Lf$

Heat Kernel: $H(x,y,t) = \sum e^{-\lambda_i t} \varphi_i(x) \otimes \underset{\varphi_i(y)}{\overset{}{\varphi_i^*(y)}}$

Heat operator: $e^{-tL} \cdot g = \int_M H(x,y,t) \cdot g(y)$

$$\frac{\partial}{\partial t}\left(e^{-tL} \cdot g\right) = \int_M \frac{\partial}{\partial t} H(x,y,t) \cdot g(y)$$

$$= \int_M L_x \, H(x,y,t) \cdot g(y)$$

$$= L\left(\int_M H(x,y,t) \cdot g(y)\right) = L\left(e^{-tL} \cdot g\right)$$

and $\lim_{t \to 0} e^{-tL} \cdot g = \lim_{t \to 0} \int_M H(x,y,t) \cdot g(y)$ goes to δ function

$$= g(x).$$

So $e^{-tL} g$ is the solution of (*) with integrable initial condition g.

Trace of heat operator / heat kernel:

$$\operatorname{tr} e^{-tL} = \sum e^{-\lambda_i t} = \int_M \operatorname{tr} H(x,x,t)$$

$$= \dim \ker L + \sum_{\lambda_i \neq 0} e^{-\lambda_i t}$$

if $P^* P \varphi_i = -\lambda_i \varphi_i$ then

$P P^* (P \varphi_i) = -\lambda_i (P \varphi_i)$ (notice that $P\varphi_i \neq 0$)

this gives a bijection of λ_i-eigenspace of P^*P and PP^*.

So get McKean–Singer Formula

$$\begin{aligned}
\operatorname{Index} P &:= \dim \ker P - \dim \operatorname{Coker} P \\
&= \dim \ker P^*P - \dim \ker PP^* \\
&= \operatorname{tr} e^{-tP^*P} - \operatorname{tr} e^{-tPP^*} \\
&= \int_M \left(\operatorname{tr} H_{P^*P} - \operatorname{tr} H_{PP^*}\right).
\end{aligned}$$

this is the supert[illegible]

Generalized Harmonic Oscilator on $V = \mathbb{R}^n$

R skew sym. $n \times n$
F $N \times N$
> coeff in $\mathcal{A}$, a comm. algebra

$$H := -\sum_i \nabla_i^2 + F$$

$$= -\sum_i \left(\partial_i + \tfrac{1}{4} R_{ij} x_j\right)^2 + F$$ acts on $\Gamma(V, \mathcal{A} \otimes \mathrm{End}(\mathbb{C}^N))$

then the heat kernel is

$$h(x,t) = \frac{1}{(4\pi t)^{n/2}} \det{}^{1/2}\left(\frac{tR/2}{\sinh(tR/2)}\right) \cdot e^{-\frac{1}{4t}\langle x \,|\, \frac{tR}{2}\coth\frac{tR}{2} \,|\, x\rangle} \cdot e^{-tF}$$

i.e. $\left(\frac{\partial}{\partial t} + H\right) h = 0$ and $\lim_{t\to 0} h = \delta(x)$.

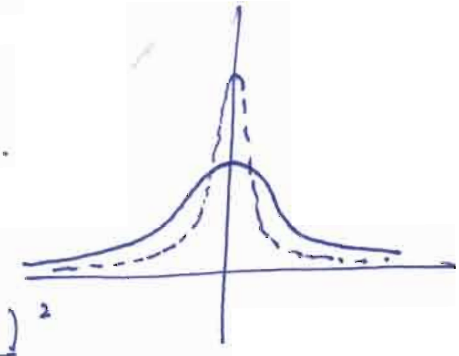
Starting Point: Gaussian integral

$$A = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} : \quad A^2 = 2\pi \int_0^\infty e^{-r^2} r\,dr$$
$$= 2\pi \left(-\tfrac{1}{2} e^{-r^2}\right)\Big|_0^\infty$$
$$= \pi$$

Scaling:

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{4t}}\, d\left(\frac{x}{\sqrt{4t}}\right) = \sqrt{\pi}$$

i.e.
$$\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4t}}\,dx = 1.$$



$$\lim_{t\to 0} \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{x^2}{4t}} = \delta_0(x)$$

let $p(x,y,t) = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{(x-y)^2}{4t}}$

then $\lim_{t\to 0} p(x,y,t) = \delta_y(x) = \delta(x-y)$

It satisfies the heat eq'n:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) p = 0.$$

p is called the "heat kernel" of $\frac{\partial^2}{\partial x^2}$.

Same computation works for n-dim:

$$p = (4\pi t)^{-n/2} \cdot e^{-\frac{|x-y|}{4t}}$$

then $\frac{\partial}{\partial t} p - \Delta p = 0$.

Computation:

$$\rho = (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}}$$

$$\rho_t = -\tfrac{1}{2}(4\pi t)^{-\frac{3}{2}} \cdot 4\pi\, e^{-\frac{(x-y)^2}{4t}} \to -\rho \cdot \frac{1}{2t}$$

$$+ \rho \cdot + \tfrac{1}{4}\frac{(x-y)^2}{t^2}$$

$$\left(\rho_x = -\rho \cdot \frac{(x-y)}{2t} \right)$$

$$\rho_{xx} = \rho \cdot \frac{(x-y)^2}{4t^2} - \rho \cdot \frac{1}{2t}$$

i.e. $\boxed{\rho_t - \rho_{xx} = 0}$

define $e^{-t\Delta} g(x) := \int_{\mathbb{R}} p(x,y,t)\, g(y)\, dy$

then $\left(\frac{\partial}{\partial t} - \Delta\right)\left(e^{-t\Delta} g\right) = 0$

and $\left(\lim_{t \to 0} e^{-t\Delta} g\right)(x) = g(x)$ $(*)$

require only that $g \in L^1$ for $e^{-t\Delta} g$ well-defined, but at least C^0 to get $(*)$

?!

Harmonic Oscilator: Mehler's formula.

let $H = -\frac{d^2}{dx^2} + x^2$

heat kernel $P_t(x,y)$ st. $\left(\frac{\partial}{\partial t} + H_x\right) P = 0$

Guess Method:

let $P = e^{A\frac{x^2}{2} + Bxy + A\frac{y^2}{2} + C}$

A, B. C are functions of t only. notice P is sym. in x and y
(bec. H is a self-adjoint op.)

$$\left(\frac{\partial}{\partial t} + H_x\right) P = \left[A'\frac{x^2}{2} + B'xy + A'\frac{y^2}{2} + C' - (Ax+By)^2 - A + x^2 \right] P \underset{\text{want}}{=} 0$$

$$\begin{cases} P_x = (Ax + By)P \\ P_{xx} = (Ax+By)^2 P + Ap \end{cases}$$

$$\Rightarrow \begin{cases} \frac{A'}{2} - A^2 + 1 = 0 & (x^2) \\ B' - 2AB = 0 & (xy) \\ \frac{A'}{2} - B^2 = 0 & (y^2) \\ C' - A = 0 & (\text{const}) \end{cases}$$

Solve ODE: $A' = 2(A^2 - 1) \Rightarrow \underline{A(t) = -\coth(2t+c)}$ see back side

$B^2 = \frac{A'}{2} = \operatorname{csch}^2(2t+c)$

$\Rightarrow \underline{B(t) = \operatorname{csch}(2t+c)}$ ($\pm$ sign is contained in c)

$\underline{C(t)} = \int A(t)\,dt = \underline{-\frac{1}{2}\log\sinh(2t+c) + D}$

$$\begin{cases} \int \frac{\cosh}{\sinh}\,dt = \log\sinh(t). \\ \left(\frac{\cosh}{\sinh}\right)' = \frac{\sinh^2 - \cosh^2}{\sinh^2} = -\operatorname{csch}^2 \end{cases}$$

for initial condition: put $y=0$.
then $t \to 0$. should go to " $\frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ "

$$A(t) = -\coth(2t+c) \sim -\frac{1}{2t} \Rightarrow c=0.$$

and $e^{c(t)} = (\sinh 2t)^{-1/2} e^{D} \sim (4\pi t)^{-1/2}$

$$\Rightarrow \underline{D = \log[(2\pi)^{-1/2}]}.$$

So the heat kernel is given by: Mehler's formula

$$\boxed{p = p(x,y,t) = (2\pi \sinh 2t)^{-1/2} \cdot e^{-\frac{1}{2}\coth 2t \cdot (x^2+y^2) - \operatorname{csch} 2t \cdot xy}}$$

Change of variable. (let $y=0$) $t \mapsto \frac{tr}{4}$; $x \mapsto \sqrt{\frac{r}{4}}\, x$.

$$(*) \quad \left(\frac{\partial}{\partial t} \underline{- \frac{\partial^2}{\partial x^2} + \frac{r^2}{16}x^2} + f\right) P_t(x) = 0$$

has solution:

$$\frac{1}{\sqrt{4\pi t}} \cdot \left[\left(\frac{tr/2}{\sinh(tr/2)}\right)^{1/2}\right] e^{-\left[\frac{tr}{2}\coth\left(\frac{tr}{2}\right)\right] \cdot \frac{x^2}{4t}} e^{-tf}$$

proof for $H = -\sum \nabla_i^2 + F = -\sum\left(\partial_i + \frac{1}{4}\underline{R_{ij}} x_j\right)^2 + F$:
↑ skew-sym.

$$P_t(x,R,F) := \frac{1}{\sqrt{4\pi t}^{\,n}} \det\left(\frac{tR/2}{\sinh tR/2}\right)^{1/2} \cdot e^{-\frac{1}{4t}\langle x | \frac{tR}{2}\coth\frac{tR}{2} | x\rangle} \cdot e^{-tF}$$

Need to show that $\frac{\partial P_t}{\partial t} = -HP_t$ (algebraic formula!)

both side are analytic functions of R_{ij}.
i.e. purely formal
$\Rightarrow$ may assume that $R_{ij} \in \mathbb{R}$
why not simply diagonalize it in $\mathbb{C}$?
May compute this in any coor. system of $\mathbb{R}$"
say. the system st. $R = \begin{bmatrix} \begin{matrix} 0 & -r \\ r & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & -s \\ s & 0 \end{matrix} & \\ & & \ddots \end{bmatrix}$ block form

The problem is reduced to the 2-dim'l case. ok. they are of diff scales

! But how about F

In this case, get

$$H = -(\partial_1^2+\partial_2^2) - \frac{1}{2}\underbrace{R_{ij}\,\partial_i(x_j\,\cdot\,)}_{\text{since } i\neq j} - \left(\frac{1}{4}r\right)^2(x_1^2+x_2^2) + F$$

$$\| \quad -\frac{1}{2}r\,(x_2\partial_1 - x_1\partial_2)$$

$$P_t(x_1,x_2) = \frac{1}{4\pi t}\cdot\frac{tr/2}{\sin tr/2}\cdot e^{-\frac{tr}{2}\cot\left(\frac{tr}{2}\right)\cdot\frac{\|x\|^2}{4t}}\cdot e^{-tF}$$

only here has nontrivial matrix part.

Reason: using the trick of cpx number (see back side lemma)

replace r by ir, get the 2-dim'l analogue of (*) except that get one more term

$$-\frac{1}{2}r\,(x_2\partial_1 - x_1\partial_2) \quad \text{- infinitesimal rotation}$$

but this term acts on $\|x\|^2 = 0$

this proves that $\frac{\partial}{\partial t}P_t = -HP_t$. $\square$

Main Idea for Local Index Formula!

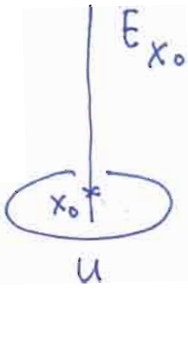
M-S: $\operatorname{index} P = \operatorname{tr} e^{-tP^*P} - \operatorname{tr} e^{-tPP^*}$

$$= \int_X \operatorname{tr} P_{\text{for } P^*P} - \operatorname{tr} P_{\text{for } PP^*}$$

for "generalized Laplace" has "Weitzenböck type formula":

$$\boxed{\Delta = \operatorname{tr}\nabla^2 + F}$$

some term comes from curvature of g and of certain bundle E.



for a pt $x_0 \in X$. pick a nbd U. trivialize all bundles ($T_X|_U$, $E|_U$)
Knowing the precise value $F(x_0)$ will determine the heat kernel at x_0.

This can be done by rescaling!

let $P : \Gamma(E) \to \Gamma(F)$ with adjoint operator
$P^* : \Gamma(F) \to \Gamma(E)$

form the operator $D : \Gamma(E \oplus F) \to \Gamma(E \oplus F)$

by $D = \begin{bmatrix} 0 & P^* \\ P & 0 \end{bmatrix}$

then D is self-adjoint

$D^2 = \begin{bmatrix} P^*P & 0 \\ 0 & PP^* \end{bmatrix}$

$$\begin{aligned} \text{index } P &:= \dim \ker P - \dim \operatorname{coker} P \\ &= \dim \ker P - \dim \ker P^* \\ &= \dim \ker P^*P - \dim \ker PP^* \\ \text{McKean-Singer: } &= \operatorname{tr} e^{-tP^*P} - \operatorname{tr} e^{-tPP^*} \\ &=: \operatorname{str}(e^{-tD^2}) \quad \leftarrow \text{super trace on } \mathbb{Z}_2\text{-graded spaces} \\ &= \int_M \operatorname{str} k_t(x,x) \end{aligned}$$

This formula is independent of $t \in \mathbb{R}^+$

$k_t(x,x)$ = heat kernel of D^2 restrict to diagonal

$\in (\operatorname{End} E_x \oplus \operatorname{End} F_x) \otimes \underline{\Lambda^n_x}$

which depends on t. coupled with the dV.

Quantum Idea:

- McKean-Singer $\leftrightarrow$ $t \to \infty$
  get discrete data
- Local Index theorem $\leftrightarrow$
  find a special value t (eg. $t \to 0^+$)
  so that $k_t(x,x)$ is an
  explicitly computable diff form.

Clifford Algebra, module & connections

V real v.s. $Q = \langle , \rangle$ quad. form

$c(V, Q) := T(V)/\mathcal{J}$ ← ideal gen. by $v \otimes v + Q(v,v)$

a $\mathbb{Z}_2$-grad algebra.

e.g. $Q = 0$ get $\Lambda(V)$ ← in fact, deformation of $\Lambda(V)$

Q = pos. def denoted by $c(V)$.

Let $c(V) = c^+(V) \oplus c^-(V)$ (so $V \subset c^-(V)$)

$E = E^+ \oplus E^-$ is a Clifford module if

E is a $c(V)$ module with $\mathbb{Z}_2$-grad action. i.e. $c^+ E^\pm \subset E^\pm$, $c^- E^\pm \subset E^\mp$.

Example: $E = \Lambda(V)$, $c(v) := v\wedge - v\lrcorner$

⇝ Symbol Map $\sigma : c(V) \xrightarrow{\sim} \Lambda(V)$, the module action on 1. with obvious inverse!

Let $C_0(V) \subset C_1(V) \subset \cdots C_i(V) \subset \cdots$ ← span by $v_1 \dots v_k$ $\forall v_i \in V$, $k \le i$

then $gr_i\, c(V) := C_i(V)/C_{i-1}(V)$

$\cong \Lambda^i(V)$ get $\sigma_i : C_i(V) \to \Lambda^i(V)$

then if $a \in C_i(V)$, $\Rightarrow \sigma(a)_{[i]} = \sigma_i(a)$.

- Thm let V be even dim. oriented, $Q = \langle , \rangle > 0$

(1) then $\exists!$ Clifford module $S = S^+ \oplus S^-$ ← (half-) Spinor modules

st $\boxed{c(V) \otimes \mathbb{C} \cong \mathrm{End}(S)}$

(2) And all other Clifford modules over $\mathbb{C}$ $E \cong S \otimes W$ ← trivial $c(V)$ action but W is still $\mathbb{Z}_2$ graded!

pf: Endow V a almost cpx str J st $Q = \langle , \rangle$ is J-inv. then

$V \otimes \mathbb{C} \cong P \oplus \bar{P}$ with $Q(w,w) = 0$ $\forall w \in P$. *

define: $S := \Lambda(P)$ ← i.e. "hol. diff. form"

action of V on S: for $v = \underset{P}{w} + \underset{\bar{P}}{\bar{w}}$

$$\begin{cases} c(w)\cdot s = \sqrt{2}\, w \wedge s \\ c(\bar{w})\cdot s = -\sqrt{2}\, \bar{w} \lrcorner s \end{cases}$$ ← this is a Clifford action by *

$\Rightarrow c(V \otimes \mathbb{C}) \overset{c}{\hookrightarrow} \mathrm{End}(S)$. isom. by counting dim. get (1).

← notice that $S = \Lambda P = \Lambda^e P \oplus \Lambda^o P = S^+ \oplus S^-$.

for (2) simply take

$W := \mathrm{Hom}_{c(V)}(S, E)$ ← in fact

Every mod. of matrix alg $\mathrm{End}(S)$ is of the form $S \otimes W$ □

For (M, g), $n = 2m$ dim. Riem. mfd. oriented

get Clifford bundle $C(M)$: $C(M)_p := C(T_p^*M)$

Now S only exists locally (~~globally iff M is spin~~)
i.e. TM has str. gp $\begin{matrix} spin(n) \\ \downarrow \\ SO(n) \end{matrix}$

then a Clifford module E is locally $S \otimes W$. (S is unique)

For local index density, it is OK to assume all these.

let $c : C(M) \longrightarrow End(E)$ be the module str.

<u>Levi-Civita connection on $C(M)$</u> : ∇^{LC} st

$$\nabla^{LC}_X (ab) = (\nabla^{LC}_X a) b + a (\nabla^{LC}_X b),$$ in fact ∇^{LC} is defined on S.

<u>Clifford connection on E</u> :

$$\nabla^E_X (c(a) s) = c(\nabla^{LC}_X a) s + c(a) \nabla^E_X s.$$

pf of $\exists$: locally $E = S \otimes W$, take $\nabla^{LS} \otimes id + id \otimes \nabla^W$ any. then use partition of 1.

Dirac operator on (E, ∇^E) : $D := c \circ \nabla^E$.

$$\Gamma(M, E) \xrightarrow{\nabla^E} \Gamma(M, T^*M \otimes E) \xrightarrow{c} \Gamma(M, E)$$

i.e. locally, $D = \sum_i c(dx^i) \cdot \nabla^E_{\partial_i}$.

<u>Fact</u> ①. <u>D is elliptic</u> : for $\xi = \xi_i dx^i \in T_p^*M \neq 0$

symbol $\underline{\sigma_1(D)} = \sum_i c(dx^i) \cdot \xi_i = \underline{c(\xi)} \neq 0$

$\sigma_1(D)$ in the sense of diff op. $\qquad$ $\sigma_1(\xi)$ in the sense of clifford alg !

Now let E be hermitian v.b. and ∇^E be metrical and Clifford.

<u>Fact</u> ②. If $c(a)$ is skew-adjoint on E, then <u>D is self-adjoint</u> :

this is not really used later.

$$\langle Ds, t \rangle = \langle c_i \nabla_i s, t \rangle = - \langle \nabla_i s, c_i t \rangle = - d_i \langle s, c_i t \rangle + \langle s, \nabla_i c_i t \rangle$$

$$\nabla_i c_i t = c_i \nabla_i t + c(\nabla_i e_i) t$$

$= \langle s, Dt \rangle - tr(\nabla X)$ i.e. $div X$, so $\int_M = 0$.

for X the v.f. given by $a(X) := \langle s, c(a) t \rangle$ check !

$\Rightarrow$ May Apply prev. McKean-Singer Formula.

Supertrace v.s. Clifford symbol.

for $E = S \otimes W$, $k(x,y,t)$ heat kernel of D^2 $\to E_x \otimes E_y^*$

"ind (D)" $= \int_M \operatorname{str} k_t(x,x)$

$S^+ \otimes W \rightleftarrows S^- \otimes W$

$\searrow \operatorname{End} E_x = S_x \otimes W_x \otimes S_x^* \otimes W_x^*$

$= \operatorname{End}(S_x) \otimes \operatorname{End}(W_x)$, $\operatorname{End}(S_x) \cong C(M)_x$

$\|$

$\left(\text{str on } \operatorname{End}(S)\right) \cdot \left(\text{str on } \operatorname{End} W\right)$

Lemma: $\operatorname{str} a = c \cdot T\sigma(a)$. ie. coeff in $e_1 \wedge \cdots \wedge e_n$.

$\operatorname{Hom}(S^+ \oplus S^-, S^+ \oplus S^-)$

$$D^2 = \begin{pmatrix} \boxed{+\,+} & X \\ X & \boxed{-\,-} \end{pmatrix}$$

Step 1:

pf: On $C(V)$ have $\mathbb{Z}_2$-commutator

$[u,v] = uv - (-1)^{|u|\cdot|v|} vu$

then $\operatorname{str}[u,v] = 0$ $\forall u,v$. (simple check by case)

Step 2:

$C_{n-1}(V) = [c(V), c(V)]$:

let $e_1 \cdots e_n$ ONB, if $|I| < n$ say $j \notin I$.

then $c(e_I) = -\frac{1}{2}[c_j, c_j c_I]$.

Step 3:

str must be propotional to $C(V) \xrightarrow{\sigma} C(V)/C_{n-1}(V) \xrightarrow{T} \mathbb{R}$.

consider the chirality element $\epsilon := i^p e_1 \cdots e_n \in C(V)$

$\epsilon^2 = 1$. we are here $\to$ $\begin{cases} p = n/2 & \text{if } n \text{ even} \\ p = (n+1)/2 & \text{if } n \text{ odd} \end{cases}$

then $S^{\pm} = \{v \mid \epsilon v = \pm v\}$.

so $\operatorname{str} \epsilon = \dim S^+ + \dim S^- = \dim S = 2^{n/2}$

But $T\sigma(\epsilon) = T(i^p e_1 \wedge \cdots \wedge e_n) = i^p = i^{n/2}$,

so $c = (-2i)^{n/2}$. $\square$

Rmk: ind D really means ind D^+. or it means dim ker D by viewing ker D as a $\mathbb{Z}_2$-graded (super) space.

Statement of Local Index Theorem

Let $E = S \otimes W$ be a Clifford module locally with D the Dirac op.

Let $k_t(x,y)$ be the heat kernel of D^2.

Since
$$\operatorname{ind}(D) = \int_M \operatorname{str} k_t(x,x)$$

$$\Gamma(M, \operatorname{End} E \otimes |\Lambda^n|) \cong \Gamma(M, C(M) \otimes \operatorname{End} W \otimes |\Lambda^n|)$$

$$\downarrow \sigma \text{ symbol map}$$

It turns out

$$\Lambda^{\cdot}(M) \otimes \operatorname{End} W$$

a) $$k_t(x,x) \sim \frac{1}{\sqrt{4\pi t}^{\,n}} \sum_{i=0}^{\infty} t^i k_i(x)$$

and $k_i \in \Gamma(M, Cl_{2i} \otimes \operatorname{End} W) \otimes |\Lambda^n|$

i.e. degree $\leq 2i$ (so for k_j, $j > \frac{n}{2}$ it is out of control)

Since "$\operatorname{str} \doteq \sigma$" on $C(M)$:

b) For $\sigma(k) := \sum_{i=0}^{n/2} \sigma_{2i}(k_i) \in \Lambda^{\cdot}(\operatorname{End} W)$

$$\Rightarrow \quad \sigma(k) = \det{}^{1/2}\left(\frac{R/2}{\sinh R/2}\right) \cdot e^{-F^W}$$

I.e. the total negative degree piece has a meaning!

<u>Idea:</u> $\sigma(k)$ is singled out (at a point x_0) via rescaling procedure δ_u: $t \mapsto ut$, $x \mapsto \sqrt{u}\, x$

(notice k_t has density $|\Lambda^{n/2}|$ part)

then take $\lim_{u \to 0} \sqrt{u}^{\,n} \delta_u k_t (t=1, x=x_0) \equiv \sigma(k)$

So finally $\operatorname{str} k_t(x,x) = \operatorname{tr} \sigma(k)_{[n]} = \left(\det{}^{1/2}(-) \cdot \operatorname{tr} e^{-F^W}\right)_n$

Remark on Sign: usually use

$$\frac{\sqrt{-1}}{2\pi} F = \frac{-1}{2\pi i} F$$

omitted here.

$= \hat{A}(M,g) \cdot ch(W,\nabla))_n$

eg. H generalized Laplace on E

at a point $(U, x) \ni p$ coor of $p = 0$.

heat kernel $k(t,x)$, want "$\underline{k(t,0)}$"

the negative degree part

rescaling: $t \mapsto ut$

$x \mapsto \sqrt{u}\, x$ $\quad u \in (0,1]$

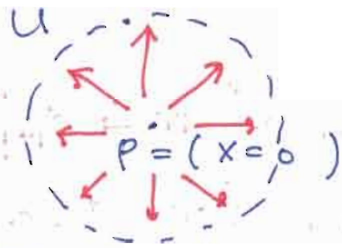
get $k(u,t,x) := \underline{u^{n/2}} \cdot k(ut, \sqrt{u}\, x)$

to keep $\lim_{t \to 0} k(u,t,x) = \delta_x$

Rmk: $q_t(x) := \dfrac{1}{\sqrt{4\pi t}^{\,n}} \cdot e^{-\frac{|x|^2}{4t}}$ is inv. under this

for the asymptotic expansion

$k(t,x) \sim \underline{q_t(x)} \sum_{i \geq 0} t^i \phi_i(x)$



get

$k(u,t,x) \sim \underline{q_t(x)} \sum_{i \geq 0} (ut)^i \phi_i(\sqrt{u}\, x)$

Lichnerowicz Formula.

$$D^2 = \underbrace{-\mathrm{tr}\,\nabla^{E\,2}}_{\text{associated Laplacian}} + c(F^W) + \overbrace{\frac{r}{4}}^{\text{scalar curvature}}$$

where $c(F) = \sum_{i<j} F(e_i, e_j)\, c(e_i)\, c(e_j)$

with ∇^E Clifford conn. on $E = S \otimes W$.

(Sketch of pf:) at p. Pick N.C. get e_i, $c^i := c(e^i)$

$$\begin{aligned} D^2 &= c^i \nabla_{e_i} c^j \nabla_{e_j} \\ &= c^i c^j \nabla_{e_i} \nabla_{e_j} + c^i c(\underbrace{\cancel{\nabla_{e_i} e^j}}_{=0 \text{ at } p}) \nabla_{e_j} \\ &= -\nabla_{e_i}^2 + \sum_{i<j} c^i c^j \underbrace{(\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i})}_{F^E(e_i, e_j)} \end{aligned}$$

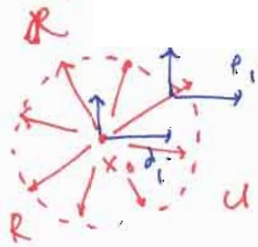
ex 1. by commutation

$$= \underbrace{-\frac{1}{8} R_{klij}\, c^i c^j c^k c^l}_{\frac14 r.} + F^W(e_i, e_j)\, c^i c^j$$

Bianchi id   ex 2.

$\frac{1}{4} r$. □

Now fix $x_0 \in U = B(x_0)$. trivialize E by parallel translate $\tau(x_0, x) : E_x \longrightarrow E_{x_0}$.
$\{\partial_i\}$ ONF at x_0, $\{e_i\}$ = ONF by parallel along geodesics.
$c^i := c(dx^i) \in \mathrm{End}(E_{x_0})$.



$\boxed{\text{FACT}}$ the function $c(e^i) \in \mathrm{End}(E_{x_0}) = \text{const} = c^i$.

pf: $R \cdot c(e^i) = \nabla^E_R\, c(e^i) \equiv [\nabla^E_R, c(e^i)] = c(\nabla_R e^i) = 0$

property of parallel transl.   cliff conn.   def of e^{i*}

Let $p_t(x, x_0)$ be heat kernel of D^2,

$k(t, \mathbf{x}) := \tau(x_0, x)\, p_t(x, x_0) \in \mathrm{End}(E_{x_0})$

with $x = \exp_{x_0}(\mathbf{x})$

$$c(T^*_{x_0}M) \otimes \mathrm{End}\, W \;\overset{\sigma}{\downarrow}\; \Lambda(T^*_{x_0}M) \otimes \mathrm{End}\, W$$

view as $\underline{\Lambda_{x_0} \otimes \mathrm{End}\, W}$ -valued on U.

$c(T^*_{x_0}M)$ acts on Λ_{x_0} by

$\boxed{c(\alpha)\beta = \varepsilon(\alpha)\beta - \iota(\alpha)\beta}$ as usual.

bec. of def. of symbol map σ.

<u>PROPOSITION</u>:

$(\partial_t + L)\, k(t, \mathbf{x}) = 0$. $L := -\sum\left(\nabla^{E\,2}_{e_i} - \nabla^E_{\nabla_{e_i} e_i}\right) + \frac{r}{4} + c(F^W)$.

Rescaling Procedure: δ_u

$$k_t \sim (4\pi t)^{-\frac{n}{2}} \left(k_0 + k_1 t + \cdots + k_{\frac{n}{2}} t^{\frac{n}{2}} + \cdots \right)$$

$q_t(x) := e^{-\frac{|x|^2}{4t}}$

at $x_0 \in U$

Say $U = B_R(x_0)$

Assume the existence of such expansion.

$$(k_1 \cdot \underline{u^{-1}}) \cdot ut \qquad \left(k_{\frac{n}{2}} \cdot \underline{u^{-\frac{n}{2}}} \right) (ut)^{\frac{n}{2}}$$

2-form part $\qquad$ n term part

$(\alpha:\ i\text{-form} \longrightarrow \sqrt{u}^{-i}\,\alpha)$

notice that $q_t(x)$ is inv. under scaling.

for higher order part u-degree > 0

hence $\sqrt{u}^{\,n} \delta_u k_t$ as $(t=1,\ x=0)$ $u \to 0$ get

$$(4\pi)^{-\frac{n}{2}} \left(k_{0\,[0]} + k_{1\,[2]} + \cdots + k_{\frac{n}{2}\,[n]} \right)$$

is the term we want to show to be

$$\sigma(\cdots) = \det{}^{1/2}\left(\frac{R/2}{\sinh R/2} \right) \cdot e^{-F^W}$$

Now the rescaled heat kernel

$$r(u, t, x) = u^{n/2} (\delta_u k_t)(t, x)$$

is (in) fact the heat kernel of heat eq'n:

rescaled heat eq'n: $\boxed{\frac{\partial}{\partial t} + u\,\delta_u L\,\delta_u^{-1}} \equiv \frac{\partial}{\partial t} + L(u)$ $\quad$ defined on $B_{R/u}$

Will see that $L(u) = K + O(\sqrt{u})$

harmonic oscilator

hence $r(0, t, x)$ can be read off from the heat kernel of harmonic oscilator

$$\frac{\partial}{\partial t} + K \qquad \text{defined on } B_\infty = \mathbb{R}^N \text{ the whole space.}$$

CLAIM: $K = -\sum \left(\partial_i + \frac{1}{4} R_{ij} x_j \right)^2 + F^W(x_0)$.

Let $\alpha \in \Gamma(\mathbb{R}^+ \times U, \Lambda^\cdot(\mathrm{End}\, W)) \equiv \mathcal{A}$

$$\delta_u \alpha := \sum_{i=0}^{n} \sqrt{u}^{-i} \, \alpha(ut, \sqrt{u}\, x)_{[i]}$$

the i-th comp.

then δ_u acts on operators on $\mathcal{A}$ via

All trivial

$$\left\{\begin{aligned} &\delta_u \phi(x) \delta_u^{-1} = \phi(\sqrt{u}\, x) \cdot \qquad \text{for } \phi \in C^\infty(U) \\ &\delta_u \tfrac{\partial}{\partial t} \delta_u^{-1} = u^{-1} \tfrac{\partial}{\partial t} \quad (*) \\ &\delta_u \tfrac{\partial}{\partial x_i} \delta_u^{-1} = \sqrt{u}^{-1} \tfrac{\partial}{\partial x_i} \\ &\delta_u \epsilon(\alpha) \delta_u^{-1} = \sqrt{u}^{-1} \epsilon(\alpha) \qquad \alpha \in T^* = \Lambda^1 \\ &\delta_u \iota(\alpha) \delta_u^{-1} = \sqrt{u}\, \iota(\alpha) \quad \star \end{aligned}\right.$$

Rescaled heat kernel

$$r(u, t, x) := \sqrt{u}^{\,n} \cdot (\delta_u k)(t, x)$$

then

$$\Big(\underbrace{\tfrac{\partial}{\partial t}}_{\overset{(*)}{=}\, u \delta_u \frac{\partial}{\partial t} \delta_u^{-1}} + \underbrace{u \delta_u L \delta_u^{-1}}_{= L(u)} \Big) r(u, t, x) = 0$$

i.e. $u \delta_u \left(\frac{\partial}{\partial t} + L \right) \delta_u^{-1} r = 0$.

$$\boxed{\begin{aligned} &\text{here } L = -\sum_i \left({\nabla^E_{e_i}}^2 - \nabla^E_{\nabla_{e_i} e_i} \right) + \frac{r}{4} + \sum_{i<j} F^W_{ij}\, c^i c^j \\ &\text{on } U \text{ (under trivialization of } E|_U \text{).} \end{aligned}}$$

$L(u) = L_1(u) + L_2(u)$:

$$\left\{\begin{aligned} &L_1(u) = -\sum_i \left(\sqrt{u}\, \delta_u \nabla^E_{e_i} \delta_u^{-1} \right)^2 + \sum_{i<j} F^W_{ij}(\sqrt{u}\, x) \cdot \sqrt{u}\left(\sqrt{u}^{-1}\epsilon^i - \sqrt{u}\, \iota^i\right) \cdot \sqrt{u}\left(\sqrt{u}^{-1}\epsilon^j - \sqrt{u}\, \iota^j\right) \\ &L_2(u) = \frac{1}{4} u\, r(\sqrt{u}\, x) + \sqrt{u} \cdot \left(\sqrt{u}\, \delta_u \nabla^E_{\nabla_{e_i} e_i} \delta_u^{-1} \right) \end{aligned}\right.$$

lemma: ½ from Lie alg identification, at point x_0

Now
$$\boxed{\nabla^E_{\partial_i} = \partial_i + \frac{1}{4}\sum_{\substack{k<l\\ j}} R_{klij}\, x^j c^k c^l + \sum_{k<l} \underbrace{f_{ikl}(x)}_{=O(|x|^2)} c^k c^l + \underbrace{g_i(x)}_{=O(|x|)}}$$

So $\nabla^{E,u}_{\partial_i} := \sqrt{u}\,\delta_u \nabla^E_{\partial_i}\,\delta_u^{-1}$

$$= \partial_i + \frac{1}{4}\sum_{\substack{k<l\\ j}} R_{klij}\, \sqrt{u}\cdot\sqrt{u}\cdot x^j \cdot\left(\sqrt{u}^{-1}\epsilon^k - \sqrt{u}\,\iota^k\right)\left(\sqrt{u}^{-1}\epsilon^l - \sqrt{u}\,\iota^l\right)$$
$$+ \overset{u^{-1}}{\sqrt{u}} \sum_{k<l} f_{ikl}(\sqrt{u}x)\left(\overset{\sqrt{u}}{\sqrt{u}^{-1}}\epsilon^k - \sqrt{u}\,\iota^k\right)\left(\overset{\sqrt{u}}{\sqrt{u}^{-1}}\epsilon^l - \sqrt{u}\,\iota^l\right)$$
$$+ \sqrt{u}\, g_i(\sqrt{u}x)$$

$$f_{ikl}(\sqrt{u}x) = O(|\sqrt{u}x|^2) = u\,O(|x|^2)$$

$\Rightarrow$ as $u\to 0$. $\nabla^{E,u}_{\partial_i}$ has limit $= \partial_i + \frac{1}{4} R_{klij}\, x^j \epsilon^k \epsilon^l$
$$= \left(\partial_i + \frac{1}{4} R_{ij}\, x^j\right)$$

Clearly: $\lim_{u\to 0} L_2(u) = 0$.

curvature matrix with 2-form entries.

bec. $\sqrt{u}\,\delta_u \nabla^E_{\nabla_{e_i}e_i}\,\delta_u^{-1}$ has a limit.

Finally, RHS of $L_1(u) \xrightarrow[u\to 0]{} \sum_{i<j} F^W_{ij}(0)\,\epsilon^i\epsilon^j \equiv F^W(x_0)$

LHS of $L_1(u) \longrightarrow -\sum_i\left(\nabla^{E,0}_{\partial_i}\right)^2 = -\left(\partial_i + \frac{1}{4}R_{ij}x^j\right)^2$

both because when $u\to 0$, $e_i \to e_i(0) = \partial_i$ $\square$.

By Mehler's formula: get
$$\lim_{u\to 0} r(u,t,x) = \frac{1}{\sqrt{4\pi t}^{\,n}} \det{}^{1/2}\left(\frac{tR/2}{\sinh tR/2}\right)\cdot e^{-\frac{1}{4t}\langle x \mid \frac{tR}{2}\coth\frac{tR}{2} \mid x\rangle} \cdot e^{-tF}$$

Put $x=0$, $t=1$. get
$$\sigma(k) = \frac{1}{\sqrt{4\pi t}^{\,n}} \det{}^{1/2}\left(\frac{R/2}{\sinh R/2}\right)\cdot e^{-F}$$

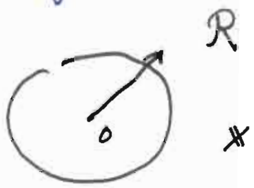
This finishes the pf of Local index thm for D.

End.

Supplementary proofs of lemmas: I. Taylor of ω.

$R := \sum_i x^i \partial_i$. radial vector

E trivialized via parallel transl.



$\nabla = d + \omega$, $\omega_j^k = \sum_i \Gamma_{ij}^k \, dx^i$ conn. 1-form matrix

$\iota(R)\omega = \omega$ evaluate at radial (geod.) direction $= 0$.

$$\Rightarrow \quad \underline{L_R \omega} = (\iota(R) d^\nabla + d^\nabla \iota(R))\omega = \iota(R) d^\nabla \omega$$

$$= \iota_R (d\omega + \omega \wedge \omega) = \underline{\iota_R F}$$

claim: Taylor expansion of $\omega_i = (\Gamma_{ij}^k \, dx^i)_{j,k}$ at $x = 0 = x_0$

is $$\omega_i(x) \doteq -\frac{1}{2} \sum_j F(\partial_i, \partial_j)(x_0)\, x^j + \sum_{|\alpha| \geq 2} \partial^\alpha \omega_i(x_0) \frac{x^\alpha}{\alpha!}.$$

pf: $L_R \omega = \iota_R F$ take Taylor expansion

$$\sum_\alpha (|\alpha|+1)\, \partial^\alpha \omega_i(x_0) \frac{x^\alpha}{\alpha!} = \sum \partial^\alpha F(\partial_k, \partial_i)(x_0) \frac{x^k x^\alpha}{\alpha!}$$

since $R = \sum x^i \frac{\partial}{\partial x^i}$ take derivative and then multiply back

from R

from plug in $\frac{\partial}{\partial x_i}$ in ω

pick $\alpha = j$, get $2 \cdot \partial_j \omega_i(x_0) = F(\partial_j, \partial_i)(x_0)$ #

II Communications in Lichnerowicz formula.