群與表示論簡介

Introduction to Groups and Representations

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近代數學導論

(對象: 數學系與物理系高年級大學部學生)

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摘要

群表示理論是現代數學中最重要的方法之一. 他在數論與理論物理都有重要的地位. 因此愈早開始接觸它, 建立良好的基本觀念, 對於學習數學與物理都是刻不容緩的事. 有鑑於此, 這個介紹性課程只假設同學具有線性代數, 微積分, 以及最基本的群論知識. 我們希望透過具體的例子與計算闡釋群表示理論中最為基本的元素, 並導引同學日後追求一般性理論的方向.

內容

CHAPTER III
REPRESENTATION THEORY OF FINITE GROUPS

REPRESENTATIONS

V finite dim vector space $/\mathbb{C}$

$\rho : G \longrightarrow GL(V)$ group homomorphism

is called a representation of G

G-invariant inner product ; let G be finite

$$\langle v, w \rangle' := \sum_{g \in G} \langle \rho(g) v, \rho(g) w \rangle$$

$$\Rightarrow \langle \rho(g) v, \rho(g) w \rangle' = \langle v, w \rangle' \quad \forall g \in G$$

So in fact, $\rho : G \longrightarrow U(n)$, $n = \dim V$

Basic constructions

G acts on V_1, V_2 then

G acts on $V_1 \oplus V_2$: $\rho_1 \oplus \rho_2$

$$g(v_1, v_2) = (g v_1, g v_2)$$

G acts on $V_1 \otimes V_2$: $\rho_1 \otimes \rho_2$

$$g(v_1 \otimes v_2) = g v_1 \otimes g v_2$$

Basic examples

I. 1-dimensional repr'n $\rho : G \longrightarrow S^1 = \{ |z| = 1 \}$

trivial repr'n $\rho : G \longrightarrow \{1\}$

II. Permutation repr'n : Let G acts on X,

$\Rightarrow$ G acts on $\mathbb{C}^{|X|}$ by permuting basis.

Regular repr'n $\underset{\overset{\shortparallel}{r_G}}{\rho_{reg}}$: the case $X = G$.

IRREDUCIBLE REPRESENTATIONS

$\rho : G \longrightarrow GL(V)$ is irreducible if the only G-invariant subspace is V or $\{0\}$.
This is equivalent to $V \neq V_1 \oplus V_2$ as repr'n s of G

Pf : If $W \subset V$ is G-inv then $W^{\perp}$ is also G-inv.

Complete reducibility

Every repr'n is a direct sum of irreducible repr'ns.

$$V = V_1^{\oplus a_1} \oplus V_2^{\oplus a_2} \oplus \cdots \oplus V_k^{\oplus a_k}$$

PROBLEMS

I. Describe all the irreducible repr'ns of G

II. Find the direct sum decomposition of any repr'n V together with multiplicities

III. Generalized Clebsch – Gordan Problem :

Decompose $V \otimes W$, $\mathrm{Sym}^k(V)$, $\Lambda^k(V)$ into irreducibles

(may assume that V, W are irreducible)

Trivial case — Abelian groups

G abelian $\Rightarrow$ all irred. repr'ns are 1-dim'l

Pf : $gh = hg \Rightarrow \rho(g), \rho(h) \in GL(V)$ are simultanuously diagonalizable.

SCHUR'S LEMMA

— The key to the study of irred. repr'ns

Let $T : V_1 \longrightarrow V_2$ be a G-linear map of two irred. repr'ns of G, then

- $V_1 \cong V_2$ as G repr'ns and $T = \lambda I$, $\lambda \in \mathbb{C}$
- or if $V_1 \not\cong V_2$ then $T \equiv 0$

pf: Since $\ker T$, $\operatorname{Im} T$ are all G-inv.

so $T \not\equiv 0 \Rightarrow V_1 \cong V_2$

Now let λ be an eigenvalue of T

then $\ker(T-\lambda I) \neq 0$, $\Rightarrow T - \lambda I \equiv 0$.

Main consequences

Given any linear map $H : V_1 \longrightarrow V_2$

$$T = \frac{1}{|G|} \sum_{g\in G} \rho_2(g)^{-1} H \rho_1(g) \quad \text{is } G\text{-linear}$$

(since $\rho_2(h)^{-1} T \rho_1(h) = T \Rightarrow T\rho_1(h) = \rho_2(h) T$)

so $V_1 \not\cong V_2 \Rightarrow T = 0$

$V_1 = V_2 = V \Rightarrow T = \frac{1}{n} Tr(H) \cdot I_n$

Recall that

$$Tr \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} := \sum_{k=1}^{n} a_{kk}$$

and $Tr(AB) = Tr(BA)$,

Hence that $Tr(S^{-1} A S) = Tr(A)$.

Now $Tr(T) = \frac{1}{|G|} \sum_{g\in G} Tr\left(\rho(g)^{-1} H \rho(g)\right) = Tr(H)$

$T = \lambda I_n \Rightarrow \lambda = \frac{1}{n} Tr(H)$.

Explicit matrix form

Let $\rho_1(g) = A = [A_{ij}(g)]$; $H = [H_{ij}]$

$\rho_2(g) = B = [B_{ij}(g)]$, since unitary

so $\rho_2(g)^{-1} = \bar{B}^t = [\bar{B}_{ji}(g)]$, then

$$0 = \frac{1}{|G|} \sum_{g \in G} \bar{B}^t_{is} H_{st} A_{tj} = 0 \qquad H \text{ arbitrary}$$

$$\Rightarrow \boxed{\frac{1}{|G|} \sum_{g \in G} \bar{B}_{si}(g) A_{tj}(g) = 0 \qquad \forall\ s, i, t, j}$$

In the case $V_1 = V_2 = V$, for any H

$$\frac{1}{n} Tr(H) I = \frac{1}{|G|} \sum_{g \in G} \bar{A}_{si}(g) H_{st} A_{tj}(g)$$

$$\Rightarrow \boxed{\begin{array}{ll} \frac{1}{|G|} \sum_{g \in G} \bar{A}_{si}(g) A_{tj}(g) = 0 & \text{if } i \neq j \\ & \text{or } s \neq t \\ \frac{1}{|G|} \sum_{g \in G} \bar{A}_{ji}(g) A_{ji}(g) = \frac{1}{n} & \end{array}}$$

Inner product on $\mathbb{C}^{|G|}$

Let φ, ψ two functions $G \longrightarrow \mathbb{C}$

$$(\varphi, \psi) := \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)}$$

$$\Rightarrow \begin{array}{l} (A_{ij}, B_{kl}) = 0 \\ (A_{ij}, A_{kl}) = \frac{1}{n} \delta_{ik} \delta_{jl} \end{array}$$

the orthogonality of irred. repr'ns !

How to make use of it ?

CHARACTERS

$$\rho : G \longrightarrow GL(V)$$

$$\chi_\rho = \mathrm{Tr} \circ \rho : G \longrightarrow \mathbb{C} \quad \text{character of } \rho$$

Naive point of view

Tr(A) does not determine A, but

$\mathrm{Tr}(A), \mathrm{Tr}(A^2), \mathrm{Tr}(A^3), \cdots$ does !

And now χ_ρ is a function on G

Actual reason

Orthogonality $\Rightarrow$ for irred. characters

$$\boxed{\begin{array}{ll} (\chi_{\rho_1}, \chi_{\rho_2}) = 0 & \text{if } \rho_1 \not\cong \rho_2 \\ (\chi, \chi) = 1 & \text{for isomorphic irred. repr}^{\text{n}}\text{s} \end{array}}$$

Consequences

- In an irreducible decomposition
  $$V = V_1^{\oplus a_1} \oplus V_2^{\oplus a_2} \oplus \cdots \oplus V_k^{\oplus a_k}$$
  $\underline{a_i = (\chi_\rho, \chi_i)}$ where χ_i = char of V_i.
- Hence two representations are isomorphic $\Leftrightarrow$ they have the same character
- In general, $(\chi_\rho, \chi_\rho) = \sum_{i=1}^{k} a_i^2 \in \mathbb{N}$
  and ρ is irreducible $\Leftrightarrow$ $\underline{(\chi_\rho, \chi_\rho) = 1}$.
  since $\chi_\rho = a_1 \chi_1 + \cdots + a_k \chi_k$
- Since $|G| < \infty$, the number of irred. characters are finite. Call them $\chi_1, \cdots, \chi_h$. Will show that
  h = # of conjugacy classes of G. ($\leq$ is clear)

EXAMPLES

$G = S_3$: 3 conj classes $[1], [(12)], [(123)]$

1-dim'l repr'ns

trivial repr $\longrightarrow (1,1,1) = \chi_U$

Alternating repr $\longrightarrow (1,-1,1) = \chi_{U'}$

Higher dim'l repr'ns

Permutation repr $\longrightarrow (3,1,0) = \chi$

S_3 acts on $\{e_1, e_2, e_3\} = \mathbb{C}^3$

not irreducible since

$$(\chi, \chi) = \frac{3^2 + 3\cdot 1^2 + 2\cdot 0^2}{|S_3|} = \frac{12}{6} = 2 \neq 1$$

Geometrically :

$\mathbb{C}^3 = U \oplus V$, V 2-dimension.

U is the invariant line : $e_1 + e_2 + e_3$

$$\Rightarrow \chi_V = \chi_{\mathbb{C}^3} - \chi_U = (3,1,0) - (1,1,1)$$
$$= (2, 0, -1)$$

Character Table :

	S_3	1 1	3 (12)	2 (123)
trivial	U	1	1	1
alternating	U'	1	-1	1
standard	V	2	0	-1

That's all since $h \le 3$ or $1^2+1^2+2^2=6$

see next page.

It is easy to decompose $V^{\otimes n}$:

Solve $\chi_{V^{\otimes n}} = a\chi_U + b\chi_{U'} + c\chi_V$

DECOMPOSITION OF THE REGULAR REPRESENTATION

Recall $\rho_{reg} : G \longrightarrow \mathbb{C}^{|G|} = \bigoplus_{g \in G} \mathbb{C} e_g$

such that $\rho(h) e_g = e_{hg}$

Let $r_G = \mathrm{Tr}\, \rho_{reg} : G \longrightarrow \mathbb{C}$

since diagonal of $\rho(h) \equiv 0$ if $h \neq e$. So

$$r_G(1) = |G|$$
$$r_G(h) = 0 \quad h \neq 1$$

Consequences

Every irred. repr'n V_i is contained in ρ_{reg} with multiplicity $= \dim V_i =: n_i$, Hence

$$\boxed{\mathbb{C}[G] = n_1 V_1 + \cdots + n_h V_h}$$

so $\sum_{i=1}^{h} n_i^2 = |G|$ and $\sum_{i=1}^{h} n_i \chi_i(g) = 0$, $g \neq 1$.

pf: $(r_G, \chi_i) = \frac{1}{|G|} \sum_{g \in G} r_G(g) \overline{\chi_i(g)} = \frac{1}{|G|} r_G(1) \overline{\chi_i(1)}$

$$= \overline{\chi_i(1)} = \dim V_i \ . \ \square$$

In fact, $\dim V_i = n_i \mid |G|$. (Harder to prove)

CLASS FUNCTIONS

$f : G \to \mathbb{C}$ is a class function if

$f(h^{-1} g h) = f(g) \quad \forall g, h \in G$

Let $\mathcal{H}$ be the space of class functions

THEOREM: $\chi_1, \dots, \chi_h$ form an orthonormal basis of $\mathcal{H}$. In particular, $h = \#$ of conjugacy classes of G.

EXAMPLE : $G = S_4$

# in conj class		1	6	8	6	3
S_4		1	(12)	(123)	(1234)	(12)(34)
trivial	U	1	1	1	1	1
Alt.	U'	1	−1	1	−1	1
Std.	V	3	1	0	−1	−1
$V \otimes U' =$	V'	3	−1	0	1	−1
unknown	W	2	0	−1	0	2

V: $\chi_{\mathbb{C}^4} = (4, 2, 1, 0, 0)$

$\chi_V = \chi_{\mathbb{C}^4} - \chi_U = (3, 1, 0, -1, -1)$

V': Irred $\otimes$ 1-dim'l must be irred because $\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1} \cdot \chi_{\rho_2}$ and by the irred criterion $(\rho, \rho) = 1$.

W: $|S_4| = 24 = 1^2 + 1^2 + 3^2 + 3^2 + 2^2$

W is in principle determined by the orthogonality relation, but ...

Here is the place to use regular repr'n : (to get a formula not related to it !)

$$\sum_i n_i \chi_i(g) = 0 \quad g \neq 1$$

e.g. For (12)(34), get

$$0 = 1\cdot 1 + 1\cdot 1 + 3\cdot(-1) + 3\cdot(-1) + 2\,\chi_W(g)$$

$$\Rightarrow \chi_W([(12)(34)]) = 2 .$$

CHAPTER IV
ELEMENTARY REPRESENTATION THEORY OF COMPACT LIE GROUPS

LIE GROUPS

A Lie group G is a group such that G is a differentiable manifold and the product map $(g,h) \mapsto gh$:

$$G \times G \longrightarrow G$$

and the inverse map $G \longrightarrow G : g \mapsto g^{-1}$ are all C^∞ mappings

Matrix groups

$GL_n(\mathbb{R})$, $O_n(\mathbb{R})$, $SL_n(\mathbb{R})$, $SO(n)$
$GL_n(\mathbb{C})$, $U(n)$, $SL_n(\mathbb{C})$, $SU(n)$
are all Lie groups

THEOREM (Ado)

Every compact Lie group can be realized as closed subgroup of Matrix groups $O(n)$.

Lie algebra (of matrix groups)

Let $G = O(n)$

$T_e G$ = tangent space of G at $e = I_n$

$A \in T_e G$ is given by a curve

$\alpha : (-1,1) \longrightarrow O(n)$, st $A = \alpha'(0)$, $\alpha(0) = I_n$

ie. $\alpha(t)^t \cdot \alpha(t) = I_n$

$\Rightarrow \alpha'(t)^t\alpha(t) + \alpha(t)^t\alpha'(t) = 0$

$\Rightarrow \quad A^t \cdot I_n + I_n \cdot A^t = 0$

So $\underline{\mathrm{Lie}(G) = \mathcal{G} = \{A \in M_n(\mathbb{R}) \mid A^t + A = 0\}}$

with $[A, B] := AB - BA$

For $SO(n)$, $\det \alpha(t) = 1 \Rightarrow \mathrm{Tr}\, A = 0$

$$\underline{\mathrm{Lie}(SU(n)) = \{A \in M_n(\mathbb{C}) \mid \bar{A}^t + A = 0, \mathrm{Tr}\, A = 0\}}$$

Left G invariant metric on compact group

Let $A, B \in T_e G$

$$\langle A, B \rangle := \mathrm{Tr}(\bar{B}^t A) = \text{Euclidean metric in } \mathbb{C}^{n \times n}$$

G invariance :

$$\langle gA, gB \rangle = \mathrm{Tr}(\bar{B}^t \bar{g}^t g A) = \mathrm{Tr}(\bar{B}^t A) = \langle A, B \rangle$$

$\Rightarrow$ Riemannian metric on G

$\Rightarrow$ Left invariant measure (Haar measure)

$$(**) \quad \int_G \cdot \, dg : C^\infty(G) \longrightarrow \mathbb{C}$$

play the role as in the finite group case average over G.

Representations of G

is a continuous group homomorphism

$$\rho : G \longrightarrow GL(V)$$

V a finite dim'l v.s. $/\mathbb{C}$. may assume in $U(N)$ by using G-inv inner product

$$\langle v, w \rangle' = \int_G \langle gv, gw \rangle \, dg$$

Schur's lemma
Complete reducibility
Characters
Orthogonality relation
irreducibility criterion

$\Rightarrow$ OK, with the same proof

Regular representation ? ∞-dim'l
All irreducible representation ? ∞-many
Formal analogue $\longrightarrow$ Peter-Weyl Thm
as generalized Fourier analysis

Explicit construction
$G = SU(2)$:
V_0 = trivial repr'n on $\mathbb{C}$
V_1 = standard repr'n on $\mathbb{C}^2$
Let $V_n = \mathrm{Sym}^n(V_1) \cong \mathbb{C}^{n+1}$
More explicitly :
V_n = homogeneous polynomials of deg $= n$
in two variables z_1, z_2

$$= \langle z_1^n, z_1^{n-1}z_2, \dots, z_1 z_2^{n-1}, z_2^n \rangle$$

for $P \in \mathbb{C}[z_1, z_2]$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $z = (z_1, z_2)$

$$(gP)(z) := P(z \cdot g) = P(az_1 + cz_2, bz_1 + dz_2)$$

THEOREM: V_n are all irreducible repr'ns.
And every irreducible (unitary) repr'n of $SU(2)$ is isomorphic to one of the V_n.

Pf: V_n is irreducible:

Enough to show that if $A: V_n \longrightarrow V_n$ is $SU(2)$ equivariant, then $A = \lambda I$.

Let $P_k(z_1, z_2) = z_1^k z_2^{n-k}$ $\quad 0 \leq k \leq n$

I. $g_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in SU(2)$, then

$g_a \underline{P_k} = a^{2k-n} \underline{P_k}$, also

$g_a \underline{(AP_k)} = A g_a P_k = A a^{2k-n} P_k = a^{2k-n} \underline{(AP_k)}$

Pick a st a^{2k-n}, $0 \leq k \leq n$ all distinct

then $\Rightarrow$ $AP_k = c_k P_k$ $\quad c_k \in \mathbb{C}$

We need to show that $c_0 = c_1 = \cdots = c_n$

II. $r_t := \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in SU(2)$ $\quad t \in \mathbb{R}$

$$A r_t P_n = A(z_1 \cos t + z_2 \sin t)^n$$

$$\| \quad = \Sigma_k C_k^n \cos^k t \cdot \sin^{n-k} t \cdot A P_k$$

$$\quad = \Sigma_k C_k^n \cos^k t \cdot \sin^{n-k} t \cdot \underline{c_k} P_k$$

$$r_t A P_n = \Sigma_k C_k^n \cos^k t \cdot \sin^{n-k} t \cdot \underline{c_n} P_k$$

$\Rightarrow c_k = c_n$ $\forall k$. done.

CLAIM: χ_n the character of V_n are uniformly dense in the space of class functions on $SU(2)$.

Any element of $SU(2)$ is conjugate to

III. $e(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$

Moreover, $e(s)$ conjugate to $e(t)$

$\Leftrightarrow$ $s \equiv \pm t \pmod{2\pi}$

So if $f : SU(2) \longrightarrow \mathbb{C}$ is a class function

$\Rightarrow f \circ e : \mathbb{R} \longrightarrow \mathbb{C}$

is an even periodic function, period $= 2\pi$

In the same way

C^0 class function $\longleftrightarrow$ even 2π-periodic function

$$\chi_n \circ e(t) = \sum_{k=0}^{n} e^{i(n-2k)t}$$

$$\left(= e^{+nit} + e^{(n-1)it} \cdot e^{-it} + \cdots + e^{-nit} \right)$$

But this implies that

$\chi_0 \circ e(t), \cdots, \chi_n \circ e(t)$ generate the same space

as $1, \cos t, \cos(2t), \cdots \cos(nt)$

which have the density property by

classical Fourier analysis. $\Box$

Now we are almost done:

Let χ be the character of W which

is irreducible and $\ncong V_n \ \forall n$

then $\langle \chi, \chi_n \rangle = 0$

$\langle \chi, \chi \rangle = 1$ $\Rightarrow$ $\times$

since χ_n generate an uniformly dense

subspace. Q.E.D.

EXAMPLE : $G = SO(3)$

There is a double covering map

$$SU(2) = Sp(1) \xrightarrow{\pi} SO(3)$$

$Sp(1) =$ norm $= 1$ elements in $\mathbb{H}$ (quaternions)

Topologically, this is just $S^3 \longrightarrow \mathbb{R}P^3$

via $\pi(g)(v) = g v g^{-1}$

by viewing $\mathbb{R}^3 = \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k \subset \mathbb{H}$

$\ker \pi = \{I, -I\} \subset SU(2)$, So

There is a 1—1 correspondences

$$\begin{pmatrix} \text{irred. } SO(3) \\ \text{repr'ns} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{irred. } SU(2) \\ \text{repr'ns st.} \\ -I \text{ acts trivially} \end{pmatrix}$$

So get $W_n \longleftrightarrow V_{2n}$ $\quad \dim W_n = 2n+1$

Geometrical realization of W_n :

Spherical Harmonics

$P_\ell :=$ cpx v.s. of homogeneous polynomials in x_1, x_2, x_3 (functions on $\mathbb{R}^3$)

$\Delta =$ Laplace on $\mathbb{R}^3$

$$= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

$H_\ell := \{ f \in P_\ell \mid \Delta f = 0 \}$ harmonic poly of degree $= \ell$

$H_\ell \restriction_{S^2}$ called spherical harmonics.

Since Δ commutes with $SO(3)$ action on $\mathbb{R}^3 \Rightarrow H_\ell \subset P_\ell$ is $SO(3)$ invariant

THEOREM : $H_\ell \cong W_\ell$

Notice first that the dimension is correct: $\dim H_\ell = 2\ell+1$ since if

$$f = \sum_{k=0}^{\ell} \frac{x_1^k}{k!} f_k(x_2, x_3)$$

$$\Delta f = 0 \Leftrightarrow f_{k+2} = -\left(\frac{\partial^2 f_k}{\partial x_2^2} + \frac{\partial^2 f_k}{\partial x_3^2}\right) \text{ homog. of deg } \ell-k$$

so $f \in H_\ell$ is uniquely determined by f_0, f_1 (deg ℓ, deg $\ell-1$)

so $\dim H_\ell = (\ell+1) + \ell = 2\ell+1$.

The rest is a character computation using the map $\pi : SU(2) \longrightarrow SO(3)$. $\square$

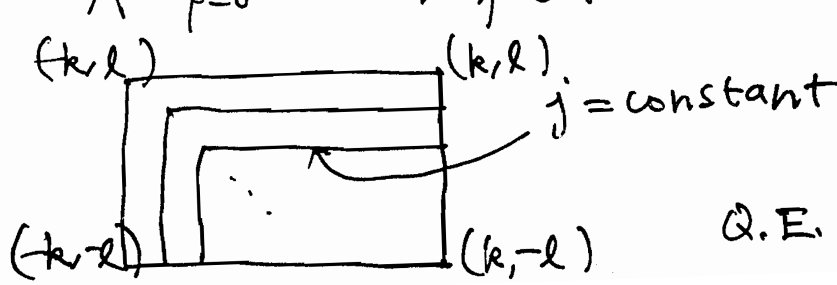
Clebsch–Gordan Formula : In $SU(2)$ case

$$V_k \otimes V_\ell = \bigoplus_{j=0}^{q} V_{k+\ell-2j} \quad q = \min(k, \ell)$$

Pf : Only need to check it for characters since that only depends on $e(t)$. so reduce to :

$$\left(\sum_{\alpha=0}^{k} x^{k-2\alpha}\right)\left(\sum_{\beta=0}^{\ell} x^{\ell-2\beta}\right) = \sum_{j=0}^{q} \sum_{i=0}^{k+\ell-2j} x^{k+\ell-2j-2i}$$

then use
(let $\ell \le k$)



Q.E.D.

CHAPTER V
SEMI-SIMPLE LIE ALGEBRAS

Lie Algebra

A Lie algebra is a vector space $\mathfrak{g}$ with a bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ st.

$$[X,Y] = -[Y,X] \quad \text{skew-symmetric}$$

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

(Jacobi identity)

Examples

I. Abelian Lie algebra : $[X,Y] = 0 \quad \forall X, Y$

II. $\mathfrak{g} = \mathbb{R}^3$ with cross product

$[v, w] := v \times w$

(why the Jacobi id is true ?)

III. Matrix algebra : $\mathfrak{g} = M_n(\mathbb{R}) = \mathrm{End}(\mathbb{R}^n)$

$[A, B] = AB - BA$

or use $\mathbb{C}$

Homomorphism

A Lie algebra homomorphism $\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a linear mapping of vector spaces st

$$\varphi([X,Y]) = [\varphi(X), \varphi(Y)]$$

Representation

is simply a Lie algebra homomorphism to a matrix algebra of some vector space

$$\varphi : \mathfrak{g} \longrightarrow \mathrm{End}(V)$$

Relations between Lie groups and Lie algebras

Let $\rho : G \longrightarrow H$ be a homomorphism of Lie groups. (For simplicity, consider G, H as subgroups of matrix groups)

then the differential of ρ induces

$$\mathfrak{g} = T_e G \xrightarrow{\rho_*} T_e H = \mathfrak{h}$$

which is in fact a Lie algebra homomorphism

$$[X, Y] = [\rho_* X, \rho_* Y] \qquad \text{why ??}$$

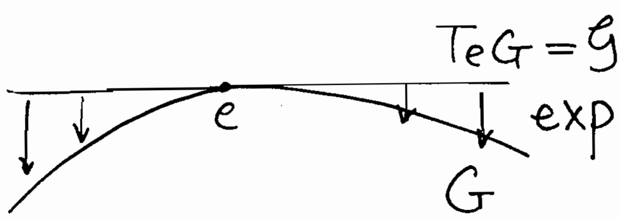
Moreover, if G is simply connected and connected, then there is a $1-1$ correspondence of

$$\begin{pmatrix} \text{Lie Alg.} \\ \text{Homomorphism} \\ \mathfrak{g} \longrightarrow \mathfrak{h} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{Lie Group} \\ \text{Homomorphism} \\ G \longrightarrow H \end{pmatrix}$$

The proof uses the so called Campbell-Hausdorff formula
(The proof is NOT easy?)

$$e^{[X,Y]} = \lim_{n \to \infty} \left(e^{\frac{X}{n}} e^{\frac{Y}{n}} e^{\frac{-X}{n}} e^{\frac{-Y}{n}} \right)^{n^2}$$

Where e is the exponential map : $T_e G \longrightarrow G$



$$e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

$$e^{x+y} = e^x \cdot e^y$$

Example: $G = O(n) \to S \ : \ S^t S = I_n$

$\mathfrak{g} = o(n) \to A \ : \ A^t + A = 0$

If $A^t = -A$, then $(e^A)^t e^A = e^{A^t} e^A = \underline{e^{-A} e^A} = I_n$

Consequences of the 1–1 corresp.

I. Lie subgroups are 1–1 correspondant to Lie subalgebras.

II. the case that $\underline{\rho : G \longrightarrow H = GL(V)}$ shows that Lie group representations are 1–1 corresp. to Lie algebra representations $\underline{\rho_* : \mathfrak{g} \longrightarrow \mathfrak{h} = End(V)}$

This gives the basic reason to consider the (easier, linealized) Lie algebra.

MOST IMPORTANT: Adjoint representation

$Ad : G \longrightarrow GL(\mathfrak{g})$ by $\quad g^t \ \ so(n) \ \ trA = 0 \ . \ A^t = -A$

$Ad(g)A := g A g^{-1} \quad (gAg^{-1})^t = g^{-t} A^t g^t$

$ad : \mathfrak{g} \longrightarrow End(\mathfrak{g})$ is given by taking differential

Let $g(t) : (-1,1) \longrightarrow G$, $g(0) = I_n$, $g'(0) = X$

$$\left(g(t) A g(t)^{-1} \right)'_{t=0} = g'(0) A g^{-1}(0) \ g(0) A \underline{g^{-1}(0) g'(0) g^{-1}(0)}$$

$$= XA - AX = [X, A] \qquad (g(t)^{-1})'$$

ie. $\underline{ad(X)Y = XY - YX = [X, Y]}$

Can check directly ad is a Lie alg. homomorphism

ie. $\quad ad([X_1, X_2]) = [ad X_1, ad X_2]$ Jacobi identity

Representations of $sl_2(\mathbb{C})$

$$sl_2(\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{C}) \;\middle|\; a+d=0 \right\}$$

is 3-dimensional, has a basis

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

with product table

$$\boxed{[H,X] = 2X, \quad [H,Y] = -2Y, \quad [X,Y] = H}$$

Let V be a finite dimensional irreducible representation of $sl_2(\mathbb{C})$

$ad(H)X = 2X$
$ad(H)Y = -2Y$

- Consider H acts on $\underline{V = \oplus V_\alpha}$, $\alpha \in \mathbb{C}$

$$v \in V_\alpha \Leftrightarrow H(v) = \alpha \cdot v$$

- Look at how X, Y acts on V_α. Since H has the power that can dig out the location "α":

Fundamental Calculation, $v \in V_\alpha$

$$\boxed{\begin{array}{l} H\big(X(v)\big) = (\alpha+2)\big(X(v)\big) \\ \text{ie. } X : V_\alpha \longrightarrow V_{\alpha+2} \end{array}}$$

pf: $HX(v) = [H,X](v) + XH(v)$
$= 2X(v) + X(\alpha v) = (\alpha+2)X(v)$.

Similarly $\boxed{Y : V_\alpha \longrightarrow V_{\alpha-2}}$

V irreducible $\Rightarrow$ all $\alpha \in \mathbb{C}$ occur are different by $2k$, $k \in \mathbb{Z}$

$$\rightleftarrows V_{n-6} \underset{Y}{\overset{X}{\rightleftarrows}} V_{n-4} \underset{Y}{\overset{X}{\rightleftarrows}} V_{n-2} \underset{Y}{\overset{X}{\rightleftarrows}} V_n \quad \leftarrow n \in \mathbb{C}$$

$$\circlearrowleft_H \qquad \circlearrowleft_H \qquad \circlearrowleft_H \qquad \circlearrowleft_H$$

Claim: Let $v \in V_n$, then $\langle v, Y(v), Y^2(v), \cdots \rangle = V$.

since V is irred. only need to check that $\{Y^k(v)\}_{k \geq 0}$ is stablized by $sl_2(\mathbb{C})$

stablized by $Y \longrightarrow$ trivial

stablized by $H \longrightarrow$ trivial

stablized by X: First of all, $Xv = 0$ $\quad \curvearrowright X$

$V_{n-2} \quad V_n \circlearrowleft$
$\quad\quad v \in V$

$$X\,Y(v) = [X,Y]v + Y\cancel{Xv}$$
$$= Hv = nv$$
$$X\,Y^2(v) = [X,Y]\,\underline{Yv} + YX(\underline{Yv})$$
$$= H(Yv) + Y(nv)$$
$$= (n-2)\,Yv + n\,Yv$$

$\Rightarrow X\,Y^m(v) = [\,n + (n-2) + (n-4) + \cdots + (n-2m+2)\,]\,Y^{m-1}(v)$

ie. $$\boxed{X\,Y^m(v) = m(n-m+1)\,Y^{m-1}(v)}$$ OK.

Moreover, pick smallest m st $Y^m(v) = 0$

then $\Rightarrow$ $n - m + 1 = 0$ ie. $\underline{m = n+1}$ so $n \in \mathbb{N}$

Consequences $\quad V_{-n} \; V_{-n+2} \; \cdots \; V_{n-2} \; V_n$

I. $\dim V_\alpha = 1$, $\forall \alpha$, $\alpha = -n, -n+2, \cdots, n-2, n$

II. $\dim V = n+1$, Call it $V^{(n)}$.

Construction $\quad\quad H = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Let V = standard repr'n $= \mathbb{C}x \oplus \mathbb{C}y = V_{+1} \oplus V_{-1}$

$V^{(n)} := \mathrm{Sym}^n(V)$

H acts on it as derivations, hence

$H(x^{n-k}y^k) = \cdots = (n-2k)\,x^{n-k}y^k$. done.

Representations of $sl_3(\mathbb{C})$, or even more...

How to generalize H, X, Y pair in $sl_2(\mathbb{C})$?

Root system in adjoint repr'n

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_\alpha \mathfrak{g}_\alpha \right)$$

Root.

maximal abelian sub Alg. (Cartan sub. alg.)

2-dim'l

common eigenspace decomposition of $\mathfrak{h}$ action

6-dim'l

$\{\alpha \in \mathfrak{h}^* \ldots\} = R$

Root system

For $H \in \mathfrak{h}$, $Y \in \mathfrak{g}_\alpha$

$[H, Y] = \operatorname{ad} H\,(Y) = \alpha(H) Y$

For a finite dim'l repr'n of $sl_3(\mathbb{C})$, V

$$V = \bigoplus_\alpha V_\alpha$$

weight decomposition wrt $\mathfrak{h}$ action

Fundamental calculations

For root system: $\operatorname{ad}(\mathfrak{g}_\alpha) : \mathfrak{g}_\beta \longrightarrow \mathfrak{g}_{\alpha+\beta}$

For weight system:

$$X \in \mathfrak{g}_\alpha : V_\beta \longrightarrow V_{\alpha+\beta}$$

pf: $H X(v) = [H, X](v) + X H(v)$

$= \alpha(H) X(v) + X \beta(H)(v)$

$= (\alpha+\beta)(H) \cdot X(v)$.

Root decomposition: $R = R_+ \cup R_-$

Highest weight vector: $v \in V_\alpha$ if V irred.

$$V = \bigoplus_{\beta \in R_-} \mathfrak{g}_\beta V_\alpha$$

In $\mathfrak{sl}_2(\mathbb{C})$ case

H spans the maximal abelian subalgebra

in fact, the eigen-decomposition of

$ad(H)$ on $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ gives

$$\mathfrak{g} = \underset{\substack{\uparrow \\ H}}{\mathfrak{h}} \oplus \big(\underset{\substack{\uparrow \\ X}}{\mathfrak{g}_2} \oplus \underset{\substack{\uparrow \\ Y}}{\mathfrak{g}_{-2}} \big)$$

because $[H,H]=0$, $[H,X]=2X$, $[H,Y]=-2Y$

In $\mathfrak{sl}_3(\mathbb{C})$ case

$$\mathfrak{g} = \underset{\substack{\uparrow \\ \text{2-dim}}}{\mathfrak{h}} \oplus \Big(\underset{\substack{\uparrow \\ \text{6-dim}}}{\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha} \Big)$$

$$\mathfrak{h} = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \,\middle|\, a_1 + a_2 + a_3 = 0 \right\}$$

so $\mathfrak{h}^* = \mathbb{C}\{L_1, L_2, L_3\}/(L_1 + L_2 + L_3 = 0)$

where L_i is the linear functional $\begin{bmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{bmatrix} \mapsto a_i$

In drawing pictures, may identify

$\mathfrak{h}^*$ with $\mathfrak{h}$ via inner product

$$\boxed{\begin{array}{l} R = \{ L_i - L_j \mid i \neq j \} \subset \mathfrak{h}^* \\ \mathfrak{g}_{L_i - L_j} = \mathbb{C} E_{ij} : \text{ 1-dim'l} \times 6 \end{array}}$$

$$E_{ij} \stackrel{i}{=} \begin{array}{c} j \\ \begin{bmatrix} \cdots & 1 & \cdots \end{bmatrix} \end{array}$$

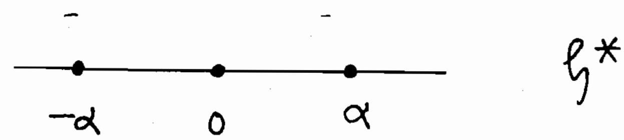
only place $\neq 0$ is the $(i,j) = 1$

pf: Clearly $L_i - L_j$ has $Tr = 0$.

Now Let D be a diagonal matrix

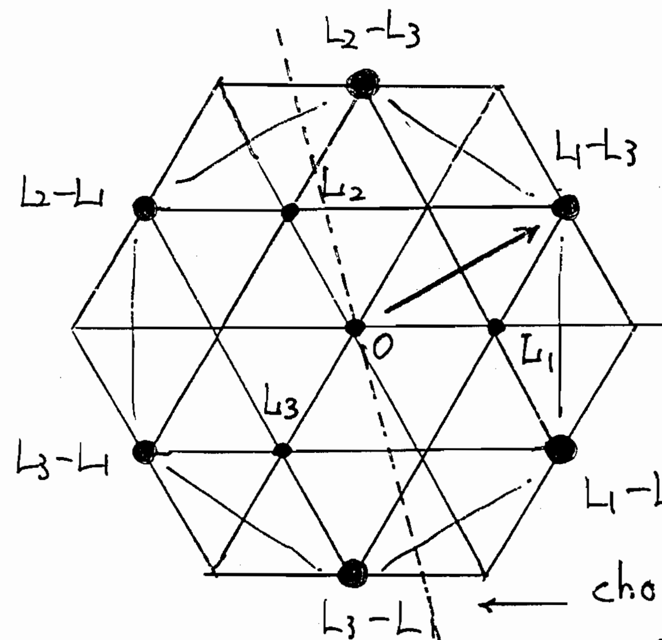
then $[D, M] = \lambda M \Rightarrow$ only one entry of $M \neq 0$.

$sl_2(\mathbb{C})$:



with $\alpha(H)=2$; $\mathfrak{g}_\alpha = \mathbb{C}X$, $\mathfrak{g}_{-\alpha} = \mathbb{C}Y$

$sl_3(\mathbb{C})$: Root system in $\mathfrak{h}^* = R$



Imagine as a plane in $\mathbb{R}^3$ given by $x+y+z=0$

not L_i we want L_i-L_j $i\neq j$

action direction of E_{13}

choose a postive direction

And the corresponding eigenspace $\mathfrak{g}_{L_i-L_j} = \mathbb{C}E_{ij}$

Let V be a representation of $sl_3(\mathbb{C})$, $V=\bigoplus_\alpha V_\alpha$

Highest weight vector

Fix a direction so that $R=R_+ \cup R_-$ as above

so for $i<j$: E_{ij} generate positive space
E_{ji} generate negative space

Let $H_{i,j} = [E_{ij}, E_{ji}] = E_{ii} - E_{jj} = \begin{bmatrix} \ddots & & 0 \\ & 1 & \\ & & -1 \\ 0 & & & \ddots \end{bmatrix}$

then there is a

$sl_2(\mathbb{C})$ piece inside $sl_3(\mathbb{C})$:

$$[\,H_{ij}\ \ E_{ij}\ \ E_{ji}\,] \leftrightarrow [H, X, Y]$$

A vector $v\in V_\alpha$ is the highest weight vector if v is killed by E_{12}, E_{13}, E_{23}.

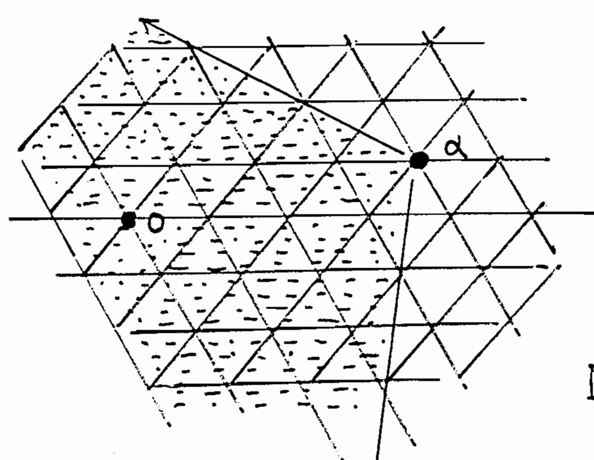
Notice that we have calculated

$$* \quad \boxed{\begin{array}{l} \operatorname{ad} \mathfrak{g}_\alpha : \mathfrak{g}_\beta \longrightarrow \mathfrak{g}_{\beta+\alpha} \\ \mathfrak{g}_\alpha : V_\beta \longrightarrow V_{\beta+\alpha} \end{array}}$$

THEOREM: Let v be a highest weight vector, then the subrepresentation of V generated by succesive applications of the three (negative) operators E_{23}, E_{31}, E_{32} is irreducible. Moreover, Any irreducible representation has a unique (up to scale) highest weight vector.

pf: Similar to the $sl_2(\mathbb{C})$ case via $*$

$V = \oplus V_\alpha$ weight lattice



α = highest weight vector

$\square$

Moreover, using the subalgebra $sl_2(\mathbb{C})$ in various different places. one may also determin the shape of the weight lattice.

Apply to the case $V = \mathfrak{g}$ itself. may even determin the Lie algebra structure of $sl_3(\mathbb{C})$.

Root System for simple Lie algebras

R is a <u>root system</u> if

(1) R is a finite set spans E

(2) $\alpha \in R \Rightarrow -\alpha \in R$ but $k\alpha \notin R$ if $k \neq \pm 1$

(3) $\alpha \in R \Rightarrow$ the reflection W_α in the hyperplane $\alpha^\perp$ maps R into R Weyl group

(4) for $\alpha, \beta \in R$, $n_{\beta\alpha} := \frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

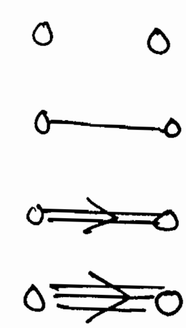
$\Rightarrow$ the <u>only possible angle</u>

$\theta = \pi/2$ no lines

$\theta = 2\pi/3$ one lines

$\theta = 3\pi/4$ two lines

$\theta = 5\pi/6$ 3 lines



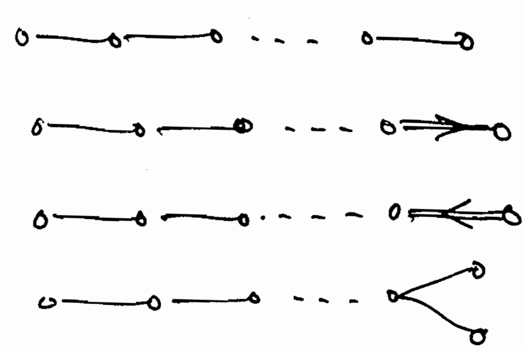
Dynkin Diagram : For positive root, assign o

A_n $n \geq 1$ $sl_{n+1}(\mathbb{C})$

B_n $n \geq 2$ $so_{2n+1}(\mathbb{C})$

C_n $n \geq 3$ $sp_{2n}(\mathbb{C})$

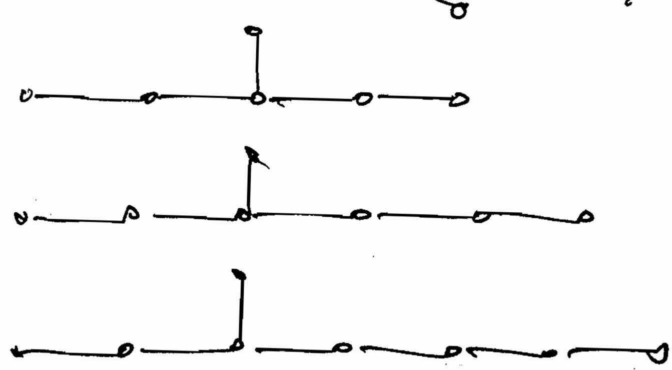
D_n $n \geq 4$ $so_{2n}(\mathbb{C})$



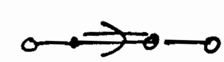
E_6

E_7

E_8



F_4



G_2