lect 1.

QFT choice of M^d, g $<$ Euclidean / Minkowski

· Fields:

Gauge th.
$$\begin{tikzcd} V \arrow[d] \\ M \arrow[u, dashed, bend left, "x"] \end{tikzcd}$$
v.b. conn. ⟷ gauge field

Sections ⟷ matter field

σ-model. $M \xrightarrow{x} N$ map ⟷ field

Path integral := integration over "space of fields"

Q-gravity. int. over metrics g on M as well.

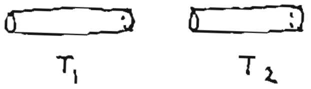
$\int \mathcal{D}x \, e^{-S(x)}$ or $e^{iS(x)}$ Action S : functional on fields

Operator formalism

$\partial M^{d \geq 1} = \coprod_i B_i$ $\mathcal{H}_i$ = Hilbert space of ∂-values (field configurations) on B_i

path int: $\bigotimes_i \mathcal{H}_i \longrightarrow \mathbb{C}$

Ex. $M = M_1 \times I$, $I = [0, T]$ $U(T) : \mathcal{H} \longrightarrow \mathcal{H}^* \simeq \mathcal{H}$



$U(T_2)\,U(T_1) = U(T_2 + T_1)$

$\Rightarrow U(T) = e^{-TH}$

"QFT exists only for $d \leq 6$" Almost rigorous up to $d \leq 1$.

Mirror sym : $d = 2$.

QFT in $d=0$

$x : M = pt \longrightarrow \mathbb{R}$ is just a "variable"

$$Z = \int dx\, e^{-S(x)} \underset{\text{example}}{=\!=} \int dx\, e^{-\left(\frac{\alpha}{2}x^2 + i\varepsilon x^3\right)} =: Z(\alpha,\varepsilon)$$

$Z(\alpha,0) =$ Gaussian $= \sqrt{\frac{2\pi}{\alpha}}$

ε small $\underset{=}{}$ $\displaystyle\int dx \sum_{n=0}^{\infty} e^{-\frac{\alpha}{2}x^2} \frac{(-i\varepsilon x^3)^n}{n!}$

Feymann Diagrams

$$f(\alpha,J) := \int e^{-\frac{\alpha}{2}x^2 + Jx} = \int e^{-\frac{\alpha}{2}(x-\frac{J}{\alpha})^2 + \frac{J^2}{2\alpha}} = \sqrt{\frac{2\pi}{\alpha}}\, e^{\frac{J^2}{2\alpha}}$$

$$\frac{\partial^r f}{\partial J^r}\Big|_{J=0} = \int dx\, e^{-\frac{\alpha}{2}x^2} \cdot x^r$$

$$= \left(\frac{1}{\alpha}\right)^{r/2} \cdot \#\text{ of contractions}$$

$$\begin{tikzcd} \sqrt{\frac{2\pi}{\alpha}}\, e^{\frac{J^2}{2\alpha}} \arrow[d, "\partial/\partial J"] \arrow[dd, bend right, dashed, "\text{pair/contracting}"'] \\ \frac{J}{\alpha} e^{\frac{J^2}{2\alpha}} \arrow[d, "\partial/\partial J"] \\ \frac{1}{\alpha} e^{\frac{J^2}{2\alpha}} + \left(\frac{J}{\alpha}\right)^2 e^{\frac{J^2}{2\alpha}} \arrow[d, "\partial/\partial J \text{ at } J=0"] \\ {} \end{tikzcd}$$

$\to 0$

1st correction term $n=2$.

$$\frac{(-i\varepsilon)^2}{2!} \int dx \cdot x^3 \cdot x^3 \cdot e^{-\frac{\alpha}{2}x^2}$$

contr. $3! + 3^2 = 15$

$$= \frac{(-i\varepsilon)^2}{2}\left(\frac{1}{\alpha}\right)^3 \cdot 15$$

each propagator $\longrightarrow$ is weighted by $1/\alpha$.

For higher corr. the graph can be not conn.

HW #1: $Z(\alpha,\varepsilon) = e^{\sum_\Gamma n_\Gamma}$ 3-valent conn. graph.

$$n_\Gamma = \frac{(-3!\,i\varepsilon)^V}{\alpha^E} \cdot \frac{1}{|\mathrm{Aut}\,\Gamma|}.$$

$V = \#$ vertex

$E = \#$ edges

Free energy $F := -\log Z$.

Multiple variables

$S(x^1, \dots, x^N ; M, C) = \frac{1}{2} M_{ij} x^i x^j + C_{ijk} x^i x^j x^k$

pos. def

$Z(M, C=0) = \int dx^1 \cdots dx^N \, e^{-\frac{1}{2} M_{ij} x^i x^j} = \frac{(2\pi)^{N/2}}{\sqrt{\det M}}$

for C small, expansion

weight $-C_{ijk}$

propagator

wt. M^{ij}

(why?)

Super symmetry

Boson ⟷ x^i $\qquad x^i \psi^a = \psi^a x^i$

Fermion ⟷ ψ^a Grassmann variables $\qquad \psi^a \psi^b = -\psi^b \psi^a$

Integration rule : $\int d\psi = 0$; $\int \psi \, d\psi = 1$ (ie. int = "diff")

multi var $\int \psi^1 \cdots \psi^n \, d\psi^1 \cdots d\psi^n = 1$.

$Z = \int \prod_i dx^i \prod_a d\psi^a \, e^{-S(x,\psi)}$

Assume Grass even

expand in ψ^a, only terms with $\psi^1 \dots \psi^n$ have contribution.

Ex. $S(\psi) = \frac{1}{2} M_{ij} \psi^i \psi^j$

$Z = \int \prod_k d\psi^k \, e^{-\frac{1}{2} M_{ij} \psi^i \psi^j} = Pf(M)$.

ie. $Pf(M)^2 = \det M$.

Here $M_{ij} = -M_{ji}$

$\neq 0$ only for even size.

The 1st non-trivial case :

$Z = \int dx \, d\psi^1 d\psi^2 \, e^{-(S_0(x) + \psi^1 \psi^2 S_1(x))}$

$= \int dx \, e^{-S_0(x)} S_1(x)$

Eg. special case $S(x, \psi_1, \psi_2) = \frac{1}{2} h'(x)^2 - h''(x) \psi_1 \psi_2$

infinitesimal SUSY : $\delta x = \varepsilon^1 \psi_1 + \varepsilon^2 \psi_2$, $\delta \psi_1 = \epsilon^2 h'$

$\delta \psi_2 = -\epsilon^1 h'$.

HW #2. Inv. of S & $dx \, \underline{d\psi_1 \, d\psi_2}$.

need Super det ! Berenzerian.

Localization via SUSY

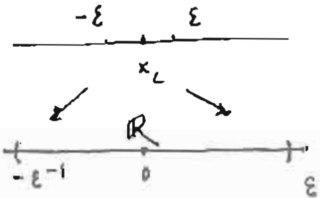
- $h'(x) \neq 0 \ \forall x \Rightarrow Z = 0$.

  idea: Choose SUSY to make ψ_1 disappear in S by making the sym. parameter to be a $\underset{\uparrow \text{ Fermion}}{\text{coor}}$. then clearly $Z = 0$.

- This is not possible if $h' = 0$ at $x = x_c$.

- $h'(x_c) = 0$ for some x_c. $\Rightarrow Z = \sum_{x_c} \frac{h''(x_c)}{|h''(x_c)|} = \{ \pm 1 = \pm \}$ or 0.

  idea: At each x_c, use scaling of x coor. (blow-up)



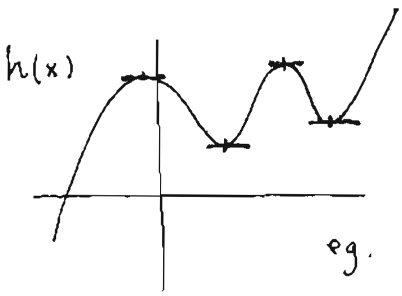
$$h(x) = h(x_c) + \frac{\alpha_c}{2}(x - x_c)^2 + \cdots \qquad \alpha_c = h''(x_c).$$

$$S = \frac{1}{2} h'^2 - h'' \psi_1 \psi_2 = \frac{1}{2}\alpha_c^2 (x - x_c)^2 - \alpha_c \psi_1 \psi_2 + \cdots$$

$$Z = \sum_{x_c} \int_{x_c - \varepsilon}^{x_c + \varepsilon} \underbrace{\frac{dx\, d\psi_1\, d\psi_2}{\sqrt{2\pi}}}_{\text{normalized measure}} e^{-\frac{1}{2}\alpha_c^2 (x - x_c)^2 + \alpha_c \psi_1 \psi_2 + \cdots} + Z_\varepsilon \ \underset{\uparrow\ x - x_c := \varepsilon^2 \tilde{x},\ \varepsilon \to 0}{=} \ \sum_{x_c} \frac{\alpha_c}{|\alpha_c|}.$$

- Explicit Check: $Z = \frac{1}{\sqrt{2\pi}} \int dx\, e^{-\frac{1}{2} h'^2} h''$ let $y = h'(x)$

$$= \deg h' \cdot \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dy\, e^{-\frac{1}{2} y^2} = \underset{\text{count with sign}}{\deg h'}.$$



eg. $\deg h' = 1$.

Lect 2. 3/2, 2011

QFT in d = 1 (quant. mechanics)

$$X : \underset{\overset{\shortparallel}{I,\ \mathbb{R}\ \text{or}\ S^1}}{M} \to \mathbb{R}$$

$$S = \int L\,dt = \int\left(\tfrac{1}{2}\dot x^2 - V(x)\right)dt$$

$$\delta S = \int\Big(\dot x\,\underset{\overset{\shortparallel}{\frac{d}{dt}(\delta x)}}{\delta\dot x} - \tfrac{dV}{dx}\,\delta x\Big)\,dt \underset{\substack{\uparrow\\ \text{suitable}\\ \partial\text{-cond.}}}{=} -\int \underset{\overset{\shortparallel}{0}}{\big(\ddot x + V'(x)\big)}\cdot\delta x\,dt \qquad \text{Euler-Lag. Eq'n}$$

Noether's procedure:

S has translation sym. in $t \mapsto t + \alpha$

variation of parameter $\alpha(t)$

$$x_s = x(t+s\alpha) \Rightarrow \delta x = \tfrac{d}{ds}x_s\Big|_{s=0} = \dot x\,\alpha \Rightarrow (\delta x)^{\cdot} = \ddot x\,\alpha + \dot x\,\dot\alpha$$

$$\Rightarrow \delta S \underset{\substack{\uparrow\\ \text{at } x \text{ solving E-L eq'n}}}{=} \int \dot x\left(-V'(x)\,\alpha + \dot x\,\dot\alpha\right) - V'(x)\,\dot x\,\alpha$$

$$= 2\int dt\,\dot\alpha\left(\tfrac{1}{2}\dot x^2 + V(x)\right)$$

Thus $H := \frac{1}{2}\dot x^2 + V(x)$ is const.

Noether's charge wrt t ≡ Hamiltonian

$$Z(x_2,t_2;x_1,t_1) = \int DX(t)\,e^{iS(x)}$$ say, def via partition of intervals $\to \infty$.

$$\Rightarrow Z_{t_2;t_1} : \mathcal{H} \to \mathcal{H} = L^2(\mathbb{R},\mathbb{C})$$

$$(Z_{t_2;t_1}f)(x_2) = \int_{\mathbb{R}} Z(x_2,t_2;x_1,t_1)\,f(x_1)\,dx_1$$

t-inv $\Rightarrow$ $Z_t = e^{-itH}$

Thm: $H = \frac{1}{2}p^2 + V(x)$ with $p = \frac{\partial L}{\partial \dot x} = \dot x \longmapsto p = -i\frac{d}{dx}$

conj. momentum (classical) (quantum).

$$x \longmapsto x\,\cdot$$

with $[x,p] = xp - px = i$

(classical Poisson $\{x,p\} = 1$)

We check this by example: $L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2$

$$H = \frac{1}{2}(p^2+x^2) = \frac{1}{2}(p+ix)(p-ix) + \frac{1}{2} =: a^\dagger a + \frac{1}{2}$$

on $M = S^1_\beta$. And $t \mapsto -i\tau$ (Wick rotation)

So $Z(\beta) = \int_{\mathbb{R}} Z_\beta(x_1,x_1)\,dx_1 = \mathrm{tr}\, e^{-\beta H}$

eg. eigen fcn expansion

From H: $[a,a^\dagger] = aa^\dagger - a^\dagger a$

$$= \frac{1}{2}(p-ix)(p+ix) - (p+ix)(p-ix) = i(px-xp) = 1$$

$$[H,a] = \underline{a^\dagger a}\,a - \underline{aa^\dagger}\,a = -a \searrow$$

$$[H,a^\dagger] = a^\dagger \underline{a a^\dagger} - a^\dagger \underline{a^\dagger a} = a^\dagger \nearrow$$

Then $H\psi = \lambda\psi \Rightarrow Ha\psi = (aH - a)\psi = (\lambda-1)a\psi$

energy ≥ 0 $\quad Ha^\dagger\psi = (\lambda+1)a^\dagger\psi$

$|0\rangle$ ground state $:= a|0\rangle = 0$, hence $H|0\rangle = \frac{1}{2}|0\rangle$

$\mathcal{H}$ is spanned by $|n\rangle = (a^\dagger)^n|0\rangle$ with $\lambda = E_n = n + \frac{1}{2}$

$$\Rightarrow \mathrm{Tr}\, e^{-\beta H} = \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})} = \frac{e^{-\beta/2}}{1-e^{-\beta}} = \frac{1/2}{\sinh(\beta/2)}.$$

Rmk: $a\psi_0 = 0 \Leftrightarrow \left(-i\frac{d}{dx} - ix\right)\psi_0(x) = 0$ i.e. $\psi_0(x) = A\, e^{-x^2/2}$.

From $Z(\beta) = \int_{x(t+\beta)=x(t)} Dx(t)\; e^{-S_E(x)}$; $S_E(x) = \frac{1}{2}\int dt\,(\dot{x}^2 + x^2)$

"call τ by t" $\quad = \frac{1}{2}\int dt\; x\underbrace{\left(-\frac{d^2}{dt^2}+1\right)}_{= \textcircled{H}}x$

$\textcircled{H} f_n = \lambda_n f_n$; $\lambda_n = 1 + \left(\frac{2\pi n}{\beta}\right)^2$, $n \in \mathbb{Z}$

in the "Fourier coor. system" $\quad x(t) = \sum x_n f_n(t)$

$$Z(\beta) = \int \text{“}\prod_n \frac{dx_n}{\sqrt{2\pi}}\text{”}\; e^{-\frac{1}{2}\sum \lambda_n x_n^2} = \text{“}\prod_n \frac{1}{\sqrt{\lambda_n}}\text{”}$$

$$= \text{“}\prod_{n=1}^{\infty}\left(\frac{2\pi n}{\beta}\right)^{-2}\text{”} \cdot \prod_{n=1}^{\infty}\left(1+\left(\frac{2\pi n}{\beta}\right)^{-2}\right)^{-1}$$

zeta fcn regularization. $\quad \text{“} = \text{”}\ \frac{1}{\beta} \qquad\qquad \text{“} = \text{”}\ \frac{\beta/2}{\sinh \beta/2}$. done.

σ - model on S^1_R. so $X \sim X + R$ $\quad (S^1_\beta \to S^1_R)$

$$S(x) = \int \frac{1}{\alpha} \dot{x}^2 \, dt \quad , \quad H = \frac{1}{2} p^2 = - \frac{1}{\alpha} \frac{d^2}{dX^2}$$

$$\Rightarrow \Psi_n(x) = e^{2\pi i n X / R} \quad , \quad E_n = \frac{2\pi^2 n^2}{R^2}$$

$$Z(\beta) = \operatorname{Tr} e^{-\beta H} = \sum_{n=-\infty}^{\infty} e^{-\frac{2\pi\beta n^2}{R^2}} .$$

path integral : let $X_m(\tau)$ be of winding # $= m$

then $\quad X_m(\tau) = \frac{m\tau R}{\beta} + X_0(\tau)$

$$Z(\beta) = \int DX \, e^{-\int_0^\beta \frac{1}{2} \dot{x}^2 d\tau} = \sum_{m=-\infty}^{\infty} \int DX_m \, e^{-S_E(X_m)}$$

$$= \sum_{m=-\infty}^{\infty} e^{-\frac{m^2 R^2}{2\beta}} \int DX_0 \, e^{-\int_0^\beta X_0 \left(-\frac{1}{\alpha} \frac{d^2}{d\tau^2} \right) X_0 \, d\tau}$$

This leads to Jacobi's $\vartheta(t) = t^{1/2} \vartheta(1/t)$, $\vartheta(t) = \sum_{-\infty}^{\infty} e^{-\pi n^2 t}$.

and a hint to T - duality.

Difficulties for general σ - models :

$$S = \frac{1}{\alpha} \int dt \; \underbrace{g_{ij}(x)}_{} \frac{dx^i}{dt} \frac{dx^j}{dt}$$

the Taylor exp. is not quadratic.

Supersymmetric QM :

$$L = \frac{1}{2} \dot{x}^2 - \frac{1}{2} h'^2 + \frac{i}{2} \left(\bar{\psi} \dot{\psi} - \dot{\bar{\psi}} \psi \right) - h'' \bar{\psi} \psi$$

$\psi = \psi_1 + i \psi_2$

cpx Fermion

let

$$\delta x = \epsilon \bar{\psi} - \bar{\epsilon} \psi$$

$$\delta \psi = \epsilon (i\dot{x} + h')$$

$$\delta \bar{\psi} = \bar{\epsilon} (-i\dot{x} + h')$$

for later use $\epsilon = \epsilon(t)$.

$$\begin{aligned} \Rightarrow \delta L = {} & \underline{\dot{x} (\epsilon \dot{\bar{\psi}}}_{5} - \underline{\bar{\epsilon} \dot{\psi}}_{6}) - h' h'' (\overset{1}{\epsilon \bar{\psi}} - \overset{2}{\bar{\epsilon} \psi})_{-h'+2h'} + \underline{\dot{x} (\dot{\epsilon} \bar{\psi} - \dot{\bar{\epsilon}} \psi)} \\ & + \frac{i}{2} \Big(\bar{\epsilon} (\underset{+}{-} i \dot{x} + h') \dot{\psi}_{6} + \bar{\psi} \dot{\epsilon} (i\dot{x} + h') + \bar{\psi} \epsilon (i\ddot{x} \underset{4}{\pm} h'' \dot{x}) \\ & \quad - \dot{\bar{\epsilon}} (-i\dot{x} + \overset{-h'+2h'}{h'}) \psi - \bar{\epsilon} (-i\ddot{x}_{3} \pm h'' \dot{x}) \psi - \dot{\bar{\psi}} \epsilon (\underset{5}{-i\dot{x}} + h') \Big) \\ & - h''' i (\epsilon \bar{\psi} - \bar{\epsilon} \psi) \bar{\psi} \psi \; \underset{3}{\underline{- h'' \bar{\epsilon} (-i\dot{x}} + \overset{2}{h'}) \psi} - \underset{4}{\underline{h'' \bar{\psi} \epsilon (i\dot{x}}} + \overset{1}{h'}) \end{aligned}$$

$$= \frac{d}{dt} (\cdots) \quad \underset{Q}{\underset{\parallel}{- i \dot{\epsilon} \bar{\psi} (i\dot{x} + h')}} \quad \underset{\bar{Q}}{\underset{\parallel}{- i \dot{\bar{\epsilon}} \psi (- i\dot{x} + h')}}$$

super charges.

check $[\delta_1, \delta_2] = 2i(\epsilon_1\bar{\epsilon}_2 - \epsilon_2\bar{\epsilon}_1)\frac{d}{dt}$ via E-L eq'n

such Fermionic transf. is called SUSY.

Quantization: Conjugate momenta $p = \frac{\partial L}{\partial \dot{x}} = \dot{x}$; $\pi = \frac{\partial L}{\partial \dot{\psi}} = i\bar{\psi}$

$\uparrow$ int. by parts $L = \dots + i\bar{\psi}\dot{\psi}$

$$\text{Poisson } \{,\} \begin{cases} \rightarrow \text{Boson } [\,,\,] \text{ comm} \\ \rightarrow \text{Fermion } \{a,b\} \text{ anti-comm} \\ \qquad\qquad\qquad\qquad = ab + ba \end{cases}$$

It is required that $[x,p] = i$
as "operators." $\{\psi, \pi\} = i \Rightarrow \{\psi, \bar{\psi}\} = 1$ *

$$H = p\dot{x} + \pi\dot{\psi} - L \longmapsto \frac{1}{2}p^2 + \frac{1}{2}h'^2 + \frac{1}{2}h''\underbrace{(\bar{\psi}\psi - \psi\bar{\psi})}_{\text{op. with rel } *}$$

$= [\bar{\psi}, \psi]$

Representation: Hilbert space of states

$$\mathcal{H} = \mathcal{H}^B \oplus \mathcal{H}^F = L^2(\mathbb{R}, \mathbb{C})\,|0\rangle \oplus L^2(\mathbb{R}, \mathbb{C})\,\bar{\psi}\,|0\rangle$$

$|0\rangle$ a vector with $\psi|0\rangle = 0$.

$$x \mapsto x\cdot, \quad p \mapsto -i\frac{d}{dx}, \quad \psi \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \bar{\psi} \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Now $Q \mapsto Q = \bar{\psi}(ip + h')$, $\bar{Q} \mapsto \bar{Q} = \psi(-ip + h')$

- HW: $[H, Q] = 0 = [H, \bar{Q}]$

Also, for $F := \bar{\psi}\psi$; $[F, \psi] = F\psi - \psi F = \bar{\psi}\cancel{\psi}\psi - \psi\bar{\psi}\psi = -\psi$
$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad [F, \bar{\psi}] = \bar{\psi}$ $\quad \psi\bar{\psi} = (1 - \bar{\psi}\psi)$

$\Rightarrow [F, Q] = Q \quad \Rightarrow \quad Q, \bar{Q}$ exchange $\mathcal{H}^B, \mathcal{H}^F$
$[F, \bar{Q}] = -\bar{Q}$ $\quad$ obvious ?

Now, $\{Q, Q\} = 0 = \{\bar{Q}, \bar{Q}\}$, and $\{Q, \bar{Q}\} = 2H$. (Key, check)

$\longrightarrow$ "Hodge theory" $\dim \mathcal{H}^B_{(0)} - \dim \mathcal{H}^F_{(0)} = \operatorname{str} e^{-\beta H}$ $= \operatorname{tr} (-1)^F e^{-\beta H}$

$Q^\dagger = \bar{Q} \Rightarrow \mathcal{H}_{(n)} := \lambda_n$-eigen space of $H \geq 0$

this is Witten index.

$$\Downarrow$$

$$\mathcal{H}^B_{(n)} \xrightarrow[\sim]{Q + \bar{Q}} \mathcal{H}^F_{(n)}$$

if $n \neq 0$ since $(Q + \bar{Q})^2 = 2H$.

Fact: Witten index is inv. under deformations which preserve SUSY.

Lect 3. 3/9

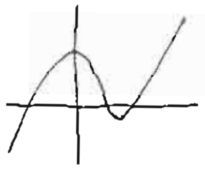
QFT in $d=1$ (conti.)

SUSY ground states : $H\Psi = 0 \Leftrightarrow Q\Psi = 0 = \bar{Q}\Psi$.

$\Psi = f_1(x)|0\rangle + f_2(x)\bar{\psi}|0\rangle \Rightarrow f_1' + h'f_1 = 0 = -f_2' + h'f_2$

ie. $f_1(x) = c_1 e^{-h(x)}$; $f_2(x) = c_2 e^{h(x)}$, Need L^2 sol.

I.



$\mathrm{Tr}(-1)^F = 0$

II

$\Psi = e^{-h(x)}|0\rangle$

$\mathrm{Tr}(-1)^F = 1$

III

$\Psi = e^{h(x)}\bar{\psi}|0\rangle$

$\mathrm{Tr}(-1)^F = -1$

eg. Harmonic oscillator : $h(x) = \frac{\omega}{2}x^2$, $H = \frac{1}{2}p^2 + \frac{\omega^2}{2}x^2 + \frac{\omega}{2}[\bar{\psi},\psi]$

$\Psi_{\omega>0} = e^{-\frac{1}{2}\omega x^2}|0\rangle$, $\Psi_{\omega<0} = e^{-\frac{1}{2}|\omega|x^2}\bar{\psi}|0\rangle$.

Perturbative Analysis for general $h(x)$: $h(x) \mapsto \lambda h(x)$, $\lambda \nearrow$

$$H = \frac{1}{2}p^2 + \frac{\lambda^2}{2}h'^2 + \frac{\lambda}{2}h''[\bar{\psi},\psi]$$

Principle : $\lambda \gg 0$, the lowest energy states concentrat at x_i with $h'(x_i) = 0$.

Assume $h''(x_i) \neq 0$ (Morse condi.)

rescaling $x - x_i = (\tilde{x} - \tilde{x}_i)/\sqrt{\lambda}$

$$h(x) = h(x_i) + \frac{1}{2}h''(x_i)\frac{(\tilde{x}-\tilde{x}_i)^2}{\lambda} + \frac{*}{\lambda^{3/2}} + \cdots$$

$$\Rightarrow H = \lambda\underbrace{\left(\frac{1}{2}\tilde{p}^2 + \frac{1}{2}h''(x_i)^2(\tilde{x}-\tilde{x}_i)^2 + \frac{1}{2}h''(x_i)[\bar{\psi},\psi]\right)}_{=:H_0 \text{ SUSY har. osc. } \omega = h''(x_i)} + \lambda^{1/2}* + * + O(\lambda^{-1/2})$$

in fact, $\Psi_i = e^{-\frac{\lambda}{2}h''(x_i)(x-x_i)^2}|0\rangle + \cdots$ or $e^{-\frac{\lambda}{2}|h''(x_i)|(x-x_i)^2}\bar{\psi}|0\rangle + \cdots$

Sigma model for QFT $d=1$

$\phi : T \to M$ Boson $(x^i(t)) = \phi$

Fermion $\psi, \bar\psi \in \Gamma(T, \phi^* TM \otimes \mathbb{C})$, $\psi = \psi^i \frac{\partial}{\partial x^i}$

$$L = \frac{1}{2} g_{ij} \dot x^i \dot x^j + \frac{i}{2} g_{ij} \left(\bar\psi^i D_t \psi^j - D_t \bar\psi^i \psi^j \right) - \frac{1}{2} R_{ijkl} \psi^i \bar\psi^j \psi^k \bar\psi^l$$

$\nearrow$ covariant diff $= \dot\psi^j + \Gamma^j_{lm} \dot x^l \psi^m$

SUSY : $\delta x^i = \epsilon \bar\psi^i - \bar\epsilon \psi^i$

$\delta \psi^i = \epsilon (i \dot x^i - \Gamma^i_{jk} \bar\psi^j \psi^k)$ HW. ① $\Rightarrow \delta \int L \, dt = 0$

$\delta \bar\psi^i = \bar\epsilon (- i \dot x^i - \Gamma^i_{jk} \bar\psi^j \psi^k)$

$\Rightarrow$ conserved supercharges $Q = i g_{ij} \bar\psi^i \dot x^j$; $\bar Q = - i g_{ij} \psi^i \dot x^j$

phase rotation $\psi^i \mapsto e^{i\gamma} \psi^i$ fixes L $\Rightarrow$ charge $F = g_{ij} \bar\psi^i \psi^j$

Quantization : $p_i = \frac{\partial L}{\partial \dot x^i} = g_{ij} \dot x^j + \Gamma^j_{im} g_{kj} \bar\psi^k \psi^m$; $\pi_i = \frac{\partial L}{\partial \dot\psi^i} \overset{\uparrow}{=} i g_{ij} \bar\psi^j$

so ② $Q = i \bar\psi^i p_i$; $\bar Q = - i \psi^i p_i$ (why?) same reason as $M = \mathbb{R}$ case

canonical relations : $\left\{ \begin{array}{l} [\hat x^i, \hat p_j] = i \delta^i_j \\ \{ \hat\psi^i, \hat\pi_j \} = i \delta^i_j \Leftrightarrow \{ \hat\psi^i, \hat{\bar\psi}^j \} = g^{ij} \end{array} \right.$

where $\mathcal{H} = \Omega(M) \otimes \mathbb{C}$, $(\omega_1, \omega_2) = \int_M \bar\omega_1 \wedge * \omega_2$

$\hat x_i = x_i \cdot$ $\qquad \hat\psi^i = g^{ij} \iota_{\partial/\partial x^j}$ $\searrow$

$\hat p_i = - i \nabla_{\partial/\partial x^i}$ $\qquad \hat{\bar\psi}^i = dx^i \wedge$ $\nearrow$

Now $|0\rangle = 1$, $F = dx^i \wedge \iota_{\partial/\partial x^i} \equiv p$ on $\Omega^p(M)$

$Q = dx^i \wedge \nabla_i \equiv d$ (e.g. use normal coor.)

$\bar Q = g^{ij} \iota_{\partial/\partial x^j} \nabla_i \equiv d^*$ $(:= *d*)$

Hamiltonian $H \underset{③}{=} \frac{1}{2} \{ Q, \bar Q \} = \frac{1}{2} (d d^* + d^* d) = \frac{1}{2} \Delta$.

HW: Show $\underset{\underset{X(M)}{\parallel}}{\operatorname{tr} (-1)^F e^{-\beta H}} = \frac{1}{(2\pi)^{n/2}} \int_M \operatorname{Pf}(-R)$ Gauss-Bonnet-Chern

for cpt M , dim $M = n$. by let $\beta \to 0$.

With potential term (Witten 1982 JDG)

$L = L_0 + \Delta L \ , \quad \Delta L = -\frac{1}{2} g^{ij} \partial_i h \, \partial_j h - \underbrace{\nabla_i(\partial_j h)}_{= \ \partial_i \partial_j h - \Gamma_{ij}^k \partial_k h} \bar{\psi}^i \psi^j$

SUSY: $\delta x^i = \epsilon \bar{\psi}^i - \bar{\epsilon} \psi^i$

$\delta \psi^i = \epsilon (i \dot{x}^i - \Gamma^i_{jk} \bar{\psi}^j \psi^k + g^{ij} \partial_j h)$

$\delta \bar{\psi}^i = \bar{\epsilon} (-i \dot{x}^i - \Gamma^i_{jk} \bar{\psi}^j \psi^k + g^{ij} \partial_j h)$

Super charges: $Q = \bar{\psi}^i (i p_i + \partial_i h) \ , \quad \bar{Q} = \psi^i (-i p_i + \partial_i h)$

Fermion rotation: charge $F = g_{ij} \bar{\psi}^i \psi^j$

Quantization: $Q = d + dh \wedge = e^{-h} d e^{h} =: d_h \ ; \ \bar{Q} = d_h^*$

$H \triangleq \frac{1}{2} \{ Q, \bar{Q} \} = \frac{1}{2} \Delta_h \qquad H^*_Q \simeq H^*_{DR}(M) \ , \ \forall h .$

Now let h be a Morse function, $\mathrm{Crit}(h) = x_1, \cdots, x_N$

rescaling $h \mapsto \lambda h$, then

$2H_\lambda = \Delta_\lambda = \Delta + \lambda^2 |\nabla h|^2 + \lambda \nabla_i \partial_j h [\bar{\psi}^i, \psi^j]$

Perturbation theory at x_i: $h = h(x_i) + \sum c_I (x^I)^2 + \cdots$

c_I: eigen values of h_{ij}

$H_\lambda \sim \frac{1}{2} \sum_{I=1}^{n} \left(p_I^2 + \lambda^2 c_I^2 (x^I)^2 + \lambda c_I [\bar{\psi}^I, \psi^J] \right)$

For $H_\lambda \Psi = 0 \rightsquigarrow \Psi_i^{(0)} = e^{-\lambda \sum |c_I| (x^I)^2} \prod_{c_J < 0} \bar{\psi}^J |0\rangle$

$\Rightarrow \Psi_i \in \Omega^{\mu_i}(M) \otimes \mathbb{C} \ , \quad \mu_i = $ Morse index at x_i

perturbative 0 energy state, not nec. $Q \Psi_i = 0$.

"Theorem (Witten, Floer)"

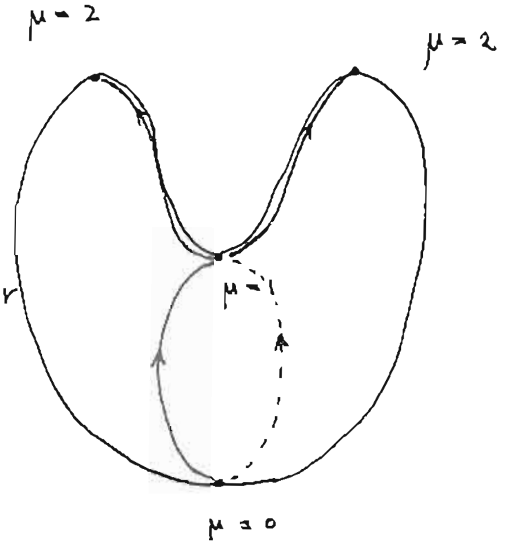
Let $C^\mu = \bigoplus_{\mu_i = \mu} \mathbb{C} \Psi_i$. Then

$0 \to C^0 \xrightarrow{Q} C^1 \xrightarrow{Q} \cdots \xrightarrow{Q} C^n \to 0$

is given by

$$Q \Psi_i = \sum_{\mu_j = \mu_i + 1} \Psi_j \underbrace{\langle \Psi_j , Q \Psi_i \rangle}_{= \ \sum_\gamma n_\gamma e^{-\lambda(h(x_j) - h(x_i))}} + \text{smaller}$$

$n_\gamma = \pm 1$ dep on ori. of $\bar{\Psi}_j \wedge * Q \bar{\Psi}_i$.



path integral argument : (very sketch)

Lemma : $\langle \Psi_j , Q \Psi_i \rangle = \dfrac{1}{h(x_i) - h(x_j) + O(1/\lambda)} \lim_{T \to \infty} \langle \Psi_j , e^{-TH} [Q, h] e^{-TH} \Psi_i \rangle$.

since $[Q, h] = dh \wedge$

$$\parallel$$

$$\int_{\substack{\phi(-\infty) = x_i \\ \phi(\infty) = x_j}} D\phi \, D\psi \, D\bar{\psi} \, e^{-S_E} \, \bar{\psi}^I \partial_I h \Big|_{\tau = 0}$$

$$S_E = \int_{-\infty}^{\infty} d\tau \left(\underline{\tfrac{1}{2} |\dot{\phi}|^2 + \tfrac{\lambda^2}{2} |\nabla h|^2} + \underline{g_{IJ} \bar{\psi}^I D_\tau \psi^J + \lambda (\nabla_I \partial_J h) \bar{\psi}^I \psi^J} + \tfrac{1}{2} R_{IJKL} \psi^I \bar{\psi}^J \psi^K \bar{\psi}^L \right)$$

Boson part $S_B = \int_{-\infty}^{\infty} \frac{1}{2} | \dot{\phi} \pm \lambda \nabla h |^2 \mp \int_{-\infty}^{\infty} \lambda \dot{\phi} \cdot \nabla h \qquad > 0$

$\qquad\qquad\qquad\qquad = \mp \lambda (h(x_j) - h(x_i))$

minimizer =: instanton $\qquad\qquad$ "$\pm$" make this > 0

How many ? $\qquad D_\pm (\delta\phi) := D_\tau (\delta\phi) \pm \lambda H_h (\delta\phi) = 0$

Hessian op : $T_x M \to T_x M$

Notice the fermion bi-linear part

$$S_{\bar{\psi}\psi} = \int_{-\infty}^{\infty} d\tau \langle \bar{\psi}, D_+ \psi \rangle = - \int_{-\infty}^{\infty} d\tau \langle D_- \bar{\psi}, \psi \rangle$$

path-int $\neq 0 \Rightarrow$ $\bar{\psi}$: 0-mode $-$ ψ : 0-mode $\equiv$ index D_- $= 1$

localize to instantons : reason

S_E inv under SUSY ($t \mapsto -i\tau$)

$[Q, h] = \bar{\psi}^I \partial_I h$ inv under δ_ϵ (ie. $\bar{\epsilon} = 0$) gen by Q

ie. $\delta_\epsilon x^I = \epsilon \bar{\psi}^I$; $\delta_\epsilon \psi^I = \epsilon \left(- \frac{dx^I}{d\tau} + \lambda g^{IJ} \partial_J h - \Gamma^I_{JK} \bar{\psi}^J \psi^K \right) \overset{\shortparallel}{0}$

the δ_ϵ fixed loci $\Rightarrow$ $\dot{\phi} = \lambda \nabla h$.

Lemma : index $D_- = \mu_j - \mu_i$. (HW. § 10.5.2)

Choose generic h st $\ker D_+ = \operatorname{coker} D_- = 0$ along any

γ : instanton $x_i \rightsquigarrow x_j$ with $\mu_j - \mu_i = 1$.

then $\ker D_- = 1$ gives only time shift in $\tau \mapsto \tau + \tau_1$.

$\gamma \mapsto \gamma_{\tau_1}$

Now calculate path-int. using "mode exp".

$\bar{\psi}_0$: zero mode gives $\quad e^{-\lambda(h(x_j) - h(x_i))} \int_{-\infty}^{\infty} d\tau_1 \int d\bar{\psi}_0 \, \bar{\psi}^I \partial_I h \Big|_{\tau = 0}$

coming from $\bar{\psi}_0$ transl.

$$\parallel$$

$$\int_{-\infty}^{\infty} d\tau_1 \frac{d \, h(\gamma(\tau_1))}{d\tau_1} = h(x_j) - h(x_i) .$$

lect 4. 3/16, 2011

QFT in 1+1 dim. free theory.

$$\Sigma = \underset{t}{\mathbb{R}} \times \underset{s}{S^1} \longrightarrow \mathbb{R} = M \qquad S = \frac{1}{2\pi}\int_\Sigma \underset{\overset{\shortparallel}{L}}{\frac{1}{2}\left[\left(\frac{\partial x}{\partial t}\right)^2 - \left(\frac{\partial x}{\partial s}\right)^2\right]} dt\,ds$$

$$s \sim s + 2\pi \qquad x(t,s)$$

$$\delta S = \frac{1}{2\pi}\int_\Sigma \left[\frac{\partial x}{\partial t}\frac{\partial}{\partial t}(\delta x) - \frac{\partial x}{\partial s}\frac{\partial}{\partial s}(\delta x)\right] dt\,ds = \frac{-1}{2\pi}\int_\Sigma \delta x\left(\frac{\partial^2 x}{\partial t^2} - \frac{\partial^2 x}{\partial s^2}\right) dt\,ds$$

EL eq'n: $(\partial_t^2 - \partial_s^2)\, x = 0 \Rightarrow x(t,s) = f(t-s) + g(t+s)$

why? (change variables) left wave etc.

Noether charges:

at eq'n of motion, it is also a conserv. eq'n $\partial_\mu j^\mu = 0$ i.e. div j

where $j^t = \partial_t x$, $j^s = -\partial_s x$ (currents)

$\Rightarrow p = \frac{1}{2\pi}\int_{S^1} j^t\, ds$ = const in t.

This is from shift in x: $\delta x = \alpha(t,s)$; so p = target space momentum

(t,s) tras. sym $\frac{d}{d\epsilon} x(t+\epsilon c^t, s+\epsilon c^s)\big|_{\epsilon=0} = x_\mu c^\mu =: \delta_c x$

(*) $\Rightarrow \delta S =: \frac{1}{2\pi}\int_\Sigma T^\nu_\mu\, \partial_\nu c^\mu = 0 \quad \forall\, c = (c^\mu) \Leftrightarrow \partial_\nu T^\nu_\mu = 0$

get conserv. charges by $\int_{S^1}$:

$H = \frac{1}{2\pi}\int_{S^1} T^t_t\, ds = \frac{1}{2\pi}\int_{S^1}\frac{1}{2}(x_t^2 + x_s^2)\, ds$ Hamiltonian

$P = \frac{1}{2\pi}\int_{S^1} T^t_s\, ds = \frac{1}{2\pi}\int_{S^1} x_t x_s\, ds$ Momentum (world sheet)

How to Quantize?

Idea: treat string S^1 as ∞ many deg of freedom: Fourier series

$$x(t,s) = x_0(t) + \sum_{n\neq 0} x_n(t)\, e^{ins}, \quad x_{-n} = \bar{x}_n$$

$$S = \int dt\left[\frac{1}{2}\dot{x}_0^2 + \sum_{n=1}^{\infty}\left(|\dot{x}_n|^2 - n^2|x_n|^2\right)\right] \quad \text{by Parseval}$$

Sector x_0: $p_0 = \dot{x}_0$ let $p_0|k\rangle_0 = k|k\rangle_0$ notice $p_0 \equiv p$

$H_0 = \frac{1}{2}p_0^2$ $|k\rangle_0$ has energy $\frac{1}{2}k^2$.

$\left(p_0 = -i\frac{d}{dx_0},\ |k\rangle_0 = e^{ikx_0},\ L^2 \text{ condi} \underset{?}{\Rightarrow} M = S^1_R\right)$

$$(*) : \delta_C S = \frac{1}{2\pi}\int_\Sigma x_t \frac{\partial}{\partial t}(x_\mu c^\mu) - x_s \frac{\partial}{\partial s}(x_\mu c^\mu)$$

$$\underline{x_t\, x_{tt}\, c^t} \quad \underline{+ x_t x_{ts}\, c^s} \quad \underline{- x_s\, x_{st}\, c^t} \quad \underline{- x_s\, x_{ss}\, c^s}$$

$$x_t\, x_t\, c^t_t + x_t x_s\, c^s_t - x_s x_t\, c^t_s - x_s\, x_s\, c^s_s$$

$$\int -\frac{1}{2}(x_t^2 - x_s^2)\, c^t_t \qquad \int \frac{d}{ds}\, \frac{1}{2}(x_t^2 - x_s^2)\, c^s$$

$$\int \frac{d}{dt}\, \frac{1}{2}(x_t^2 - x_s^2)\cdot c^t \qquad = \int -\frac{1}{2}(x_t^2 - x_s^2)\, c^s_s$$

$$= \frac{1}{2\pi}\int_\Sigma \underbrace{\frac{1}{2}(x_t^2 + x_s^2)}_{T^t{}_t} c^t_t + \underbrace{x_t x_s}_{T^t{}_s} c^s_t - \underbrace{x_s x_t}_{T^s{}_t} c^t_s - \underbrace{\frac{1}{2}(x_t^2 + x_s^2)}_{T^s{}_s} c^s_s$$

$n \geq 1$,

Sector x_n: $L_n = \left(\frac{1}{2}\dot{x}_{1n}^2 - \frac{n^2}{2}x_{1n}^2\right) + \left(\frac{1}{2}\dot{x}_{2n}^2 - \frac{n^2}{2}x_{2n}^2\right)$ $\qquad x_n := \frac{1}{\sqrt{2}}(x_{1n} + i x_{2n})$

2 harmonic oscillators $\qquad p_{in} = \dot{x}_{in} \quad (i = 1, 2)$

$$H_n = \left(\frac{1}{2}p_{1n}^2 + \frac{n^2}{2}x_{1n}^2\right) + \left(\frac{1}{2}p_{2n}^2 + \frac{n^2}{2}x_{2n}^2\right)$$

$$= n\left(a^\dagger_{1n} a_{1n} + \frac{1}{2}\right) + n\left(a^\dagger_{2n} a_{2n} + \frac{1}{2}\right)$$

$a^\dagger_{in} := \frac{1}{\sqrt{2}}\left(\frac{p_{in}}{\sqrt{n}} + i\sqrt{n}\, x_{in}\right)$

can. relation $[a_{in}, a^\dagger_{jn}] = \delta_{ij}$

others $= 0$

get creation / annihilation operators.

Equiv. we use the cpx coor. form:

$$\alpha_n := \sqrt{\frac{n}{2}}(a_{1n} + i a_{2n}) = \frac{\sqrt{n}}{2}\left[\frac{p_{1n} + i p_{2n}}{\sqrt{n}} - i\sqrt{n}(x_{1n} + i x_{2n})\right]$$

$$= \frac{1}{2}(\underline{p_n - i n\cdot x_n}) \quad ; \quad \text{so } \alpha_{-n} = \frac{1}{2}(\bar{p}_n + i n \bar{x}_n) = \sqrt{\frac{n}{2}}(a^\dagger_{1n} - i a^\dagger_{2n}) = \alpha^\dagger_n$$

$$\tilde{\alpha}_n := \sqrt{\frac{n}{2}}(a_{1n} - i a_{2n}) \quad ; \quad \tilde{\alpha}_{-n} = \tilde{\alpha}^\dagger_n = \sqrt{\frac{n}{2}}(a^\dagger_{1n} + i a^\dagger_{2n}) = \frac{1}{2}(\underline{p_n + i n x_n})$$

Now the can. rel. is: $[\alpha_n, \alpha_{-n}] = n = [\tilde{\alpha}_n, \tilde{\alpha}_{-n}]$, others $= 0$

$$H_n = \alpha^\dagger_n \alpha_n + \tilde{\alpha}^\dagger_n \tilde{\alpha}_n + n$$

$|0\rangle_n$ = vector killed by $\alpha_n, \tilde{\alpha}_n$ $\Rightarrow$ $H_n |0\rangle_n = n|0\rangle_n$

- Combine all $n \geq 0$: $\mathcal{H} = \bigotimes_{n \geq 0} \mathcal{H}_n$, $|k\rangle := |k\rangle_0 \otimes \bigotimes_{n\geq 1} |0\rangle_n$

$$H = \sum_{n \geq 0} H_n = \frac{1}{2}p_0^2 + \sum_{n\geq 1}(\alpha_{-n}\alpha_n + \tilde{\alpha}_{-n}\tilde{\alpha}_n) + \sum_{n \geq 1} n$$

$\sum_{n\geq 1} n$ "=" $\zeta(-1) = \frac{-1}{12}$: energy of $|0\rangle$.

general states are gen by $\alpha_{-n}, \tilde{\alpha}_{-n}$ on $|k\rangle$.

Since $[H, x_0] f = -\frac{1}{2}(x_0 f)'' + \frac{1}{2} x_0 f'' = -f' = -i p_0 \cdot f$

$-i \frac{\partial x_0}{\partial t} = [H, x_0] = -i p_0$ & $[H, p_0] = 0 \Rightarrow x_0(t) = x_0 + t p_0$

\ the Schrödinger eq'n for "operators"

Also as before, $\underset{-i \frac{\partial \alpha_n}{\partial t}}{\underset{\shortparallel}{[H, \alpha_n]}} = -n \alpha_n \Rightarrow \alpha_n(t) = e^{-int} \alpha_n$

$\tilde{\alpha}_n(t) = e^{-int} \tilde{\alpha}_n$

$$x_n = \frac{\tilde{\alpha}_{-n} - \alpha_n}{\sqrt{2}\, i n} \Rightarrow x(t,s) = x_0 + t p_0 + \frac{i}{\sqrt{2}} \sum_{n \neq 0} \frac{1}{n} \Big(\alpha_n e^{-in(t-s)} + \tilde{\alpha}_n e^{-in(t+s)} \Big)$$

"op. exp." right move left move

Def'n: Normal ordering, for $n \geq 1$

$:\alpha_{-n} \alpha_n: \; = \; :\alpha_n \alpha_{-n}: \; = \alpha_{-n} \alpha_n$ etc. ie. <u>annihilation</u> to RHS

$:x_0 p_0: \; = \; :p_0 x_0: \; = x_0 p_0$ to avoid extra contri.

HW: (Vertex operator) work out details in § 11.1.3

eg. $:e^{ikx(t,s)}: \; = U^{+} e^{ikx_0} e^{ikp_0} U$; where

$U \triangleq e^{ik \frac{i}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_n z^{-n} + \tilde{\alpha}_n \tilde{z}^{-n})}$, $z = e^{i(t-s)}$, $\tilde{z} = e^{i(t+s)}$

$\lim_{t \to -\infty} :e^{ik x(t,s)}: |0\rangle = e^{ikx_0} |0\rangle = |k\rangle$, has target space momentum $= k$.

Notice that

$$P = \frac{1}{2\pi} \int_{S^1} \partial_t x \, \partial_s x \, ds = -i \sum_n n \overset{p_n}{\overset{\shortparallel}{\dot{x}_n}} x_{-n}$$

$$= -\sum_{n=1}^{\infty} \alpha_{-n} \alpha_n + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \tilde{\alpha}_n \qquad \text{Since } p_n = \frac{\tilde{\alpha}_{-n} + \alpha_n}{\sqrt{2}}$$

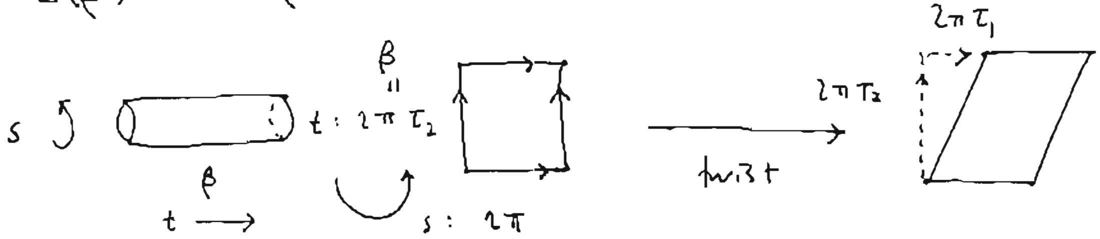
(This does not need normal ordering, why?)

check $H = \frac{1}{2\pi} \int_{S^1} \frac{1}{2} \left((\partial_t x)^2 + (\partial_s x)^2 \right) ds$ agrees with the prev. expression (after quantization).

Def'n: $H_R = \frac{1}{2}(H - P) = \frac{1}{4} p_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \alpha_n - \frac{1}{24}$

$H_L = \frac{1}{2}(H + P) = \frac{1}{4} p_0^2 + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \tilde{\alpha}_n - \frac{1}{24}$.

$$Z(\beta) = \mathrm{tr}\, e^{-\beta H}$$



$$\tau = \tau_1 + i\tau_2$$
$$q = e^{2\pi i \tau}$$

$$Z(\tau_1,\tau_2) = \mathrm{tr}\, e^{-2\pi i \tau_1 P}\, e^{-2\pi \tau_2 H}$$ "since P comes from transl. in s-direction"

$$= \mathrm{tr}\, e^{2\pi i \tau H_R}\, e^{-2\pi i \bar\tau H_L}$$

$$\Rightarrow \quad Z(\tau,\bar\tau) = \mathrm{tr}\, q^{H_R}\, \bar q^{H_L} \quad \text{on} \quad \mathcal{H} = \mathcal{H}_0 \otimes \bigotimes_{n=1}^{\infty} \mathcal{H}_n^R \otimes \bigotimes_{n=1}^{\infty} \mathcal{H}_n^L$$

Now $(\alpha_{-n}\alpha_n)\,\alpha_{-n}^{\ell}|0\rangle_n = \ell n \cdot \alpha_{-n}^{\ell}|0\rangle$ by $[\alpha_n, \alpha_{-n}] = n$

$$\mathrm{Tr}\, q^{\alpha_{-n}\alpha_n}\big|_{\mathcal{H}_n^R} = \sum_{\ell=0}^{\infty} q^{\ell n} = \frac{1}{1-q^n}$$

Similarly $\mathrm{Tr}\, \bar q^{\tilde\alpha_{-n}\tilde\alpha_n}\big|_{\mathcal{H}_n^L} = \frac{1}{1-\bar q^n}$

$$\Rightarrow \quad Z(\tau,\bar\tau) = (q\bar q)^{-1/24}\, \mathrm{Tr}\,(q\bar q)^{p_0^2/4}\big|_{\mathcal{H}_0} \prod_{n=1}^{\infty} \frac{1}{|1-q^n|^2}$$

$$= \frac{\text{"}V\text{"}}{|\eta(\tau)|^2}\cdot\frac{1}{\sqrt{\tau_2}} \qquad \overset{\shortparallel}{e^{-2\pi\tau_2\left(-\frac{1}{2}\frac{d^2}{dx^2}\right)}}$$

continuous spectrum * gives a divergent factor $V = \mathrm{vol}\,\mathbb{R}$ e^{ikx} not L^2

Dedekind's eta function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \quad \text{is "modular"}$$

- indep of point of view of (t,s) under $SL(2,\mathbb{Z})$
- Even does not dep. on area of Σ (metric ?) only on the cpx str. → conformal field theory. For σ model, this put strong condi. on M.

Rmk: $\eta(\tau+1) = e^{\pi i/12}\eta(\tau)$; $\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau)$.

$$* : \quad \int_{\mathbb{R}} e^{-2\pi\tau_2\cdot\frac{k^2}{2}}\,dk = \frac{\sqrt{2\pi}}{\sqrt{2\pi\tau_2}} = \frac{1}{\sqrt{\tau_2}}.$$

QFT in 1+1 dim (conti.)

$\Sigma = \mathbb{R} \times S^1 \longrightarrow S^1_R \;:\; X \sim X + 2\pi R$ <u>quantized by QM.</u>

$\downarrow$

target momentum $p = \frac{1}{2\pi}\int_{S^1} \partial_t X \, ds = \dot{x}_0(t) \mapsto p_0 = -i\frac{d}{dx_0}$

now have discrete spectrum = quant. #: $p = \frac{l}{R}$, $l \in \mathbb{Z}$

another target "top" charge. i.e. <u>quantized classically.</u>

$w = \frac{1}{2\pi}\int_{S^1} \partial_s X \, ds = mR$, $m \in \mathbb{Z}$: winding #

$\mathcal{H} = \bigoplus_{l,m} \mathcal{H}_{(l,m)}$; $\mathcal{H}_{(l,m)}$ gen by: α_{-n}, $\tilde{\alpha}_{-n}$ acting on $|l,m\rangle$

$|l,m\rangle$ killed by $\alpha_n, \tilde{\alpha}_n$ $\forall n > 0$.

$e^{i\frac{l}{R}x_0}$. shifts moment.

$[x_0, p_0] = i$

$\exists\, e^{imR\hat{x}_0}$. shifts winding #

$[\hat{x}_0, w_0] = i$

Then $X(t,s) = X_R(t-s) + X_L(t+s)$ even for "0-sector"

$$= \frac{1}{2}(x_0 - \hat{x}_0) + \frac{t-s}{\sqrt{2}}\frac{p_0 - w_0}{\sqrt{2}} + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\frac{\alpha_n}{n}e^{-in(t-s)}$$

$$+ \frac{1}{2}(x_0 + \hat{x}_0) + \frac{t+s}{\sqrt{2}}\frac{p_0 + w_0}{\sqrt{2}} + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\frac{\tilde{\alpha}_n}{n}e^{-in(t+s)}$$

$p_R := \frac{p_0 - w_0}{\sqrt{2}}$

$p_L := \frac{p_0 + w_0}{\sqrt{2}}$

Hence $H_R = \frac{1}{2}(H - P) = \frac{1}{2}p_R^2 + \sum_{n=1}^{\infty}\alpha_{-n}\alpha_n - \frac{1}{24}$

$H_L = \frac{1}{2}(H + P) = \frac{1}{2}p_L^2 + \sum_{n=1}^{\infty}\tilde{\alpha}_{-n}\tilde{\alpha}_n - \frac{1}{24}$

$$\Rightarrow Z(\tau, \bar{\tau}; R) = \frac{1}{|\eta(\tau)|^2}\sum_{l,m} q^{\frac{1}{4}\left(\frac{l}{R} - mR\right)^2}\bar{q}^{\frac{1}{4}\left(\frac{l}{R} + mR\right)^2}$$

T-duality $R \mapsto 1/R$: $Z(\tau, \bar{\tau}; R) = Z(\tau, \bar{\tau}; 1/R)$

Moreover, $\mathcal{H}_{(l,m)} \longmapsto \hat{\mathcal{H}}_{(m,l)}$ (w. to switch $l \leftrightarrow m$

$(p_R, p_L) \longmapsto (-\hat{p}_R, \hat{p}_L)$

$(\alpha_n, \tilde{\alpha}_n) \longmapsto (-\hat{\alpha}_n, \hat{\tilde{\alpha}}_n)$ (check)

Thus: $\hat{X}(t,s) = -X_R(t-s) + X_L(t+s)$ in quantum level.

This concludes the operator formalism.

Path integral :

$(\Sigma_g, h) \xrightarrow[x]{} S^1_R$ let $\varphi := \frac{x}{R}$ periodic 2π

R.S. of genus g   now assume Riemannian

$$S(\varphi) := \frac{1}{4\pi}\int_\Sigma R^2 |d\varphi|^2_h$$

$$S'(\varphi, B) := \frac{1}{4\pi}\int_\Sigma \frac{1}{R^2}|B|^2_h + \underline{\frac{i}{2\pi}\int_\Sigma B\wedge d\varphi}$$

1-form

then $S(\varphi) = S'(\varphi, B)$ for $B = iR^2 * d\varphi$ (since $a\wedge(*a) = |a|^2$)

DO $\int DB \int D\varphi\, e^{-S'(\varphi, B)}$; $d\varphi = d\varphi_0 + \sum_{i=1}^{2g} 2\pi n_i \omega^i$

over φ_0 & n_i

with ω^i basis of $H^1(\Sigma, \mathbb{Z})$

φ_0 (single valued) fun on Σ.

γ_i dual basis in $H_1(\Sigma, \mathbb{Z})$

φ shifts by $2\pi n_i$ along γ_i.

$\int_\Sigma B\wedge d\varphi_0 = \int_\Sigma dB\wedge\varphi_0$

in order be inv. under $\varphi_0 \mapsto \varphi_0 + \psi_0$

$\Rightarrow dB = 0 \Rightarrow B = d\vartheta_0 + \sum_{i=1}^{2g} a_i\omega^i$

$$\Rightarrow \int_\Sigma B\wedge d\varphi = 2\pi \sum_{i,j} a_i n_j \underbrace{\int_\Sigma \omega^i\wedge\omega^j}_{\Downarrow} =: J^{ij} \text{ invertible } \mathbb{Z}\text{-matrix}$$

$n^i := n_j J^{ij} \in \mathbb{Z}$ (Poincaré duality)

Poisson summation :

$$\sum_{n\in\mathbb{Z}} e^{ian} = 2\pi \sum_{m\in\mathbb{Z}} \delta(a - 2\pi m)$$

Fourier transf of e^{iax} (why ?)

B has contribution in $\int DB$ only when $\underline{a_i \in \mathbb{Z}\cdot 2\pi}$

So $B = d\vartheta_0 + 2\pi\sum_{i=1}^{2g} m_i\omega^i =: d\vartheta$, ϑ periodic 2π

$e^{-S'(\varphi, B)} \longmapsto e^{-S'(\vartheta)}$; $S'(\vartheta) = \frac{1}{4\pi}\int_\Sigma \frac{1}{R^2}|d\vartheta|^2_h$

This is the T-duality : $R \mapsto 1/R$

with $R\,d\varphi = \frac{i}{R} * B = i\left(\frac{1}{R}\right) * d\vartheta$ $(*^2 = -1)$

$\partial_t\varphi / \partial_s\varphi \longleftrightarrow \partial_s\vartheta / \partial_t\vartheta$ exchange momentum & winding # measurement

HW : (p. 252) Show that the vertex operator $e^{i\vartheta}$ creates the shift of unit winding number.

- σ-model on T^2 : $\Sigma \xrightarrow{X} T^2 = M$

If $T^2 = S^1_{R_1} \times S^1_{R_2}$ rectangular tori $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$

parameter (R_1, R_2) is equiv. to $A = \frac{\text{Area}}{(2\pi)^2} = R_1 R_2$; $\sigma = i \frac{R_1}{R_2}$

T-duality for the 2nd factor :

$$(A, \operatorname{im}\sigma) \mapsto (R_1/R_2, R_1 R_2) = (A', \operatorname{im}\sigma') = (\operatorname{im}\sigma, A)$$

Kähler str cpx str. exchange symp / cpx str.

General case : cpx str: $\sigma = \sigma_1 + i\sigma_2 \in \mathbb{C}$ (moduli)

$\otimes \mathbb{C}$ - Kähler str : $\rho = \frac{B}{2\pi} + iA \in \mathbb{C}$ $B \in H^2(M, \mathbb{R}) / H^2(M, \mathbb{Z}) \cdot 2\pi$

and $Z := \int DX \, e^{-S + i\int_\Sigma X^* B}$

HW : Formulate S correctly for T^2 a general torus. Compute Z with B-field and show inv. under T-duality exchanges σ / ρ.

Free Dirac Fermion (Spinor / $\mathbb{C}$) ; A Summary

$$\begin{tikzcd} Cl^{\mathbb{C}}_{1,1} \arrow[d] \\ \Sigma = \mathbb{R} \times S^1 \end{tikzcd}$$

$Cl^{\mathbb{C}}_{1,1}$: Clifford algebra (bundle) at $T^{\mathbb{C}}_p\Sigma = \langle e^t, e^s \rangle \otimes \mathbb{C}$

$\Sigma = \mathbb{R} \times S^1$: Minkowski $\langle , \rangle$ (t, s)

$uv + vu + 2\langle u, v\rangle = 0$

i.e. $(e^t)^2 = 1 = -(e^s)^2$

$e^t e^s = -e^s e^t$

$Cl^{\mathbb{C}}_{1,1} \simeq \operatorname{End} S$; $S = S_- \oplus S_+$ deform of $\Lambda^* T\Sigma$

Spinor repr $\simeq \mathbb{C}^2 \ni \psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \in \Gamma(\Sigma, S)$

$e^t \mapsto \gamma^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $e^s \mapsto \gamma^s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$S(\psi) = \frac{1}{2\pi} \int_\Sigma i \bar{\psi} \underline{\gamma^\mu \partial_\mu} \psi \, dt\, ds$ $\bar{\psi} := \psi^t \gamma^t = (\bar{\psi}_+, \bar{\psi}_-)$

$\not{\partial}$ op. i.e. $\langle , \rangle$ on S $= i\psi_1^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi_2$

$\Rightarrow \delta S = \frac{1}{2\pi} \int_\Sigma 2i\delta\bar{\psi} \cdot \not{\partial}\psi$ $\not{\partial}$ adjoint op.

eq'n of motion $0 = \not{\partial}\psi = \begin{pmatrix} 0 & \partial_t - \partial_s \\ \partial_t + \partial_s & 0 \end{pmatrix} \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}$.

i.e. $\psi_-(t,s) = f(t-s)$ right
$\psi_+(t,s) = g(t+s)$ left
move.

(Rotations : $\psi_\pm \mapsto e^{-i\alpha}\psi_\pm$ vector $\rightarrow F_V = \frac{1}{2\pi}\int_{S^1} (\bar{\psi}_- \psi_- + \bar{\psi}_+ \psi_+) \, ds$
$\psi_\pm \mapsto e^{\mp i\beta}\psi_\pm$ axial $\rightarrow F_A = \frac{1}{2\pi}\int_{S^1} (-\bar{\psi}_- \psi_- + \bar{\psi}_+ \psi_+) \, ds$.)

Space-time translation :

$$H = \frac{1}{2\pi}\int_{S^1}\left(-i\bar\psi_- \partial_s\psi_- + i\bar\psi_+\partial_s\psi_+\right) ds$$

$$P = \frac{1}{2\pi}\int_{S^1}\left(i\bar\psi_- \partial_s\psi_- + i\bar\psi_+\partial_s\psi_+\right) ds$$

> no ∂_t by Dirac eq'n.

Again, decomp into Fourier modes on S^1. For periodic bd. condi.

$\psi_- = \sum \psi_n(t)\, e^{ins}$ $\quad \psi_-^\dagger = \bar\psi_- = \sum \bar\psi_n(t)\, e^{ins}$ $\quad \Rightarrow \quad \bar\psi_n = \psi_{-n}^\dagger$

$\psi_+ = \sum \tilde\psi_n(t)\, e^{ins}$ $\quad \psi_+^\dagger = \bar\psi_+ = \sum \tilde{\bar\psi}_n(t)\, e^{ins}$ $\quad \Rightarrow \quad \tilde{\bar\psi}_n = \tilde\psi_{-n}^\dagger$

$$\Rightarrow S = \int \sum_{n\in\mathbb{Z}}\left(i\bar\psi_{-n}(\partial_t + in)\psi_n + i\tilde{\bar\psi}_{-n}(\partial_t + in)\tilde\psi_n\right) dt$$

Q: why not just say $\psi_n = \psi_{-n}$?

$$\Rightarrow \quad \pi_n = \frac{\partial L}{\partial(\partial_t\psi_n)} = i\bar\psi_{-n}$$

Quantization : $\{\psi_n, \bar\psi_m\} = \delta_{n+m,0}$ & $\{\tilde\psi_n, \tilde{\bar\psi}_m\} = \delta_{n+m,0}$.

Now each n, $\psi_n, \bar\psi_{-n}$ is rep in a 2 dim v.s. (as in Boson.) but ~~even~~ $n=0$.

- $H_{n(+)} = n\bar\psi_{-n}\psi_n$: in this sector, $|0\rangle_n$: killed by ψ_n if $n>0$
- $H_{n(-)} = n\tilde{\bar\psi}_{-n}\tilde\psi_n$: similarly get $\widetilde{|0\rangle}_n$.

" $\bar\psi_{-n}$ if $n<0$

e.g. $\tilde\psi_0\widetilde{|0\rangle}_0 = 0$ | any one of the 2 if $n=0$

say $\psi_0|0\rangle_0 = 0$.

$$|0\rangle := \bigotimes_{n\geq 0} |0\rangle_n \otimes \widetilde{|0\rangle}_n$$

$$H = \sum_{n\in\mathbb{Z}}\left(n\bar\psi_{-n}\psi_n + n\tilde{\bar\psi}_{-n}\tilde\psi_n\right) = \sum_{n\in\mathbb{Z}} n:\bar\psi_{-n}\psi_n: + n:\tilde{\bar\psi}_{-n}\tilde\psi_n: \; + 1/6$$

since $\sum_{n=1}^{\infty}(-2n) = 1/6$.

So $|0\rangle$ has energy $E_0 = 1/6$

But now there are 4 such ground states : $\psi_0|0\rangle, \tilde{\bar\psi}_0|0\rangle, \psi_0\tilde{\bar\psi}_0|0\rangle$.

Easier : $P = \sum_{n\in\mathbb{Z}} -n:\bar\psi_{-n}\psi_n: + n:\tilde{\bar\psi}_{-n}\tilde\psi_n:$.

For twisted bd. condi.

$$\psi_-(t, s+2\pi) = e^{2\pi i a}\psi_-(t,s)$$

$$\psi_+(t, s+2\pi) = e^{-2\pi i\tilde a}\psi_+(t,s)$$

Periodic : Ramond sector

Anti-P : Neveu-Schwarz sector

e.g. $(a,\tilde a) = (0,0), (0,\frac{1}{2}), (\frac{1}{2},0), (\frac{1}{2},\frac{1}{2})$

R-R R-NS NS-R NS-NS

$$H_R = \frac{1}{2}(H-P) = \sum_{r\in\mathbb{Z}+a} r:\bar\psi_{-r}\psi_r: + \frac{1}{2}\left(\{a\}-\frac{1}{2}\right)^2 - \frac{1}{24}$$

$$H_L = \frac{1}{2}(H+P) = \sum_{r\in\mathbb{Z}+\tilde a} r:\tilde{\bar\psi}_{-r}\tilde\psi_r: + \frac{1}{2}\left(\{\tilde a\}-\frac{1}{2}\right)^2 - \frac{1}{24}$$

Partition function : need bd condi on $t \mapsto t + 2\pi\tau_2$: $e^{2\pi i b}$; $e^{-2\pi i\tilde b}$

$$Z = \mathrm{Tr}\left(e^{-2\pi i(b-\frac{1}{2})F_R}\, e^{2\pi i(b-\frac{1}{2})F_L}\, e^{-2\pi i\tau_1 P}\, e^{2\pi\tau_2 H}\right)$$

$\tau = \tau_1 + i\tau_2$, $\tau_1 = 1$

$$F_R = \frac{1}{2}(F_V - F_A) = \sum_{r\in\mathbb{Z}+a} :\bar\psi_{-r}\psi_r: + \left(\{a\}-\frac{1}{2}\right)$$

$$F_L = \frac{1}{2}(F_V + F_A) = \sum_{r\in\mathbb{Z}+\tilde a} :\tilde{\bar\psi}_{-r}\tilde\psi_r: + \left(\{\tilde a\}-\frac{1}{2}\right)$$

when $(a,b) = (\tilde a, \tilde b) = (0,0), (0,\frac{1}{2}), (\frac{1}{2},0), (\frac{1}{2},\frac{1}{2})$

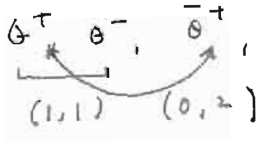
$$\Rightarrow Z_{[*,*]} = \frac{|\vartheta_{[*,*]}(0,\tau)|^2}{|\eta(\tau)|^2}.$$

$N = (2,2)$ SUSY   Lect 6. 3/30 2011

Systematic way to get SUSY Lagrangian.

$\mathbb{R}^2 \ni (t,s) = (x^0, x^1)$
Minkowski

$\theta^+, \theta^-, \bar\theta^+, \bar\theta^-$ cpx Fermion coor. ie. spinors



This is (2,2) superspace.

Superfields: $\mathcal{F}(x,\theta) = f_0 + \theta^+ f_+ + \theta^- f_- + \bar\theta^+ f'_+ + \bar\theta^- f'_-$
$+ \theta^+\theta^- f_{+-} + \cdots$ ($2^4 = 16$ terms)
$f_* = f_*(x^0, x^1)$.

$x^\pm := x^0 \pm x^1$, $\partial_\pm = \frac{\partial}{\partial x^\pm} := \frac{1}{2}\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right)$

$Q_\pm := \frac{\partial}{\partial\theta^\pm} + i\bar\theta^\pm\partial_\pm$; $\bar Q_\pm := -\frac{\partial}{\partial\bar\theta^\pm} - i\theta^\pm\partial_\pm$

$D_\pm := \frac{\partial}{\partial\theta^\pm} - i\bar\theta^\pm\partial_\pm$; $\bar D_\pm := -\frac{\partial}{\partial\bar\theta^\pm} + i\theta^\pm\partial_\pm$

then $\{Q_\pm, \bar Q_\pm\} = -2i\partial_\pm$, $\{D_\pm, \bar D_\pm\} = 2i\partial_\pm$, All others $= 0$.

chiral superfield Φ : $\bar D_\pm \Phi = 0$.

HW 1: $\Phi = \phi(y^\pm) + \theta^\alpha\psi_\alpha(y^\pm) + \theta^+\theta^- F(y^\pm)$ < use left >
with $y^\pm := x^\pm - i\theta^\pm\bar\theta^\pm$ (Decide Fermion diff order and chain rule)

other choices: anti-chiral $D_\pm \bar\Phi = 0$ ($\Leftrightarrow \bar{\bar\Phi}$ chiral)
twisted chiral $\bar D_+ U = 0 = D_- U$ ($\bar U$ tw-anti)

SUSY Action: let $\delta = \epsilon_+ Q_- - \bar\epsilon_+ \bar Q_- - \epsilon_- Q_+ + \bar\epsilon_- \bar Q_+$

- D-term: $\int d^2x\, d^4\theta\, K(F_i)$; $d^4\theta := d\theta^+ d\theta^- d\bar\theta^- d\bar\theta^+$.

  $\delta(-)$: $\bar\epsilon_-$ coeff : $\bar\epsilon_- \bar Q_+ K$
  $= -\bar\epsilon_-\left(\frac{\partial K}{\partial\bar\theta} + i\theta^+\frac{\partial K}{\partial x^+}\right) \xrightarrow{\int} 0$
  no $\bar\theta$ total derivative

- F-term: $\int d^2x \underbrace{d^2\theta}_{= d\theta^+ d\theta^-} W(\Phi_i)$ holo chiral s.f.
  this means setting $\bar\theta^\pm = 0$.

  $\delta(-)$: $\bar\epsilon_-$ coeff : notice $\bar Q_- = \bar D_- - 2i\theta^-\partial_-$
  get $-2i\bar\epsilon_-\theta^-\frac{\partial W}{\partial x^-}$ total deri. $\xrightarrow{\int} 0$

- twisted F-term: $\int d^2x\, d^2\tilde\theta\, \tilde W(U_i)$; $d^2\tilde\theta = d\bar\theta^- d\theta^+$, $\tilde W$ hol. U_i : anti-chiral

Basic Examples

① One chiral s.f. $\Phi = \phi(y^\pm) + \theta^\alpha \psi_\alpha(y^\pm) + \theta^+\theta^- F(y^\pm)$

Taylor ⇒ $\underline{\Phi} = \phi - i\theta^\pm\bar\theta^\pm\partial_\pm\phi - \theta^+\theta^-\bar\theta^-\bar\theta^+\partial_+\partial_-\phi$

$y^\pm = x^\pm - i\theta^\pm\bar\theta^\pm$ $\quad + \theta^\pm\psi_\pm - i\theta^\pm\theta^\mp\bar\theta^\mp\partial_\mp\psi_\pm + \theta^+\theta^- F$

$(\psi_1\psi_2)^\dagger = \psi_2^\dagger\psi_1^\dagger \Rightarrow \underline{\bar\Phi} = \bar\phi + i\theta^\pm\bar\theta^\pm\partial_\pm\bar\phi - \theta^+\bar\theta\,\bar\theta^-\bar\theta^+\partial_+\partial_-\bar\phi$

$- \bar\theta^\pm\bar\psi_\pm - i\bar\theta^\pm\theta^\mp\bar\theta^\mp\partial_\mp\bar\psi_\pm + \bar\theta^-\bar\theta^+\bar F$

- Kenitic D term:

$$S_{kin} = \int d^2x\, d^4\theta\, \bar\Phi\Phi = \int d^2x \Big(-\bar\phi\partial_+\partial_-\phi + \partial_\pm\bar\phi\partial_\mp\phi - \partial_+\partial_-\bar\phi\phi$$

pick coeff of $\theta^4 = \theta^+\theta^-\bar\theta^-\bar\theta^+$

$$+ i\bar\psi_\pm\partial_\mp\psi_\pm - i\partial_\mp\bar\psi_\pm\psi_\pm + |F|^2 \Big)$$

int. by parts $= \int d^2x \left(\left\{ |\partial_0\phi|^2 - |\partial_1\phi|^2 \right\}\frac{1}{2} + 2i\,\bar\psi_\pm\partial_\mp\psi_\pm + \underline{|F|^2} \right)$

free scalar   free Dirac fermion   auxiliary field

- F-term: $S_W = \int d^2x\, d^2\theta \left(W(\Phi) + \bar W(\bar\Phi) \right)$ to make it real.

$$= \int d^2x \left(\underline{W'(\phi)F} - W''(\phi)\psi_+\psi_- + \underline{\bar W'(\bar\phi)\bar F} - \bar W''(\bar\phi)\bar\psi_-\bar\psi_+ \right)$$

$S = S_{kin} + S_W = \int d^2x \big($ scalar + Dirac $- |W'(\phi)|^2 - (W''(\phi)\psi_+\psi_- + \bar W''(\bar\phi)\bar\psi_-\bar\psi_+)\big)$

set $F = -\bar W'(\bar\phi)$ $+ |F + \bar W'(\bar\phi)|^2$ potential   Yukawa

Rmk: Can δ on $\bar\Phi$ be written in $\delta\phi, \delta\psi_\pm, \delta\bar\psi_\pm, \delta F$?

$\bar D_\pm\Phi = 0$, this is possible only if $\bar D_\pm\delta\Phi = 0$ as well.

this is OK since $Q_\pm, \bar Q_\pm$ anti-comm with $D_\pm$.

conserved current & charges:

HW2: $Q_\pm = \int dx^1\, G^0_\pm = \int dx^1 \left(2(\partial_\pm\bar\phi)\psi_\pm \mp i\bar\psi_\mp \bar W'(\bar\phi) \right)$

$\bar Q_\pm = \int dx^1\, \bar G^0_\pm = \int dx^1 \left(2\bar\psi_\pm(\partial_\pm\phi) \pm i\psi_\mp W'(\phi) \right)$

Rotation Symmetries   U(1) sym.

Axial: $\theta^\pm \mapsto e^{\mp i\alpha}\theta^\pm$; $F_A = \int dx^1 J_A^0 = \int dx^1 (\bar\psi_+\psi_+ - \bar\psi_-\psi_-)$

Vector: $\theta^\pm \mapsto e^{-i\alpha}\theta^\pm$; D-term is OK. (θ^4 is inv.)

$\theta^2 \mapsto e^{-2i\alpha}\theta^2 \Rightarrow$ F-term inv only if $W(\Phi) \mapsto e^{2i\alpha}W(\Phi)$.

$$\Rightarrow F_V = \int dx^1 J_V^0 = \int dx^1 \left\{ \frac{2i}{k}\left((\partial_0\bar\phi)\phi - \bar\phi\partial_0\phi\right) - \left(\frac{2}{k} - 1\right)(\bar\psi_+\psi_+ - \bar\psi_-\psi_-) \right\}$$

Φ^k monomial

② One twisted chiral s.f. U

$$S = -\int d^2x\, d\theta^4\, \bar{U} U + \int d^2x\, \underset{d\bar{\theta}^- d\theta^+}{\overset{}{d^2\tilde{\theta}}} \left(\tilde{W}(U) + \bar{\tilde{W}}(\bar{U}) \right)$$

Notice chiral s.f. $\longleftrightarrow$ twisted chiral s.f. $\Leftrightarrow$ " $\theta^- \longleftrightarrow -\bar{\theta}^-$ "

in particular, $Q_- \leftrightarrow \bar{Q}_-$; $F_V \leftrightarrow F_A$: $(\bar{D}_- \longleftrightarrow D_-)$

$$\text{Axial}: \left(\begin{array}{l} \theta^+ \mapsto e^{-i\alpha}\theta^+ \\ \theta^- \mapsto e^{i\alpha}\theta^- \quad \rightsquigarrow \quad \bar{\theta}^- \mapsto e^{i\alpha}\bar{\theta}^- \text{ ie } \theta^- \mapsto e^{-i\alpha}\theta^- \end{array} \right) \text{Vector}$$

$N = (2,2)$ SUSY QFT :

start with a classical SUSY FT (for a few fields)

$\rightsquigarrow$ 4 supercharges $Q_\pm, \bar{Q}_\pm$

Noether charges for time $\partial/\partial x^0$ H

spatial $\partial/\partial x^1$ P

Lorentz rotation $x^0\partial_1 + x^1\partial_0$ M

R-rotations F_V, F_A.

If the symmetries are not lost in quantum theory (no anomally)

then conserved charges $\mapsto$ sym. transf. in QT

eg. $\delta O = [\hat{\delta}, O]$ with $\hat{\delta} = i\epsilon_+ Q_- - i\bar{\epsilon}_+\bar{Q}_- - i\epsilon_- Q_+ + i\bar{\epsilon}_-\bar{Q}_+$

$Q_\pm^2 = \bar{Q}_\pm^2 = 0$, $\{Q_\pm, \bar{Q}_\pm\} = H \pm P$, others $= 0$

$[iF_A, Q_\pm] = \mp i Q_\pm$, $[iF_V, Q_\pm] = -iQ_\pm$, also on $\bar{Q}_\pm$.

another version : without $F_V, (F_A)$. then can allow

$\{\bar{Q}_+, \bar{Q}_-\} = Z$, $(\{\bar{Q}_+, Q_-\} = \tilde{Z})$: central charges

Def'n: The above defines $N=(2,2)$ SUSY algebra.

(*) $\mathbb{Z}_2$ outer auto. $Q_- \longleftrightarrow \bar{Q}_-$; $F_V \leftrightarrow F_A$; $Z \leftrightarrow \tilde{Z}$.

Def'n: Two $N=(2,2)$ SUSY alg are mirror to each other if they are Isom as QFT, which is induced by (*).

Non-linear σ models (classical theory)

let $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K(\Phi^k, \bar{\Phi}^k) > 0$ $\quad \Phi^1, \dots, \Phi^n$ chiral multiplet

i.e. $ds^2 = g_{i\bar{j}} dz^i d\bar{z}^j$ metric on (local) $\mathbb{C}^n$

fact: Levi-Civita $\Gamma^i_{jk} = g^{i\bar{\ell}} \partial_j g_{k\bar{\ell}} = \overline{\Gamma^{\bar{i}}_{\bar{j}\bar{k}}}$ (no others)

$L_{kin} = \int d^4\theta \, K(\Phi, \bar{\Phi})$ in terms of component fields

$$= -g_{i\bar{j}} \partial^\mu \phi^i \partial_\mu \overline{\phi^j} + i g_{i\bar{j}} \bar{\psi}^{\bar{j}}_{\mp} D_\pm \psi^i_\mp + R_{i\bar{j}k\bar{\ell}} \psi^i_+ \psi^k_- \bar{\psi}^{\bar{j}}_- \bar{\psi}^{\bar{\ell}}_+$$

HW 3:

$$+ g_{i\bar{j}} \underline{(F^i - \Gamma^i_{\ell k} \psi^\ell_+ \psi^k_-)(\bar{F}^{\bar{j}} - \Gamma^{\bar{j}}_{\bar{i}\bar{k}} \bar{\psi}^{\bar{i}}_- \bar{\psi}^{\bar{k}}_+)}$$

modulo total diff. $\quad$ notation: $\bar{\psi}^{\bar{j}} \equiv \overline{\psi^j}$

This can be globally defined for

$\phi : \Sigma \to M$ Kähler mfd. $\qquad \psi_\pm \in \Gamma(\Sigma, \phi^* T \otimes S_\pm)$

$\qquad \bar{\psi}_\pm \in \Gamma(\Sigma, \phi^* \bar{T} \otimes S_\pm)$

But the SUSY can only be checked locally.

$$L_W = \frac{1}{2} \int d^2\theta \left(W(\Phi) + \overline{W(\Phi)} \right) = \frac{1}{2}\underline{\left(F^i \partial_i W + \bar{F}^{\bar{i}} \partial_{\bar{i}} \bar{W}\right)} - \frac{1}{2} \partial_i \partial_j W \psi^i_+ \psi^j_- - \frac{1}{2} \partial_{\bar{i}} \partial_{\bar{j}} \bar{W} \bar{\psi}^{\bar{i}}_- \bar{\psi}^{\bar{j}}_+$$

holo, $\exists$ only if M non-cpt

Set $F^i = \Gamma^i_{jk} \psi^j_+ \psi^k_- - \frac{1}{2} g^{i\bar{\ell}} \partial_{\bar{\ell}} \bar{W}$; $\quad \bar{F}^{\bar{j}} = \overline{\Gamma^j_{ik}} \bar{\psi}^{\bar{i}}_- \bar{\psi}^{\bar{k}}_+ - \frac{1}{2} g^{\bar{j}\ell} \partial_\ell W$

$$\Rightarrow \quad L = -g_{i\bar{j}} \partial^\mu \phi^i \partial_\mu \overline{\phi^j} + i g_{i\bar{j}} \bar{\psi}^{\bar{j}}_\mp D_\pm \psi^i_\mp + R_{i\bar{j}k\bar{\ell}} \psi^i_+ \psi^k_- \bar{\psi}^{\bar{j}}_- \bar{\psi}^{\bar{\ell}}_+$$

$$- \frac{1}{4} g^{\bar{i}j} \partial_{\bar{i}} \bar{W} \partial_j W - \frac{1}{2} D_i(\partial_j W) \psi^i_+ \psi^j_- - \frac{1}{2} D_{\bar{i}}(\partial_{\bar{j}} \bar{W}) \bar{\psi}^{\bar{i}}_- \bar{\psi}^{\bar{j}}_+ .$$

$$\delta \int d^2x \, L = \int d^2x \left(\partial_\mu \epsilon_+ G^\mu_- - \partial_\mu \epsilon_- G^\mu_+ + \partial_\mu \bar{\epsilon}_- \bar{G}^\mu_+ - \partial_\mu \bar{\epsilon}_+ \bar{G}^\mu_- \right)$$

A direct generalization of HW 2 works for supercharges

$$Q_\pm = \int dx^1 \, G^0_\pm = \int dx^1 \left(2 g_{i\bar{j}} (\partial_\pm \overline{\phi^j}) \psi^i_\pm \mp \frac{i}{2} \bar{\psi}^{\bar{i}}_\mp \partial_{\bar{i}} \bar{W} \right)$$

$$\bar{Q}_\pm = \int dx^1 \, \bar{G}^0_\pm = \dots$$

For R-rotations: notice θ^4 inv. but not $\theta^2 \overset{V}{\mapsto} e^{-2i\alpha} \theta^2$

D term: S_{kin}, always $U(1)_V$, $U(1)_A$-inv by setting Φ^i charge 0

if $K(\Phi^i, \bar{\Phi}^i)$ dep only on $|\Phi^i|^2$, can do any charge.

F term: S_W, $U(1)_A$-inv ok by setting 0 R-charge to Φ^i.

For $U(1)_V$: An assignment is possible $\Leftrightarrow$ $W(\lambda^{q_i} \Phi^i) = \lambda^2 W(\Phi^i)$.

Anomally. Toy model

$$S = \int_{T^2} d^2z \left(i\bar{\Psi}_+ D_{z} \Psi_+ + i\bar{\Psi}_- D_{\bar{z}} \Psi_- \right) \qquad \begin{array}{l} \Psi_\pm \in \Gamma(T^2, E \otimes S_\pm) \\ \bar{\Psi}_\pm \in \Gamma(T^2, E^* \otimes S_\pm) \end{array}$$

Dirac with her. conn on E

inv. under $V : e^{-i\alpha}$, $A : \Psi_\pm \longmapsto e^{\mp i\beta} \Psi_\pm$

Atiyah-Singer index thm

$$\dim\ker D_{\bar{z}} - \dim \operatorname{coker} D_{\bar{z}} = \int_{T^2} \underset{\overset{''}{c_1}}{ch(E)} \cdot \underset{\overset{''}{1}}{\hat{A}(T^2)} = k$$: actually for any Σ.

if $k > 0$, then $\int D\Psi D\bar{\Psi}\, e^{-S[\Psi,\bar\Psi]} = 0$ on zero modes.

but $\langle \Psi_-(z_1) \cdots \Psi_-(z_k) \bar{\Psi}_+(w_1) \cdots \bar{\Psi}_+(w_k) \rangle \neq 0 \xmapsto{A} e^{2ik\beta}$. not inv.

σ-model : $\phi : \underset{\Sigma}{\|}T^2 \to M$, $E = \phi^* T^{1,0}_M$

anomally free requires $\langle \phi^* c_1(M) . \Sigma \rangle = 0$. eg. Calabi-Yau.

Thus	CY σ-model :	$U(1)_A$	$U(1)_V$
	σ-model $c_1 \neq 0$:	×	$U(1)_V$
	LG model on CY with general W :	$U(1)_A$	×
	LG model on CY with quasi-homog W :	$U(1)_A$	$U(1)_V$

This suggests CY/LG correspondence as "Mirror Sym"

on T^2 can be checked on ground states via T-duality.

Renormalization

σ model $(\Sigma, h) \xrightarrow{\phi} (M, g)$ Kähler

classical action $S = \int_\Sigma \sqrt{h}\, d^2x \left(g_{i\bar{j}} h^{\mu\nu} \partial_\mu \phi^i \partial_\nu \bar{\phi}^j + i g_{i\bar{j}} \bar{\Psi}^j \gamma^\mu D_\mu \Psi^i + R_{i\bar{j}k\bar{l}} \Psi^i_+ \Psi^k_- \bar{\Psi}^j_- \bar{\Psi}^l_+ \right)$

inv. under scale transf.

$h_{\mu\nu} \mapsto \lambda^2 h_{\mu\nu}$ via $\phi^i \mapsto \phi^i$, $\gamma^\mu \mapsto \lambda^{-1} \gamma^\mu$, $\Psi_\pm \mapsto \lambda^{-1/2} \Psi_\pm$

since $\{\gamma^\mu, \gamma^\nu\} = -2h^{\mu\nu}$

Clifford relation

Quantum level ?

still require $c_1(M) = 0$ to keep sym.

Rmk: Infact, $[\omega] \mapsto [\omega] - \log\lambda \cdot c_1(M)$ for $\omega = \omega_g$.

Bosonic σ-model on Riem mfd :

$$S = \frac{1}{2}\int g_{IJ}(\phi)\, \partial_\mu \phi^I \partial^\mu \phi^J \, d^2x \sqrt{h}$$

$\phi^I = \phi^I_0 + \xi^I$ expansion near a pt $\phi_0 \in M$

$$g_{IJ}(\phi) = \delta_{IJ} - \frac{1}{3} R_{IKJL}(\phi_0)\xi^K \xi^L + O(\xi^3) \qquad \text{RNC at } \phi_0$$

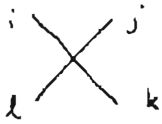
- Recall 0-dim QFT :

$$\langle O \rangle = \frac{1}{Z(M,C)} \int d^n X \, e^{-\frac{1}{2} X_i M_{ij} X_j + C_{ijkl} X_i X_j X_k X_l} \cdot O(X_1, \cdots, X_n)$$

2 pt functions at $C = 0$ (propagator)

$$\langle x_i x_j \rangle_{(0)} = \frac{1}{Z(M,0)} \int d^n X \, e^{-\frac{1}{2} X_k M_{kl} X_l} X_i X_j = M^{ij}$$

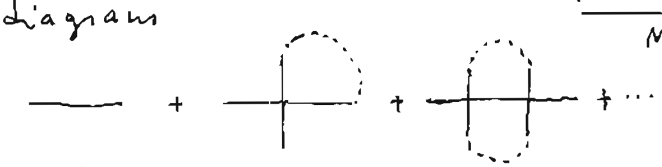
ie. $M_{ij} \langle X_j X_k \rangle_{(0)} = \delta_{ik}$ "2pt fcn = Green fcn"



$-C_{ijkl}$ vertex

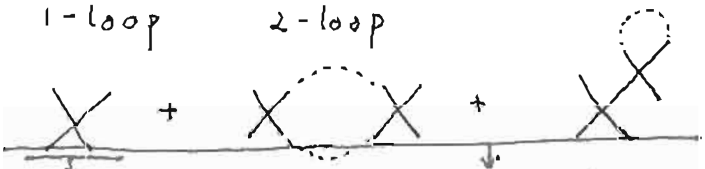
Feymann diagrams

$\langle x_i x_j \rangle =$



0-loop 1-loop 2-loop

$\langle x_i x_j x_k x_l \rangle =$



$\langle x_i x_j x_k x_l \rangle_{(0)}$ $\langle x_i x_j x_k x_l \rangle_{(1)}$

* This means sum up to 1-loop

HW 7.1. Calculate 2pt, 4pt fcn at 1-loop level.

- analogue in 2D QFT : sketch of idea

$$\Delta \langle \xi^I(x), \xi^J(y) \rangle_{(0)} = \delta(x-y)\,\delta^{IJ} \ ; \qquad \Delta = -\partial^\mu \partial_\mu$$

F.T.

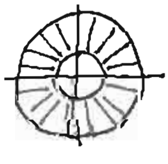
$$\Rightarrow \langle \xi^I(x), \xi^J(y) \rangle_{(0)} = \int \frac{d^2k}{(2\pi)^2} \frac{e^{ik\cdot(x-y)}}{k^2} \delta^{IJ} \left(= -\frac{1}{2\pi} \log|x-y| \cdot \delta^{IJ} \right)$$

$$\left(\text{Now } S = \int \left(\frac{1}{2} \partial^\mu \xi^I \partial_\mu \xi^I - \frac{1}{6} R_{IKJL}(\phi_0)\, \partial^\mu \xi^I \partial_\mu \xi^J \xi^K \xi^L + O(\xi^5) \right) d^2x \sqrt{h} \right)$$

$$\langle \xi^I(x), \xi^J(y) \rangle_{(1)} = \int \frac{d^2p}{(2\pi)^2} \frac{e^{ip\cdot(x-y)}}{p^2} \left(\delta^{IJ} + \frac{1}{3} \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2} R_{IJ}(\phi_0) \right)$$

why? $\Big\downarrow$ same F.T.

log divergence at $k \to 0, \infty$. $\Downarrow$



Cut off $\mu \leq |k| \leq \Lambda_{UV}$: $\frac{1}{2\pi} \log \frac{\Lambda_{UV}}{\mu}$

ultraviolet

Similarly, HW 7.2 : give details for ① + ②

$$\langle \xi^{I_1}(x_1)\, \xi^{I_2}(x_2)\, \xi^{I_3}(x_3)\, \xi^{I_4}(x_4) \rangle_{(1)}$$

① $$= -\frac{1}{3} \int \prod_{i=1}^{4} \frac{d^2 p_i}{(2\pi)^2} \frac{e^{i p_i x_i}}{p_i^2} \cdot (2\pi)^2 \delta(p_1 + p_2 + p_3 + p_4)$$

$$\times \Big((p_3 \cdot p_4) \Big\{ R_{I_1 I_2 I_3 I_4} + \frac{1}{6\pi} \log \frac{\Lambda_{UV}}{\mu} \underbrace{(R_4 R_2)}_{\text{contraction}}{}_{I_1 I_2 I_3 I_4} \Big\} + \cdots \Big)$$

Now let $g_{IJ} = \delta_{IJ} \mapsto \tilde{g}_{IJ} = \delta_{IJ} + a_{IJ}$

(at ϕ_0) $\xi^I \mapsto \tilde{\xi}^I = \xi^I + b^I_J \xi^J$

get $$\tilde{S} = \int \Big[\frac{1}{2} (\delta + a + 2b)_{IJ}\, \partial_\mu \xi^I \partial^\mu \xi^J - \frac{1}{6} (R_4 + R_4 b)\, \partial_\mu \xi^I \partial^\mu \xi^J \xi^K \xi^L + \cdots \Big] d^2x \cdot \sqrt{h}$$

Q : Can we find $a, b \sim \log \frac{\Lambda_{UV}}{\mu}$ st the new divergence factors cancel the prev ones ?

A : yes : Renormalization $a_{IJ} = \frac{1}{2\pi} \log \frac{\Lambda_{UV}}{\mu} R_{IJ}$

$b^I_J = -\frac{1}{6\pi} \log \frac{\Lambda_{UV}}{\mu} R^I_J$ ② .

[Ken Wilson : Collection of fields $\phi(x)$, coupling constants g $\mapsto S(\phi, g)$

field at cut off scale Λ_{UV} : $\phi_0(x) := \int_{|k| \leq \Lambda_{UV}} \frac{d^2k}{(2\pi)^2} e^{ikx} \hat{\phi}(k)$

In $Z = \int D\phi_0\, e^{-S(\phi_0, \overset{g}{\overset{\shortparallel}{g_0}})}$, the UV div. disappears

Decompose $\phi_0(x) = \phi_L(x) + \phi_H(x)$ $\phi_L = \int_{0 \leq |k| \leq \mu}$

energy level $\phi_H = \int_{\mu \leq |k| \leq \Lambda_{UV}}$

$$e^{-S_{\text{eff}}(\phi_L, g_0)} := \int D\phi_H\, e^{-S(\phi_L + \phi_H, g_0)}$$

Goal : change the description at low energy scale μ to make the eff action "regular" under $\frac{\Lambda_{UV}}{\mu} \to \infty$.

In many cases, $g_0 = g_0(g, \frac{\Lambda_{UV}}{\mu})$

$\phi_0(x) = Z(g, \frac{\Lambda_{UV}}{\mu}) \phi(x) + \phi_H(x)$

$\phi(x), g$ are new variables.

$$\beta(g) := \mu \frac{d}{d\mu} g(\underset{\text{old}}{g_1}, \underset{\text{new}}{\frac{\mu}{\mu_1}}) \Big|_{\substack{g_1 = g \\ \mu_1 = \mu}}$$ beta function for coupling const. g .]

For σ-model, clearly at 1-loop level

$$\beta_{IJ} = -\mu \frac{d}{d\mu} \hat{g}_{IJ} = \frac{1}{2\pi} R_{IJ} \qquad \text{Ricci is crucial}$$

- $R_{ij} > 0$: Asymptotically free as $\Lambda_{UV} \to \infty$
  since $\hat{g}_{IJ} = g_{IJ} + \frac{1}{2\pi} \log \frac{\Lambda_{UV}}{\mu} R_{IJ} \nearrow$
  the perturbation theory becomes better
- $R_{ij} = 0$: Scale invariance at 1-loop level
  In fact, for 2-loop only involve ∇R_{IJ}
  hence also $= 0$. But $\beta \neq 0$ for 4-loops.
- $R_{ij} < 0$: Ultra violet singularity
  the σ-model is NOT a well-defined theory

The F-term non-renormalization theorem

e.g. $W(\bar{\Phi}, m, \lambda) = m\bar{\Phi}^2 + \lambda \bar{\Phi}^3 \longmapsto W_{eff}(\bar{\Phi}) = ?$

Seiberg: promote m, λ to chiral s.f. M, Λ get $W(\bar{\Phi}, M, \Lambda)$
this is now quasi-homog. Take $K_M + K_\Lambda = (\bar{M}M + \bar{\Lambda}\Lambda)/\varepsilon$
assigne vector R charge $(1, 0, -1)$ to $\hat{\Phi}, M, \Lambda$ get W R-charge 2
another $U(1)$ symmetry : $(1, -2, -3)$ fixes W, anomaly free
constraints for $W_{eff}(\bar{\Phi}, M, \Lambda)$: same sym. holo. asymp. behavior to classical value

$\Rightarrow \quad W_{eff}(\bar{\Phi}, M, \Lambda) = M\bar{\Phi}^2 f(t) \qquad t = \Lambda\bar{\Phi}/M$

when $\Lambda = \alpha\Lambda_*$, $M = \alpha M_*$, $\alpha \to 0$ get $t_* = \Lambda_* \bar{\Phi}/M_*$

$W_{eff} \to M\bar{\Phi}^2 f(t_*)$ classical value $\Rightarrow f(t) = 1 + t$. Now $\varepsilon \to 0$.

HW 7.3 Do the LG model for multi-variables.

CFT := fixed pt of RG flow, i.e. scale-inv. QFT

Rmk: in current (1+1)-D case, get ∞-dim gp of sym. (Virasoro)
whose generators $L_n = -z^n \cdot z\frac{d}{dz}$ (z = coor. on Σ) $= \frac{1}{2}\sum_m :\alpha_m \alpha_{n-m}:$
in free field th.

$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12}(n^3 - n)\,\delta_{n+m,0}$

Conj 1. CY dim D, Ricci flat $\xrightarrow[\text{flows}]{}$! CFT with $c = 3D$.
1-loop

2. LG to be CFT $\Rightarrow$ W is quasi-homog. $\overbrace{\text{nec}}^{\text{not Ricci flat}}$.
with suitable D-term $\xrightarrow[\text{flows}]{}$! CFT.

lect 8. 4/27 2011 (In "eg": let $\Sigma = \mathbb{R} \times S^1$) p. 29

- chiral rings. let $Z = \tilde{Z} = 0$ (central charge)

$Q = (Q_B := \bar{Q}_+ + \bar{Q}_-)$ or $(Q_A := \bar{Q}_+ + Q_-)$ $\Rightarrow$ $Q^2 = 0$

"eg." Q-coh of states $\simeq$ SUSY ground states (cf. ch. 13.3)

in Kähler σ-models M, $\simeq \mathbb{H}^*(M)$. har. forms

Q-coh on chiral op / tw-chiral op $\mathcal{O}$: $[Q_B, \mathcal{O}] = 0$ / $[Q_A, \mathcal{O}] = 0$ i.e. commuting

"eg." let chiral s.f. $\Phi = \phi + \theta^\alpha \psi_\alpha + \theta^+\theta^- F$ on $y^\pm$

then $[\bar{Q}_+, \phi] = 0 = [\bar{Q}_-, \phi]$. Similarly for tw. ch. s.f.

direct check: $\shortparallel\ \left(-\frac{\partial}{\partial\bar{\theta}^+} - i\theta^+\partial_+\right)\left(\phi - i\theta^\pm\bar{\theta}^\pm\partial_\pm\phi - \theta^+\theta^-\bar{\theta}^-\bar{\theta}^+\partial_+\partial_-\phi\right)$

Q comm with $\mathcal{O}_1, \mathcal{O}_2$ $\Rightarrow$ Q comm with $\mathcal{O}_1\mathcal{O}_2$ how ϕ at $x^\pm$.

Def'n: (tw)-chiral ring $C(Q)$ = Q-coh ring of (tw-) chiral op.

"eg." World sheet transl. leads to Q-boundary:

$$i\partial_-\mathcal{O} = \frac{i}{2}\left(\frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^1}\right)\mathcal{O} = [H-P, \mathcal{O}] = [\{Q_-, \bar{Q}_-\}, \mathcal{O}]$$

$$= -\{[\bar{Q}_-, \mathcal{O}], Q_-\} - \{[\mathcal{O}, Q_-], \bar{Q}_-\} \longmapsto \{Q_B, [Q_-, \mathcal{O}]\}.$$

$$\{\overset{\shortparallel}{[\bar{Q}_+, \mathcal{O}]}, Q_-\} = -\{[\mathcal{O}, Q_-], \bar{Q}_+\} - [\underset{\shortparallel\ \tilde{Z} = 0}{\{Q_-, \bar{Q}_+\}}, \mathcal{O}]$$

- Twisting. How to proceed if Σ is not flat?

Recall SUSY action $\delta_\epsilon S = \int_\Sigma \left((\nabla_\mu\epsilon_+)G^\mu_- - (\nabla_\mu\bar{\epsilon}_+)\bar{G}^\mu_- - (\nabla_\mu\epsilon_-)G^\mu_+ + (\nabla_\mu\bar{\epsilon}_-)\bar{G}^\mu_+\right)\sqrt{h}\,d^2x$

eg. $\bar{G}_+ = \bar{G}^\mu_+\partial_\mu \in \Gamma(\Sigma, \phi^* T^{01} \otimes S_+)$

require $\epsilon_+, \bar{\epsilon}_+ \in \Gamma(S_+)$ $\quad S_+ \simeq \bar{K}^{1/2}$, $S_- \simeq K^{1/2}$

$\epsilon_-, \bar{\epsilon}_- \in \Gamma(S_-)$ but it's impossible to have cov. const. sections

if one of $U(1)_A$ or $U(1)_V$ exists; if (Σ, h) is not flat

	$U(1)_V$	$U(1)_A$	$U(1)_E$	$\mathcal{L}$	A twist by F_V		B twist by F_A	
					$U(1)_E'$	$\mathcal{L}$	$U(1)_E'$	$\mathcal{L}$
ϕ	0	0	0	$\mathbb{C}$	0	$\mathbb{C}$	0	$\mathbb{C}$
$\psi_- \sim Q_-$	-1	1	1	$K^{1/2}$	0	$\mathbb{C}$	2	K
$\bar{\psi}_+ \sim \bar{Q}_+$	1	1	-1	$\bar{K}^{1/2}$	0	$\mathbb{C}$	0	$\mathbb{C}$
$\bar{\psi}_- \sim \bar{Q}_-$	1	-1	1	$K^{1/2}$	2	K	0	$\mathbb{C}$
$\psi_+ \sim Q_+$	-1	-1	-1	$\bar{K}^{1/2}$	-2	$\bar{K}$	-2	$\bar{K}$

i.e. replace $U(1)_{\text{Euclidean}}$ by diagonal $U(1)_E'$ in $U(1)_E \times U(1)_R$

Topological nature of twisted th.

(Hilbert space (hence operators) is same as original
(physical observable $\longleftrightarrow$ Q-coh classes $\longleftrightarrow$ (tw) chiral ring

- correlation fun $\langle \{Q, O_1] O_2 \cdots O_n \rangle = \langle [Q, O_1 \cdots O_n] \rangle = 0$

world sheet metric $\delta_h \langle \prod O_i \rangle \underset{\uparrow}{=} \langle \frac{1}{4\pi} \int \sqrt{h}\, d^2x \, \underline{\delta h^{\mu\nu} \{Q, G_{\mu\nu}\}} \prod O_i \rangle \underset{0}{\shortparallel}$

generate SUSY

from path int. pf omitted

deformation of D-term : eg. for B twist :

$\delta\langle \cdots \rangle \mapsto$ insertion $\langle \cdots \int d^4\theta\, \Delta K \rangle$ in path int.

why ? $\{ \bar{Q}_+, [\bar{Q}_-, \int d\theta^+ d\theta^- \Delta K |_{\bar{\theta}^{\pm}=0}]\} = \{Q, [\cdots]\}$

Only dep on chiral parameters holomorphically .

- A-twist of non-linear σ-model ; $W = 0$

$\phi : \Sigma \to X$ Kähler ; change spin $\psi_-, \bar{\psi}_+$ scalar $\mapsto \chi^i, \chi^{\bar{i}}$

locally, $\Phi^i = \phi^i + \theta^\alpha \psi^i_\alpha + \cdots$ $\quad \psi_+$ in $\bar{K}$, $\bar{\psi}_-$ in K $\quad \searrow \rho^i_{\bar{z}} \quad \searrow \rho^{\bar{i}}_z$

$$S = \int d^2z \Big(g_{i\bar{j}} h^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^{\bar{j}} \sqrt{h} - i g_{i\bar{j}} \rho^{\bar{j}}_z D_{\bar{z}} \chi^i + i g_{i\bar{j}} \rho^i_{\bar{z}} D_z \chi^{\bar{j}} - \tfrac{1}{2} R_{i\bar{k}j\bar{l}} \rho^i_{\bar{z}} \chi^j \rho^{\bar{k}}_z \chi^{\bar{l}} \Big)$$

cov. diff. contains $\partial_\mu \phi^i$ etc.

$\delta = \bar{\epsilon}_- \bar{Q}_+ + \epsilon_+ Q_-$ (set $\bar{\epsilon}_- = \epsilon_+ = \epsilon$ for $Q_A = \bar{Q}_+ + Q_-$ variation)

is given by $\delta\phi = \epsilon\chi$, $\delta\chi = 0$, $\delta\rho^i_{\bar{z}} = 2i\bar{\epsilon}_- \partial_{\bar{z}} \phi^i + \epsilon_+ \Gamma^i_{jk} \rho^j_{\bar{z}} \chi^k$

We restrict ourself to operators made up of ϕ, χ only, no ρ in the quantum theory. We set $\phi^i \leftrightarrow z^i$, $\chi^i \leftrightarrow dz^i$, $\chi^{\bar{i}} \leftrightarrow d\bar{z}^i$

(cf. QFT in $d=1$) $\quad Q_- \leftrightarrow \partial$, $\bar{Q}_+ \longleftrightarrow \bar{\partial}$, $Q = \partial + \bar{\partial} = d$

$\{$ scalar op's, $Q \} \simeq H^*_{dR}(X)$ in sp level

correlation : $\langle O_1 \cdots O_s \rangle = \sum_{\beta \in H_2(M,\mathbb{Z})} \langle \prod O_i \rangle_\beta = \sum_\beta \int_{\phi_*[\Sigma] = \beta} D\phi\, D\chi\, D\rho\, e^{-S} O_1 \cdots O_s$

let $O_i \leftrightarrow \omega_i \in H^{p_i, q_i}(X)$, then $\langle \prod O_i \rangle_\beta \neq 0$ only if

$\{$ $q_V = -p_i + q_i$ sym fixed ie $\sum p_i = \sum q_i$

$\{$ $q_A = p_i + q_i$ sym broken ie. $\sum (p_i + q_i) = 2 \cdot \underset{\chi : 0\text{-mode} - \rho : 0\text{-mode}}{\underset{\shortparallel}{\text{index}\, \bar{\partial}}} = \underset{2k}{\underset{\shortparallel}{2\big(c(X) \cdot \beta + \dim X (1-g)\big)}}$

$\Rightarrow$ $\sum p_i = \sum q_i = k$, the exp. dim from R.R.

localization to Q fixed pts $\mapsto$ $\partial_{\bar z}\phi^i = 0$ & $\chi = 0$

$\langle \prod O_i \rangle_\beta = e^{-(\omega - iB)\beta} \int_{M_\Sigma(X,\beta)} e(V)\cdot\prod ev_i^*\omega_i$ $\qquad D_i = PD(\omega_i)$

$= n_{\beta, D_1, \dots, D_s}$ $\qquad \| $ e.g. δ_{D_i}

Boson & Fermion det cancel.

$V = 0$, if no ρ-zero mode (generic case).

otherwise they form bundle V' and get $Pf(F_V) = e(V)$.

- B-twist of LG model

M is a non-cpt CY, $W: M \to \mathbb{C}$ holo

change spin: scalar $\bar\psi^i_\pm \mapsto \psi^{\bar i} := \bar\psi^{\bar i}_-$, $\bar\psi^{\bar i} := \bar\psi^{\bar i}_+$ (all $\bar i$)

$\rho^i_z := \psi^i_-$ in K, $\rho^i_{\bar z} := \psi^i_+$ in $\bar K$ (all i)

$$S = \int d^2z \Big(\sqrt{h}\,|D\phi|^2 - i g_{i\bar j}\psi^{\bar j} D_{\bar z}\rho^i_z + i g_{i\bar j}\bar\psi^{\bar j} D_z \rho^i_{\bar z} - \tfrac{1}{2} R_{i\bar k j\bar l}\,\rho^i_{\bar z}\rho^j_z \psi^{\bar k}\bar\psi^{\bar l} + \tfrac{1}{4}|dW|^2 + \tfrac{1}{2}(D_i\partial_j W)\rho^i_{\bar z}\rho^j_z + \tfrac{1}{2}(D_{\bar i}\partial_{\bar j}\bar W)\psi^{\bar i}\bar\psi^{\bar j} \Big)$$

under $\delta = \bar\epsilon_-\bar Q_+ - \bar\epsilon_+\bar Q_-$, set $\bar\epsilon_+ = -\bar\epsilon_- = \bar\epsilon$, $Q_B = \bar Q_+ + \bar Q_-$

Then for M flat, δ is given by: $\delta\phi^i = 0$, $\delta\bar\phi^{\bar i} = -\bar\epsilon(\psi^{\bar i} + \bar\psi^{\bar i})$

$\delta(\psi^{\bar i} - \bar\psi^{\bar i}) = \bar\epsilon\, g^{\bar i j}\partial_j W$, $\delta(\psi^{\bar i} + \bar\psi^{\bar i}) = 0$

$\delta\rho^i_z = 2i\bar\epsilon\,\partial_z\phi^i$, $\delta\rho^i_{\bar z} = -2i\bar\epsilon\,\partial_{\bar z}\phi^i$

localization to Q-fixed pts $\mapsto$ $\partial_z\phi^i = 0 = \partial_{\bar z}\phi^i$ & $\partial_i W = 0$

i.e. const. map to Crit(W): assume isolated, non-deg. $y_1, \dots, y_N$.

correlation: O_p = hol. fun in ϕi, i.e. in M, $f \mapsto O_f$ $\qquad$ why?

$$\langle \prod_{i=1}^{s} O_{f_i} \rangle = \int D\phi\, D\psi\, D\rho \cdot e^{-S}\cdot \prod O_{f_i} = \sum_{i=1}^{N} \langle O_{f_1}, \dots, O_{f_s} \rangle \big|_{y_i}$$

At each y_i, const. mode kills kinetic term

Boson/Fermion non-const. modes has det canceled

but const. modes not paired:

$$\Rightarrow \int d^{2n}\phi\; e^{-\frac{1}{4} g^{i\bar j}\partial_i W \partial_{\bar j}\bar W} = \det W_{ij}^{-2}(y_i) \quad \text{by CVF } u_i = \partial_i W$$

normalized measure by $(1/\sqrt{2\pi})^{2n}$.

$$\int d^n\bar\psi\, d^n\psi\; e^{-\frac{1}{2}\overline{W_{ij}}\psi^{\bar i}\bar\psi^{\bar j}} = \overline{\det W_{ij}}(y_i)$$

fermion integral, notice $\psi^{i''}\overline{\psi}^{j''}$

$$\int d^{ng}\rho\, d^{ng}\bar\rho\; e^{-\frac{1}{2} W_{ij}\rho^i_{\bar z}\rho^j_z} = (\det W_{ij})^g (y_i) \quad \text{since } h^0(K) = g$$

i.e. $$\langle O_{f_1} \cdots O_{f_s} \rangle_g = \sum_{i=1}^{N} f_1(y_i)\cdots f_s(y_i)\cdot\big(\det W_{ij}(y_i)\big)^{g-1}.$$

for $g(\Sigma) = g = 0$, we get

3pt fcns $C_{ijk} = \sum_{dW=0} \frac{f_i f_j f_k}{\det W''}$

top. metric $\eta_{ij} = \sum_{dW=0} \frac{f_i f_j}{\det W''}$

to define the chiral ring

HW 8.1 This gives the Jacobi ring $J(W) = \mathbb{C}[\phi^1, \cdots, \phi^n]/(\partial_i W)$.
for $X = \mathbb{C}^n$. $(Q = ?)$

- B-twist of Calabi-Yau σ-model, $W = 0$

convenient variables $\eta^{\bar{i}} := -(\psi^{\bar{i}} + \bar{\psi}^{\bar{i}})$, $g^{\bar{i}j}\theta_j := \psi^{\bar{i}} - \bar{\psi}^{\bar{i}}$

q_B variation: $\delta\phi^i = 0$, $\delta\bar{\phi}^{\bar{i}} = \bar{\epsilon}\eta^{\bar{i}}$, $\delta\theta_i = 0$, $\delta\eta^{\bar{i}} = 0$

$$\delta\rho_\mu^i = \pm 2i\bar{\epsilon}\partial_\mu\phi^i .$$

physical operators: $\eta^{\bar{i}} \longleftrightarrow d\bar{z}^{\bar{i}}$; $\theta_i \longleftrightarrow \frac{\partial}{\partial z^i}$ Fermion

$q = Q_B = \bar{\partial}$

every time to quantize scalar field

get Dolbeault complex: $\Omega^{0,q}(M, \Lambda^p T) \xrightarrow{\bar{\partial}} \Omega^{0,q+1}(M, \Lambda^p T)$

correlation: By localization only at $\partial_\mu\phi^i = 0$ i.e. const. map

so $\langle O_1 \cdots O_s \rangle$ should go to some "obj" over "M", with

ferm. int. dep on a choice of Ω^n on M. so in fact a section of bundle over cpx moduli

eg. For $n=3$, $O_i \leftrightarrow \mu_i \in H^1(T_M)$, $\langle O_1 O_2 O_3 \rangle = \int_M \mu_1^i \wedge \mu_2^j \wedge \mu_3^k \Omega_{ijk} \wedge \Omega$.

- Variations of vacuum bundle : $V = \ker Q \cap \ker Q^\dagger$

top basis of states $\phi_i|0\rangle = |i\rangle$

connection $(A_i)_j^k = \langle k|\partial_i|j\rangle$

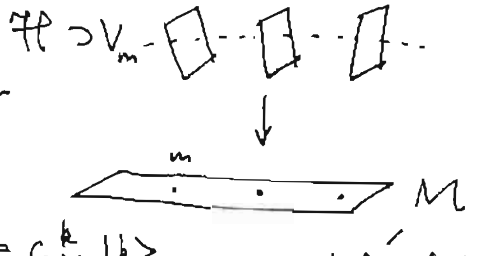
Top. dep. Fact: $(A_{\bar{i}})_j^k = 0$

$$\partial_{\bar{k}}\eta_{ij} = 0, \quad \partial_{\bar{l}} C_{ij}^k = 0.$$

Theorem (tt* equations)
Let $D_i = \partial_i - A_i$ then
the improved conn

$$\nabla^\alpha = D + \alpha C, \quad \bar{\nabla}^\alpha = \bar{D} + \alpha^{-1}\bar{C}$$

is flat.



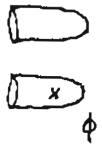
moduli of QFT
cpx mfd

exist by axiom of QFT
⇓
so $\phi_i|j\rangle = C_{ij}^k|k\rangle$

in fact,

$|0\rangle$ = path int. on

$|\phi_i\rangle$ = ...



top metric $\eta_{ij} = \langle i|j\rangle$

her. metric $g_{i\bar{j}} = \langle i|\bar{j}\rangle$

$$\Rightarrow A_i = g^{-1}\partial_i g$$

lect 9. 5/4, 2011

SUSY Gauge th.

cpx $U(\phi) = \frac{e^2}{2}\left(\sum_i^N |\phi_i|^2 - r\right)^2$

Scalar field case $L = -\sum_{i=1}^{N} |\partial_\mu \phi_i(x)|^2 - U$ inv under $\phi_i(x) \mapsto e^{i\alpha}\phi_i(x)$

but if $\alpha = \alpha(x)$, then Need $D_\mu \phi_i := \partial_\mu \phi_i + i v_\mu \phi_i$ (Gauge field)

with $v_\mu \mapsto v_\mu - \partial_\mu \alpha$ then $L = -\sum |D_\mu \phi|^2 - U$ is inv. real v

since now $\partial_\mu \phi_i \mapsto e^{i\alpha}(\partial_\mu + i\partial_\mu \alpha)\phi_i$

HW 1: The massless mode is $S^{2N-1}/U(1) \simeq \mathbb{C}P^{N-1}$ with FS metric, show $v_\mu = \frac{i}{2} \cdot \frac{\sum_{i=1}^{N} \bar{\phi}_i \partial_\mu \phi_i - (\partial_\mu \bar{\phi}_i)\phi_i}{\sum_{i=1}^{N} |\phi_i|^2}$

Apply the idea to chiral s.f. Φ

$$L = \int d^4\theta\, \bar{\Phi}\Phi \quad \text{under } \Phi \mapsto e^{iA}\Phi, \text{ A also a chiral s.f.}$$

$$\searrow \bar{\Phi} e^{-i\bar{A}+iA}\Phi$$

, consider real s.f.

$$V(x^\mu, \theta^\pm, \bar{\theta}^\pm) \longmapsto V + i(\bar{A} - A)$$

HW2: Under suitable gauge and $L = \int d^4\theta\, \bar{\Phi} e^V \Phi$ is inv.

(= Wess-Zumino), $V = \theta^\mp \bar{\theta}^\mp (v_0 \mp v_1) - \theta^- \bar{\theta}^+ \sigma + i\theta^- \theta^+ (\bar{\theta}^- \bar{\lambda}_- + \bar{\theta}^+ \bar{\lambda}_+) + \theta^- \theta^+ \bar{\theta}^+ \bar{\theta}^- D$

real 1-form, cpx scalar, +c.c., cpx Dirac ferm., real

The SUSY $\delta = \pm\epsilon_\pm Q_\mp \mp \bar{\epsilon}_\pm \bar{Q}_\mp$ on comp fields of Φ & V is determined:

The s.f. strength (curvature) of V is $\Sigma := \bar{D}_+ D_- V$, Σ is tw-chiral!

$$\text{and} \quad \Sigma = \sigma(\tilde{y}) \pm i\theta^\pm \text{"}\bar{\lambda}_\pm\text{"}(\tilde{y}) + \theta^+\bar{\theta}^-\left(D(\tilde{y}) - i v_{01}(\tilde{y})\right)$$

with $\tilde{y}^\pm = x^\pm \mp i\theta^\pm \bar{\theta}^\pm$ & $v_{01} = \partial_0 v_1 - \partial_1 v_0$ (curv. of v).

Now the SUSY Gauge-inv Lagrangian:

$$L = \int d^4\theta \left(\bar{\Phi} e^V \Phi - \frac{1}{2e^2}\bar{\Sigma}\Sigma\right) + \frac{-t}{2} \cdot \int d^2\tilde{\theta}\, \Sigma + c.c.$$

L_{kin}, L_{gauge}, e coupling const., tw chiral F term, $t := r - i\theta$ - theta angle, Fayet-Iliopoulos parameter

under (0,2) change to Σ, get $U(1)_V \times U(1)_A$ sym for classical system

As before, eliminating F and D from eq of motion, get

$$L = -D^\mu \bar{\phi} D_\mu \phi + i\bar{\psi}_\mp (D_0 \pm D_1)\psi_\mp - \left(|\sigma|^2|\phi|^2 + \frac{e^2}{2}(|\phi|^2 - r)^2\right)$$

$$-\bar{\psi}_- \sigma \psi_+ - \bar{\psi}_+ \bar{\sigma} \psi_- \mp i\bar{\phi}\lambda_\mp \psi_\pm \pm i\bar{\psi}_\pm \bar{\lambda}_\mp \phi$$

$$+ \frac{1}{2e^2}\left(-\partial^\mu \bar{\sigma}\partial_\mu \sigma + i\bar{\lambda}_\pm (\partial_0 \pm \partial_1)\lambda_\mp + v_{01}^2\right) + \theta v_{01}$$

$=: U$ potential for scalar fields ϕ & σ

In general for $\Phi_1, \cdots, \Phi_N$ under $U(1)^k = \prod_{a=1}^{k} U(1)_a$: $\Phi_i \mapsto e^{iQ_{ia}A_a}\Phi_i$

get
$$L = \int d^4\theta \left(\sum_{i=1}^{N} \bar{\Phi}_i e^{Q_{ia}V_a}\Phi_i - \sum_{a,b=1}^{k} \frac{1}{2e_{ab}^2}\bar{\Sigma}_a \Sigma_b\right) + \frac{1}{2}\left(\int d^2\tilde{\theta} \sum_{a=1}^{k} -t_a \Sigma_a + c.c.\right)$$

If $\exists\, W(\bar{\Phi}_i)$ poly gauge-inv. then can have F-term

$$L_W = \int d^2\theta\, W(\bar{\Phi}_i) + c.c.$$

Eliminating D_a and F_i get $\quad$ inverse of e_{ab}^2

$$U = \sum_{a,\, i=1}^{N} |Q_{ia}\sigma_a|^2 |\phi_i|^2 + \sum_{\substack{a,b=1\\ i,j}}^{k} \frac{(e^{a,b})^2}{2}\left(Q_{ia}|\phi_i|^2 - r_a\right)\left(Q_{jb}|\phi_j|^2 - r_b\right) + \sum_{i=1}^{k}\left|\frac{\partial W}{\partial\phi_i}\right|^2$$

Quantum theory: consider $k=1$, $N=1$, charge $Q_{ia}=1$

effective th at scale μ; i.e. int over $\mu \le |k| \le \Lambda_{UV}$

The part on L related to D field before substitute eq of motion is: $\quad \frac{1}{2e^2}D^2 + D(|\phi|^2 - r_0)$

$$\int D\phi \;\longmapsto\; \frac{1}{2e^2}D^2 + D\left(\log\frac{\Lambda_{UV}}{\mu} - r_0\right)$$

FI parameter

For Λ_{UV}, r_0 fixed, $\Rightarrow$ $\overset{..}{=} r$ renormalized FI. ie. $r(\mu) = \log\frac{\mu}{\Lambda}$

Anomaly of $U(1)_A$: the sym is broken due to

$$-2i\bar{\Psi}_- D_{\bar{z}}\Psi_- + 2i\bar{\Psi}_+ D_z \Psi_+ \quad \left(\text{Euclidean version of } i\bar{\Psi}_\mp (D_0 \pm D_1)\Psi_\mp\right)$$

Wick rotation $x^0 \mapsto -ix^2$; $\pm D_1 - iD_2$

$$k = \text{"}\Psi_-\text{ zero"} - \text{"}\bar{\Psi}_-\text{ zero"} = c_1(E) \neq 0 \quad (v \text{ is a conn. of } E)$$

$$\text{So } D\Psi D\bar{\Psi} \longmapsto e^{-2ki\alpha} D\Psi D\bar{\Psi}$$

$\overset{..}{O}(-k)$ here.

$$\frac{i}{2\pi}\int (\theta\, v_{12})\, dx^1 dx^2 = +ik\theta$$

$>$ equiv to $\theta \mapsto \theta - 2\alpha$

General case: $b_a := \sum_{i=1}^{N} Q_{ia}$; $\quad r_a(\mu) = b_a \log\frac{\mu}{\Lambda} + \tilde{r}_a$; $\theta_a \mapsto \theta_a - 2b_a\alpha$

If $b_a = 0\ \forall a$ then $U(1)_A$ anomaly free & $t_a = r_a - i\theta_a$ are FI-theta parameters of the quantum theory.

Non-linear σ models from Gauge theory.

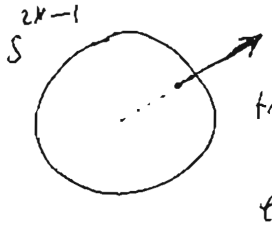
(1) $\mathbb{C}P^{N-1}$. This is the case $U(1)^{k=1}$ and $N=N$.

$$U = \sum_{i=1}^{N} |\sigma|^2|\phi_i|^2 + \frac{e^2}{2}\left(\sum_{i=1}^{N}|\phi_i|^2 - r\right)^2$$

only $r>0$ to admit classical SUSY vacua $\cong S^{2N-1}/U(1)$

$\sigma = 0$, $\phi_i = $ constant.

mass := eigenvalue of $\partial_i\partial_j U(\phi)$



transv. mass $\overset{?}{=}$ $\left.\frac{\partial^2}{\partial\rho^2}U(\rho)\right|_{\rho=\sqrt{r}} = e^2 4r$

$e\sqrt{2r}$ in the book p. 357. Tangent of vacuum mfd is massless.

The gauge field also has mass $e\sqrt{2r}$.

If $\psi_{i\pm}, \bar{\psi}_{i\pm}$ satisfy

$$\sum_{i=1}^N \bar{\phi}_i \psi_{i\pm} = 0 = \sum_{i=1}^N \bar{\psi}_{i\pm} \phi_i \quad (\text{tangent of } \mathbb{C}P^{N-1} \text{ at } \phi_i)$$
, it has mass 0

other modes and $\lambda, \bar{\lambda}$ (Fermion in V) has mass $e\sqrt{2r}$.

Now let $e \to \infty$, system decouple

classical theory reduced to massless modes only

claim: This is the non-linear σ model to $\mathbb{C}P^{N-1}$

classical: A direct check on L, eg. $ds^2 = \frac{r}{2\pi} g^{FS}$.

quantum: The effective th of massless mode is by int out the massive mode $M = e\sqrt{2r}$ (if $\mu \ll e\sqrt{2r}$) & massless mode in $\mu < |k| < \Lambda_{UV}$

From $r(\mu) = \underset{\overset{\shortparallel}{N}}{\left(\sum_{i=1}^N Q_i\right)} \log \frac{\mu}{\Lambda} \Rightarrow r = r' + N \log \frac{\mu}{\mu'}$ (*)

Recall the RG flow for metric in NLSM: $\hat{g}_{I\bar{J}} = g_{I\bar{J}} + \frac{1}{2\pi} \log \frac{\Lambda_{UV}}{\mu} R_{I\bar{J}}$

Now $R_{i\bar{j}}$ for $\frac{r}{2\pi} g^{FS}$ is indep of r, $= N g^{FS}_{i\bar{j}}$ up to 1-loop

So $g'_{i\bar{j}} = \frac{1}{2\pi}\left(r - N \log \frac{\mu}{\mu'}\right) g^{FS}_{i\bar{j}}$, this agrees with (*)

Moreover, in this special case $[\omega - iB] = \left[\frac{r - i\theta}{2\pi} \omega^{FS}\right] = \frac{t}{2\pi}[\omega^{FS}]$

ie. the cpxified Kähler class is a tw-chiral parameter "t".

(2) Tonic Manifolds: $U(1)^k$, $N = N$, $e^2_{ab} := \delta_{ab} e_a^2$, no W.

The Vacuum mfd $X_r = \left\{(\phi_1, \cdots, \phi_N) \,\middle|\, \underset{\mu_a := \,\shortparallel}{\sum_{i=1}^N Q_{ia} |\phi_i|^2 - r_a} \underset{\overset{\shortparallel}{0}}{}, \ a = 1 \cdots k \right\} / U(1)^k$ moment map

as in (1), $e \to \infty$ get NLSM on X_r.

$\omega_{\mathbb{C}^N}$ descend to a sympl. form ω on X_r (sympl. reduction)

cpx str. $X_r \simeq X_P = \underset{\text{G.I.T quotient}}{(\mathbb{C}^N \setminus P) /\!/ (\mathbb{C}^\times)^k}$: P = set of pt whose $(\mathbb{C}^\times)^k$ orbit has no sol. $\{\mu_a = 0\}$

P depends on $r = (r_a)$.

Chapter 7:

P, X_P can be constructed from a fan Σ.

let Δ_Σ = convex hull of $\Sigma(1)$. Then X_Σ is Fano iff Δ_Σ is reflexive.

eg. $X = P(Q_1, \cdots, Q_N)$: $U(1)^{k=1}$, $Q_i \in \mathbb{N}$. weighted proj space

eg. $\sum Q_i > 0$ but only $Q_1, \cdots, Q_l > 0 \ni X = \left[\bigoplus_{j=l+1}^N \mathcal{L}^{Q_j} \to P(Q_1, \cdots, Q_l)\right]$ and r large.

eg. $\sum_{i=1}^{N} Q_i = 0$ then FI para r does not run. both $\begin{matrix} r>0 \\ r<0 \end{matrix}$ ok.
X is also a bundle space.

- $\mathcal{O}(-N) \to P^{N-1}$ vs $\mathbb{C}^N/\mathbb{Z}_N$: $\bar{\Phi}_1, \cdots, \bar{\Phi}_N$, $Q_i = 1$, $P : Q_p = -N$.
  $N|p|^2 = -r + \sum_{i=1}^{N} |\phi_i|^2$ for $r \ll 0$ $\longmapsto$ $\mathbb{C}^N/\mathbb{Z}_N$.
- $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \longrightarrow P^1$ in 2 ways. $\bar{\Phi}_1, \bar{\Phi}_2, \bar{\Phi}_3, \bar{\Phi}_4$, $Q_i = 1, 1, -1, -1$
  vacuum eq'n: $|\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 - |\phi_4|^2 = r$.

(3). Hypersurfaces (& complete int) in $\mathbb{C}P^{N-1}$:

Now we "turn on" some superpotential W in $U(1)$ Gauge th

let $W = G(\bar{\Phi}_1, \cdots, \bar{\Phi}_N) \cdot P$ $\quad$ $N+1$ chiral s.f. of charge $1, \cdots 1, -d$
generic homog poly deg $= d$ $\quad$ is Gauge-inv

$$L = \int d^4\theta \left(\sum \bar{\Phi}_i e^V \Phi_i + \bar{P} e^{-dV} P - \frac{1}{2e^2} \bar{\Sigma} \Sigma \right) + \frac{1}{2} \int d^2\tilde{\theta} \left(-t \Sigma \right) + c.c. + \frac{1}{2} \int d^2\theta \, P G(\bar{\Phi}_i) + c.c.$$

potential for scalar fields

$$U = |\sigma|^2 \sum |\phi_i|^2 + |\sigma|^2 d^2 |p|^2 + \frac{e^2}{2} \left(\sum |\phi_i|^2 - d|p|^2 - r \right)^2 + \frac{1}{4} \left| G(\phi_i) \right|^2 + \frac{1}{4} \sum |p|^2 |\partial_i G|^2$$

$r > 0$: $\quad U = 0 \Rightarrow \exists i, \phi_i \neq 0 \Rightarrow \sigma = 0 \Rightarrow p = 0$ (otherwise $G = 0 = \partial G_i$ $\Rightarrow \phi_i = 0$ $\ast$)

i.e. $\left\{ \sum |\phi_i|^2 = r , G(\phi_i) = 0 \right\} / U(1) \simeq M$: hyp surf def by $G = 0$.

Some fields have mass $e\sqrt{r}$ or a_I (coeff of W)

in a scaling st. $e, a_I \to \infty$ then the system goes to nonlinear σ model on M, $[\omega - iB] = \frac{t}{2\pi} [\omega^{FS}]|_M$.

$r < 0$: $\quad U = 0 \Rightarrow p \neq 0 \Rightarrow \sigma = 0 \Rightarrow \phi_i = 0$ $\Rightarrow$ $|p| = \sqrt{|r|/d}$ circle

$\Rightarrow$ vacuum mfd $= 1$ pt. $\quad$ let $\langle p \rangle := \sqrt{|r|/d}$ a vacuum value

$e \to \infty$ get LG theory $\quad$ $U(1)$ sym breaks to $\mathbb{Z}_d$

with $W = \langle p \rangle G(\bar{\Phi}_1, \cdots, \bar{\Phi}_N)$ with $\mathbb{Z}_d$ Gauge. i.e. LG orbifold.

$r = 0$: $\Rightarrow$ complex σ plane.

Quantum theory : renormalization of "FI para. r".

- $d < N$: $r(\mu) = (N-d) \log \frac{\mu}{\Lambda}$. take $\frac{e}{\Lambda}, \frac{a_I}{\Lambda} \to \infty$
  the system $\longmapsto$ non-linear σ-model on M. $c_1(M) = (N-d) H|_M$.
- $d = N$: The theory is para by $t = r - i\theta$; $c_1(M) = 0$.
  $r \gg 0 \longmapsto$ CY NLSM, t para cpxified Kähler class.
  $r \ll 0 \longmapsto e\sqrt{|r|} \to \infty$ to LG orbifold.
- $d > N$??.

Lect 10. 5/10 2011

"Physics Proof" of Mirror Symmetry by Hori-Vafa '2000.

Step 1. T-duality on a charged field

$$\text{GLSM } *: L = \int d^4\theta \Big(\underset{\text{chiral}}{\bar{\Phi}}\, e^{\underset{\text{real s.f.}}{2QV}} \Phi - \underset{\Sigma = \bar{D}_+ D_- V}{\tfrac{1}{2e^2}\bar{\Sigma}\Sigma} \Big) + \frac{1}{2}\Big(\int d^2\tilde{\theta}\, \big(\overset{r-i\theta}{-t}\,\Sigma\big) + c.c.\Big) \quad \text{tw. chiral}$$

Vacuum mfd $\{\phi \in \mathbb{C} \mid |\phi|^2 - r = 0\}/U(1) \simeq pt$

consider $L_0 = \int d^4\theta \big(e^{\underset{\in \mathbb{Z}}{2QV} + B} - \frac{1}{2}(\underset{\text{tw chiral}}{Y + \bar{Y}})\, \underset{\text{real s.f.}}{B} \big)$ Im Y is periodic in 2π

- $\int DY \Rightarrow \delta Y$ takes the form $\bar{D}_+ D_- Z \Rightarrow \bar{D}_+ D_- B = 0 = D_+ \bar{D}_- B$

HW 10.1 (Ex 12.1.3 $\partial\bar{\partial}$-lemma's for s.f.) $\Rightarrow B = \Psi + \bar{\Psi}$ chiral sf.

get $L_1 = \int d^4\theta\, \bar{\Phi} e^{2QV} \Phi \qquad \Phi := e^{\Psi}$

- $\int DB \Rightarrow \int [e^{2QV + B} - \frac{1}{2}(Y + \bar{Y})]\, \delta B = 0 \Rightarrow B = -2QV + \log \frac{Y + \bar{Y}}{2}$

$\int d^4\theta\, (Y + \bar{Y}) = 0 \Rightarrow$ get $L_2 = \int d^4\theta \Big(QV(Y + \bar{Y}) - \frac{Y + \bar{Y}}{2} \log(Y + \bar{Y}) \Big)$

but $\int d^4\theta\, V.Y = -\frac{1}{2}\int d\theta^+ d\bar{\theta}^-\, \bar{D}_+ D_- V.Y = \frac{1}{2}\int d^2\tilde{\theta}\, \Sigma Y$ tw. chiral comp.

So L is T-dual to

$$\tilde{L} = \int d^4\theta \Big(-\frac{1}{2e^2}\bar{\Sigma}\Sigma - \frac{Y + \bar{Y}}{2} \log(Y + \bar{Y}) \Big) + \frac{1}{2}\Big(\int d^2\tilde{\theta}\, \Sigma(QY - t) + c.c. \Big)$$

Quantum theory: for $\delta = \epsilon_+ Q_- + \bar{\epsilon}_- \bar{Q}_+$

From B, get $Y + \bar{Y} = 2\bar{\Phi} e^{2QV} \Phi$ eg. $\bar{\chi}_+ = 2\bar{\psi}_+ \phi$, $\bar{\chi}_- = -2\phi^\dagger \psi_-$

$y = \rho - i\vartheta$, $\bar{\chi}_+, \chi_-$ Fermi. $\phi = \rho e^{i\varphi}$, ψ_+, ψ_- Fermi., F

in Euclidean signature v, σ, λ, D

Fermion variation $= 0 \Longrightarrow \sigma = 0$, $D_{\bar{z}}\phi = 0$, $F_{12} = e^2(|\phi|^2 - r_0)$

Vortices (= instanton) $k := \frac{1}{2\pi}\int F_{12}\, d^2x$ top. number $\in \mathbb{Z}$

Dynamical generation of (twisted --) superpotential via vortices:

$U(1)_A$: $e^{i\alpha}$ acts on the top sector by $e^{2ik\alpha}$?
tw. potential has axial charge 2 $\succ\!\!\rightarrow$ only $k = 1$ contributes

Fermion zero modes for $k=1$ $\Rightarrow$ (heavy calculations)

$0 \neq \langle \chi_+(x)\, \bar{\chi}_-(y) \rangle$. This comes from $\int d^4\tilde{\theta} \cdot e^{-Y}$:

Since $\Delta\tilde{W}$ is hol in T, periodic in theta angle, R-sym.
with certain asymp. behavior.

So $\tilde{W} = \Sigma(QY - t) + e^{-Y}$.

Step 2. The mirror of toric varieties (here we do WPS)

for $U(1)^{k=n}$, $N=n$ $(\bar{\Phi}_1, \cdots, \bar{\Phi}_n)$

Get dual "effective superpotential" $\tilde{W} = \sum_{i=1}^{n} \left(Q_i Y_i - t_i\right)\Sigma_i + e^{-Y_i}$

Now keep only the diagonal action (set corr $\frac{1}{e_{ab}^2} = 0$, D-term vari)

This does not affect F-term

reduce to $U(1)^{k=1}$, $\tilde{W} = \left(\sum_{i=1}^{n} Q_i Y_i - t\right) \Sigma + \sum_{i=1}^{n} e^{-Y_i}$

$\Sigma_i = \Sigma \ \forall i$; $t = \sum t_i$

Now $\int D\Sigma$: i.e. solve $\partial_\Sigma \tilde{W} = 0$, get constraint $\sum_{i=1}^{n} Q_i Y_i = t$

with potential $\tilde{W} = \Sigma_{i=1}^{n} e^{-Y_i}$

The low energy limit is

thus NLSM on $P(Q_1, \cdots, Q_n)$ $\xleftrightarrow{\text{dual}}$ LG theory on variables Y_i.

Eg. $\mathbb{C}P^{n-1}$, $Q_i = 1\ \forall i$. Let $X_i = e^{-Y_i}$ then $\tilde{W} = X_1 + \cdots + X_n$ on $\prod_{i=1}^{n} X_i = e^{-t}$.

recall, t = FI-theta parameter = Kähler moduli of $\mathbb{C}P^{n-1}$

equiv. $\tilde{W}(X_1, \cdots, X_{n-1}) = X_1 + \cdots + X_{n-1} + \frac{e^{-t}}{X_1 \cdots X_{n-1}}$ (on $(\mathbb{C}^\times)^{n-1}$)

$D\tilde{W} = 0 \iff 1 - \frac{e^{-t}}{X_i(X_1 \cdots X_{n-1})} = 0 \iff X_i = \omega\, e^{-t/n} \ \forall i$

$\omega^n = 1$

\# crit $\tilde{W} = n \longleftrightarrow$ coh. basis of $H^*(\mathbb{C}P^{n-1})$.

identify: $H \in H^2(\mathbb{C}P^{n-1}) \longleftrightarrow -\partial_t \tilde{W}$

get $QH(\mathbb{C}P^{n-1}) \xleftrightarrow{\sim}$ tw. chiral ring

isom. as QFT chiral.

$H^n \longleftrightarrow \left(e^{-t} / (X_1 \cdots X_{n-1})\right)^n = e^{-t}$.

Rmk: $-t = -(r - i\theta) = i\theta - r = i(\theta + ir)$. $\mathrm{Re}\, t = r \to \infty$ get $H^n = 0$

(since $\mathbb{C}P^{n-1} \to$ flat).

Step 3. The hypersurface (or c.i.) case.

(*) consider GLSM onto non-cpt toric variety $r \gg 0$

$n+2$ chiral s.f. $(P, \bar{\Phi}_1, \cdots, \bar{\Phi}_{n+1})$ charge $= (-d, 1, \cdots, 1) \mapsto [\mathcal{O}(-d) \to \mathbb{C}P^n]$

low energy limit $\mapsto$ NLSM

$x_0 = e^{-P}$, $x_i = e^{-Y_i}$, $i=1 \dots n+1$

$\tilde{W} = x_0 + \dots + x_{n+1}$ with $x_0^{-d} x_1 \dots x_{n+1} = e^{-t}$

Redefine $\tilde{x}_i = x_i^{1/d}$, $1 \leq i \leq n+1$. $\Rightarrow$ $x_0 = e^{t/d} \tilde{x}_1 \cdots \tilde{x}_{n+1}$

So $\tilde{W} = \tilde{x}_1^d + \dots + \tilde{x}_{n+1}^d + e^{t/d} \tilde{x}_1 \cdots \tilde{x}_{n+1}$

with orbifold structure by $\mathbb{Z}_d^n \subset \mathbb{Z}_d^{n+1}$ preserving $\tilde{x}_1 \cdots \tilde{x}_{n+1}$

(since x_0 is well-defined)

eg. For $d=n+1$: Original = non-cpt CY.

$[\ \mathcal{O}(-(n+1)) \to \mathbb{C}P^n\ ] \xrightarrow{\text{dual}}$ LG with homg. $\tilde{W}$.

- To get cpt hyp. surface need to add potential $W = P \cdot G_d(\Phi_i)$, same d. view as perturbation terms of $(*)$.

$M = (G=0)$

for A-twist (Q-coh), $Q_A = \bar{Q}_+ + Q_-$ dep only on tw. chiral op.
hence does not dep on W-variation. (in lect 8. we set $W=0$)
but the "topology" for fields space may change .

$\mathcal{O}(-d) \to \mathbb{C}P^n$ coh class $1, k, k^2, \dots, k^n$; $|k\rangle_{nc} = [\mathbb{C}P^n] \leftrightarrow |1\rangle_c$

LG-Periods (= BPS mass) on dual (mirror) side

$$\Pi_{nc}^{\gamma} := \langle \gamma | 1 \rangle_{nc} = \int_\gamma d\tilde{Y}_1 \cdots d\tilde{Y}_{n+1} e^{-\tilde{W}} = \int_\gamma \frac{d\tilde{X}_1}{\tilde{X}_1} \cdots \frac{d\tilde{X}_{n+1}}{\tilde{X}_{n+1}} e^{-\tilde{W}}$$

$$\langle \gamma | 1 \rangle_c = \langle \gamma | k \rangle_{nc} = -\partial_t \Pi_{nc}^{\gamma} = \int_\gamma \frac{e^{td}}{d} \cdot d\tilde{x}_1 \cdots d\tilde{x}_{n+1} e^{-\tilde{W}}$$

the variables become $\tilde{X}_i$.

This is linearly related to Calabi-Yau periods when $d=n+1$

defined by $\tilde{W}=0$