GEOMETRY

(Honor Course, NTU 2016 Fall)

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Course Manuscript

CONTENTS

(based on Modern Geometry I by Dubrovin, Fomenko and Novikov)

complex analysis in surface theory

$\mathbb{R}^{2n} = \mathbb{C}^n \quad \begin{cases} x^k + \sqrt{-1}\, y^k = z^k \\ x^k - \sqrt{-1}\, y^k = \bar{z}^k \end{cases} \qquad dl^2 = \sum_{i=1}^n dz^k d\bar{z}^k$

$\frac{\partial}{\partial z^k} := \frac{1}{2}\left(\frac{\partial}{\partial x^k} - i \frac{\partial}{\partial y^k}\right)$ so that $\frac{\partial}{\partial z}(z) = 1 \quad \frac{\partial}{\partial z}(\bar{z}) = 0$

$\frac{\partial}{\partial \bar{z}^k} := \frac{1}{2}\left(\frac{\partial}{\partial x^k} + i \frac{\partial}{\partial y^k}\right) \qquad \frac{\partial}{\partial \bar{z}}(z) = 0 \quad \frac{\partial}{\partial \bar{z}}(\bar{z}) = 1$

Thm: A C^1 diff function $f(x,y) = u + iv$ is diff in cpx sense ie holomorphic $\Leftrightarrow$ $\frac{\partial}{\partial \bar{z}} f = 0$ ie. $u_x = v_y$, $v_x = -u_y$ (C-R eq^{n})

Hard Thm: holo in $\Omega \subset \mathbb{C}$ $\Rightarrow$ (cpx) analytic in Ω.

Fact: A real analytic $p(x^1, y^1; \dots, x^n, y^n) = q(z^1, \bar{z}^1, \dots, z^n, \bar{z}^n)$ is indep of $\bar{z}^k$ $\Leftrightarrow$ $\frac{\partial}{\partial \bar{z}^k} \cdot p \equiv 0$ look at the lowest degree term!

Def^{n}: f is (cpx) analytic in $\Omega \subset \mathbb{C}^n$ if f is real differentiable and analytic in each z^k.

$\Rightarrow$ The notion of cpx coordinate change

Lemma: For analytic F, $(z_1, \dots z^n) \xmapsto[F]{} (w^1(z^1, \dots, z^n), \dots, w^n(z^1, \dots, z^n))$ to be analytic

$$J_{\mathbb{R}} := \det(DF) = |J_{\mathbb{C}}|^2 \text{ where } J_{\mathbb{C}} = \det\left(\frac{\partial w^i}{\partial z^j}\right).$$

$\overset{\shortparallel}{A} = D^{\mathbb{C}} F$

pf: $$\det \begin{pmatrix} \frac{\partial w}{\partial z} & \frac{\partial \bar{w}}{\partial z} \\ \frac{\partial w}{\partial \bar{z}} & \frac{\partial \bar{w}}{\partial \bar{z}} \end{pmatrix} = \det \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} = |J_{\mathbb{C}}|^2$$

related to DF up to $(x^k, y^k) \leftrightarrow (z^k, \bar{z}^k)$ #

Corollary: Inverse and implicit function theorem for complex analytic functions.

pf: We only need to show the "real inverse" is cpx analytic but this is clear since $D^{\mathbb{C}} F^{-1} = (D^{\mathbb{C}} F)^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & \bar{A}^{-1} \end{pmatrix}$ #

Example: 2D surfaces as cpx curves in $\mathbb{C}^2$:

$C = \{ (w, z) \in \mathbb{C}^2 \mid f(w, z) = 0 \}$ f cpx analytic

$p = (w_0, z_0) \in C$ is non-singular if $\nabla^{\mathbb{C}} f := \left(\frac{\partial f}{\partial w}, \frac{\partial f}{\partial z}\right) \neq 0$ at p

say $f_w \neq 0$, then $\exists$ $w = w(z)$ uniquely near p st

$f(w(z), z) = 0$. $f_w \frac{dw}{dz} + f_z = 0$; $\frac{dw}{d\bar{z}} \equiv 0$.

Most important (classical) case: $f \in \mathbb{C}[w, z]$ polynomials.

1st fund. form induced from $\mathbb{C}^2$, Euclidean / Hermitian metric

$$dl^2 = |dw|^2 + |dz|^2 = \left(1 + \left|\frac{dw}{dz}\right|^2\right)|dz|^2 = h(z,\bar{z})\,dz\,d\bar{z}$$

2D –

$$= \lambda^2(x,y)\,(dx^2+dy^2)$$

Lemma : Conformal coor change $\Leftrightarrow$ holo or anti-holo.

pf: $(x,y) \overset{}{\underset{F}{\mapsto}} (u,v)$ $DF = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ is a 2×2 conformal matrix

linear algebra $\Rightarrow$ $= \rho\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \Rightarrow u_x = v_y \quad v_x = -u_y \Rightarrow \frac{\partial F}{\partial \bar{z}} = 0$

or $\rho\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \Rightarrow u_x = -v_y \quad v_x = u_y \Rightarrow \frac{\partial F}{\partial z} = 0$ #

Cor : Under conf. coor,

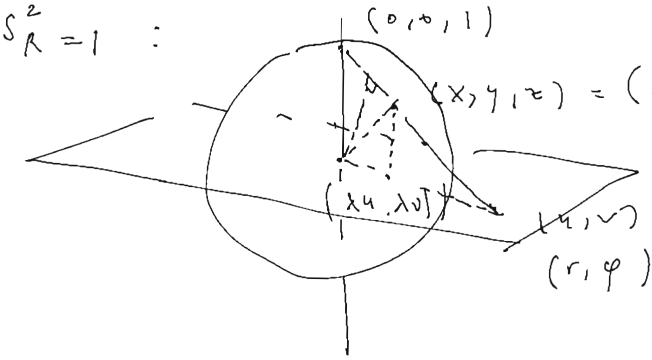
$dl^2 = \lambda^2(du^2+dv^2)$, get $K = -\frac{1}{AB}\left(\left(\frac{B_u}{A}\right)_u + \left(\frac{A_v}{B}\right)_v\right) = \frac{-1}{2\lambda^2}\Delta\log\lambda^2$

i.e. $A = B = \lambda$ in Ex 8.4 #7.

and $\sqrt{g} = \sqrt{EG-F^2} = \lambda^2$. $= -2\sqrt{g}^{-1}\frac{\partial^2}{\partial z\,\partial\bar{z}}\log\sqrt{g}$.

Now we are ready to analyse the Non-Euclidean geometry on S^2 and L^2 more closely!

$S^2_{R=1}$:



λ: conformal factor

$(x,y,z) = (\lambda u, \lambda v, 1-\lambda)$ Solve $\Rightarrow \lambda = \frac{2}{u^2+v^2+1}$

$$dl^2 = d\theta^2 + \sin^2\theta\, d\varphi^2 \qquad r = \cot\frac{\theta}{2}$$

$$= 4\sin^4\frac{\theta}{2}\,(dr^2 + r^2 d\theta^2)$$

$$= \frac{4}{(u^2+v^2+1)^2}(du^2+dv^2)$$

$$= \frac{4|dz|^2}{(1+|z|^2)^2}$$

CA $\Rightarrow$ Any analytic $w(z)$ on

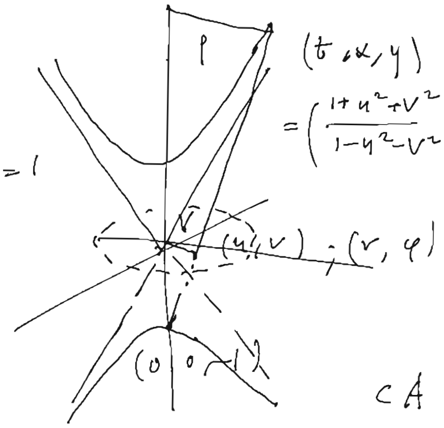
$S^2 \cong \mathbb{CP}^1 = \mathbb{C}\cup\{\infty\}$ is a Mobius transf $w = \frac{az+b}{cz+d}$ say $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$.

Easy to check: it preserves $dl^2 \Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)/\pm 1$

$L^2_{R=1}$:

$t^2 - (x^2+y^2) = 1$

in $\mathbb{R}^{1,2}$



$$= \left(\frac{1+u^2+v^2}{1-u^2-v^2}, \frac{2u}{1-u^2-v^2}, \frac{2v}{1-u^2-v^2}\right)$$

$dl^2 = d\chi^2 + \sinh^2\chi\, d\varphi^2 \quad (0 + \chi \in \mathbb{R})$

$K \equiv -1$ easily

geodesic $\Leftrightarrow$ χ = const

$$= \frac{4(du^2+dv^2|}{(1-u^2-v^2)} = \frac{4|dz|^2}{(1-|z|^2|^2}$$

in $\mathbb{D} = \{|z|<1\}$ Poincaré model.

CA Schwatz lemma $\Rightarrow$ $\mathrm{Aut}(\mathbb{D}) = \{e^{i\theta}\frac{z-a}{1-\bar{a}z}\}$

Easy to see this preserves dl^2.

Q: geodesics in Poincaré model ?

Möbius transf. $w = \frac{az+b}{cz+d} : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ det by 3 pts

$\alpha, \beta, \gamma \longmapsto 0, 1, \infty$

Set $w(z) := \frac{\beta-\gamma}{\beta-\alpha} \cdot \frac{z-\alpha}{z-\gamma}$

Easy to see $\frac{4\,|dz|^2}{(1+|z|^2)^2} = \frac{4\,|dw|^2}{(1+|w|^2)^2} \Longleftrightarrow \begin{pmatrix} a & b \\ c & a \end{pmatrix} \in SU(2)/\pm 1$

so the difficult part is : why $f : \mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^1 \Rightarrow f$ Möbius ?

Thm (Liouville) : f bounded hol on $\mathbb{C} \Rightarrow f = \text{const}$.

Pf : Cauchy $f'(a) = \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{(z-a)^2}\, dz \to 0$ as $R \to \infty$ ※

Cor. Any mero fcn on $\mathbb{P}^1$ is rational, $\underline{\text{degree } 1 \Rightarrow \text{Mobius}}$.

$\uparrow$ do this part is easier

Next we study $\text{Conf}^+(\mathbb{D})$:

Lemma : Mobius $f : \mathbb{D} \xrightarrow{\sim} \mathbb{D} \iff f = e^{i\theta} \frac{z-\alpha}{1-\bar{\alpha}z}$ $\alpha \in \mathbb{D}, \theta \in \mathbb{R}$

Pf : $|z-\alpha| < |1-\bar{\alpha}z| \iff |z|^2 + |\alpha|^2 < 1 + |\alpha|^2|z|^2$

i.e. $(1-|\alpha|^2)(1-|z|^2) < 0$ ※

Thm (Schwarz Lemma) (i) (ii)

$f : \mathbb{D} \to \mathbb{D}$, $f(0) = 0 \Rightarrow |f(z)| \le |z|$, $|f'(0)| \le 1$, "$=$" $\iff f = e^{i\theta} z$.

Pf : $\frac{f(z)}{z} : \mathbb{D} \to \mathbb{C}$ well defined at $z=0$ as $f'(0)$ $\left|\frac{f(z)}{z}\right|_{D_R} \le \frac{1}{R}$ $\forall R < 1$ by max principle

$R \to 1 \Rightarrow$ (i), for (ii) at $z_0 \in D_{R<1}$

$\Rightarrow f(z) = az$, $|a| = 1$ as expected ※

Cor : $\text{Conf}^+(\mathbb{D}) = \left\{ e^{i\theta} \frac{z-\alpha}{1-\bar{\alpha}z} \right\} = \text{Isometry}(\mathbb{D}) = SU(1,1)/\pm 1$

called oriented (directed, proper) isometries.

Klein model : $\mathbb{H} = \{ \text{Im}\, w > 0 \}$, $d\ell^2 = \frac{|dw|^2}{(\text{Im}\, w)^2}$

$$\begin{tikzcd} \mathbb{H} \arrow[d, "S"'] \\ \mathbb{D} \end{tikzcd}$$

$z = \frac{i-w}{i+w}$ $\text{Conf}^+(\mathbb{H}) = SL(2,\mathbb{R})/\pm 1$

Cor : $SU(1,1)/\pm 1 \simeq SL(2,\mathbb{R})/\pm 1 \simeq SO(1,2)^0$. (conn. component).

Thm: $dl^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2 = f(u,v)\left(du^2 + v^2\right)$
assuming E, F, G are real analytic in p, q.

"pf:" We try to find $\lambda(p,q)$ st $(g := EG - F^2)$, $\exists\, u, v$ with

$$\lambda\left(\sqrt{E}\,dp \pm \frac{F + i\sqrt{g}}{\sqrt{E}}\,dq\right) = du \pm i\,dv$$

i.e. $\lambda\sqrt{E} = u_p + i v_p$

$$\lambda\,\frac{F + i\sqrt{g}}{\sqrt{E}} = u_q + i v_q \quad \Rightarrow \quad (F + i\sqrt{g})\cdot(u_p + i v_p) = E\,(u_q + i v_q)$$

i.e. $F u_p - \sqrt{g}\,v_p = E u_q$ & $\sqrt{g}\,u_p + F v_p = E v_q$

so $u_p, u_q \longleftrightarrow v_p, v_q$ determine each other

eg. $v_p = \frac{1}{\sqrt{g}}\left(F u_p - E u_q\right)$

$$v_q = \frac{\sqrt{g}}{E}u_p + \frac{F}{E}v_p = \frac{(EG - F^2) + F^2}{E\sqrt{g}}u_p - \frac{F}{\sqrt{g}}u_q = \frac{1}{\sqrt{g}}\left(G u_p - F u_q\right)$$

$v_{pq} = v_{qp} \Rightarrow Lu = 0$ where $L := \partial_q\left(\frac{F\partial_p - E\partial_q}{\sqrt{g}}\right) + \partial_p\left(\frac{F\partial_q - G\partial_p}{\sqrt{g}}\right)$

is the Laplace-Beltrami operator $\square$

Rmk: for general dl^2, $\Delta_{dl^2} := \sum_{i,j} \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}\,g^{ij}\partial_j\right)$.

Liouville theory:

for $dl^2 = e^{\varphi}|dz|^2$, $K = -\frac{1}{2}\frac{1}{e^{\varphi}}\Delta\log e^{\varphi} \Rightarrow \Delta\varphi = -2Ke^{\varphi}$

Thm: A Surface of constant K is locally isometrically to S^2_R, $\mathbb{R}^2$, or L^2_R, where $R = 1/\sqrt{|K|}$.

Pf: We compute $\frac{\partial}{\partial\bar{z}}\left(\varphi_{zz} - \frac{1}{2}\varphi_z^2\right) = \varphi_{zz\bar{z}} - \varphi_z\varphi_{z\bar{z}}$

$$= -\frac{K}{2}\varphi_z e^{\varphi} + \frac{K}{2}\varphi_z e^{\varphi} = 0$$

$\Rightarrow \psi(z) := \varphi_{zz} - \frac{1}{2}\varphi_z^2$ is analytic

Lemma: Under $z = f(w)$ analytic change, get

$$\tilde{\varphi} = \varphi + \log|f'|^2 \quad \& \quad \tilde{\psi} = \psi\cdot f'^2 + \overbrace{\left(\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right)}^{\{f, z\}}.$$

Ex. Show that (1) $\{f, z\} = (f')^2 \{z, f\}$ and

(2) $\varphi'' - I\varphi = 0$ sol $\varphi_1, \varphi_2 \Rightarrow \{\varphi_1/\varphi_2, z\} = \frac{1}{2} I$.

Hence $\tilde{\varphi} \equiv 0$ above is solvable by some f.

Now $$(e^{-\tilde{\varphi}/2})_{ww} = \left(e^{-\tilde{\varphi}/2} \tfrac{-1}{2} \tilde{\varphi}_w\right)_w = \tfrac{-1}{2}\left(\tilde{\varphi}_{ww} - \tfrac{1}{2}\tilde{\varphi}_w^2\right) = \tfrac{-1}{2}\tilde{\Psi} \equiv 0$$

As a function in $w = u + iv$ this means that

$$(e^{-\tilde{\varphi}/2})_{uu} = (e^{-\tilde{\varphi}/2})_{vv} \quad \text{and} \quad (e^{-\tilde{\varphi}/2})_{uv} = 0$$

ie. $$e^{-\tilde{\varphi}/2} = a(u^2+v^2) + \beta u + \beta' v + c \qquad a, \beta, \beta', c \in \mathbb{R}$$
$$= a w\bar{w} + bw + \bar{b}\bar{w} + c \qquad b \in \mathbb{C}.$$

and $$d\ell^2 = e^{\tilde{\varphi}} |dw|^2 = \frac{|dw|^2}{(a|w|^2 + bw + \bar{b}\bar{w} + c)^2}$$

$$\Rightarrow K = -2 e^{-\tilde{\varphi}} \frac{\partial^2}{\partial w \partial \bar{w}} \log \frac{1}{(\cdots)^2} = 4 (\cdots)^2 . \partial_w \partial_{\bar{w}} \log(\cdots) = 4(ac - |b|^2)$$

Now it's easy find Möbius transf to get the result $\Rightarrow$ (Ex)

Transf. gp. as surfaces in $\mathbb{R}N$ (Classical Lie gps) or $\mathbb{C}^N$:

$G := GL(n, \mathbb{R}) \hookrightarrow \mathbb{R}^{n^2}$ by $\det A \neq 0$ $\quad \left\{ \begin{array}{ll} G \times G \longrightarrow G & (A,B) \mapsto AB \\ G \longrightarrow G & A \mapsto A^{-1} \end{array} \right.$

algebraic, hence analytic

$|A|^2 = \sum |a_{ij}|^2$

$|A + B| \leq |A| + |B|$

$|A \cdot B| \leq |A| \cdot |B| : \sum \left| \sum_k a_{ik} b_{kj} \right|^2 \leq \sum \sum_k |a_{ik}|^2 \sum_k |b_{kj}|^2$

$(1 - A)^{-1} = 1 + A + A^2 + A^3 + \cdots$ if $|A| < 1$: in fact, abs.

$\underset{n \to \infty}{\lim} B_n$ Cauchy in $\mathbb{R}^{n^2}$, converges

$(1 - A) . B_n = 1 - A^{n+1} \longrightarrow 1$ hence $\lim B_n = (1 - A)^{-1}$.

Rmk : I_n has nbd $I_n + X$ with $|X| < 1$ $\quad X = (x_{ij})$

B_0 has nbd $B_0(I_n + X) = B_0 + B_0 X = B_0 + Y \Rightarrow |Y| < |B_0|$

However for $|Y| < |B_0|$ it is NOT true that $|X| < 1$

from $X = B_0^{-1} Y$, we need $|Y| < |B_0^{-1}|^{-1}$ to get $|X| < 1$.

Prop : $GL(n, \mathbb{R}) \supset SL(n, \mathbb{R}) \supset SO(n)$ are non-singular surfaces with tangent plane at $e = I_n$: $\mathbb{R}^{n^2} \supset (\operatorname{tr} X = 0) \supset (X^T + X = 0)$.

Prop: $GL(n,\mathbb{C}) \begin{array}{l} \supset SL(n,\mathbb{C}) \supset \\ \supset U(n) \supset \end{array} SU(n)$ $\quad \mathbb{C}^{n^2} \begin{array}{l} \supset \operatorname{tr} X = 0 \\ \supset \bar{X}^T + X = 0 \end{array}$

Exponential Map: $T_eG \xrightarrow{\exp} G \quad X \mapsto e^X := \sum_{n=1}^{\infty} \frac{X^n}{n!}$ abs. conv.

Facts: $XY = YX \Rightarrow e^{X+Y} = e^X \cdot e^Y$

$e^X \in G$, ie. invertible, $(e^X)^{-1} = e^{-X}$.

Lemma: Behavior of exp on subgroups holds.

eg. $\det e^X = e^{\operatorname{tr} X}$ for SL by Jordan form or upper Δ.

for $O(n)$: $X^T + X = 0 \Rightarrow [X^T, X] = 0 \Rightarrow (e^X)^T e^X = e^{X^T+X} = I$

Lemma: $d\exp_0 = Id : T_eG \to T_eG \Rightarrow \exp$ 1-1, onto near e.

pf: $d\exp_0(X) = \frac{d}{dt} e^{tX}\Big|_{t=0} = X$ ※ ← 1 para. subgps.

Rmk: Globally, exp might not be 1-1, nor onto, even for connected G! (Ex. $G = SL(2,\mathbb{R})$)

$\mathbb{R}, \mathbb{C}, \mathbb{H}$ = Quaternions $\ni q = a + bi + cj + dk = z_1 + z_2 j$

skew field $\quad i^2 = j^2 = k^2 = -1, \quad ij = k$

Lemma: $A : \mathbb{H} \longrightarrow M(2,\mathbb{C})$ is a ring mono.

$$q \longmapsto A(q) := \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$$

Pf: For basis $A(i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $A(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $A(k) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

it's clear that $A(q_1 q_2) = A(q_1) A(q_2)$, hence done ※

Fact: $\bar{q} := a - bi - cj - dk \Rightarrow \overline{q_1 q_2} = \bar{q}_2 \bar{q}_1$. $q\bar{q} = |q|^2$ norm

$|q|^2 = 1 \iff |z_1|^2 + |z_2|^2 = 1$ ie. $\mathbb{H}_1 \simeq SU(2) \quad (\simeq S^3)$

Symplectic group $Sp(n)$ on $\mathbb{H}^n$, row vectors, acting on Right.

$$\begin{tikzcd} A + Bj = \Lambda \in GL(n,\mathbb{H}) \arrow[d, mapsto] \arrow[d, shift left=20, "c"] \\ \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} = c(\Lambda) \in GL(2n,\mathbb{C}) \end{tikzcd}$$

which preserves $\langle \xi_1, \xi_2 \rangle_{\mathbb{H}^n} := \sum_{k=1}^{n} q_1^k \bar{q}_2^k$.

$= \sum (x_1^k + y_1^k j)(\bar{x}_2^k + \bar{j}\bar{y}_2^k)$

$= \sum (x_1^k \bar{x}_2^k + y_1^k \bar{y}_2^k) + \sum (y_1^k x_2^k - x_1^k y_2^k) j$

ie. $U(2n)$ and preserves symp form $/\mathbb{C}$.

Liouville's thm on conformal maps

$\varphi : (U, x^i, g_{\alpha\beta}(x)) \longrightarrow (V, y^i, g'_{\alpha\beta}(y))$ is conformal

if $\varphi^* \sum g'_{\alpha\beta}(y)\, dy^\alpha dy^\beta = \rho^2(x) \sum g_{\gamma\delta}(x)\, dx^\gamma dx^\delta$ ← Linear Alg fact: $|Av| = \rho |v|$.

$$\overset{\shortparallel}{\sum \Big(\sum_{\alpha,\beta} g'_{\alpha\beta}(\varphi(x)) \frac{\partial \varphi^\alpha}{\partial x^\gamma} \frac{\partial \varphi^\beta}{\partial x^\delta} \Big) dx^\gamma dx^\delta}$$

(or anti -)

dim $=1$: no condition, dim $=2 \Leftrightarrow \varphi$ analytic, $\exists$ a lot!

Thm (Liouville)

for $\varphi : U \to V$ in $\mathbb{R}^{n \geq 3}$, $\varphi \in \langle$isometries, dilations, inversions$\rangle$.

pf: Consider behavior of <u>constant v.f.'s</u> under φ.

Let A = matrix function for $d\varphi = \left(\frac{\partial y^\alpha}{\partial x^\gamma} \right) = (\varphi_1, \cdots, \varphi_n)$

choose $\eta_1, \cdots, \eta_n \perp \Rightarrow \varphi_1, \cdots, \varphi_n \perp$. know $\langle A\eta, A\xi \rangle = \rho^2 \langle \eta, \xi \rangle$.

$$\begin{aligned} & 0 = \partial_k \langle \varphi_i, \varphi_j \rangle = \langle \varphi_{ik}, \varphi_j \rangle + \langle \varphi_i, \varphi_{jk} \rangle \\ + \; & 0 = \partial_j \langle \varphi_i, \varphi_k \rangle = \langle \varphi_{ij}, \varphi_k \rangle + \langle \varphi_i, \varphi_{kj} \rangle \\ - \; & 0 = \partial_i \langle \varphi_j, \varphi_k \rangle = \langle \varphi_{ji}, \varphi_k \rangle + \langle \varphi_j, \varphi_{ik} \rangle \end{aligned}$$

$\Rightarrow \langle \varphi_{jk}, \varphi_i \rangle = 0 \quad \forall\, i \neq j, k$

i.e. $\varphi_{jk} = \nu_j \varphi_j + \nu_k \varphi_k$

$$\nu_j = \frac{\langle \varphi_{jk}, \varphi_j \rangle}{\langle \varphi_j, \varphi_j \rangle} = \frac{1/2}{\langle \varphi_j, \varphi_j \rangle} \partial_k \langle \varphi_j, \varphi_j \rangle = \frac{1}{2} \frac{2\rho}{\rho^2} \rho_k = (\log \rho)_k =: \mu_k$$

j, k switched.

$(\mu := \log \rho)$

$\varphi_{jk} = \mu_k \varphi_j + \mu_j \varphi_k \Rightarrow \quad \varphi_{jki} = \mu_{ki} \varphi_j + \mu_k \varphi_{ji} + \mu_{ji} \varphi_k + \mu_j \varphi_{ki}$

$\Rightarrow \mu_{kj} \varphi_i + \mu_i \varphi_{jk}$ is sym in i, j, k

$$= \mu_{kj} \varphi_i + \mu_i \mu_j \varphi_k + \mu_i \mu_k \varphi_j \overset{\downarrow}{=} \mu_{ki} \varphi_j + \mu_j \mu_i \varphi_k + \mu_j \mu_k \varphi_i$$

$\cdot \varphi_i \Rightarrow \quad \mu_{kj} \rho^2 = \mu_j \mu_k \rho^2$ i.e. $\mu_{kj} - \mu_k \mu_j = 0$. From $\mu = \log \rho$ $= -\log \rho^{-1}$

$$\Rightarrow 0 = -\frac{(\rho^{-1})_{kj} \rho^{-1} - (\rho^{-1})_k (\rho^{-1})_j}{(\rho^{-1})^2} - \frac{(\rho^{-1})_k}{\rho^{-1}} \cdot \frac{(\rho^{-1})_j}{\rho^{-1}} = -\frac{(\rho^{-1})_{kj}}{\rho^{-1}}$$

i.e. $D^2 \rho^{-1}(\eta_k, \eta_j) = 0$ for any 2 orthogonal vectors (preserving $\perp$)

As in the original linear algebra fact $\Rightarrow \rho^{-1}_{\alpha\beta}(x) = \sigma(x)\, \underset{\substack{\uparrow \\ \text{constant in } x \\ \text{(Euclidean)}}}{g_{\alpha\beta}}(x)$.

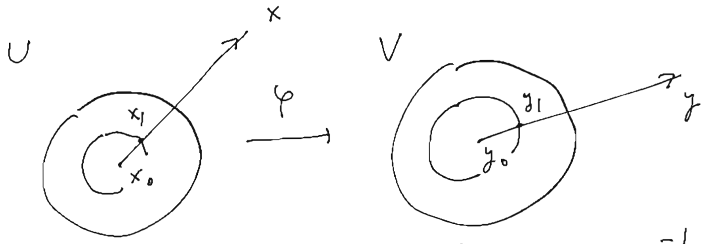
$$\rho^{-1}_{\alpha\alpha\gamma} = \sigma_\gamma \underset{\substack{\neq \\ 0}}{g_{\alpha\alpha}} = \sigma_\alpha \underset{\substack{\shortparallel \\ 0}}{g_{\gamma\alpha}} \Rightarrow \sigma_\gamma = 0 \Rightarrow \sigma = \text{const}$$

Hence $\rho^{-1} = a_1 |x - x_0|^2 + b_1$

Conversely, φ^{-1} is conformal $\Rightarrow \rho = a_2 |y - y_0|^2 + b_2$

i.e. $(a_1 |x - x_0|^2 + b_1)(a_2 |y - y_0|^2 + b_2) = 1 \Rightarrow$ spheres at x_0 $\longleftrightarrow$ spheres at y_0

algebraic dependence



conformal $\Rightarrow$ line to line

$$\varphi(\overrightarrow{x_1 x}) = \overrightarrow{y_1 y}$$

let τ = arc length in $\overrightarrow{x_1 X}$

$$\tau \in [t_1, t]$$

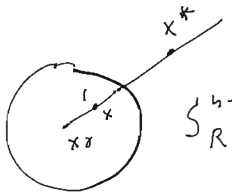
$$|y - y_1| = \int_{t_1}^{t} \rho \, d\ell_U = \int_{t_1}^{t} (a_1 \tau^2 + b_1)^{-1} d\tau = \text{algebraic in } |x - x_0| = t - t_1$$

$$\Longleftrightarrow a_1 = 0 \text{ or } b_1 = 0 .$$

$a_1 = 0 \Rightarrow$ dilation

$b_1 = 0 \Rightarrow \rho = a_1^{-1} \dfrac{1}{|x - x_0|^2}$ this is the conformal factor arising from Mobius inversion :

$$M : x \longmapsto x^* := \frac{x - x_0}{|x - x_0|^2}$$



$S^{n-1}_{R=1}(x_0)$

By composing M first, we reduce to the dilation case. Done $\Box$

Ex. check this is conf. but with $J < 0$!

Rmk: for $n=2$, $\mathbb{R}^2 \simeq \mathbb{C}$, $z^* = \dfrac{z}{|z|^2} = \dfrac{1}{\bar{z}}$.

Vector (= tangent vector to a curve)

$$v = \sum_i \frac{dx^i}{dt} e_i = \sum_{j,i} \frac{\partial x^i}{\partial y^j} \frac{dy^j}{dt} e_i = \sum_j \frac{dy^j}{dt} \left(\sum_i \frac{\partial x^i}{\partial y^j} e_i \right) = \sum_j \frac{dy^j}{dt} \tilde{e}_j$$

$$\xi^i = \sum \frac{\partial x^i}{\partial y^j} \tilde{\xi}^j \qquad \sum \frac{\partial x^i}{\partial y^j} e_i = \tilde{e}_j \qquad \tilde{\xi}^j$$

co-vector (= gradient of a function, or total differential)

$$\nabla f = \sum_i \frac{\partial f}{\partial x^i} e^i = \sum_{j,i} \frac{\partial f}{\partial y^j} \frac{\partial y^j}{\partial x^i} e^i = \sum_j \frac{\partial f}{\partial y^j} \left(\sum_i \frac{\partial y^j}{\partial x^i} e^i \right) = \sum_j \frac{\partial f}{\partial y^j} \tilde{e}^j$$

$$\xi_i = \sum \tilde{\xi}_j \frac{\partial y^j}{\partial x^i} \qquad \sum_i \frac{\partial y^j}{\partial x^i} e^i = \tilde{e}^j \qquad \tilde{\xi}_j$$

It will be very useful to set notations $e_i = \frac{\partial}{\partial x^i}$; $e^i = dx^i$

Duality : $df(v) = \sum \frac{\partial f}{\partial x^i} v^i$ $\qquad v = \sum v^i \frac{\partial}{\partial x^i} \qquad df = \sum \frac{\partial f}{\partial x^i} dx^i$

can be interpreted as directional derivative of f in $v = v(f)$

Defn : An (p,q) tensor on V is an element $T \in \otimes^p V \otimes \otimes^q V^*$; $e^j(e_i) = \delta^j_i$

i.e. $T = \sum T^{i_1 \cdots i_p}_{j_1 \cdots j_q} e_{i_1} \otimes \cdots \otimes e_{i_p} \otimes e^{j_1} \otimes \cdots \otimes e^{j_q}$

For tensor fields T, it is a collection of functions $\left(T^{i_1 \cdots i_p}_{j_1 \cdots j_q}(x) \right)$ in each coor. x which transf in the correct way

$$T^{i_1 \cdots i_p}_{j_1 \cdots j_q}(x) = \sum \tilde{T}^{k_1 \cdots k_p}_{\ell_1 \cdots \ell_q}(y) \frac{\partial x^{i_1}}{\partial y^{k_1}} \cdots \frac{\partial x^{i_p}}{\partial y^{k_p}} \times \frac{\partial y^{\ell_1}}{\partial x^{j_1}} \cdots \frac{\partial y^{\ell_q}}{\partial x^{j_q}}$$

contra-variant part $\qquad$ co-variant part

Write $T = \sum_{|I|=p, |J|=q} T^I_J(x)\, e_I \otimes e^J$

$$e_I \otimes e^J = \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q}$$

Examples :

(0,2) tensor field : $\sum g_{ij}(x)\, dx^i \otimes dx^j = \sum g_{ij}\, dx^i dx^j$ (notation when $g_{ij} = g_{ji}$)

(2,0) tensor field : Let $(g^{ij}) = (g_{ij})^{-1}$ and define inner product of co-vectors as

$$\langle \xi, \eta \rangle := \langle \xi_i dx^i, \eta_j dx^j \rangle = \sum \xi_i \eta_j \langle dx^i, dx^j \rangle = \sum \xi_i \eta_j g^{ij}(x)$$

then $g^{ij} = \langle dx^i, dx^j \rangle = \langle \frac{\partial x^i}{\partial y^k} dy^k, \frac{\partial x^j}{\partial x^\ell} dy^\ell \rangle = \tilde{g}^{k\ell}(y) \frac{\partial x^i}{\partial y^k} \cdot \frac{\partial x^j}{\partial y^\ell}$ must transf correctly - but why ?

Ex. This is how we identify $V \xrightarrow{\sim} V^*$ under given $\langle , \rangle$. and check

(1,1) tensor :

This is just a "field of" linear maps $A = \sum_{i,j} a^i_j\, e_i \otimes e^j$

Algebra on tensors :

(i) permutations $\sigma \in S_q \Rightarrow \sigma T := \left\{ \tilde{T}^{i_1 \cdots i_p}_{j_1 \cdots j_q} := T^{i_1 \cdots i_p}_{j_{\sigma(1)} \cdots j_{\sigma(q)}} \right\}$

(ii) contraction (trace) $\otimes^{p,q} V \to \otimes^{p-1,q-1} V$ sum over $i_k = j_\ell = i$

(iii) tensor product $\otimes^{p,q} V \otimes \otimes^{k,\ell} V \xrightarrow{\sim} \otimes^{p+k,q+\ell} V$ order is important.

Eg. $A^k_\ell, \xi^i \longmapsto T^{ki}_\ell := A^k_\ell \xi^i \longmapsto \Sigma_i A^k_i \xi^i =: \eta^k$ trivial example!

Tensors of type $(0,k)$: forms

$$T = \sum T_{i_1 \cdots i_k} e^{i_1} \otimes \cdots \otimes e^{i_k} = \sum T_I \, dx^{\otimes I} \quad \text{(ordered)}$$

Most important special cases :

$$\mathrm{Sym}^k V^* \subset \otimes^{0,k} V = \otimes^k V^* \ni T \quad \text{st.} \quad \sigma(T) = T \quad \forall \sigma \in S_k$$

$$\Lambda^k V^* \equiv \mathrm{Alt}\, V^* \subset \otimes^k V^* \ni T \quad \text{st} \quad \sigma(T) = \mathrm{sgn}(\sigma)\, T$$

fact : for $k=2$, $\otimes^2 V^* \simeq \mathrm{Sym}^2 V^* \oplus \Lambda^2 V^*$ $\qquad T = T^{sym} + T^{alt}$:

$$\sum_{i,j} T_{ij}\, dx^i \otimes dx^j = \sum \underbrace{\tfrac{1}{2}(T_{ij} + T_{ji})}\, dx^i \otimes dx^j + \sum \underbrace{\tfrac{1}{2}(T_{ij} - T_{ji})\, dx^i \otimes dx^j}$$

$$\overset{\parallel}{T^{sym}_{ij}\, dx^i dx^j} \qquad \qquad \overset{\parallel}{\sum_{i<j} T^{alt}_{ij}\, dx^i \wedge dx^j}$$

st. $dx^i dx^j = dx^j dx^i$

for $\Lambda^2 V^*$: better basis $dx^i \wedge dx^j := dx^i \otimes dx^j - dx^j \otimes dx^i$.

General $k \in \mathbb{N}$: $\mathrm{Sym}^k V^* \simeq$ polynomial of degree k in $v = \sum_{i=1}^n x^i e_i \in V$

$\Rightarrow$ Set $P_T(x^1, \cdots, x^n) := T(v, \cdots, v)$. Q : Converse ?

Ex. Derive the polarization formula for $p \mapsto T_p$.

for $\Lambda^k V^*$: $dx^{i_1} \wedge \cdots \wedge dx^{i_k} := \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, e^{\sigma(i_1)} \otimes \cdots \otimes e^{\sigma(i_k)}$

may choose $i_1 < \cdots < i_k$ since $dx^{\tau(i_1)} \wedge \cdots \wedge dx^{\tau(i_k)} = (-)^{|\tau|} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$

Rmk : notation $\mathrm{sgn}(\sigma) = (-)^{|\sigma|} = \varepsilon_{\sigma(1) \cdots \sigma(n)}$, $\varepsilon_{i_1 \cdots i_n} := 0$ if not in S_n .

Def^{n} : exterior product $T = \sum T_I \, dx^I \in \Lambda^p$, $S = \sum S_J \, dx^J \in \Lambda^q$,

$$T \wedge S := \sum T_I S_J \, dx^I \wedge dx^J \;=\!=\!= \sum_{k_1 < \cdots < k_{p+q}} R_{k_1 \cdots k_{p+q}} \, dx^{k_1} \wedge \cdots \wedge dx^{k_{p+q}}$$

where $R_{k_1 \cdots k_{p+q}} = \sum_{\sigma \in S_{p+q}} \frac{(-)^{|\sigma|}}{p!\, q!} T_{\sigma(k_1 \cdots k_p)} S_{\sigma(k_{p+1} \cdots k_{p+q})}$.

Fact : $T \wedge S = (-)^{pq} S \wedge T$. $(\Lambda(V^*), \wedge)$ is called the exterior algebra.

The Art of Raising / Lowering indices

Given non-deg $(0,2)$ tensor g_{ij} (Sym $\Rightarrow$ Riemannian or pseudo-Riem metric)

$$\flat : \otimes^{p,q} V \longrightarrow \otimes^{p-1,q+1} V \qquad \text{via} \quad g_{\underline{i_1} k}\, T^{k \cdots i_p}_{\;\; j_1 \cdots j_q} =: T^{\;\; i_2 \cdots i_p}_{\underline{i_1}\, j_1 \cdots j_q}$$

Then . $\qquad \otimes$ product , then trace

$$\# : \otimes^{p,q} V \longrightarrow \otimes^{p+1,q-1} V \qquad \text{via} \quad g^{\underline{j_1} k}\, T^{i_1 \cdots i_p}_{k\, j_2 \cdots j_q} =: T^{\underline{j_1}\, i_1 \cdots i_p}_{\;\; j_2 \cdots j_q}$$

$\flat, \#$ are inverse since $g^{ik} g_{kj} = \delta^i_j$.

Lemma : $\flat : V \longrightarrow V^*$ is an isometry .

pf : Let $v = \sum v^i e_i$, $w = \sum w^i e_i$, then $v_\flat = \sum \underline{\overset{v_i}{\overset{\parallel}{g_{ik} v^k}}}\, e^i$ etc.

$$\Rightarrow \langle v_\flat, w_\flat \rangle = g^{ij} v_i w_j = g^{ij} g_{ik} v^k g_{jl} w^l = \delta^j_k g_{jl} v^k w^l = \langle v, w \rangle \quad \#$$

Example : $A = (a^i_{\ j})$ vs. $a_{ij} = g_{ik} a^k_{\ j}$.

- A is self-adjoint $\Leftrightarrow$ a_{ij} is sym. since $\langle Av, w \rangle = g_{ik} a^i_j v^j v^k = a_{kj} v^j v^k$
- eigenvalues : $(a^i_j - \lambda \delta^i_j) v^j = 0 \Leftrightarrow (a_{ij} - \lambda g_{ij}) v^j = 0$

For (b^i_j) = matrix for $-dN$ then b_{ij} is the 2nd fund. form.

In particular, $K = \det(b^i_j) = \dfrac{\det(b_{ij})}{\det(g_{ij})}$

$$H = tr(b^i_j) = b^i_i = g^{ij} b_{ij}$$

holds for any hypersurface in $\mathbb{R}^n$.

Volume form and Hodge $*$ operator on Riem, pseudo-Riem "space"

Thm: $d\sigma = \sqrt{|g|}\, dx^1 \wedge \cdots \wedge dx^n$ $\quad g := \det(g_{ij})$ is a top form ONLY among cor. change with $J > 0$.

pf: $g_{ij} = \tilde{g}_{kl} \dfrac{\partial y^k}{\partial x^i} \dfrac{\partial y^l}{\partial x^j} \Rightarrow g = \tilde{g} J^2 \Rightarrow \sqrt{|g|} = \sqrt{|\tilde{g}|} \cdot |J|$

but $dx^1 \wedge \cdots \wedge dx^n = \left(\sum_{\sigma \in S_n} (-1)^{|\sigma|} \dfrac{\partial x^1}{\partial y^{\sigma(1)}} \cdots \dfrac{\partial x^n}{\partial y^{\sigma(n)}} \right) dy^1 \wedge \cdots \wedge dy^n = J\, dy^1 \wedge \cdots \wedge dy^n$ $\#$

Def$^{\underline{n}}$: Hodge $*$ (orthogonal comp) : $\Lambda^k \longrightarrow \Lambda^{n-k}$ linear op st

$$(*T)_{i_{k+1} \cdots i_n} := \frac{1}{k!} \sqrt{|g|}\, \varepsilon_{i_1 \cdots i_n} T^{i_1 \cdots i_k}$$
$$'' \; g^{i_1 j_1} \cdots g^{i_k j_k} T_{j_1 \cdots j_k}.$$

The idea is :

for $(V, \langle , \rangle)$ an inner product space with ONB $e_1, \cdots, e_n$

V^* has ONB $e^1, \cdots, e^n$ and $\Lambda^k(V^*)$ has ONB $e^{i_1} \wedge \cdots \wedge e^{i_k}$ $\quad i_1 < \cdots < i_k$

Then $* : \Lambda^k \to \Lambda^{n-k}$ simply sends $e^1 \wedge \cdots \wedge e^k \mapsto e^{k+1} \wedge \cdots \wedge e^n$ and extends by linearity.

Proposition: (Exercise) Let $T, S \in \Lambda^k$

(1) $*(*T) = (-1)^{k(n-k)} \operatorname{sgn}(g)\, T$

(2) $T \wedge *S = \langle T, S \rangle\, d\sigma$ where $\langle , \rangle$ is the induced inner product on Λ^p. Hence $* : \Lambda^k \to \Lambda^{n-k}$ is an isometry of v.s.

Will develop differential / integral calculus of forms in the whole course.

Nevertheless, as a side remark; it is also useful to study :

Integrals wrt Fermion (i.e. anti-commuting) variables on $\mathbb{R}^n$

Differential Calculus of tensors (Lie aspect)

Problem: Given $\xi = \xi^i \frac{\partial}{\partial x^i}$, $\frac{\partial}{\partial x^j}\xi^i =: \xi^i_j$ does not form a (1,1) tensor since in coor. transf get 2nd derivatives ! WHAT HAPPENS ?!

Pull back (restriction) of forms under a map

$f : (V, y^1, \dots, y^n) \to (U, x^1, \dots, x^n)$, $(0,k)$ tensor field on $U \xrightarrow{f^*} (0,k)$ on V

$$f^* T := f^*\left(T_{i_1 \cdots i_k}(x)\, dx^{i_1} \otimes \cdots \otimes dx^{i_k} \right)$$
$$= T_{i_1 \cdots i_k}(f(y)) \frac{\partial x^{i_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{j_k}} dy^{j_1} \otimes \cdots \otimes dy^{j_k} .$$

Push forward of tangent vectors, ie. tangent map

$f_{*p} : T_p V \longrightarrow T_{f(p)} U \;:\; T^i \mapsto T^j \underset{\uparrow \text{invertible}}{\frac{\partial x^i}{\partial y^j}}$

Rmk: in textbook, it prefers notation: $y^j = "x^{i'}"$.

However, vector field $\mapsto$ vf only if f is a diffeomorphism at p, ie f^{-1} exists locally and C^∞ at p.

In that case, we may pull back vf's, in fact any tensors !

Fundamental Theorem of ODE: Smooth dependence

Given a C^∞ v.f. ξ on $U \subset \mathbb{R}^n$, consider integral curve $x(t)$

$$\begin{cases} x'(t) = \xi(x(t)) \\ x(0) = x_0 \end{cases} \quad \text{defines} \quad F_t(x_0) = x(t, x_0) \text{ is } C^\infty \text{ in } (t, x_0) .$$

Def'n (Flow): At any x_0, $\exists\, (-\epsilon, \epsilon) \ni t$ st F_t is a local diffeomorphism

Moreover, $\{F_t\}$ form a local 1-para. gp. of diffeo

$F_{t+s} = F_t \circ F_s$, $F_{-t} = F_t^{-1}$ (easy part, uniqueness)

Differentiability $\Rightarrow$ $x(t, x_0) = x_0 + t\, x'(0) + o(t) = x_0 + t\, \xi(x_0) + o(t)$

Proposition (Jacobian): $\frac{\partial x^i(t)}{\partial x_0^j} = \delta^i_j + t \frac{\partial \xi^i}{\partial x_0^j} + o(t)$. Take inverse get

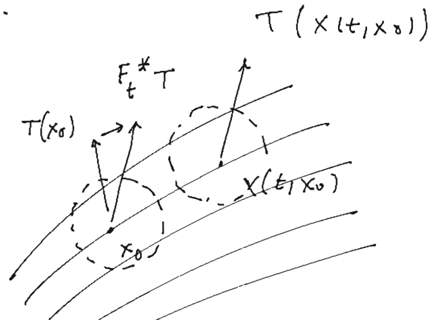
$$\frac{\partial x_0^i}{\partial x^j} = \delta^i_j - t \frac{\partial \xi^i}{\partial x_0^j} + o(t) .$$

Lie Derivative on tensors along v.f. ξ :

$T \in \otimes^{p,q} \Rightarrow F_t^* T \in \otimes^{p,q}$ is well-defined

via "pull back" to coor system x_0 :

$$(F_t^* T)^{i_1 \cdots i_p}_{j_1 \cdots j_q} = T^{k_1 \cdots k_p}_{l_1 \cdots l_q} \frac{\partial x^{l_1}}{\partial x_0^{j_1}} \cdots \frac{\partial x^{l_q}}{\partial x_0^{j_q}} \cdot \frac{\partial x_0^{i_1}}{\partial x^{k_1}} \cdots \frac{\partial x_0^{i_p}}{\partial x^{k_p}}$$



Then $L_\xi T := \frac{d}{dt}(F_t^* T)\Big|_{t=0}$ (at the space $V = T_{x_0}\mathbb{R}^n$)

it can be computed component-wise ! : $(L_\xi T)^I_J = \frac{d}{dt}(F_t^* T)^I_J \Big|_{t=0}$.

Explicit formulae : up to $o(t)$,

$$(F_t^* T)^{i_1 \cdots i_p}_{j_1 \cdots j_q} = T^{k_1 \cdots k_p}_{l_1 \cdots l_q}(x) \left(\delta^{l_1}_{j_1} + t \frac{\partial \xi^{l_1}}{\partial x_0^{j_1}} \right) \cdots \left(\delta^{l_q}_{j_q} + t \frac{\partial \xi^{l_q}}{\partial x_0^{j_q}} \right) \left(\delta^{i_1}_{k_1} - t \frac{\partial \xi^{i_1}}{\partial x_1^{k_1}} \right) \cdots \left(\delta^{i_p}_{k_p} - t \frac{\partial \xi^{i_p}}{\partial x_1^{k_p}} \right)$$

$$= T^{i_1 \cdots i_p}_{j_1 \cdots j_q}(x) + t \left(T^{i_1 \cdots i_p}_{l \, j_2 \cdots j_q} \frac{\partial \xi^l}{\partial x_0^{j_1}} + \cdots + T^{i_1 \cdots i_p}_{j_1 \cdots j_{q-1} l} \frac{\partial \xi^l}{\partial x_0^{j_q}} - T^{k \, i_2 \cdots i_p}_{j_1 \cdots j_q} \frac{\partial \xi^{i_1}}{\partial x_0^k} - \cdots - T^{i_1 \cdots k}_{j_1 \cdots j_q} \frac{\partial \xi^{i_p}}{\partial x_0^k} \right)$$

Thm: $$(L_\xi T)^I_J = \frac{\partial T^I_J}{\partial x^s} \xi^s + T^I_{l \cdots} \frac{\partial \xi^l}{\partial x^{j_1}} + \cdots + T^I_{\cdots l} \frac{\partial \xi^l}{\partial x^{j_q}} - T^{k \cdots}_J \frac{\partial \xi^{i_1}}{\partial x^k} \cdots - T^{\cdots k}_J \frac{\partial \xi^{i_p}}{\partial x^k}$$

in coordinate $x = x_0$.

Examples :(a) $L_\xi f = \xi^i \frac{\partial f}{\partial x^i} = \xi f \equiv \partial_\xi f$ no correction needed.

(b) On vector field η, " $L_\xi \eta^i$ " $= (L_\xi \eta)^i = \xi^j \frac{\partial \eta^i}{\partial x^j} - \eta^j \frac{\partial \xi^i}{\partial x^j}$ $(= - L_\eta \xi \; !)$

Thm : $L_\xi \eta = [\xi, \eta] := \xi \eta - \eta \xi$ as bracket of diff op's.

pf : $$[\xi^i \partial_i , \eta^j \partial_j] f = \xi^i \partial_i (\eta^j \partial_j f) - \eta^j \partial_j (\xi^i \partial_i f)$$

$$= \xi^i (\partial_i \eta^j) \partial_j f + \cancel{\xi^i \eta^j \partial_i \partial_j f} - \eta^j (\partial_j \xi^i) \partial_i f - \cancel{\eta^j \xi^i \partial_j \partial_i f}$$

$(i,j) \leftrightarrow (j,i)$ $$= \left(\xi^j \frac{\partial \eta^i}{\partial x^j} - \eta^j \frac{\partial \xi^i}{\partial x^j} \right) \frac{\partial}{\partial x^i} f = (L_\xi \eta) f \quad ※$$

Constructing coor system with given "directions"

Cor : Given $\xi_1, \cdots, \xi_m$ vfs in $\mathbb{R}^n$. If $\exists$ coor system $y^1, \cdots, y^n$ st ξ_j is tangent to coor axis y^j, $j = 1, \cdots, m$, then $[\xi_j, \xi_k] = \alpha_{jk} \xi_j + \beta_{jk} \xi_k$.

pf : $\xi_j = f_j \partial_j \Rightarrow [\xi_j, \xi_k] = f_j (\partial_j f_k) \partial_k - f_k (\partial_k f_j) \partial_j$ ※

Rmk : The converse holds (Frobenius thm). A simplified version $[X_i, X_j] = 0$ $i = 1, \cdots, n$ is given in Exercise.

General properties of 'L_ξ' : Leibniz rule on $\otimes$, $\wedge$ etc (Exercise)

(c) On one forms $T = T_j dx^j$, $(L_\xi T)_j = \xi^l \partial_l T_j + T_l \frac{\partial \xi^l}{\partial x^j}$

In particular, for $T = df$, get $L_\xi df = d(L_\xi f)$ comm. with d

Rmk: After we introduce exterior derivative d, will see this in general.

(d) For (0,2) tensor g_{ij}. Say the metric.

" $L_\xi g_{ij}$ " $= \xi^s \partial_s g_{ij} + g_{kj} \partial_i \xi^k + g_{ik} \partial_j \xi^k$

measures the change of g_{ij} in the small deformation F_t along ξ.

ξ is a Killing vector field if $L_\xi g = 0$

For Euclidean case, $g_{ij} = \delta_{ij}$, $L_\xi g_{ij} = \partial_i \xi^j + \partial_j \xi^i$

Q : what are all the solutions ?

Examples of (0,2) tensor in $\mathbb{R}^{1,3}$

① $F = F_{ij}\, dx^i \wedge dx^j =: E_\alpha\, dx^0 \wedge dx^\alpha + H^1\, dx^2 \wedge dx^3 + H^2\, dx^3 \wedge dx^1 + H^3\, dx^1 \wedge dx^2$

skew-sym case $\qquad (\mathbb{E}, \mathbb{H}) = (E_\alpha, H^\beta)$ is the "classical notation" with Euclidean product

Def'n: The invariants of F are coefficients of $\quad p(\lambda) := \det(F_{ik} - \lambda g_{ik})$

$$= -\lambda^4 + (|\mathbb{E}|^2 - |\mathbb{H}|^2)\lambda^2 + \langle \mathbb{E}, \mathbb{H}\rangle^2$$

Fact: They are inv. under Lorentz transf. (i.e. isometry of $\mathbb{R}^{1,3}$)

Lemma: $\quad * F = -(E_1\, dx^2 \wedge dx^3 + E_2\, dx^3 \wedge dx^1 + E_3\, dx^1 \wedge dx^2) + \sum_{\alpha=1}^{3} H^\alpha\, dx^0 \wedge dx^\alpha$.

Pf: Using $\omega \wedge * \omega = |\omega|^2 d\sigma$ and $dx^i \wedge dx^j$ form ONB of $\Lambda^2(\mathbb{R}^{1,3})$

Or by direct computations:

$(*F)_{ik} = \frac{1}{2} \varepsilon_{iklm} F^{lm} = \frac{1}{2} \varepsilon_{iklm} g^{lp} g^{mq} F_{pq}$ since $\varepsilon_{lmik} = \varepsilon_{iklm}$

$\Rightarrow (*F)_{01} = \frac{1}{2} \varepsilon_{0123} g^{22} g^{33} F_{23} + \frac{1}{2} \varepsilon_{0132} g^{33} g^{22} F_{32} = \frac{1}{2} H_1 - \frac{1}{2}(-H1) = H^1$ ⋕

Lemma: $*^2 = -1$ on Λ^2, hence a $\mathbb{C}$-v.s. str. is defined via

$$(a + b\sqrt{-1})\,\omega := a\omega + b * \omega .$$

Then F is identified with $F = \mathbb{H} + \sqrt{-1}\,\mathbb{E}$ since $*F = -\mathbb{E} + \sqrt{-1}\,\mathbb{H}$

Let $Z^\alpha := H^\alpha + \sqrt{-1}\, E_\alpha$, $\quad \alpha = 1, 2, 3$.

$$\langle F, F\rangle := -*(F \wedge (*F) + \sqrt{-1}\, F \wedge F) = -\frac{1}{2}(F_{ik}F^{ki} + \sqrt{-1}\, \varepsilon^{ijkl} F_{ij} F_{kl})$$

$$= (|\mathbb{H}|^2 - |\mathbb{E}|^2) + 2\sqrt{-1}\langle \mathbb{E}, \mathbb{H}\rangle = (Z^1)^2 + (Z^2)^2 + (Z^3)^2$$

This leads to $SO(1,3) \hookrightarrow O(3, \mathbb{C})$ and $\exists$ canonical form of skew-sym (0,2) tensor F, cf. Thm 21.1.5, in $\mathbb{R}^{1,3}$. (Ex. Read it)

② Sym case. $\quad T = T_{ik}\, dx^i dx^k$. $\qquad$ Fact: $(T_{ik} - \lambda g_{ik})\xi^k = 0$

Set $p(\lambda) = \det(T_{ik} - \lambda g_{ik})$

eigenvalues $\lambda_0, \lambda_1, \lambda_2, \lambda_3$

vectors ξ_i are still orthogonal.

Q: canonical form of T in $\mathbb{R}^{1,3}$?

Thm ($\mathbb{R}^{1,1}$ case): (Thm 21.2.2)

(i) $\lambda_0 \neq \lambda_1$ real $\Rightarrow T \sim \begin{pmatrix} \lambda_0 & \\ & -\lambda_1 \end{pmatrix}$

(ii) $\lambda_0, \lambda_1 = \alpha \pm \beta i$, $\beta \neq 0 \Rightarrow T \sim \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$

(iii) $\lambda_0 = \lambda_1 = \lambda \Rightarrow T \sim \begin{pmatrix} \lambda + \mu & -\mu \\ -\mu & -\lambda + \mu \end{pmatrix}$ in any coor system, some μ.

$\sim$ means under $O(1,1)$ Lorentz transformation.

The pf is straightforward, (ii) occurs since $g^{ij} T_{jk}$ might not be sym.

E.g. Given F_{ik} anti-sym, define

$$T_{ik} := \frac{1}{4\pi}\left(-g^{lm} F_{il} F_{km} + \frac{1}{4} F_{lm} F^{lm} g_{ik}\right) \quad \text{a sym tensor}$$

If F is the electromagnetic field, T is called its energy-stress tensor

Lie Algebra $(V, [\,,\,])$ st. $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$
vector space, skew-sym, Jacobi identity

Rmk: if we define $(\mathrm{ad}(a))\,x := [a,x]$ then this is equiv to
$\mathrm{ad}(a)[b,c] = [\mathrm{ad}(a)\,b, c] + [b, \mathrm{ad}(a)\,c]$ i.e. $\mathrm{ad}(a)$ is a Derivation.

Examples:
(a) $\times$ product in $\mathbb{R}^3$

(b) A space of linear operators like "matrices" or "vector fields"
$[a,b] := a \circ b - b \circ a$ or in fact any associative alg.

(c) Killing vfs form a Lie subalgebra:
$[L_\xi, L_\eta]\,g = L_\xi L_\eta g - L_\eta L_\xi g = 0 - 0 = 0$ obviously!

(d) classical Lie alg = matrix sub Lie alg of $gl(n, F)$.
with $[A,B] = AB - BA$ as in (b) $F = \mathbb{R}$ or $\mathbb{C}$.

Lemma: For $G \subset GL(n,F)$, $\mathfrak{g} := T_eG$ is closed under $[\,,\,]$.

Pf: $A, B \in \mathfrak{g} \Rightarrow f(s) := e^{A} \cdot e^{sB} e^{-A} \in G \Rightarrow \mathfrak{g} \ni f'(0) = e^{A} B e^{-A}$
with $f(0) = e$

Now $h(t) := e^{tA} B\, e^{-tA} \in \mathfrak{g} \Rightarrow \mathfrak{g} \ni h'(0) = AB - BA$ ✠

Example (1): $su(2) \simeq so(3,\mathbb{R}) \simeq (\mathbb{R}^3, \times)$ as Lie algebras over $F = \mathbb{R}$.
of course this follows from $SU(2) \simeq SO(3)$ as Lie group.

Explicitly. $su(2)$ is spanned by $\qquad$ $so(3,\mathbb{R})$ is spanned by

$$S_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ S_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ S_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}; \qquad X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},\ X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\ X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$[S_1, S_2] = 2S_3$, $[S_2, S_3] = 2S_1$, $[S_3, S_1] = 2S_2$ $\qquad$ $[X_1, X_2] = X_3$, $[X_2, X_3] = X_1$, $[X_3, X_1] = X_2$

Example (2): $sl(2,\mathbb{R}) \simeq so(1,2)$.

Pf: This again was seen from isometry gp of $\mathbb{H} \simeq L^2$ (non-Euclidean geom)
Alternatively, use adjoint representation $\mathrm{ad}: \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$
which comes from $\mathrm{Ad}: G \longrightarrow \mathrm{Aut}(\mathfrak{g})$ $\quad Y \mapsto gYg^{-1}$

$\mathrm{Ad}(g)$ preserve the quadratic form, under basis Y_i:

$$Y_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ Y_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ Y_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which is the Minkowski-Lorentz metric!

$$|Y|^2 := \det Y = \det(y^i Y_i)$$
$$= \begin{vmatrix} y_1 & y_2 + y_0 \\ y_2 - y_0 & -y_1 \end{vmatrix} = y_0^2 - y_1^2 - y_2^2$$

Taking $g = \exp(tX) \Rightarrow \mathrm{ad}(X) \in so(1,2)$ ✠

Relation between Lie bracket of matrices & of differential operators

Def'n: Denote T_X the linear v.f. $T_X(x) := -Xx \cong -X^i_k x^k \frac{\partial}{\partial x^i}$.

Thm: $[T_X, T_Y] = T_{[X,Y]}$.

pf: $[T_X, T_Y]^i = X^j_k x^k \frac{\partial}{\partial x^j}(Y^i_\ell x^\ell) - Y^j_k x^k \frac{\partial}{\partial x^j}(X^i_\ell x^\ell)$

$= X^j_k x^k Y^i_j - Y^j_k x^k X^i_j = -([X,Y]x)^i$ ※

This applies to general left inv. v.f.s on a Lie group:

Def'n: Let $X \in \mathfrak{g} = \mathrm{Lie}\, G$, for $A \in G$ get $l_A : G \to G : g \mapsto Ag$

hence $dl_{A,e} : T_e G \to T_A G$ is simply $X \mapsto AX$

This defines a v.f. L_X on G via $L_X(A) = AX$

L_X is left inv: $dl_{B*}(L_X(A)) = dl_{B*}\, dl_{A*} X = dl_{BA*} X = BAX = L_X(BA)$.

Cor: The integral curve $A'(t) = L_X(A(t)) = A(t)X$ is $A(t) = A(0)\, e^{tX}$.

Also, $[L_X, L_Y] = L_{[X,Y]}$. (same pf.) "notice left/right"

ie. The Lie alg. $\mathfrak{g} = \mathrm{Lie}\, G \cong$ Lie alg of l.i.v.f. on G.

Now we may extend any bi-linear form $\langle\,,\rangle_e$ on $\mathfrak{g} = T_e G$

to any $T_A G$, $A \in G$ by $\langle v, w\rangle_A := \langle dl_{A^{-1}*} v, dl_{A^{-1}*} w\rangle_e$.

ie. for $v = L_X(A)$, $w = L_Y(A) \in T_A G$, we set $\langle v, w\rangle_A = \langle L_X(A), L_Y(A)\rangle_A = \langle X, Y\rangle_e$

if $\langle\,,\rangle_e$ is non-degenerate, we get left-inv metric on G: $d\ell^2 = \langle A^{-1}dA, A^{-1}dA\rangle_e$.

Key Example: $G = SO(n, \mathbb{R}) \hookrightarrow M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$

Euclidean metric on $\mathbb{R}^{n^2}$: $\langle X, Y\rangle = \sum x^i_j y^i_j = tr(XY^T)$

then $X \in SO(n) \Rightarrow |x|^2 = tr(XX^T) = tr(I_n) = n \Rightarrow SO(n) \hookrightarrow S^{n^2-1}_{\sqrt{n}}$.

It induces a metric $\langle\,,\rangle$ on $T_A G$, $\forall A \in G$.

Claim: $\langle\,,\rangle$ is bi-invariant: Let $X, Y \in so(n, \mathbb{R})$, $A \in SO(n, \mathbb{R})$:

pf: $\langle L_X(A), L_Y(A)\rangle = tr(AX(AY)^T) = tr(\underline{A}\,\underline{XY^TA^T}) = tr(XY^TA^TA) = tr(XY^T)$. R_X the same ※

Definition (Remark): A quad. form $\langle\,,\rangle_0$ on $\mathfrak{g}$ satisfies

$\langle [X,Y], Z\rangle_0 + \langle Y, [X,Z]\rangle_0 = 0$ is called a Killing form (or metric)

A standard choice is $\langle X, Y\rangle_0 := -tr(\mathrm{ad}X\, \mathrm{ad}Y)$. $\mathfrak{g}$ is semi-simple if $\langle\,,\rangle_0$ is non-dege.

Thm: $\langle\,,\rangle$ on $so(n, \mathbb{R})$ is a Killing form, and non-degenerate.

pf: $X^T = -X$, $\forall X \in so(n, \mathbb{R})$, hence $\langle X, Y\rangle = tr(XY^T) = -tr(XY)$.

$\langle [X,Y], Z\rangle + \langle Y, [X,Z]\rangle = -tr\, \underline{XYZ} + tr\, YXZ - tr\, YXZ + tr\, \underline{YZX} = 0$.

Ex (i) Via $u(n) \subset so(2n, \mathbb{R})$, we get Killing form $\langle X, Y\rangle = \mathrm{Re}\, tr\, X\bar{Y}^T = -\mathrm{Re}\, tr\, XY$.

(ii) compare the def'n via ad X.

Cartan's d operator on Λ^k

ef'': $\overset{x}{d}\omega = \overset{x}{d}(h \; dx^{i_1} \wedge \cdots \wedge dx^{i_k})$

$:= dh \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$

Basic properties : d satisfies

(1) df = total differential

(2) $d^2 = 0$

(3) Leibniz' rule $d(\omega \wedge \eta) = (d\omega)\wedge\eta + (-1)^p \omega \wedge d\eta$ if $\omega \in \Lambda^p$

In fact, d is unique under (1), (2), (3).

The pf is easy. But the key point is "why does it indep of coor system"?

Recall the pull back f^*:

$F^* \overset{x}{d}\omega = F^*(dh \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k})$

$= d(h\circ F) \wedge d(x^{i_1}\circ F) \wedge \cdots \wedge d(x^{i_k}\circ F)$

$\longrightarrow$ d on function is indep of coor.

$df = \frac{\partial f}{\partial x^i} dx^i = \frac{\partial f}{\partial y^j} dy^j$

" $\frac{\partial f}{\partial y^j} \frac{\partial y^j}{\partial x^i} dx^i$ "

Thm (Functoriality) for $F : (V, y^\cdot) \to (U, x^\cdot)$, $\overset{y}{d} F^* \omega = F^* \overset{x}{d} \omega$ holds.

Pf : $F^*\omega = (h\circ F)\, d(x^{i_1}\circ F) \wedge \cdots \wedge d(x^{i_k}\circ F)$

$$\overset{y}{d} F^* \omega = d\left[(h\circ F) \frac{\partial x^{i_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{j_k}}\right] dy^{j_1} \wedge \cdots \wedge dy^{j_k}$$

$$= d(h\circ F) \wedge d(x^{i_1}\circ F) \wedge \cdots \wedge d(x^{i_k}\circ F) \longrightarrow$$ this terms gives $F^* d^x \omega$ as recalled above.

$$+ \; (h\circ F)\; d\left(\frac{\partial x^{i_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{j_k}}\right) \wedge dy^{j_1} \cdots dy^{j_k}$$

Now, a typical term is $d\left(\frac{\partial x^{i_1}}{\partial y^{j_1}}\right) \wedge dy^{j_1} \wedge \underbrace{dx^{i_2} \wedge \cdots \wedge dx^{i_k}}$

skew-sym $\Rightarrow$ $\sum_{l, j_1}$ " $\frac{\partial^2 x^{i_1}}{\partial y^{j_1} \partial y^l} dy^l \wedge dy^{j_1}$ " $= 0$ #

Thm : Cartan's intrinsic formula : for sm v.f.s $X_0, \cdots, X_k$

$$(d\omega)(X_0, \cdots, X_k) = \sum_{i=0}^{k} (-1)^i \partial_{X_i}(\omega(X_0 \cdots \hat{X_i} \cdots X_k)) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0 \cdots \hat{X_i} \cdots \hat{X_j} \cdots X_k)$$

Rmk: The sign and $(k+1)$ in the book is wrong!

eg. $k=0$, $df(X) = \partial_X f$ OK.

$k=1$, $\omega \in \Lambda^1$, $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$?

Let $\omega = h\, dx^k$ $\quad d\omega = \frac{\partial h}{\partial x^j} dx^j \wedge dx^k$ (we do not set $j<k$!)

for $X = a^i \partial_i$ $\; Y = b^j \partial_j$ $\Rightarrow$ $d\omega(X,Y) = \partial_j h\, a^j b^k - \partial_j h\, b^j a^k$ = LHS

$$\text{RHS} = a^l \partial_l (h \underset{\cdots}{b^k}) - b^l \partial_l (h \underset{\cdots}{a^k}) - h\left(a^i \underset{\cdots}{(\partial_i b^j)} \partial_j - b^j \underset{\cdots}{(\partial_j a^i)} \partial_i\right)^k = \text{LHS}.$$

Q : A pf for general k is similar but lengthy !!

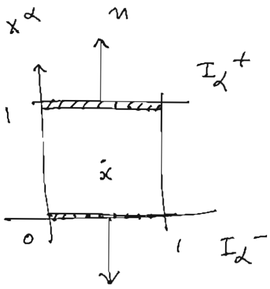
Rmk : (Tensor notation) for $\omega = \sum T_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$; $(d\omega)_{i_1 \cdots i_{k+1}} = \sum_{q=1}^{k+1} (-1)^{q-1} \partial_{x^{i_q}} T_{i_1 \cdots \hat{i_q} \cdots i_{q+}}$

Stokes' Fund. Thm. of Calculus

Def'n: $f : U \subset \mathbb{R}^k \to \mathbb{R}^n$, $\omega \in \Lambda^k$ then

$$\int_{f(U)} \omega := \int_U f^*\omega := \int_U \Big(\sum T_{i_1 \cdots i_k} J^{i_1 \cdots i_k}_{1 \cdots k} \Big) dx^1 \wedge \cdots \wedge dx^k$$

as Riem integral in $\mathbb{R}^k$.

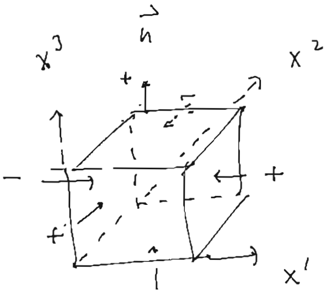


"singular" cube $\sigma : I^k \longrightarrow \mathbb{R}^n$ C^∞

$\overset{\shortparallel}{[0,1]^k}$ $\partial I^k := \bigcup_{\alpha=1}^{k} (I_\alpha^+ \cup I_\alpha^-)$

Set orientation at $(\vec{n}, v_1, \cdots, v_{k-1})$ is the same as the one in $x \in \operatorname{Int} I^k$.

Eg. when $\vec{n}$ is in the α-th direction get $(-1)^{\alpha-1}$ sign:



Thm: $\varphi \in \Lambda^{k-1}(\Omega)$, $\sigma : I^k \to \Omega \subset \mathbb{R}^n$.

Then $\int_{\partial\sigma} \varphi = \int_\sigma d\varphi$.

pf: this is just $\int_{\partial I^k} \sigma^*\varphi = \int_{I^k} \overset{= d(\sigma^*\varphi)}{\sigma^* d\varphi}$ in $\mathbb{R}^k$.

sign: $+, -, + \cdots$
$\alpha = 1, 2, 3, \cdots$

so we may assume $\varphi = h \, dx^1 \wedge \cdots \wedge \widehat{dx^\alpha} \wedge \cdots \wedge dx^k$

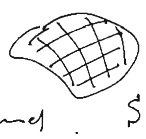
$d\varphi = \frac{\partial h}{\partial x^\alpha} (-1)^{\alpha-1} dx^1 \wedge \cdots \wedge dx^k$

$x^\beta = 0$ or 1
$\Rightarrow dx^\beta = 0$.

Fubini $\Rightarrow \int_{I^k} d\varphi = (-1)^{\alpha-1} \Big(\int_{I_\alpha^+} \varphi - \int_{I_\alpha^-} \varphi \Big) \underset{\uparrow}{=} \int_{\partial I^k} \varphi$

get zero contribution from other $\beta \neq \alpha$. ⌘

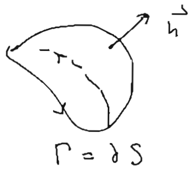
Corollary (Stokes' Thm): $\int_{\partial S} \varphi = \int_S d\varphi$ holds for general k-dim'l surface by partitioning S into union of singular cubes.



Rmk: No metric is involved in the def'n. Called integral of 2nd kind. integral of 1st kind means volume element $d\sigma$ is used.

Example 1. In $\mathbb{R}^3$, All integral (of 2nd kind) can be viewed as 1st kind.

$\varphi \in \Lambda^0$: $\int_\gamma dh = h \Big|_{\gamma(a)}^{\gamma(b)}$ for $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ h any function.



$\Gamma = \partial S$

$\varphi \in \Lambda^1$: $\int_S d\varphi = \int_{\partial S} \varphi = \int_{\partial S} P\,dx + Q\,dy + R\,dz = \int_{\partial S} \vec{F} \cdot d\vec{r}$

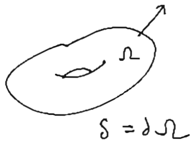
$= \int_S (R_y - Q_z)\, dy \wedge dz + (P_z - R_x)\, dz \wedge dx + (Q_x - P_y)\, dx \wedge dy$

$= \int_S \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ P & Q & R \end{vmatrix} \cdot \vec{n}\, d\sigma = \int_S \langle \operatorname{curl} \vec{F}, n \rangle \, d\sigma$

Fact: $\vec{n}\, d\sigma =$
$(dy \wedge dz, dz \wedge dx, dx \wedge dy)$

$\varphi \in \Lambda^{\overset{2}{\shortparallel}}{}^{n-1}$: $\int_\Omega d\varphi = \int_{\partial\Omega} \varphi = \int_S P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy = \int_S \vec{F} \cdot \vec{n}\, d\sigma$

$\shortparallel \int_\Omega (P_x + Q_y + R_z)\, dx \wedge dy \wedge dz = \int_\Omega \operatorname{div} \vec{F}\, d\mathrm{vol}$



$S = \partial\Omega$

Example 2. In $\mathbb{R}^{1,3}$, Maxwell eq'n: $dF = 0$ i.e. $\operatorname{div} \mathbb{H} = 0 = \operatorname{curl} \mathbb{E} + \frac{1}{c} \frac{\partial \mathbb{H}}{\partial t}$

$\delta F := \overset{-1}{*} d * F = \frac{4\pi}{c} \rho (c, \vec{v})$ i.e. $\operatorname{div} \mathbb{E} = 4\pi\rho$; $\operatorname{curl} \mathbb{H} + \frac{1}{c} \frac{\partial \mathbb{E}}{\partial t} = \frac{4\pi}{c} \rho \vec{v}$.

ρ = charge density

Ex. Explain meaning after $\int$.

Covariant Differentiation / Christoffel symbol

Def^{n}: Ⓣ (Gauge transf). In $z^{1'}, \dots, z^{n'}$: $\Gamma^{k'}_{p'q'}(z') = \frac{\partial z^{k'}}{\partial z^k} \Gamma^k_{pq} \frac{\partial z^p}{\partial z^{p'}} \frac{\partial z^q}{\partial z^{q'}} + \frac{\partial z^{k'}}{\partial z^k} \frac{\partial^2 z^k}{\partial z^{p'} \partial z^{q'}}$

In particular, Γ is NOT a tensor of type $(1,2)$. Only the difference is.

Thm: Given $\Gamma^k_{ij}(x)$ in any coor x, the modified "co-variant derivative"

$$T^k{}_{;j} := \frac{\partial T^k}{\partial x^j} + \Gamma^k_{ij} T^i \text{ is a } (1,1) \text{ tensor} \iff Ⓣ \text{ holds.}$$

Pf: $$T^k_{;j} = \frac{\partial x^{j'}}{\partial x^j} \frac{\partial}{\partial x^{j'}} \left(T^{k'} \frac{\partial x^k}{\partial x^{k'}} \right) + \Gamma^k_{ij} T^{i'} \frac{\partial x^i}{\partial x^{i'}}$$

$$= \frac{\partial T^{k'}}{\partial x^{j'}} \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^k}{\partial x^{k'}} + T^{k'} \frac{\partial x^{j'}}{\partial x^j} \frac{\partial^2 x^k}{\partial x^{j'} \partial x^{k'}} + \Gamma^k_{ij} T^{i'} \frac{\partial x^i}{\partial x^{i'}} \qquad k' \leftrightarrow i' \text{ in the middle}$$

$$= \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^k}{\partial x^{k'}} \left[\frac{\partial T^{k'}}{\partial x^{j'}} + T^{i'} \left(\frac{\partial x^{k'}}{\partial x^k} \frac{\partial^2 x^k}{\partial x^{j'} \partial x^{i'}} + \Gamma^k_{ij} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^{k'}}{\partial x^k} \frac{\partial x^j}{\partial x^{j'}} \right) \right] \quad ※$$

Cor: Let $\{\Gamma^k_{ij}\}$ be a Christoffel symbol (or connection, or co-var. deri)

Then $T^k_{ij} := \Gamma^k_{ij} - \Gamma^k_{ji}$ is a $(1,2)$ tensor, called the torsion tensor.

Def^{n}: Given Γ^k_{ij} is equiv. to say $\nabla_i \frac{\partial}{\partial x^j} = \Gamma^k_{ji} \frac{\partial}{\partial x^k}$ notice the (i,j) order

ie. $\nabla_i (T^j \partial_j) = (\partial_i T^j) \partial_j + T^j \Gamma^k_{ij} \partial_k = (\partial_i T^k + \Gamma^k_{ij} T^j) \partial_k$

(i) By duality, $\nabla_i dx^j = -\Gamma^j_{ki} dx^k$.

$0 = \partial_i \left(dx^j, \frac{\partial}{\partial x^k} \right) = \left(\nabla_i dx^j, \frac{\partial}{\partial x^k} \right) + \left(dx^j, \Gamma^l_{ki} \frac{\partial}{\partial x^l} \right)$ $= \Gamma^j_{ki}$

Rmk: ← This is also Leibniz' rule for contraction of tensors

(ii) Extending by Leibniz' rule get ∇_i on all (p,q) tensors.

$$\nabla_i T = \nabla_i \left(T^{i_1 \cdots i_p}_{j_1 \cdots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \cdots \otimes dx^{j_q} \right)$$

$$= \frac{\partial T^I_J}{\partial x^i} + T^{i_1 \cdots i_p}_{j_1 \cdots j_q} \Gamma^k_{i_1 i} \frac{\partial}{\partial x^k} \otimes \cdots - T^{i_1 \cdots i_p}_{j_1 \cdots j_q} \frac{\partial}{\partial x^I} \otimes \Gamma^{j_1}_{ki} dx^k \otimes \cdots$$

$$= \left(\partial_i T^I_J + \sum_{s=1}^{p} T^{i_1 \cdots k \cdots i_p}_J \Gamma^{i_s}_{ki} - \sum_{s=1}^{q} T^I_{j_1 \cdots k \cdots j_q} \Gamma^k_{j_s i} \right) \frac{\partial}{\partial x^I} \otimes dx^J \quad ※$$

Parallel Transport of Tensors along a curve:

for $X \in T_p S$, define $\nabla_X T = \sum_k \xi^k \nabla_{\partial_k} T$ for any tensor (field) T

for $\gamma : [a,b] \to S$ be a curve $\nabla_{\gamma'} T$ is defined, with $\xi^k = \frac{dx^k}{dt}$.

ODE $\Rightarrow$ Given $T_a \in \otimes^{p,q}(T_{\gamma(a)} S)$, $\exists!$ T along γ st $\nabla_{\gamma'} T = 0$

eg. for vector fields: $(\nabla_{\gamma'} T)^i = \frac{dx^k}{dt} \left(\frac{\partial T^i}{\partial x^k} + \Gamma^i_{jk} T^j \right) = \frac{dT^i}{dt} + \Gamma^i_{jk} \frac{dx^k}{dt} T^j = 0$

Def^{n} (Geodesic): γ is a geodesic wrt given conn. Γ^k_{ij} iff $\nabla_{\gamma'} \gamma' = 0$.

ie. $\frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^k}{dt} \frac{dx^j}{dt} = 0$ non-linear 2nd order ODE

det. by initial condi. $\gamma(0)$, $\gamma'(0)$.

It depends only on $\Gamma^i_{(jk)} := \Gamma^i_{jk} + \Gamma^i_{kj}$, which is a torsion-free conn.

Notice that: Parallel transport / geodesics are notions indep. of coor. systems.

Question: Do connection really exist? say on a surface $S^m \hookrightarrow \mathbb{R}^n$?

Answer: YES. For any ∇ on $\mathbb{R}^n$ (say $\Gamma_{ij}^k \equiv 0$, or any matrix function) ~ since there is only one chart.

Set $\bar{\nabla}_X := \nabla_X^T$ the tangent part.

The standard $\nabla_i := \partial_i$ has a better property: $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ (*)

for $X, Y, Z \in C^\infty(TS) \Rightarrow X\langle Y, Z\rangle = \langle \bar{\nabla}_X Y, Z\rangle + \langle Y, \bar{\nabla}_X Z\rangle$ as well. This motivates:

Connection ∇ compatible with $g_{ij}\,dx^i \otimes dx^j \triangleq g_{ij}$ is parallel, i.e. $\nabla_k g_{ij} \equiv 0$.

That is, $\nabla_k g_{ij} \equiv g_{ij;k} = \partial_k g_{ij} - g_{lj}\Gamma_{ik}^l - g_{il}\Gamma_{jk}^l = 0$ (**)

equivalently, (*) holds for ∇.

Fundamental Thm of metric geom (Levi-Civita):

$\exists!$ torsion-free conn ∇ comp. with given metric. called ∇^{LC}.

In fact: $\Gamma_{ij}^k = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$. pf: cyclic trick for (**) $\Rightarrow$ unique. Check ① holds $\#$

Cor. 1: All other connections are $\nabla^{LC} + T$ where T is any (1,2) tensor.

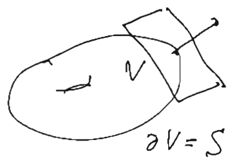
Cor. 2: $\langle S, T\rangle = $ const along $r(t)$ if S, T parallel along $r(t)$.

In particular, $|T| = $ const. eg. r geodesic $\Rightarrow |r'| = c$, constant speed.

Example 1. Divergence of a v.f.: $\operatorname{div} T = \nabla_i T^i = \frac{\partial T^i}{\partial x^i} + \Gamma_{ki}^i T^k$

$$\Gamma_{k\,i}^{\,i} = \frac{1}{2} g^{il}(\cancel{\partial_i g_{lk}} + \partial_k g_{li} - \cancel{\partial_l g_{ki}}) = \frac{1}{2g}\partial_k g = \partial_k \log\sqrt{|g|}$$

$$\Rightarrow \operatorname{div} T = \partial_i T^i + \frac{1}{2g}(\partial_k g)\,T^k = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^k}\left(\sqrt{|g|}\,T^k\right)$$



$\Rightarrow$ Thm (Divergence thm for Riem spaces) (Ex 29.5 #14)

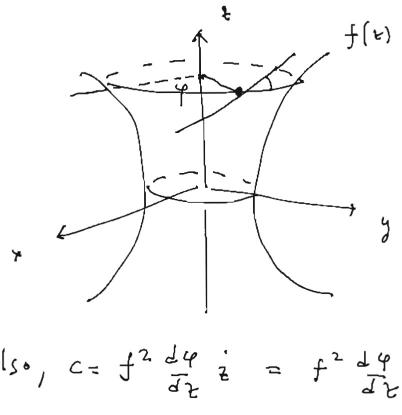
$$\int_{\partial V} \langle T, \vec{n}\rangle\, dS = \int_V \operatorname{div} T\, dV.$$

* Q: why is this true?

pf: $\text{LHS} \overset{*}{=} \int_{\partial V} T^i\, dS_i$ with $dS_i := (-1)^{i-1}\sqrt{|g|}\, dx^1 \wedge \dots \widehat{dx^i} \dots \wedge dx^n$

$$= \int_{\partial V} T^i (-1)^{i-1}\sqrt{|g|}\, dx^1 \wedge \dots \widehat{dx^i} \dots \wedge dx^n = \int_V \frac{\partial}{\partial x^i}\left(T^i\sqrt{|g|}\right) dx^1 \wedge \dots \wedge dx^n = \text{RHS} \quad \#$$

Example 2: Geodesics on surfaces of revolution. i.e. $r(t)$, $\nabla_{r'} r' = (r'')^T = 0$.



$r(\varphi, z) = (f(z)\cos\varphi, f(z)\sin\varphi, z)$

$r_1 = (-f\sin\varphi, f\cos\varphi, 0)$

$r_2 = (f'\cos\varphi, f'\sin\varphi, 1)$

$dl^2 = f^2 d\varphi^2 + (1 + (f')^2)\, dz^2$ i.e. $1 = f^2\dot{\varphi}^2 + (1 + f'^2)\dot{z}^2$

$\ddot{\varphi} + \frac{2ff'}{f^2}\dot{\varphi}\dot{z} = 0 \Rightarrow (f^2\dot{\varphi})' = f^2\ddot{\varphi} + 2ff'\dot{z}\dot{\varphi} = 0$

i.e. $f^2\dot{\varphi} = $ constant $\Rightarrow$ Clairaut relation: $f\cos\psi = c$

$= \langle r_1\dot{\varphi} + r_2\dot{z}, r_1\rangle = \langle \dot{r}, r_1\rangle = f\cos\psi$

Also, $c = f^2\frac{d\varphi}{dz}\dot{z} = f^2\frac{d\varphi}{dz}\sqrt{\frac{1 - f^2\dot{\varphi}^2}{1 + f'^2}} = f\frac{d\varphi}{dz}\sqrt{\frac{f^2 - c^2}{1 + f'^2}} \Rightarrow \frac{d\varphi}{dz} = \frac{c}{f}\sqrt{\frac{1 + f'^2}{f^2 - c^2}} \Rightarrow \varphi = \int \frac{c}{f}\sqrt{\frac{1 + f'^2}{f^2 - c^2}}\, dz$ $\#$

Riemann's Curvature Tensor $R^i_{jk\ell}$:

let $T = T^i \partial_i$ be a v.f. Want to compare $\nabla_k \nabla_\ell T^i - \nabla_\ell \nabla_k T^i$

i.e. $T^i_{;\ell k} - T^i_{;k\ell}$

$$\nabla_k(\nabla_\ell T^i) = \nabla_k\left(\partial_\ell T^i + \Gamma^i_{q\ell} T^q\right) \equiv \nabla_k T^i_{;\ell}$$

$$= \partial_k\left(\partial_\ell T^i + \Gamma^i_{q\ell} T^q\right) + \Gamma^i_{pk} T^p_{;\ell} - \Gamma^p_{\ell k} T^i_{;p}$$

$$= \underline{\partial_k \partial_\ell T^i} + \underline{\partial_k \Gamma^i_{q\ell}} T^q + \Gamma^i_{q\ell} \partial_k T^q + \underline{\Gamma^i_{pk}}\left(\partial_\ell T^p + \underline{\Gamma^p_{q\ell} T^q}\right) - \Gamma^p_{\ell k}\underline{\left(\partial_p T^i + \Gamma^i_{qp} T^q\right)}$$

$$\Rightarrow \nabla_k \nabla_\ell T^i - \nabla_\ell \nabla_k T^i = \left(\underline{\partial_k \Gamma^i_{q\ell}} - \partial_\ell \Gamma^i_{qk} + \underline{\Gamma^i_{pk} \Gamma^p_{q\ell}} - \Gamma^i_{p\ell} \Gamma^p_{qk}\right) T^q$$

$$+ \cancel{\Gamma^i_{q\ell} \partial_k T^q} - \cancel{\Gamma^i_{pk} \partial_\ell T^p} - \left(\underline{\Gamma^p_{\ell k}} - \Gamma^p_{k\ell}\right) \underline{T^i_{;p}}$$

$$- \cancel{\Gamma^i_{qk} \partial_\ell T^q} + \cancel{\Gamma^i_{p\ell} \partial_k T^p}$$

$$=: R^i_{qk\ell} T^q + \mathcal{T}^p_{k\ell} T^i_{;p} \qquad \mathcal{T} = \text{torsion tensor}$$

Rmk: the sign in the book is wrong !

Def / Thm : For any conn, $R^i_{qk\ell} := \dfrac{\partial \Gamma^i_{q\ell}}{\partial x^k} - \dfrac{\partial \Gamma^i_{qk}}{\partial x^\ell} + \Gamma^i_{pk} \Gamma^p_{q\ell} - \Gamma^i_{p\ell} \Gamma^p_{qk}$

is a (1,3) tensor ※

Coordinate free formulas / defin :

$$\mathcal{T}(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$$

$$R(X,Y)Z := [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]} Z \qquad \text{for any } C^\infty \text{ v.f. } X, Y, Z$$

The correction terms are added so that they are "function-linear" (check!)

so can compute RHS using any frame, eg. coor. frame ∂_i .

① $\mathcal{T}(\partial_i, \partial_j) = \nabla_{\partial_i} \partial_j - \nabla_{\partial_j} \partial_i - [\partial_i, \partial_j] = \left(\Gamma^k_{ji} - \Gamma^k_{ij}\right) \partial_k = - \mathcal{T}^k_{ij} \partial_k$ ※

② $R(\partial_k, \partial_\ell) \partial_q = \nabla_k \nabla_\ell \partial_q - \nabla_\ell \nabla_k \partial_q = \nabla_k\left(\Gamma^i_{q\ell} \partial_i\right) - \nabla_\ell\left(\Gamma^i_{qk} \partial_i\right)$

$$= \partial_k \Gamma^i_{q\ell} \partial_i + \Gamma^i_{q\ell} \Gamma^j_{ik} \partial_j - \partial_\ell \Gamma^i_{qk} \partial_i - \Gamma^i_{qk} \Gamma^j_{i\ell} \partial_j$$

$$= R^i_{qk\ell} \partial_i$$ ※

Thm (Symmetries)

Compare the sign convention with the book.

(i) $R^i_{q\underline{k\ell}} = - R^i_{q\underline{\ell k}}$ for any ∇

(ii) (1st Bianchi identity) ∇ sym $\Rightarrow R^i_{[qk\ell]} := R^i_{qk\ell} + R^i_{k\ell q} + R^i_{\ell qk} = 0$

(iii) ∇ compatible with metric $\Rightarrow R_{\underline{iq}k\ell} := g_{ip} R^p_{qk\ell}$ $\underline{\text{skew-sym}}$ in (i,q)

(iv) $\nabla = \nabla^{LC} \Rightarrow R_{\underbrace{iq}\underbrace{k\ell}} = R_{\underbrace{k\ell}\underbrace{iq}}$. $\equiv \langle R(\partial_k, \partial_\ell) \partial_q, \partial_i \rangle$

pf : (i) is by defin. (ii) : By ②, equiv to : $0 = [\nabla_k, \nabla_\ell] \partial_q + [\nabla_\ell, \nabla_q] \partial_k + [\nabla_q, \nabla_k] \partial_\ell$

expand out : $\nabla_k \underline{\nabla_\ell \partial_q} - \nabla_\ell \underline{\underline{\nabla_k \partial_q}} + \nabla_\ell \underline{\underline{\nabla_q \partial_k}} - \nabla_q \underline{\nabla_\ell \partial_k} + \nabla_q \underline{\nabla_k \partial_\ell} - \nabla_k \underline{\nabla_q \partial_\ell} = 0$.

since sym (torsion free) $\Leftrightarrow \nabla_\ell \partial_q = \nabla_q \partial_\ell$ ※

(iii) $\langle [\nabla_k, \nabla_\ell]\xi, \xi \rangle = \langle [\nabla_k, \nabla_\ell](\xi^i \partial_i), \xi^q \partial_q \rangle$ $\quad$ ξ any C^∞ v.f.

$= \langle R_p{}^i{}_{k\ell}\, \partial_i, \partial_q \rangle \xi^p \xi^q = g_{iq} R_p{}^i{}_{k\ell}\, \xi^p \xi^q = R_{qpk\ell}\, \xi^p \xi^q$

↗ use the fact that $R(X,Y)Z$ is function-linear (tensor).

Rmk: The sign and pf in the book is wrong!

Now $\partial_k \partial_\ell \langle \xi, \xi \rangle = 2 \partial_k \langle \nabla_\ell \xi, \xi \rangle = 2 \langle \nabla_k \nabla_\ell \xi, \xi \rangle + 2 \langle \nabla_\ell \xi, \nabla_k \xi \rangle$

$\Rightarrow \quad 0 = \partial_k \partial_\ell \langle \xi, \xi \rangle - \partial_\ell \partial_k \langle \xi, \xi \rangle = 2 \langle [\nabla_k, \nabla_\ell]\xi, \xi \rangle$. ※

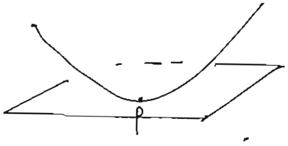
(iv) Exercise : (i) + (ii) + (iii) $\Rightarrow$ (iv).

Def$^{\text{n}}$: (Let $\nabla = \nabla^{LC}$) Ricci curvature $R_{q\ell} := R^k{}_{qk\ell} \equiv g^{ik} R_{iqk\ell}$.

scalar curvature $R := g^{\ell q} R_{q\ell} \equiv g^{q\ell} g^{ik} R_{iqk\ell}$.

Examples 1. Thm: 2D $\Rightarrow$ $R = 2K$.

For $S \hookrightarrow \mathbb{R}^{n+1}$ hyp. surface



$z = f(x^1, x^2)$, $\nabla f|_p = 0$

$r(x^1, x^2) = (x^1, x^2, f)$

$r_1 = (1, 0, f_1)$

$r_2 = (0, 1, f_2)$

$g_{ij} = \delta_{ij} + f_i f_j$ $\qquad$ $\nabla^{LC} = (D^{\mathbb{R}^3})^T$

$\partial_k g_{ij}|_p = 0 \Rightarrow \Gamma^k_{ij}|_p = 0$

$g_{ij}|_p = \delta_{ij}$ (normal coor at p)

$$\Rightarrow \quad R^i{}_{qk\ell} \underset{\text{at } p}{=} \partial_k \Gamma^i_{q\ell} - \partial_\ell \Gamma^i_{qk} = \frac{1}{2}\Big(\partial_k \partial_q g_{i\ell} + \cancel{\partial_k \partial_x g_{iq}} - \partial_k \partial_i g_{q\ell} - \partial_\ell \partial_q g_{ik} - \cancel{\partial_\ell \partial_k g_{iq}} + \partial_\ell \partial_i g_{qk} \Big)$$

Now for $n = 2$, $R_{1212} = R^1{}_{212} = \frac{1}{2}\left(\partial_1 \partial_2 g_{12} - \partial_1^2 g_{22} - \partial_2^2 g_{11} + \partial_2 \partial_1 g_{21} \right)$

$= \frac{1}{2}\left(2(f_{21}^2 + f_{11} f_{22}) - 2 f_{21}^2 - 2 f_{12}^2\right) = f_{11} f_{22} - f_{12}^2 = K(p)$

Hence $R = g^{q\ell} R^i{}_{qi\ell} \underset{\text{at } p}{=} R^2{}_{121} + R^1{}_{212} = 2K$ ※

Rmk: It is clear that $R_{ij} = R g_{ij}$, $R = 2\dfrac{R_{1212}}{g_{11} g_{22} - g_{12}^2}$

In general for a plane $\sigma = \langle X, Y \rangle \subset T_p S$

sectional curvature $K(\sigma) := \dfrac{\langle R(X,Y)Y, X \rangle}{\| X \wedge Y \|^2}$

generalizes Gauss K.

$\partial_1 \partial_2 (f_1 f_2)$ at p

$= \partial_1 (f_{21} f_2 + f_1 f_{22})$

$= f_{21}^2 + f_{11} f_{22}$

. $\partial_1 \partial_1 (f_2 f_2) = 2 f_{21}^2$

. In general we set

$R(X,Y,Z,W) := \langle R(X,Y)W, Z \rangle$

(Thm: All $K(\sigma)$ at $T_p S$ determine $R(X,Y,Z,W)$.)

Example 2: 3D, EX. Thm: $R_{\alpha\beta\gamma\delta} = -\frac{R}{2}(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}) + R_{\alpha\gamma} g_{\beta\delta} - R_{\alpha\delta} g_{\beta\gamma} + R_{\beta\delta} g_{\alpha\gamma} - R_{\beta\gamma} g_{\alpha\delta}$

idea: $R_{iqk\ell} = R_{[iq][k\ell]} = R_{AB}$ quad form on $\Lambda^2(T_p S)$ $\quad [i,q] \leftrightarrow e^i \wedge e^q$ $\quad rk = 3$

$R_{AB} = R_{BA} \Rightarrow$ only 6 components. R_{ij} also has 6 comp. sym.

Example 3: 4D. Einstein's field Eq$^{\text{n}}$: $R_{ij} - \frac{1}{2} R g_{ij} = \lambda T_{ij}$; $\quad \nabla_j T^j_i = 0$

Ex. 2nd Bianchi identity $R_{ij[k\ell;m]} = 0$ (Ex 30.5 #7)

(30.5 #7+#8) Also, Einstein eq$^{\text{n}}$ in dim $n \geq 3$, Riem case, $R_{ij} = \lambda g_{ij} \Rightarrow \lambda = $ const.

Example 4 : Lie gp G with bi-inv metric $\langle\, ,\, \rangle$.

$X, Y \in \mathfrak{g}$, L_X, L_Y l.i.v.f Let $\nabla_{L_X} L_Y := \frac{1}{2} L_{[X,Y]} = \frac{1}{2} [L_X, L_Y]$

Lemma : $\nabla = \nabla^{LC}$! super easy !! Q : why does it enough to check just for l.i.v.f.'s ?

pf : $T(L_X, L_Y) = \nabla_{L_X} L_Y - \nabla_{L_Y} L_X - [L_X, L_Y] = 0$ tor-free

$\underset{\overset{\shortparallel}{0}}{L_X \langle L_Y, L_Z \rangle} - (\langle \underset{\overset{\shortparallel}{\frac{1}{2}[L_X, L_Y]}}{\nabla_{L_X} L_Y}, L_Z \rangle + \langle L_Y, \underset{\overset{\shortparallel}{\frac{1}{2}[L_X, L_Z]}}{\nabla_{L_X} L_Z} \rangle) = 0$ by def^n of Killing condition

cor : for such (G, ∇), $R(X,Y)Z = -\frac{1}{4}[[X,Y],Z]$ at $\mathfrak{g} = T_e G$.

Hence $R(X,Y,Z,W) = \frac{1}{4}\langle [X,Y],[Z,W] \rangle$ and $R(X,Y,X,Y) = \frac{1}{4}|[X,Y]|^2 \geq 0$.

The formula hold at every $T_A G$ via l.i.v.f.s .

pf : $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

$= \frac{1}{4}[X,[Y,Z]] - \frac{1}{4}[Y,[X,Z]] - \frac{1}{2}[[X,Y],Z] = -\frac{1}{4}[[X,Y],Z]$ by Jacobi id.

$R(X,Y,Z,W) = \langle R(X,Y)W, Z \rangle = -\frac{1}{4}\langle [[X,Y],W],Z \rangle = \frac{1}{4}\langle (\mathrm{ad}\, W)[X,Y], Z \rangle$

$= -\frac{1}{4}\langle [X,Y], (\mathrm{ad}\, W) Z \rangle = \frac{1}{4}\langle [X,Y],[Z,W] \rangle$ ※

Cor : For such (G, ∇), geodesics thr e $\equiv$ 1-parameter subgroups .

In particular, the exp maps defined in both cases coincide .

Pf : $A(t) = e^{tX} \Rightarrow A'(t) = AX = L_X \Rightarrow \nabla_{A'} A' = \nabla_{L_X} L_X = \frac{1}{2}[L_X, L_X] = 0$

Conversely eq^n $\nabla_{A'} A' = 0$ with $A'(0) = X$ always has (unique) sol $A(X) = e^{tX}$ ※

Rmk : for such (G, ∇), it is geod. complete . Hopf-Rinow $\Rightarrow$ any 2 pt can be joined by a geod , ie exp is onto. Hence $SL(2, \mathbb{R})$ does not .

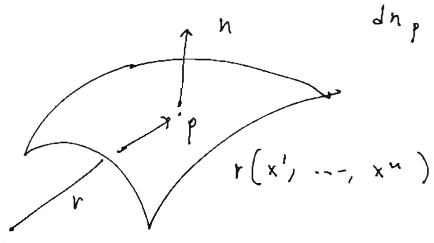
Gauss-Codazzi Equations :

For $\underset{n\text{-dim}}{S} \hookrightarrow \mathbb{R}^{n+1}$ coor $y^1, \cdots, y^{n+1}$

$$r_{ij} = (D_j r_i)^T + (D_j r_i)^N$$
$$= \nabla_j r_i + \langle r_{ij}, n \rangle n$$
$$=: \sum_{k=1}^{n} \Gamma_{ij}^k r_k + b_{ij} n$$

2nd fund form

$dn_p : T_p S \to T_{n(p)} S^n \cong T_p S$



So $b_{ij} = \langle r_{ij}, n \rangle = -\langle r_i, n_j \rangle \equiv -\langle r_j, n_i \rangle$

$dn_p : n_i = a_i^j r_j \Rightarrow a_i^j g_{jl} = \langle n_i, r_l \rangle = -b_{il}$

$= -b_i^j r_j$

compatibility :

$\underline{r_{ijk}} = \partial_k \Gamma_{ij}^l r_l + \Gamma_{ij}^l r_{lk} + \partial_k b_{ij} n - b_{ij} b_k^l r_l$

$= (\partial_k \Gamma_{ij}^l + \Gamma_{ij}^s \Gamma_{sk}^l - b_{ij} b_k^l) r_l + (\Gamma_{ij}^s b_{sk} + \partial_k b_{ij}) n \overset{\overset{\text{has to satisfy}}{\downarrow}}{=\!=} r_{i\underline{kj}}$

Thm : $\partial_k \Gamma_{ij}^l - \partial_j \Gamma_{ik}^l + \Gamma_{ij}^s \Gamma_{sk}^l - \Gamma_{ik}^s \Gamma_{sj}^l = b_{ij} b_k^l - b_{ik} b_j^l$ (Gauss eq^n)

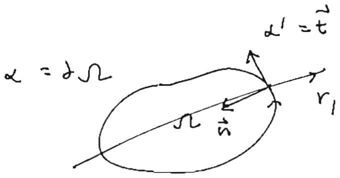
Frobenius integrability] $\partial_k b_{ij} - \partial_j b_{ik} = -\Gamma_{ij}^s b_{sk} + \Gamma_{ik}^s b_{sj}$ (Codazzi) det. $S \hookrightarrow \mathbb{R}^{n+1}$ upto $\mathbb{R}^n$-motion

※

Gauss-Bonnet for surface

pick any $e_1 \perp e_2$ then

Lemma: Orthogonal coor exists in dim 2. (for elementary pf. using level functions)



$$f = 0 \Rightarrow K = -\frac{1}{\sqrt{EG}}\left[\left(\frac{\sqrt{E}_2}{\sqrt{G}}\right)_2 + \left(\frac{\sqrt{G}_1}{\sqrt{E}}\right)_1\right]$$

$$\frac{1}{2}\frac{1}{\sqrt{EG}}\left(E_2 x' - G_1 y'\right)$$
$\|$ Q : what is it ?

$$\int_\Omega K\,dA = \int_\Omega (Q_x - P_y)\,dx\,dy = \int_{\partial\Omega}\left(\frac{\sqrt{E}_2}{\sqrt{G}}x' - \frac{\sqrt{G}_1}{\sqrt{E}}y'\right)dl$$

Def'n: $\alpha' = r_1 x' + r_2 y'$ by arc length l. geodesic curvature $\nabla_{\alpha'}\alpha' =: k_g \vec{n}$.

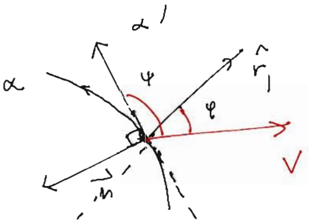
idea: k_g is the change rate of certain angle, as in the $\mathbb{R}^2$ case.

① let V be a parallel v.f. along α : $\nabla_{\alpha'}V = 0$; $|V| = 1$.

$$\langle V, \alpha'\rangle' = \langle V, \nabla_{\alpha'}\alpha'\rangle = k_g\langle V, \hat{n}\rangle = \cos\left(\frac{\pi}{2} + \psi\right) = -\sin\psi \Rightarrow k_g = \psi'$$

$\|$
$(\cos\psi)' = -\sin\varphi\cdot\psi'$ notice the "+", not "−".

② other choices of unit v.f.'s. eg. $\hat{r}_1 = r_1/|r_1| = r_1/\sqrt{E}$: let $\nabla_{\alpha'}\hat{r}_1 = \lambda\hat{r}_2$.



then similarly $\lambda = \varphi'$ when $\cos\psi = \langle V, \hat{r}_1\rangle$.

of course ψ', φ' are all indep of choices of V.

$$\nabla_{\alpha'}\hat{r}_1 = x'\nabla_1\frac{r_1}{\sqrt{E}} + y'\nabla_2\frac{r_1}{\sqrt{E}} \underset{\hat{r}_2 \text{ component}}{\longmapsto} \left(\frac{x'}{\sqrt{E}}\Gamma_{11}^2 + \frac{y'}{\sqrt{E}}\Gamma_{12}^2\right)r_2$$

$$\Gamma_{11}^2 = \frac{1}{2}g^{22}(-\partial_2 g_{11}) = \frac{1}{2}\frac{-E_2}{G} \;;\; \Gamma_{12}^2 = \frac{1}{2}g^{22}\partial_1 g_{22} = \frac{1}{2}\frac{G_1}{G}$$

$\theta_\alpha = \int_\Omega K dA$

$$\Rightarrow \lambda = \langle\nabla_{\alpha'}\hat{r}_1, \hat{r}_2\rangle = \frac{-1}{2}\frac{1}{\sqrt{EG}}\left(E_2 x' - G_1 y'\right)$$

$$\Rightarrow \int_\Omega K\,dA = -\int_{\partial\Omega}\lambda\,dl = -\int_{\partial\Omega}\varphi'\,dl = (-\varphi)\Big|_0^L =: \text{Holonomy angle } \theta_\alpha$$

eg.

③ let $\Theta := \psi - \varphi = \text{angle}(\hat{r}_1, \alpha')$

$$\Rightarrow \int_{\partial\Omega}\Theta' = \int_{\partial\Omega}\psi' + \int_{\partial\Omega}(-\varphi)' = \int_\alpha k_g + \int_\Omega K\,dA$$

Hopf: see ⊛ " $2\pi - \sum\alpha_j$ $\Rightarrow$ $2\pi = \underset{\text{0-dim curvature}}{\sum\alpha_j} + \underset{\text{1-dim curvature}}{\int_\alpha k_g} + \underset{\text{2-dim curvature}}{\int_\Omega K}$ *

Theorem (Gauss-Bonnet)

$$2\pi\cdot\chi(\Omega) = \sum\alpha_j + \int_{\partial\Omega}k_g + \int_\Omega K.$$ (Ex.. Prove it by triangulations)

Applications: (1) geodesic △

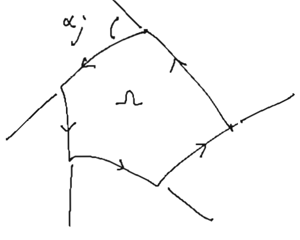
$$\theta_1 + \theta_2 + \theta_3 - \pi = \int_\Omega K.$$

(2) $K < 0 \Rightarrow$ 2 geodesics intersects at most once.

(3) $K > 0$ cpt $\Rightarrow$ any 2 geodesics intersect.

pf: $g(S) = 0$. if  $\simeq$  then $0 = \int K \underset{*}{>} 0$.

⊛ :



α_j = outer angle

1-dim'l variational problems : Action functional via Lagrangian .

$S[\gamma] = \int_P^Q L(x(t), \dot{x}(t))\, dt$ $\qquad$ $L(t, x, \xi)$ a function in $(t, x, \xi) \in [a,b] \times U \times \mathbb{R}^n$

Q : Find γ with least action. $\qquad$ $\gamma: \; x : [a,b] \to U \quad x(a) = P, \; x(b) = Q$

Thm : If γ_0 attains minimum among sm curves from P to Q

then the "variational derivative" $\frac{\delta S}{\delta x^i} := \frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \xi^i}(x, \dot{x})\right) = 0, \; \forall \, i = 1, \dots, n$.

$\nwarrow$ (Euler-Lagrange eq'n) .

pf : For any $\eta \in C^1([a,b], U)$, $\eta(a) = 0 = \eta(b)$:

$$0 = \frac{d}{d\varepsilon} S[\gamma_0 + \varepsilon\eta]\Big|_{\varepsilon=0} = \int_a^b \left(\frac{\partial L}{\partial x^i} \eta^i(t) + \frac{\partial L}{\partial \xi^i} \dot{\eta}^i(t)\right) dt$$

$$= \int_a^b \frac{\partial L}{\delta x^i} \eta^i \, dt + \frac{\partial L}{\partial \xi^i} \eta^i \Big|_a^b - \int_a^b \frac{d}{dt}\left(\frac{\partial L}{\partial \xi^i}\right) \eta^i \, dt = \int_a^b \frac{\delta S}{\delta x^i} \eta^i \, dt$$

Set $\eta^i = f \frac{\delta S}{\delta x^i}$ with $f \in C^\infty$, $f(a) = f(b) = 0$, o.w. $> 0 \Rightarrow \frac{\delta S}{\delta x^i} \equiv 0 \quad \forall i$ ※

Def'n : Write E-L eq'n as $\dot{p}_i = f_i$ where $p_i := \frac{\partial L}{\partial \xi^i}$, $f_i := \frac{\partial L}{\partial x^i}$.

momentum $\qquad$ force

Also, $E := \xi^i \frac{\partial L}{\partial \xi^i} - L$: energy .

Examples (a) $L = \frac{1}{2} m |\xi|^2 - U(x) \Rightarrow \vec{f} = \nabla^{(x)} L = -\nabla U, \; \vec{p} = \nabla^{(\xi)} L = m\xi = m\dot{x}$

so E-L $\Leftrightarrow$ Newton's equation $m\ddot{x} = -\nabla U$. Also $E = m\dot{x}^2 - L = \frac{1}{2} m \dot{x}^2 + U$.

(b) Energy of a curve in a space with metric g_{ij} :

$L = \frac{1}{2}|\xi|^2 = \frac{1}{2} g_{ij}(x)\xi^i\xi^j \Rightarrow p_k = g_{kj}\xi^j = \xi_k$; $f_k = \frac{1}{2}(\partial_k g_{ij})\xi^i\xi^j$; $E = L$!

E-L $\Leftrightarrow$ $\frac{d}{dt} p_k = \partial_i g_{kj} \dot{x}^i \dot{x}^j + g_{kj}\ddot{x}^j = f_k = \frac{1}{2}\partial_k g_{ij} \dot{x}^i \dot{x}^j$

$\cdot g^{km} \Rightarrow \ddot{x}^m + g^{km}\left(\partial_i g_{kj} - \frac{1}{2}\partial_k g_{ij}\right)\dot{x}^i\dot{x}^j \underset{\text{sym in } \Sigma_{ij}}{=} \ddot{x}^m + \Gamma_{ij}^m \dot{x}^i\dot{x}^j = 0$

i.e. the extremal curve γ_0 is a geodesic , so with parameter prop. to ℓ .

(c) Length of a curve in $d\ell^2 = g_{ij}\, dx^i dx^j$:

$L = |\xi| = \sqrt{g_{ij}\xi^i\xi^j}$. Then E-L $\Leftrightarrow$ $\frac{d}{dt}\left(\frac{\cancel{2}\, g_{kj}\xi^j}{\cancel{2}\sqrt{g_{ij}\xi^i\xi^j}}\right) = \frac{(\partial_k g_{ij})\xi^i\xi^j}{2\sqrt{g_{ij}\xi^i\xi^j}}$

This could be complicate. but if we set $t = \mu\ell$, then $|\xi| = |\dot{x}| = \mu$

and E-L eq'n goes back to case (b) . i.e. geodesics .

Rmk : arc length ℓ is the "natural parameter" since it is inv. under parametrization.

Conservation Laws : preliminary cases .

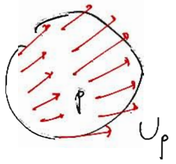
$$\frac{dE}{dt} = \frac{d}{dt}\left(\dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L\right) = \cancel{\ddot{x}^i \frac{\partial L}{\partial \dot{x}^i}} + \cancel{\dot{x}^i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)} - \frac{\partial L}{\partial t} - \cancel{\frac{\partial L}{\partial x^i}\dot{x}^i} - \cancel{\frac{\partial L}{\partial \dot{x}^i}\ddot{x}^i} = -\frac{\partial L}{\partial t}$$

Thm : Along an extremal $\gamma_0(t)$, the energy E is indep of t iff L is indep of t .

Similarly, and easier, $\dot{p}_i \equiv 0$ along γ_0 $\Leftrightarrow$ L is indep of x^i $\left(\text{i.e. } \frac{\partial L}{\partial x^i} \equiv 0\right)$.

General case: Suppose that $\exists$ 1 parameter local gp of local transformations $\forall p \in \mathbb{R}^n$, $\exists S_\tau : U_p \to \mathbb{R}^n$ C^∞. $\forall \tau \in (-\delta_p, \delta_p)$

st $S_0 = id$, $S_{\tau_1+\tau_2} = S_{\tau_1} \circ S_{\tau_2}$, $S_{-\tau} = S_\tau^{-1}$ over the region / that they are defined.



Had seen, $\{S_\tau\}$ is gen. by a v.f. $X(p) := \frac{d}{d\tau} S_\tau(p) \big|_{\tau=0}$.

Def^{n}: (time-indep case)

$\{S_\tau\}$ preserves $L(x,\xi)$ if $S_\tau^* L(x,\xi) := L(S_\tau(x), S_{\tau *}(\xi)) \equiv L$.

Write $x(\tau) := S_\tau(x)$. Then this is equiv. to

$$0 = \frac{dL}{d\tau} = \frac{\partial L}{\partial x^i} \dot{x}^i(\tau) + \frac{\partial L}{\partial \xi^i} \frac{d\xi^i(\tau)}{d\tau} \qquad \text{where } \xi^i(\tau) = \frac{\partial x^i(\tau)}{\partial x^j} \xi^j \text{ tangent map } x = x(0) \text{ to } x(\tau)$$

$$(\text{At } \tau = 0) = X^i \frac{\partial L}{\partial x^i} + \frac{\partial X^i}{\partial x^j} \xi^j \frac{\partial L}{\partial \xi^i}. \qquad (*)$$

Theorem (E. Noether): If L is preserved by S_τ gen. by X, then the momentum in the X direction is conserved. ie., $X^i p_i$ = constant.

pf: $$\frac{d}{dt}(X^i p_i) = \frac{d}{dt}\left(X^i \frac{\partial L}{\partial \dot{x}^i}\right) = \frac{d}{dt}(X^i(x)) \frac{\partial L}{\partial \dot{x}^i} + X^i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)$$

$$= \frac{\partial X^i}{\partial x^j} \dot{x}^j \frac{\partial L}{\partial \dot{x}^i} + X^i \frac{\partial L}{\partial x^i} = 0 \text{ by } (*) \text{ and E-L} \quad \divideontimes$$

Notice: X does not dep on t. but $x(t)$ does!

Example: 2-particle system in $\mathbb{R}^3$. (Example (c)+(d) in §32.2)

Lagrangian $L = \sum_{i=1}^n \frac{1}{2} m_i |\dot{x}_i|^2 - \frac{1}{2} \sum_{i \neq j} V(x_i, x_j)$ in $\mathbb{R}^{3n}$. $x_i = (x_i^1, x_i^2, x_i^3) \in \mathbb{R}^3$

translation invariance $\Rightarrow$ $V(x_i, x_j) = V(x_i - x_j)$

Cor. $P_{total} := \sum_{i=1}^n m_i \dot{x}_i \in \mathbb{R}^3$ is const. in t.

pf: $x^1 \mapsto x^1 + \tau$ in $\mathbb{R}^3$ induces a v.f. in $\mathbb{R}^{3n}$: $X = (1,0,0;1,0,0;\ldots)^T$.

$\Rightarrow$ $\sum_{i=1}^n m_i \dot{x}_i^1$ is const. in t. Similarly for x^2 and x^3 components $\divideontimes$

Case $n = 2$: $m_1 \dot{x}_1 + m_2 \dot{x}_2 = \vec{c} = 0$ (may assume this under unif. moving frame)

then may even assume $m_1 x_1 + m_2 x_2 = 0$ by choosing $\vec{0}$ = center of mass.

$\Rightarrow$ $x_2 = -\frac{m_1}{m_2} x_1$; $V(x_1 - x_2) = V\left(\left(1 + \frac{m_1}{m_2}\right) x_1\right) =: U(x_1)$

set $m^* = 1 + m_1/m_2$. Let $x = x_1$, $r = |x|$.

$\Rightarrow$ The E-L eq^{n} reduced to 1 particle case: $m^* \ddot{x}^\alpha = -\frac{\partial U(x)}{\partial x^\alpha}$.

If L is SO(3) inv, then $V(x_1 - x_2) = V(|x_1 - x_2|)$, hence $-\nabla U(r) \parallel x$

"Angular momentum" $[x, p] = m[x, \dot{x}]$ is const since $[x, \dot{x}]^{\cdot} = 0$.

ie. central force field $\Rightarrow$ plane motion.

L ind. of t $\Rightarrow$ $E = \frac{1}{2} m^* |\dot{x}|^2 + U$ = const (1st integral)

Ex. Show (1) This is completely integrable (Get eq^{n}),

(2) $\exists$ closed orbit for $U(r) = \frac{\alpha}{r}$ $(\alpha < 0, E < 0)$, αr^2 $(\alpha > 0, E \geq 0)$.

ie. periodic

(*) Rmk: These are the only 2 cases st $\exists$ open set in phase space which is filled in by periodic orbits.

Hamiltonian Formalism : < From tangent bundle to cotangent bundle >

Def^{n}: Legendre transform: $(x, \xi) \mapsto (x, p)$ is non-singular (ie. loc. invertible)

if $\det\left(\frac{\partial p_\alpha}{\partial x^\beta}\right) = \det\left(\frac{\partial^2 L}{\partial x^\alpha \partial x^\beta}\right) \neq 0$. It is strongly non-singular (ie. invertible)

if $p_\alpha = \frac{\partial L}{\partial \xi^\alpha}(x, \xi)$ determines $\xi = v(x, p)$ uniquely C^∞, in the region.

phase space for $L := \{(x,p)\}$. Energy E in this coor system (x,p) : $E(x,\xi) = H(x,p)$

$H(x,p) := p_i v^i - L(x, v)$ is called Hamiltonian.

Thm : For L strongly non-singular

$$\left(\begin{array}{c} \text{E-L Eq}^n: \ \dot{p} = \frac{\partial L}{\partial x} \text{ with } p = \frac{\partial L}{\partial \xi} \\ \text{in coor } (x, \xi) \end{array}\right) \simeq \left(\begin{array}{c} \text{Hamilton Eq}^n: \ \dot{x} = \frac{\partial H}{\partial p} \ \& \ \dot{p} = -\frac{\partial H}{\partial x} \\ \text{in coor } (x,p) \end{array}\right).$$

pf : $\Rightarrow$:

$$\frac{\partial H}{\partial p} = v + \cancel{p \frac{\partial v}{\partial p}} - \cancel{\frac{\partial L}{\partial v} \frac{\partial v}{\partial p}} = v = \xi = \dot{x}$$

$$-\frac{\partial H}{\partial x} = \cancel{-p \frac{\partial v}{\partial x}} + \frac{\partial L}{\partial x} + \cancel{\frac{\partial L}{\partial v} \frac{\partial v}{\partial x}} = \frac{\partial L}{\partial x} = \dot{p} \quad *$$

$\Leftarrow$: Conversely,

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x}(p\xi - H) = \cancel{\frac{\partial p}{\partial x}\xi} - \frac{\partial H}{\partial x} - \cancel{\frac{\partial H}{\partial p}\frac{\partial p}{\partial x}} = \dot{p}$$

$L(x,\xi) := p(x,\xi)\xi - H(x,p)$:

$$\frac{\partial L}{\partial \xi} = \cancel{\frac{\partial p}{\partial \xi}\xi} + p - \cancel{\frac{\partial H}{\partial p}\frac{\partial p}{\partial \xi}} = p \quad *$$

Remark: In fact, Hamilton eq^{n} is exactly the E-L eq^{n} for

$L(x, p, \dot{x}, \dot{p}) := p\dot{x} - H(x,p)$ on the $2n$ dim'l phase space.

Cor. Along any trajectory $(x(t), p(t))$ of Hamilton eq^{n}, $H = \text{const}$. ($=$ energy E).

Thm (Maupertuis' Principle) let $H(x,p)$ be a Hamiltonian

Any $(x(t), p(t))$ extremizing $S = \int L\,dt = \int (p\dot{x} - H)\,dt$ also extremizes the truncated action $S_0 = \int p\dot{x}\,dt = \int p\,dx$ among curves of same energy.

Examples (a) $L = \frac{1}{2}m|\dot{x}|^2 - U(x)$. Then $H(x,p) = \frac{|p|^2}{2m} + U(x)$ where $p = m\dot{x}$.

$\Rightarrow$ Any extremal of energy E has $|p| = \sqrt{2m(E - U(x))}$, is also extremal for

$S_0 = \int p\dot{x}\,dt = \int |p||\dot{x}|\,dt = \int \sqrt{g_{ij}\dot{x}^i\dot{x}^j}\,dt$ where $g_{ij} = 2m(E - U(x))\delta_{ij}$.

ie. a geodesic but with non-natural parameter t.

(b) Fermat's principle on light in continuous isotropic medium !

"Choose" $H(x,p) = c(x)|p|$. Among energy level E, $|p| = E/c(x)$

$\dot{x} = \frac{\partial H}{\partial p} = c(x)\frac{p}{|p|}$, so $\boxed{|\dot{x}| = c(x)}$ along the trajectory γ_0.

Among all γ with E :

$p\dot{x} = |p||\dot{x}| = \frac{E}{c(x)}|\dot{x}| \Rightarrow S_0 = \int p\dot{x}\,dt = E\int \frac{|\dot{x}|}{c(x)}\,dt = E\int\sqrt{g_{ij}\dot{x}^i\dot{x}^j}\,dt$

where $g_{ij} = \frac{1}{c^2(x)}\delta_{ij}$ (conformally Euclidean). γ_0 is a geodesic of it.

For ani-isotropic medium, the metric will not be conformally Euclidean.

Geometric Theory of phase Space (cotangent bundle $T^*\mathbb{R}^n$)

consider gradient flow $\dot{y} = \nabla f(y)$ where $\nabla f = df^\#$ i.e. $f^i = g^{ij} f_j$ in $\mathbb{R}^m \ni (y^1, \cdots, y^m)$ with non-deg metric g_{ij} (not nec. sym)

Lemma : for any $h(y)$, $\dot{h} := \frac{d}{dt} h(y(t)) = dh(\nabla f) = \frac{\partial h}{\partial y_i} g^{ij} \frac{\partial f}{\partial y_j} = \langle dh, df \rangle = \langle \nabla f, \nabla h \rangle$.

chain rule

since $= h_i f^i = g_{ji} h^i f^j$ #

Now set $m = 2n$, $g_{ij} = -g_{ji}$

$$\Omega = \sum_{i<j} g_{ij}\, dy^i \wedge dy^j \qquad g := \det g_{ij} \neq 0$$

Lemma : $\frac{1}{n!} \Omega^n = \sqrt{g}\, dy^1 \wedge \cdots \wedge dy^{2n}$, $g > 0$

$\Rightarrow$ $\sqrt{g}$ is a polynomial of g_{ij}, called the "Pfaffian".

pf: for each $p \in \mathbb{R}^{2n}$, work on $T_p\mathbb{R}^{2n} \cong \mathbb{R}^{2n}$. Linear algebra $\Rightarrow$

$\exists$ coor $(x^1, p_1, x^2, p_2, \cdots, x^n, p_n) = (z^1, \cdots, z^{2n})$ st $y = Az$ has $A \in SO(2n)$

$$\tilde{g}_{ij}(p) = A^T (g_{ij}(p)) A = \begin{pmatrix} \begin{matrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{matrix} & & \\ & \ddots & \\ & & \begin{matrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{matrix} \end{pmatrix}, \text{ i.e. } \Omega = \sum_{i=1}^n \lambda_i\, dx^i \wedge dp_i \text{ at } p$$

$\Rightarrow$ $\Omega^n = \Omega \wedge \cdots \wedge \Omega = n! \prod_{i=1}^n \lambda_i \cdot dx^1 \wedge dp_1 \wedge \cdots \wedge dx^n \wedge dp_n$. Since $\tilde{g} = \lambda_1^2 \cdots \lambda_n^2$, done #

Def^n: $\mathbb{R}^{2n}$ with coor $(x^i, p_i)_{i=1}^n$ and $\Omega = dx^1 \wedge dp_1 + \cdots + dx^n \wedge dp_n$ is called an abstract phase space, (x^i, p_i) the canonical coordinates.

in this space, the gradient flow for $f = H(x^1, p_1, \cdots, x^n, p_n)$ becomes Hamilton's eq^n:

$$\dot{y} = \nabla H(y) \iff \dot{x}^i = \frac{\partial H}{\partial p_i} \;\&\; \dot{p}_i = -\frac{\partial H}{\partial x^i}. \qquad i = 1, \cdots, n.$$

The Poisson Bracket : on phase space

Def : $\{f, g\} := \langle \nabla f, \nabla g \rangle = g^{ij} f_i f_j \overset{\downarrow}{=} \sum_{i=1}^n \left(\frac{\partial f}{\partial x^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x^i} \right)$.

The def^n applies to any g_{ij}.

Thm : (i) $\{f, g\}$ is $\mathbb{R}$-bilinear, skew-sym,

(ii) Jacobi identity,

(iii) $\{fg, h\} = f\{g, h\} + g\{f, h\}$, $\Rightarrow$ Get a Lie algebra of functions !

(iv) $\nabla\{f, g\} = -[\nabla f, \nabla g]$.

pf: (i), (iii) by def^n, (iv) $\Rightarrow$ (ii). Only (iv) is important ! But it is straightforward #

Thm . Given g_{ij} skew-sym. The Poisson bracket on $C^\infty(\mathbb{R}^{2n})$ forms a Lie algebra $\iff$ Ω is symplectic. ie. $d\Omega = 0$ $\iff$ locally an abstract phase space. (Darboux thm)

pf (sketch) : Jacobi for $\{\,,\}$ $\iff$ $g^{pq} \partial_q g^{ij} (\partial_p f_1 \partial_i f_2 \partial_j f_3 + [2,3,1] + [3,1,2]) = 0$

$\iff$ $\partial_p f_1 \partial_i f_2 \partial_j f_3 (g^{pq} \partial_q g^{ij} + g^{iq} \partial_q g^{jp} + g^{jq} \partial_q g^{pi}) = 0$ $\forall f_1, f_2, f_3$

sum over $g_{rp} g_{sj} g_{ti}$ $\iff$ $\partial_r g_{ts} - \partial_s g_{tr} + \partial_t g_{sr} = 0$ ie $(d\Omega)_{rst} = 0$ #

Now any $f(x, p)$ leads to $\dot{f} = \langle \nabla f, \nabla H \rangle = \{f, H\}$ along $\dot{y} = \nabla H(y(t))$.

Def^n: f is an integral function if $\dot{f} = 0$ along any trajectory, ie. $\{f, H\} = 0$.

Cor: Int. functions form a Lie subalgebra.

Proof of Darboux Thm:

Step 1. (Moser's lemma) ω_t symp st. $\frac{d}{dt}\omega_t = d\sigma_t \Rightarrow \psi_t^*\omega_t = \omega_0$ for some ψ_t local diffeomorphism

pf: If $\exists$ time-depend. v.f. X_t generates ψ_t then must have

$$0 = \frac{d}{dt}\omega_0 = \frac{d}{dt}\psi_t^*\omega_t \underset{\Delta}{=} \psi_t^*\left(\frac{d}{dt}\omega_t + L_{X_t}\omega_t\right) = \psi_t^*\left(d\sigma_t + d(\iota_{X_t}\omega_t)\right) \qquad \Delta : \text{Ex.}$$

ie. $\iota_{X_t}\omega_t = -\sigma_t + df$. can solve X_t since ω_t is non-degenerate.

Step 2. At T_pS, ω is equiv to ω_0, the standard one.

set $\omega_t = t\omega + (1-t)\omega_0 = d(t\sigma + (1-t)\sigma_0)$ by Poincaré lemma

Need to pick $U \ni p$ small st ω_t is non-degenerate on U $\forall t \in [0,1]$ ✱

Comparison in Riem and symplectic geometry: symp. has only "global inv".

Thm A: Along any integral curve, $\dot{\Omega} = L_{\nabla H}\Omega = 0$.

pf: $L_{\nabla H}\Omega \underset{\uparrow}{=} (\iota_{\nabla H} d + d\,\iota_{\nabla H})\Omega = d\,\iota_{\nabla H}\Omega$: $(d\,\iota_{\nabla H}\Omega)(X,Y)$

(Ex. 25.3 #3) Cartan's homotopy formula $= X\,\iota_{\nabla H}\Omega(Y) - Y\,\iota_{\nabla H}\Omega(X) - \iota_{\nabla H}\Omega([X,Y])$

$= X\,\Omega(\nabla H, Y) - Y\,\Omega(\nabla H, X) - \Omega(\nabla H, [X,Y])$

Notice: $\Omega(\nabla h, Y) = g_{ij}\, g^{is} h_s Y^j = -dh(Y) = Yh$ $\curvearrowright$ $= -XY(H) + YX(H) + [X,Y](H) = 0$ ✱

Cor (Liouville): The vol. form $\sqrt{g}\, dy^1 \wedge \cdots \wedge dy^{2n} = \frac{1}{n!}\Omega^n$ is inv. under any Hamiltonian system. (pf: $L_{\nabla H}\Omega^n = 0$ by Leibniz' rule).

Def'n: Canonical Transformation Φ are transf. preserving Ω: $\Phi^*\Omega = \Omega$.

Thm B: Every local 1-parameter group of canonical transf. Φ_t (set $X = \frac{d}{dt}\Phi_t\big|_{t=0}$) is locally generated by a Hamiltonian v.f. $X = \nabla H$ for some H.

pf: $\Phi_t^*\Omega = \Omega \Rightarrow 0 = L_X\Omega = \iota_X \cancel{d}\Omega + d\,\iota_X\Omega = d\,\iota_X\Omega$

Poincaré lemma $\Rightarrow \iota_X\Omega = dH$ locally, ie. $X = \nabla H$ ✱

Rmk: For simple proofs of Thm A, B using Darboux chart, see the textbook.

Lie Algebra of symplectic transformations

locally, Darboux chart $\Rightarrow$ $X = \nabla H = \left(\frac{\partial H}{\partial p_1}, -\frac{\partial H}{\partial x^1}, \dots, \frac{\partial H}{\partial p_n}, -\frac{\partial H}{\partial x^n}\right)$

Def'n: Linear canonical transf. on $\mathbb{R}^{2n}$ is called symplectic transf.

ie. $\exists$ a matrix $K \in M_{2n}(\mathbb{C})$ with $X(y) = Ky$ (linear vector field)

$$\Rightarrow H = \frac{1}{2}\sum_{i,j=1}^{n}\left(a_{ij}x^ix^j + 2b_{ij}x^ip_j + c_{ij}p_ip_j\right) \Rightarrow K = \begin{pmatrix} B^T & C \\ -A & -B \end{pmatrix}$$

$A = A^t$, $C = C^t$ under coor. $(x^1, \dots, x^n, p_1, \dots, p_n)$

Cor. This Lie algebra is exactly $sp(n) = \mathrm{Lie}\, Sp(n)$, (see §14.3, Def'n 14.3.5)

$Sp(n) \subset GL(n, \mathbb{H})$ which preserves $\langle\, ,\rangle_{\mathbb{H}}$ in $\mathbb{H}^n \simeq \mathbb{C}^{2n}$ $\qquad \zeta = a + bi + cj + dk = (a+bi) + (c+di)j$

$$\langle \zeta_1, \zeta_2 \rangle := \sum_{k=1}^n \zeta_1^k \bar{\zeta}_2^k = \sum (x_1^k + y_1^k j)\overline{(x_2^k + y_2^k j)} = \sum (x_1^k \bar{x}_2^k + y_1^k \bar{y}_2^k) + (y_1^k x_2^k - x_1^k y_2^k) j$$

ie. $Sp(n) \cong \{T \in U(2n) \mid T$ preserves the skew sym form $\sum_k (y_1^k x_2^k - x_1^k y_2^k)\}$ ✱.

Lagrange surface

Def'n: Extended phase space. Given Hamiltonian System via $H(x,p)$: $\dot{y} = \nabla H(y)$

Had seen for $\varphi(x,p,t)$, $\dot{\varphi} = \frac{\partial \varphi}{\partial t} + \langle \nabla\varphi, \nabla H \rangle \Rightarrow \dot{E} = \dot{H} = \frac{\partial H}{\partial t} + \underset{=0}{\cancel{\langle \nabla H, \nabla H \rangle}}$.

if $H(x,p,t)$ depends on t, set $x^{n+1} = t$, $p_{n+1} = E$, $\tilde{H}(x,p,t,E) := H(x,p,t) - E$

then $\frac{\partial \tilde{H}}{\partial p_{n+1}} = \frac{\partial \tilde{H}}{\partial E} = -1 = -\dot{t}$ Notice that, (t, E) are New variables!

$\frac{\partial \tilde{H}}{\partial x^{n+1}} = \frac{\partial \tilde{H}}{\partial t} = \frac{\partial H}{\partial t} = \dot{E}$ with $\tilde{\Omega} := \Omega - dt \wedge dE = \sum_{i=1}^{n} dx^i \wedge dp_i - dt \wedge dE$

Def'n: A surface Γ of $\frac{1}{2}$-dim. is Lagrangian if $\Omega|_\Gamma \equiv 0$. Eg. $(x=0)$ or $(p=0)$

Hence for Φ a canonical transf. $\Phi(\Gamma)$ is also Lagrangian. $(x,p) \mapsto (p,-x)$.

Typical Example: In Darboux chart: $T^*\mathbb{R}^n$, $\Gamma := (x, d\varphi) = (x^i, p_i = \frac{\partial \varphi}{\partial x^i})$

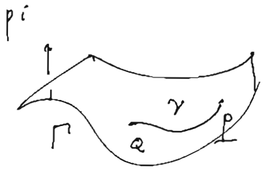
then $\sum dx^i \wedge dp_i = \sum dx^i \wedge d\left(\frac{\partial \varphi}{\partial x^i}\right) = \sum \frac{\partial^2 \varphi}{\partial x^i \partial x^j} dx^i \wedge dx^j = 0$.

Thm: Every Lagrangian surface Γ is locally the graph of a differential.

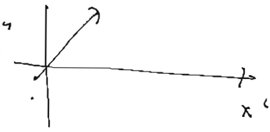
pf: locally $\Omega = d\eta$ (eg. $\eta = -p_i dx^i$) now $d\eta|_\Gamma = 0$ or $d(\eta|_\Gamma) = 0$

$\Rightarrow \eta|_\Gamma = d\varphi$ for some function φ on Γ.

suppose that p_i on Γ can be parametrized by $x^1, \cdots, x^n$,



Then $d\varphi = \frac{\partial \varphi}{\partial x^i} dx^i = \eta|_\Gamma = -p_i dx^i \Rightarrow p_i = -\frac{\partial \varphi}{\partial x^i}$ ⋕



For classical Hamilton System via H, this gives the alternative def'n:

Cor: (1) For phase space, Γ^n is Lagrangian iff $\forall Q \in \Gamma^n$ the truncated action $S_0(P) = \int_\gamma p\,dx$ is locally indep of $\gamma \subset \Gamma^n$ joining Q, P

(2) For extended phase space, Γ^{n+1} is Lagrangian iff $\forall Q \in \Gamma^{n+1}$, the truncated action $S_0(P) = \int_\gamma (p\,dx - E\,dt)$ is locally indep of $\gamma \subset \Gamma^{n+1}$ joining Q, P

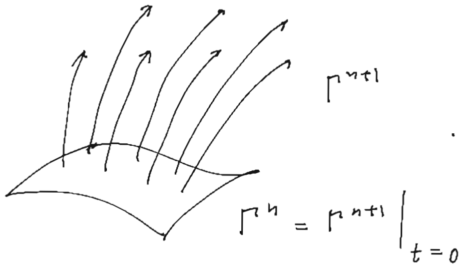
Def'n: In case (2), $S(x,t) := S_0(P)$ is called the action of the trajectory bundle.

and the eq'n $\frac{\partial S}{\partial t} + E(x,p,t) = \frac{\partial S}{\partial t} + H(x, \frac{\partial S}{\partial t}, t) = 0$ is the Hamilton-Jacobi eq'n.

Thm: Given $H(x,p)$ indep of t. $S := \{H = \text{const } E_0\}$

(i) $\nabla H = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial x}\right) \in T_p S$, (ii) $\langle \nabla H, \xi \rangle = 0 \quad \forall \xi \in T_p S$

(iii) if Γ^n is Lagrangian and with const H, then $\nabla H \in T_p S$ and, any trajectory touching Γ^n lies in Γ^n entirely.



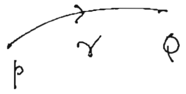
Pf: (i) + (ii): By def'n: $\langle \nabla H, \xi \rangle = dH(\xi) = 0$ on $T_p S$, in particular for $\xi = \nabla H$. since $\langle , \rangle$ is skew-sym.

(iii): $T_p\Gamma^n$ has max dim of Lagrangian directions, (ii) $\Rightarrow \nabla H \in T_p \Gamma^n$

if γ is a trajectory $\gamma(0) = p$, then $\dot{\gamma} = \nabla H \in T_p \Gamma^n \Rightarrow \gamma \subset \Gamma^n$ ⋕

Rmk: If $H(x,p,t)$ dep on t, Thm can be applied to extended phase space on Γ^{n+1}.

2nd Variations (mainly for geodesics)



Let γ satisfy E-L eq'n for $S[\gamma] = \int_\gamma L(x,\dot{x})\,dt$

for vector fields ξ, η along γ, $= 0$ at p, q

(*) $G_\gamma(\xi,\eta) := \frac{\partial^2}{\partial\lambda\,\partial\mu} S[\gamma + \lambda\xi + \mu\eta]\Big|_{\lambda=0,\,\mu=0}$

γ minimum $\Rightarrow$ $G_\gamma(\xi,\xi) \geq 0$, and $G_\gamma(\xi,\xi) > 0\ \forall \xi \not\Rightarrow \gamma$ is minimum.

Lemma: $G_\gamma(\xi,\eta) = -\int_a^b (J_{ij}\xi^j)\eta^i\,dt$ where J is the Jacobi diff. operator

$$J_{ij}\xi^j := -\left(\frac{\partial^2 L}{\partial x^i\partial x^j}\xi^j + \frac{\partial^2 L}{\partial x^i\partial \dot{x}^j}\dot{\xi}^j\right) + \frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{x}^i\partial x^j}\xi^j + \frac{\partial^2 L}{\partial \dot{x}^i\partial \dot{x}^j}\dot{\xi}^j\right).$$

↖ $\ddot{\xi}^j$ here!

pf: let $y(t) = x(t) + \lambda\xi(t)$, then

$$(*) = \frac{\partial}{\partial\lambda}\Big|_{\lambda=0}\int_a^b\left(\frac{\partial L}{\partial y^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}^i}\right)\eta^i\,dt = \int_a^b (J_{ij}\xi^j)\eta^i$$ ⋕

Def'n: ξ is called a Jacobi field (wrt. S) if "$J\xi$" $\equiv 0$. Will study geodesics in more details.

Thm: For $S[\gamma] = \int_a^b \frac{1}{2} g_{ij}\dot{x}^i\dot{x}^j\,dt$ with γ a geodesic

$$G_\gamma(\xi,\eta) = -\int_a^b\left(\nabla_{\dot{x}}^2\xi^k + R^k_{jli}\,\dot{x}^j\dot{x}^i\xi^l\right)\eta^m g_{km}\,dt = -\int_a^b\langle \underbrace{\nabla_T^2\xi + R(\xi,T)T}_{=:\,J\xi}, \eta\rangle$$

pf: $(*) = \frac{\partial}{\partial\lambda}\Big|_{\lambda=0} -\int_a^b\left(\ddot{y}^k + \Gamma^k_{ij}\dot{y}^i\dot{y}^j\right)\eta^m g_{km}\,dt$ where $y(t) = x(t) + \lambda\xi(t)$

$$= -\int_a^b\left(\ddot{\xi}^k + \underline{\frac{\partial\Gamma^k_{ij}}{\partial x^l}\xi^l\dot{x}^i\dot{x}^j} + 2\Gamma^k_{ij}\dot{\xi}^i\dot{x}^j\right)\eta^m g_{km}\,dt$$ let $T = \dot{\gamma} = \dot{x}^i\partial_i$

Now $\nabla_T\xi^k = \dot{\xi}^k + \Gamma_{li}\xi^l\dot{x}^i$

$$\nabla^2_T\xi^k = \ddot{\xi}^k + \frac{d}{dt}\left(\Gamma^k_{li}\xi^l\dot{x}^i\right) + \Gamma^k_{ps}\dot{x}^s\left(\dot{\xi}^p + \Gamma^p_{li}\xi^l\dot{x}^i\right)$$

$$\overset{\shortparallel}{\underline{\frac{\partial\Gamma^k_{li}}{\partial x^j}\xi^l\dot{x}^i\dot{x}^j}} + \Gamma^k_{li}\dot{\xi}^l\dot{x}^i + \Gamma^k_{li}\xi^l\ddot{x}^i$$

at each, pick Riem normal coor get $\Gamma^k_{ij}(p) = 0$

$$(*) = -\int_a^b\left(\nabla^2_T\xi^k + R^k_{ilj}\xi^l\dot{x}^i\dot{x}^j\right)\eta^m g_{km}\,dt$$ ←

Cor. For $(g_{ij}) > 0$, a geodesic is of shortest length when Q is close to P.

Pf: $G_\gamma(\xi,\xi) = -\int_0^l\langle\nabla_T\nabla_T\xi,\xi\rangle + \langle R(\xi,T)T,\xi\rangle = \int_0^l|\nabla_T\xi|^2 - R(\xi,T,\xi,T)$

Ex 36.2 $\Rightarrow$ This is > 0 if $\xi \not\equiv 0$ and l small ⋕

Def'n: P, Q are conjugate pts along γ if $\exists\,\xi \not\equiv 0$, $\xi(p) = \xi(q) = 0$, $J\xi = 0$ (Jacobi field).

Thm. For $(g_{ij}) > 0$: (1) $G_\gamma(\xi,\eta)$ is non-degenerate $\Leftrightarrow$ $\gamma(0)$ and $\gamma(l)$ are not conjugate.

(2) γ is no longer minimal beyond the 1st conj pt Q of P.

Pf: (1) is easy. (2) By Ex 36.1, use broken v.f. ξ Jacobi on $\overline{PQ}$:

⋕

Q: Ex: Why min energy $\Leftrightarrow$ min arc length?

Higher Dimensional Variations

$D \subset \mathbb{R}^n$, ∂D piece-wise smooth $f : D \to \mathbb{R}^N$ C^∞

Lagrangian $L = L(x^\beta;\ p^i;\ q^j_\alpha)$
$\qquad\qquad\qquad\qquad n \ + \ k \ + \ nN$

$I[f] = \int_D L(x^\beta;\ f^i;\ f^j_\alpha)\, d\sigma$

Eg. (area of a surface) $r : \overset{\mathbb{R}^2}{\overset{\cup}{D}} \longrightarrow \mathbb{R}^3$

$A[f] = \int_D \sqrt{EG - F^2}\, du\, dv = \int_D \sqrt{|r_1|^2 |r_2|^2 - (r_1 \cdot r_2)^2}\, du\, dv$

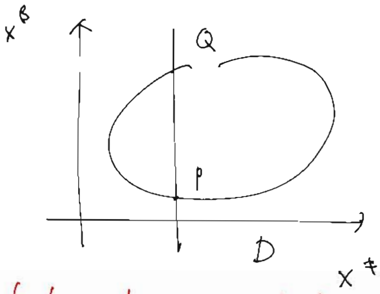
Thm: $\eta|_{\partial D} = 0 \ni I'[f](\eta) = \int_D \text{"}\frac{\delta I}{\delta f^i}\text{"}\, \eta^i\, d\sigma$ where $\frac{\delta I}{\delta f^i} := \frac{\partial L}{\partial f^i} - \sum_{\alpha=1}^n \frac{\partial}{\partial x^\alpha}\left(\frac{\partial L}{\partial f^i_\alpha}\right)$.

E-L eqn

pf: $\frac{d}{d\varepsilon} I[f + \varepsilon\eta]\Big|_{\varepsilon=0} = \int_D \frac{d}{d\varepsilon} L(x, f + \varepsilon\eta, Df + \varepsilon D\eta)\Big|_{\varepsilon=0}\, d\sigma$

$= \int_D \left(\frac{\partial L}{\partial f}\eta + \frac{\partial L}{\partial Df} D\eta\right) d\sigma = \int_D \frac{\partial L}{\partial f}\eta + \sum_{\alpha=1}^n \frac{\partial L}{\partial f_\alpha}\partial_\alpha\eta\, d\sigma$

$= \int_D \left\{\frac{\partial L}{\partial f} - \sum_{\alpha=1}^n \frac{\partial}{\partial x^\alpha}\left(\frac{\partial L}{\partial f_\alpha}\right)\right\}\eta\, d\sigma$ ※



Energy-Momentum Tensor:

If $L(p^i, q^j_\alpha)$ does not involve $x^1, \cdots, x^n$. Then

$T^k_j := f^i_j \frac{\partial L}{\partial f^i_k} - \delta^k_j L$ satisfies $\sum_k \frac{\partial}{\partial x^k} T^k_j = 0$. ($k \leftrightarrow$ tangent vector component f^i_k)

pf: $\partial_k T^k_j = \cancel{\partial_k f^i_j \frac{\partial L}{\partial f^i_k}} + f^i_j \frac{\partial}{\partial x^k}\cancel{\left(\frac{\partial L}{\partial f^i_k}\right)} - \cancel{\delta^k_j \frac{\partial L}{\partial f^i}\partial_k f^i} - \delta_{jk}\cancel{\frac{\partial L}{\partial f^i_\ell}\frac{\partial^2 f^i}{\partial x^\ell \partial x^k}} = 0$ ※

Rmk: If $n = 1$, then $x^1 = t$ and this reduces to energy E. For $n \geq 2$ $\exists$ many components!

Def^n: If D has metric g_{ij} then $T_{ik} := g_{k\ell} T^k_i$. Also, $T^{ik} := g^{ij} T^k_j$.

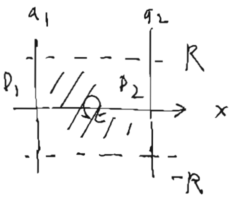
First Simple Example: Minkowski space-time $\mathbb{R}^{1,3}$.

Symmetry: If $T^{ik} \neq T^{ki}$, choose $\psi^{i\underbrace{k\ell}} = -\psi^{i\underbrace{\ell k}}$ st $\sum \frac{\partial}{\partial x^\ell}\psi^{ik\ell} = -\frac{1}{2}(T^{ik} - T^{ki})$

(✪) Then $\tilde{T}^{ik} := T^{ik} + \sum \frac{\partial}{\partial x^\ell}\psi^{ik\ell} = \frac{1}{2}(T^{ik} + T^{ki})$ and $\sum_k \frac{\partial}{\partial x^k}\tilde{T}^{ik} = 0$.

Def^n: The momentum 4 vector $p = (p^0, p^1, p^2, p^3)$ for L is, under fast decay assumption, $p^i := \frac{1}{c}\int_{x^0 = a} T^{ik}\, dS_k = \frac{1}{c}\int_{x^0 = a} T^{i0}\, dS_0$; $\left(dS_k = \frac{1}{6}\epsilon_{ijmk}\, dx^i \wedge dx^j \wedge dx^m\right)$

Prop. If $T^{ik} = O(R^{-(3+\epsilon)})$ then p is conserved (indep of const a).



pf: $0 = \int_\Omega \sum_k \frac{\partial}{\partial x^k} T^{ik}\, d\sigma = \int_{D_2 - D_1 + \mathrm{II}_R} T^{ik}\, dS_k$. Let $R \to \infty$ ※

Lemma: p is inv. under (✪). Hence may assume $T^{ik} = T^{ki}$.

pf: $\int_{x^0 = a} \partial_\ell \psi^{ik\ell}\, dS_k = \int_{x^0 = a} \mathrm{div}(\psi^{i01}, \psi^{i02}, \psi^{i03})\, dx^1 \wedge dx^2 \wedge dx^3$ ↖ since $\psi^{i00} = 0$ skew-sym.

$= \lim_{R \to \infty} \int_{S^2_R} \langle \psi^{i0}, \vec{n} \rangle\, dA = 0$ ※

Cor: Angular momentum:

If $T^{ik} = T^{ki}$, then $M^{ik} := \int x^i dp^k - x^k dp^i = \frac{1}{c}\int_{x^0 = a}(x^i T^{k\ell} - x^k T^{i\ell})\, dS_\ell$

is also indep of a. (Same pf as Prop, via $\partial_\ell(x^i T^{k\ell} - x^k T^{i\ell}) = T^{ki} - T^{ik} = 0$.)

Examples on HD variational Problems

I. Hodge Theory on compact surface S with $\partial S = \phi$

de Rham cohomology $H^k_{dR}(S, \mathbb{R}) :=$ closed k forms / exact k forms

in each class $[\omega] = \{\omega + d\eta \, , \, \eta \in \Lambda^{k-1}(S)\}$ with $d\omega = 0$

if ω minimize norm: $\|\omega\|^2$ defined via $(\alpha, \beta) := \int \alpha \wedge * \beta$

$$\langle \omega + \epsilon d\eta , \omega + \epsilon d\eta \rangle = \|\omega\|^2 + 2 \langle \omega, d\eta \rangle \epsilon + \|d\eta\|^2 \epsilon^2 \geqslant \|\omega\|^2 \quad \forall \epsilon \text{ small}$$

$\not\ni \quad 0 = \langle \omega, d\eta \rangle = \langle \delta\omega, \eta \rangle \quad \forall \eta \Rightarrow \delta\omega = 0$. In fact $>$ hold if $d\eta \neq 0$

Also the minimizer is unique if it exists.

Thm (Hodge) (1) $H^k_{dR}(S, \mathbb{R}) \simeq \mathbb{H}^k$ by harmonic forms $\Delta\omega = 0$ $(\Leftrightarrow d\omega = 0 = \delta\omega)$

(2) $\dim \mathbb{H}^k < \infty$; $\Lambda^k(S) = \mathbb{H}^k \oplus^{\perp} \Delta \Lambda^k(S) = \mathbb{H}^k \oplus^{\perp} d\Lambda^{k-1} \oplus^{\perp} \delta \Lambda^{k+1}$.

Sketch: (1) har form exists and C^∞ by elliptic PDE theory! Also $\dim \mathbb{H}^k < \infty$.

(2) Hence Δ has a spectral resolution $\mathbb{H} \oplus H_{\lambda_i > 0}$, $G := \Delta|_{\oplus H_{\lambda_i > 0}}^{-1}$ Green op.

$\lambda_i \to \infty$ as $i \to \infty$ $\not\ni$ G is a cpt operator $\not\ni$ $I = \mathbb{H} \oplus^{\perp} \Delta G$ ⁂

Cor. $\Delta \omega = \varphi$ in Λ^k if solvable $\Leftrightarrow \varphi \in \mathbb{H}^{\perp}$, i.e. $\langle \varphi, \psi \rangle = 0 \ \forall$ har. ψ ⁂

II. The Equations of an Electromagnetic Field, on $\mathbb{R}^{1,3}$

Action $S = S_m + \underline{S_{mf} + S_f} = -\sum \int mc \, d\ell - \frac{1}{c^2} \int A_i j^i \, d\sigma - \frac{1}{16 c \pi} \int F_{ik} F^{ik} \, d\sigma$

< change density = current >

$\parallel$

Focus on "fields" $\not\ni$ $S[A] = \int L(A_i, A_{ij}) \, d\sigma$; $A_{ij} = \partial_j A_i$ $\qquad \partial_i A_k - \partial_k A_i$

$A = A_i dx^i \in \Lambda^1(\mathbb{R}^{1,3})$ is the "vector potential"; $F = dA \in \Lambda^2(\mathbb{R}^{1,3})$

At ∞, L suitably decay to 0 st. the integral converges, but not A!

$$\not\ni S[A] = -\frac{1}{c} \int \left(\frac{1}{c} A(j) + \frac{1}{16\pi} \langle F, F \rangle \right) d\sigma \qquad \langle F, F \rangle d\sigma = F \wedge * F$$

$$S[A + \epsilon\eta] = S[A] - \epsilon \frac{1}{c} \int \left(\frac{1}{c} \langle \eta, j^b \rangle + \frac{1}{8\pi} \langle d\eta, F \rangle \right) d\sigma - \frac{\epsilon^2}{c} \frac{1}{16\pi} \int |F|^2 d\sigma$$

take extremal, in fact minimal $\Leftrightarrow$ $\frac{1}{8\pi} \delta F = \frac{-1}{c} j^b$

(Recall) under the correspondence $x^0 = ct$, and

$$F = F_{ij} \, dx^i \wedge dx^j = E_\alpha \, dx^0 \wedge dx^\alpha + H^1 dx^2 \wedge dx^3 + H^2 dx^3 \wedge dx^1 + H^3 dx^1 \wedge dx^2 \sim \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & H^1 & -H^2 \\ -E_2 & -H^1 & 0 & H^3 \\ -E_3 & H^2 & -H^3 & 0 \end{pmatrix}$$

$$(\Leftrightarrow) \text{ Maxwell eq'ns } \begin{cases} dF = 0 \ (\text{trivial}) \Leftrightarrow \operatorname{curl} \mathbb{E} = \frac{-1}{c} \frac{\partial \mathbb{H}}{\partial t}, \ \operatorname{div} \mathbb{H} = 0 \\ \delta F = \frac{-8\pi}{c} j^b \Leftrightarrow \operatorname{curl} \mathbb{H} = -\frac{1}{c} \frac{\partial \mathbb{E}}{\partial t} + \frac{4\pi}{c} \vec{j}, \ \operatorname{div} \mathbb{E} = 4\pi\rho \end{cases}$$

Energy-momentum tensor (here stress tensor) $\qquad$ where $j = (c\rho, \vec{j})$

for free space $(j = 0)$: notice $\exists F \neq 0$ in contrast to Hodge theory (why?)

$$T^{ik} = g^{im} A_{\ell m} \frac{\partial L}{\partial A_{\ell k}} - g^{ik} L = -\frac{g^{im}}{4\pi c} \frac{\partial A_\ell}{\partial x^m} F^{k\ell} + \frac{g^{ik}}{16\pi c} |F|^2 \xrightarrow[\text{via Maxwell eq'n } \partial_\ell F^{k\ell} = 0]{\text{symmetrized}} \frac{1}{4\pi c} \left(-g^{im} F_{m\ell} F^{k\ell} + g^{ik} |F|^2 \right)$$

⁂

Examples of HD variational problems

III. Hilbert - Einstein action in general relativity (Hilbert 1915, 1 day earlier)

$$S = \int R \, d\sigma \qquad R = g^{lj} R^{i}_{\ lij} \qquad d\sigma = \sqrt{|g|}\, dx$$

(1) $R^{k}_{\ lij} = \partial_i \Gamma^{k}_{jl} - \partial_j \Gamma^{k}_{il} + \Gamma^{k}_{i\mu}\Gamma^{\mu}_{jl} - \Gamma^{k}_{j\mu}\Gamma^{\mu}_{il}$

$\delta R^{k}_{\ lij} = \partial_i \delta\Gamma^{k}_{jl} - \partial_j \delta\Gamma^{k}_{il} + \delta\Gamma^{k}_{i\mu}\Gamma^{\mu}_{jl} + \Gamma^{k}_{i\mu}\delta\Gamma^{\mu}_{jl} - \delta\Gamma^{k}_{j\mu}\Gamma^{\mu}_{il} - \Gamma^{k}_{j\mu}\delta\Gamma^{\mu}_{il}$

Key point I : $\delta\Gamma^{k}_{ij}$ is a (1,2) tensor.

$\Rightarrow \quad \delta R^{k}_{\ lij} = \nabla_i \delta\Gamma^{k}_{jl} - \nabla_j \delta\Gamma^{k}_{il}$ (by direct observation, or at a pt can assume $\Gamma^{k}_{ij}(p) = 0$)

$\delta R = \delta(g^{lj} R^{i}_{\ lij}) = \delta g^{lj} R_{lj} + g^{lj}(\nabla_i \delta\Gamma^{i}_{jl} - \nabla_j \delta\Gamma^{i}_{il})$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

$\qquad\qquad\qquad\qquad\qquad \nabla_i g^{lj}\delta\Gamma^{i}_{jl} - \nabla_j g^{lj}\delta\Gamma^{i}_{il}$

Key point II : (cf. wikipedia) this is the divergence term $\int (\cdot)\, d\sigma = 0$ when $\delta g^{ij} \to 0$ at ∞ rapidly.

(2) $\delta g = \delta \det(g_{ij}) = g\, g^{ij} \delta g_{ij}$.

$$\delta\sqrt{|g|} = \frac{|g|\, g^{ij}}{2\sqrt{|g|}} \delta g_{ij} = -\frac{1}{2}\sqrt{|g|}\, g_{ij}\, \delta g^{ij}$$

This holds in both Riem and pseduo Riem cases.

(1) + (2) $\Rightarrow \quad \delta S = \int \delta(R\sqrt{|g|})\, dx = \int \delta R\, d\sigma + R\, \delta\sqrt{|g|}\, dx = \int \left(R_{ij} - \frac{R}{2} g_{ij}\right) \delta g^{ij}\, d\sigma$

The Euler-Lagrange eq'n $\dfrac{\delta S}{\delta g^{ij}} = R_{ij} - \frac{1}{2} R g_{ij} = 0$ gives Einstein eq'n for vacum.

The most general situation is

$S = \int \left(\frac{1}{2K}(R - 2) - \mathcal{L}_M\right) d\sigma$

λ = cosmological constant (never observed yet)

$\mathcal{L}_M$ = Lagrangian describing matter

$K = \dfrac{8\pi G}{c^4}$ $\quad$ G = Newton's gravitational constant

$T_{ij} := \dfrac{-2}{\sqrt{|g|}} \dfrac{\delta(\sqrt{|g|}\,\mathcal{L}_M)}{\delta g^{ij}} = -2 \dfrac{\delta \mathcal{L}_M}{\delta g^{ij}} + g_{ij}\mathcal{L}_M$ is the Energy - Stress tensor

$\Rightarrow \quad R_{ij} - \dfrac{R}{2} g_{ij} + \lambda g_{ij} = \dfrac{8\pi G}{c^4} T_{ij}$.

Notice : The whole deduction does NOT use dim = 4 ! can be any $n \in \mathbb{N}$.

Taking trace get $R - 2R + n\lambda = \dfrac{8\pi G}{c^4} g^{ij} T_{ij}$ $\quad$ ie. $R = n\lambda - K^2 T^{i}_{i}$

of Course for vacum (with $\lambda = 0$) get $R \equiv 0$, hence $R_{ij} = 0$.

RmK : The Ricci flat space are models for the space-time

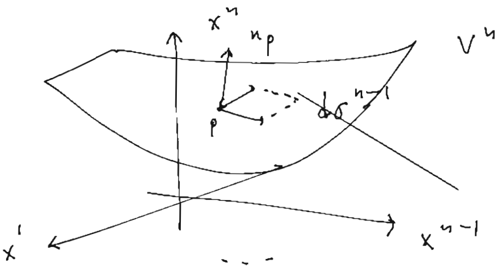
In Super Symmetric string theory, space-time $\cong \mathbb{R}^{1,3} \times M^6$

M a Ricci flat Kähler (= complex + symplectic) manifold of $\dim_{\mathbb{R}} M = 6$.

They are known as Calabi-Yau manifolds after YAU (1976) who proved the Calabi conjecture : $M^6 \Leftrightarrow c_1(M) = 0$ the 1st Chern class.

These are called "inner spaces". Q : How to classify all of them ?

IV. Soap films / minimal submanifolds



V^{n-1} graph of $x^n = f(x^1,\cdots,x^{n-1})$

$r_i = (0,\cdots,1,\cdots 0, f_i)$

clearly $n_p = (-f_1,\cdots,-f_{n-1},1)^\wedge$

$\Rightarrow d\sigma^{n-1} = \dfrac{dx^1\wedge\cdots\wedge dx^{n-1}}{\cos(e_n,n_p)} = \sqrt{1+f_1^2+\cdots+f_{n-1}^2}\; dx^{n-1}$

$S[f] = \int_D \sqrt{1+\sum_{i=1}^{n-1} f_i^2}\; dx^{n-1}$ has E-L eq'n: $\sum_{i=1}^{n-1}\dfrac{\partial}{\partial x^i}\left(\dfrac{f_i}{\sqrt{1+\sum f_i^2}}\right) = 0$.

More generally, can work for $r: D \subset \mathbb{R}^m \longrightarrow (\mathbb{R}^n, h_{kl})$

V^m

$$S[r] = \int_D \sqrt{\langle r_i, r_j\rangle}\, d^m u = \int_D \sqrt{g}\, du \; :$$

$$\delta S = \int_D \frac{g\cdot g^{ij}\delta g_{ij}}{2\sqrt{g}}\, du = \int_D d^m\sigma\; g^{ij}\langle r_i, \delta r_j\rangle = \int_D d^m\sigma\; g^{ij} h_{kl}\, r_i^k\, (\delta r^l)_j$$

may use covariant der.

$$= \int_D d^m\sigma \cancel{\left(g^{ij} h_{kl} r_i^k \delta r^l\right)_{,j}} - \left(g^{ij} h_{kl} r_i^k\right)_{,j} \underline{\delta r^l}$$

Stokes' thm

may assume normal variations $\delta r \perp TV$

$$= -\int_D d^m\sigma\; g^{ij}\langle \nabla_j r_i, \delta r\rangle = 0 \;\; \forall \delta r \iff \vec{H} := \left(g^{ij}\nabla_j r_i\right)^N \equiv 0$$

this is the mean curv. vector.

Def: 2nd fund. form $\mathrm{II}(v,w) := (\nabla_v w)^N \in N_p$

(1) $\mathrm{II}(w,v) = \mathrm{II}(v,w)$ since $(\nabla_w v - \nabla_v w)^N = [w,v]^N = 0$

(2) function linear: $\mathrm{II}(fv,w) = f\,\mathrm{II}(v,w)$, hence also in w by (1)

So in general $\nabla_{r_i} r_j = \nabla^T_{r_j} r_j + \nabla^N_{r_i} r_j \rightarrow \mathrm{II}(r_i,r_j)$, $\vec{H} := \mathrm{Tr}\,\mathrm{II}$ trace.

$V^2 \underset{r}{\hookrightarrow} \mathbb{R}^3$ is very special: can choose $(u^1,u^2) = (u,v)$ be isothermal coor.

i.e. r is conformal: $\langle r_1, r_1\rangle = \langle r_2, r_2\rangle$; $\langle r_1, r_2\rangle = 0$

Prop: (a) $\vec{H} = (\Delta r)^N$ for general cases.

(b) For isothermal coor $dl^2 = \lambda(du^2+dv^2)$ in $\mathbb{R}^3$, $\vec{H} = \Delta r = \dfrac{1}{\lambda}(r_{uu}+r_{vv})$.

Pf: (a) $\vec{H} = \mathrm{II}(g^{ij} r_j, r_i) = \dfrac{1}{\sqrt{g}}\left(\nabla_j\, g^{ij}\sqrt{g}\,\partial_i r\right)^N$ since $(r_i)^N = 0$ ($\Rightarrow$ function-linear)

$= \dfrac{1}{\sqrt{g}}\left(\partial_j\, g^{ij}\sqrt{g}\,\partial_i r\right)^N$ same reason.

(b) Need to show $\langle \Delta r, r_i\rangle = 0$:

$\langle r_1, r_1\rangle = \langle r_2, r_2\rangle \overset{\partial_1}{\Rightarrow} \langle r_{11}, r_1\rangle = \langle r_{12}, r_2\rangle$

$\langle r_1, r_2\rangle = 0 \underset{\partial_2}{\Rightarrow} \langle r_{12}, r_2\rangle + \langle r_1, r_{22}\rangle = 0$

sum $\Rightarrow \langle \Delta r, r_1\rangle = 0$ #

RmK: minimal graph is absolutely "local minimum".

But in general the minimization problem is VERY HARD! ($\exists$ sing pts)

Plateau problem: Given $\Gamma \subset \mathbb{R}^n$ Jordan curve, find minimal disk S with $\partial S = \Gamma$.

Solved by Douglas-Rado in 1936. HD case → Geom. Measure Theory.

Examples of Lagrangians from Quantum mechanics (relativistic)

On $\mathbb{R}^{1,3}$, $S(\varphi, \bar{\varphi}) = \int \left(\hbar^2 g^{ij} \frac{\partial \bar{\varphi}}{\partial x^i} \frac{\partial \varphi}{\partial x^j} - m^2 c^2 \bar{\varphi} \varphi \right) d\sigma = \int \Lambda(\varphi, \bar{\varphi}, \varphi_i, \bar{\varphi}_i)\, d\sigma$

view as independent complex scalar fields (wave functions)

$$\frac{\delta S}{\delta \varphi} = 0 = \frac{\delta S}{\delta \bar{\varphi}} = \frac{\partial \Lambda}{\partial \bar{\varphi}} - \sum_{i=0}^{3} \frac{\partial}{\partial x^i}\left(\frac{\partial \Lambda}{\partial \bar{\varphi}_i} \right) = -m^2 c^2 \varphi - \hbar^2 g^{ij} \frac{\partial^2 \varphi}{\partial x^i \partial x^j}$$

$$= -(\hbar^2 \Box + m^2 c^2) \varphi \quad \text{Klein - Gordon } 1927: \quad \Box := \frac{\partial^2}{\partial x^{0\,2}} - \sum_{\alpha=1}^{3} \frac{\partial^2}{\partial x^{\alpha\,2}}.$$

e.g. The energy $T^{00} = g^{00}\left(\varphi_0 \frac{\partial \Lambda}{\partial \varphi_0} + \bar{\varphi}_0 \frac{\partial \Lambda}{\partial \bar{\varphi}_0} \right) - \Lambda \underset{\substack{\uparrow \\ \text{via E-L}}}{=} \hbar^2 \sum_{c=0}^{3} \bar{\varphi}_c \varphi_c + m^2 c^2 \bar{\varphi} \varphi$.

(Λ is indep of x^i)

For α const, Λ is inv: $\varphi \mapsto e^{i\alpha} \varphi$, $\bar{\varphi} \mapsto e^{-i\alpha} \bar{\varphi}$ Q: Try get this from Noether's thm

get "Noether current" (inv. quantity) $J^a = i g^{ab} (\bar{\varphi} \varphi_b - \varphi \bar{\varphi}_b)$, i.e. $\sum \frac{\partial J^a}{\partial x^a} = 0$.

(check: $J^a_a = i g^{ab} (\cancel{\bar{\varphi}_a \varphi_b} + \bar{\varphi} \varphi_{ab} - \cancel{\varphi_a \bar{\varphi}_b} - \varphi \bar{\varphi}_{ab}) = i \frac{m^2 c^2}{\hbar^2} (-\bar{\varphi} \varphi + \varphi \bar{\varphi}) = 0$)

$q := \int_{t = \text{const}} J^0 dx^1 \wedge dx^2 \wedge dx^3$ is called the "charge" of φ.

Rmk: $\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right) \varphi = 0$ has obvious solutions $e^{i\langle k, x \rangle}$ st. $\langle k, k \rangle = \left(\frac{mc}{\hbar}\right)^2$.
↖ 4 vectors.

K-G: include electromagnetic field by the rule (for momentum operators)

$$p_a = i\hbar \frac{\partial}{\partial x^a} \longmapsto p_a + \frac{e}{c} A_a = i\hbar \frac{\partial}{\partial x^a} + \frac{e}{c} A_a = i\hbar \left(\frac{\partial}{\partial x^a} - \frac{e}{c\hbar} i A_a \right) =: i\hbar \nabla^A_a$$

$$\Lambda \mapsto \Lambda(\varphi, \bar{\varphi}, A) = \hbar^2 g^{ab} \overline{\nabla^A_a \varphi}\, \nabla^A_b \varphi - m^2 c^2 \bar{\varphi} \varphi - \frac{1}{16\pi c} F_{ab} F^{ab}$$

$$= \hbar^2 |\nabla^A \varphi|^2 - m^2 c^2 |\varphi|^2 - \frac{1}{16\pi c} |F|^2 \qquad F = dA$$

For $\alpha = \alpha(x)$ not const. The new action $S(\varphi, \bar{\varphi}, A)$ is inv under "Gauge transf":

$$(\varphi, \bar{\varphi}, A) \mapsto \left(e^{i\alpha} \varphi,\ e^{-i\alpha} \bar{\varphi},\ A + \frac{c\hbar}{e} d\alpha\right) \quad \left(F \mapsto d\left(A + d \frac{c\hbar}{e} \alpha\right) = dA = F \right)$$

History: Schrödinger eq'n $i\hbar \frac{\partial}{\partial t} \varphi = \hat{H} \varphi$ arising from "quantization" of Hamiltonian formalism is NOT compatible with special relativity since $\hat{H}$ is not a 1st order op. as in t. The K-G eq'n is relativistic, but it does NOT explian all particles (only those without "spin"). It was Dirac who solved it !!

Dirac (1928): Can we factorize (set $\hbar = c = 1$):

$$-(\Box + m^2) = \left(i \gamma^a \frac{\partial}{\partial x^a} + m\right)\left(i \gamma^b \frac{\partial}{\partial x^b} - m\right) \text{ ? } \quad \text{i.e. } \Box = g^{ab} \partial_a \partial_b = \not{\partial}^2 \ ; \ \not{\partial} = \gamma^a \partial_a \text{ ?}$$

with $\gamma^a =$ constant

obviously, this is equiv. to $\gamma^a \gamma^b + \gamma^b \gamma^a = 2 g^{ab}$

This is impossible for numbers: $a \neq b \Rightarrow \gamma^a \gamma^b = 0 \Rightarrow$ say $\gamma^a = 0$, $\not{\partial}$ has no ∂_a!

Solution: Need to allow γ^a being non-commutative (matrix) and φ being vectors.

Spinor Representations / Clifford Algebras

Lemma : Every algebra automorphism on $M(n,\mathbb{C})$ is inner $x \mapsto gxg^{-1}$.

pf : Let $P_i = E_{ii}$ clearly $P_i^2 = P_i$ $P_iP_j = 0$ for $i \neq j$ $P_1 + \cdots + P_n = I_n = 1$

if $h \in \mathrm{Aut}\, M(n,\mathbb{C})$, then $P_i' := h(P_i)$ $i=1,\dots,n$ have the same property.

hence $\mathbb{C}^n \simeq P_1'(\mathbb{C}^n) \oplus \cdots \oplus P_n'(\mathbb{C}^n) =: C_1' \oplus \cdots \oplus C_n'$

similarly for E_{ij} define $E_{ij}' := h(E_{ij})$ which induces $C_j' \simeq C_i'$

let e_1' gen C_1' then $e_i' := E_{i1}'(e_1')$ gen C_i' $\Rightarrow$ $e_1', \dots, e_n'$ is a basis

Define $g : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ by $g(e_i) = e_i'$. claim: $h(x) = gxg^{-1}$.

check $h E_{ij} = g E_{ij} g^{-1}$ i.e. $E_{ij}'(e_k') = g E_{ij} g^{-1}(e_k')$ clear.

but E_{ij} gen $M(n,\mathbb{C})$, hence done ✱

Def'n (Pauli matrix) : $\sigma_0 = I_2$, $\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

is a basis for $M(2,\mathbb{C})$. For $\sigma_1, \sigma_2, \sigma_3$, we compute

(i) $[\sigma_i, \sigma_j] = \sigma_i\sigma_j - \sigma_j\sigma_i = 2i\,\sigma_k$ for (i,j,k) even $\Rightarrow \langle \frac{\sigma_1}{2i}, \frac{\sigma_2}{2i}, \frac{\sigma_3}{2i} \rangle \simeq so(3) \simeq (\mathbb{R}^3, \times)$

(ii) $\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$, Clifford algebra : $v^2 - \langle v, v \rangle = 0$ in $\otimes V$.

(Dirac matrix): $\gamma^0 = \left(\begin{array}{c|c} \sigma_0 & \\ \hline & -\sigma_0 \end{array}\right)$ $(\sigma_0 = id)$, $\gamma^1 = \left(\begin{array}{c|c} & \sigma_1 \\ \hline -\sigma_1 & \end{array}\right)$, $\gamma^2 = \left(\begin{array}{c|c} & \sigma_2 \\ \hline -\sigma_2 & \end{array}\right)$, $\gamma^3 = \left(\begin{array}{c|c} & \sigma_3 \\ \hline -\sigma_3 & \end{array}\right)$.

Satisfy (iii) $\{\gamma^a, \gamma^b\} = \gamma^a\gamma^b + \gamma^b\gamma^a = 2g^{ab}$: ie $v^2 - \langle v, v \rangle = 0$ in $\mathbb{R}^{1,3}$.

Digression: Had seen $SO(3) \simeq SU(2)/\pm 1 \simeq S^3/\pm 1$ ie. $\pi_1(SO(3)) \simeq \mathbb{Z}/2$

$SO(n-1) \hookrightarrow SO(n) \longrightarrow S^{n-1}$ $\Rightarrow$ $\pi_1(SO(n)) \simeq \mathbb{Z}/2$ $\forall\, n \geq 3$

Spin(n) := the univ (double) cover of SO(n). Matrix repr of Spin(n)?

Cor : Spin repr of Spin(1,3) on $\mathbb{C}^4$ is given by $O(1,3) \longrightarrow SL(4,\mathbb{C})/\pm 1$ via

$\Lambda = (\lambda^a_b)$ acts on $M(4,\mathbb{C})$: $\gamma'^a = \lambda^a_b \gamma^b$ as algebra automorphism

$\Rightarrow$ $\Lambda \cdot x = g x g^{-1}$ for some $g = g(\Lambda) \in SL(4,\mathbb{C})/\pm 1$.

Pf: γ'^a satisfy (iii) as well. Since 1, γ^a, $\gamma^a\gamma^b$ $(a<b)$, $\gamma^a\gamma^b\gamma^c$ $(a<b<c)$

and $\gamma^0\gamma^1\gamma^2\gamma^3$ span $M(4\mathbb{C})$ (Ex. linear indep of these 16 matrices)

$\Rightarrow$ $\Lambda \in \mathrm{Aut}(M(4,\mathbb{C}))$, $\exists\, g(\Lambda)$ by Lemma, up to ± 1 ✱

Rmk: Can construct SO(3) case using σ_i in $M(2,\mathbb{C})$, check (i), (ii) for $\sigma_i' = \lambda_i^j \sigma_j$.

SO(4) case using $\gamma^0, i\gamma^1, i\gamma^2, i\gamma^3$ in $M(4,\mathbb{C})$.

Half spinor: $\mathbb{C}^4 \ni \psi = \psi_+ + \psi_-$ according to the block decomp st $\gamma^1, \gamma^2, \gamma^3$ switch them.

Def'n : $\psi^* := (\bar\psi_1, \bar\psi_2, \bar\psi_3, \bar\psi_4) = {}^t\psi^{\mathrm{bar}}$ and $\bar\psi := \psi^* \gamma^0$ is the Dirac conjugate.

Ex. The spinor repr of O(1,3) is not unitary, but $\langle \psi, \varphi \rangle := \psi^* \gamma^0 \varphi = \bar\psi \varphi$ is inv.

Finally we get Dirac's equation for 4-spinor : $\not\partial \psi := \left(i \gamma^b \frac{\partial}{\partial x^b} - m \right) \psi = 0$ ✱

Covariant derivative for the general case

$\psi(x) = (\psi^1(x), \dots, \psi^N(x)) : \mathbb{R}^4 \to \mathbb{R}^N$ (or $\mathbb{C}^N$)

$A(x)$ $\mathfrak{g}$-valued 1-form $\mathfrak{g} = \mathrm{Lie}\, G$, G some matrix gp in $GL(N, \mathbb{R})$ ~ $\mathbb{C}$

" $A_\alpha(x)\, dx^\alpha \in \Lambda^1(\mathfrak{g})$ Define $\nabla_\alpha \psi = \frac{\partial \psi}{\partial x^\alpha} + A_\alpha(x)\psi$

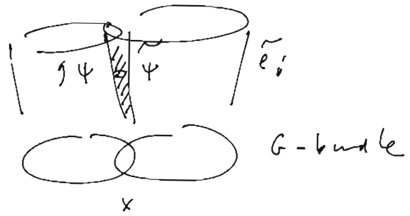
in order that the result is inv. under G-action ie. $g(\nabla_\alpha \psi) = \nabla_\alpha(g\psi)$

we require $\psi \longmapsto g(x)\psi = \tilde{\psi}$ and $A_\alpha(x) \longmapsto -\frac{\partial g}{\partial x^\alpha} g^{-1} + g A_\alpha g^{-1} = \tilde{A}_\alpha$

Gauge field Gauge transf. $\{g(x)\} =: \mathcal{G}$ Gauge group

As in the case of TS

Now this is a vector bundle $\begin{matrix} E \\ \downarrow \\ S \end{matrix}$ and $\nabla : \Lambda^0(E) \longrightarrow \Lambda^1(E)$



two local trivializations

$g^i_j \psi^j = \tilde{\psi}^i \iff$

$g^i_j \psi^j \tilde{e}_i = \tilde{\psi}^i \tilde{e}_i = \Phi$

" $\psi^j (g^i_j \tilde{e}_i) = \psi^j e_j$

curvature (operator) $R(X,Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$

for $X, Y \in T_pS$. function-linear in X, Y, skew

Also $R(\partial_i, \partial_j) f\psi = \nabla_i \nabla_j f\psi - \nabla_j \nabla_i f\psi$

$= \nabla_i\big((\partial_j f)\psi + f \nabla_j \psi\big) - \nabla_j\big((\partial_i f)\psi + f \nabla_i \psi\big)$

$= \cancel{(\partial_i \partial_j f)}\psi + \partial_j f\, \cancel{\nabla_i \psi} + (\partial_i f)\cancel{\nabla_j \psi} + f \nabla_i \nabla_j \psi - (i \leftrightarrow j)$

$= f(\nabla_i \nabla_j - \nabla_j \nabla_i)\psi = f R(\partial_i, \partial_j)\psi. \Rightarrow \tilde{F}_{ij} = g F_{ij} g^{-1}!$

ie. $F = \sum_{i<j} F_{ij}\, dx^i \wedge dx^j \in \Lambda^2(\mathfrak{g})$ in terms of a trivialization of $E|_U$

Formula: $F_{ij}\psi = \nabla_i(\partial_j \psi + A_j \psi) - \nabla_j(\partial_i \psi + A_i \psi) = \partial_i A_j - \partial_j A_i + [A_i, A_j]$; $F = dA + A \wedge A$

∇ acts on E^* and any $\otimes^{r,s} E$, eg. $E \otimes E^* \cong \mathrm{End}\, E$ accordingly by Leibniz rule.

eg. $B(x) \in \mathfrak{g} \subset E \otimes E^* \ni \nabla_\alpha b^i_j e_i \otimes e^j = \frac{\partial b^i_j}{\partial x^\alpha} e_i \otimes e^j + \left(A_\alpha{}^k_i\, e_k \otimes e^j - e_i \otimes A_\alpha{}^j_\ell\, e^\ell\right) b^i_j$

$= \partial_\alpha b^i_j + \left(A_\alpha{}^i_k\, b^k_j - b^i_\ell\, A_\alpha{}^\ell_j\right) e_i \otimes e^j = \partial_\alpha B + [A_\alpha, B]$.

Prop (Ex.) Show that $\exists!$ extension $\Lambda^0(E) \xrightarrow{\nabla} \Lambda^1(E) \xrightarrow{d^\nabla} \Lambda^2(E) \xrightarrow{d^\nabla} \cdots$

so that $d^\nabla \circ d^\nabla = R$. Moreover $d^\nabla : \Lambda^2(\mathfrak{g}) \to \Lambda^3(\mathfrak{g})$ has $d^\nabla R = 0$ (Bianchi identity)

Examples (a) $G = U(1) \simeq SO(2)$ commutative $\Rightarrow F_{ij} = \partial_i A_j - \partial_j A_i$ ie. $F = dA$

(b) Linear connection on TS, $(A_i)^k_j = \Gamma^k_{ji}$, $(F_{ij})^k_\ell = R^k{}_{\ell ij}$.

(c) Cartan connection (or affine) on TS $(\tilde{\nabla}_i \xi)^i := (\nabla_i \xi)^i + \delta^i_i$

in one chart. ∇ linear. G = affine gp = linear transf + translations

(d)* (semi-) spinors with induced Levi-Civita conn on (M^4, g_{ij}) of type $(1,3)$. Let $U = \bar{U}^T$

by $U = u^\beta \sigma_\beta(x) = \begin{pmatrix} u^0 + u^3 & u^1 + i u^2 \\ u^1 - i u^2 & u^0 - u^3 \end{pmatrix}$ with $\sigma_\beta(x)$ ON. $\forall x$. $\det U = (u^0)^2 - \sum_{i=1}^{3} (u^i)^2 \Rightarrow sL(2, \mathbb{C}) \xrightarrow{2:1} so(1,3)$ via $U \mapsto gU\bar{g}^T$

Any $A_\alpha(x) \in sl_2(\mathbb{C}) \Rightarrow \hat{\nabla}_\alpha$ on $\mathbb{C}^2 \ni \hat{\nabla}_\alpha U = \partial_\alpha U + A_\alpha U + U \bar{A}_\alpha^T$. we identify TM by $\{U\}$.

Ex. The requirement $\hat{\nabla}_\alpha U \equiv \hat{\nabla}_\alpha(\sigma_\beta u^\beta) = \sigma_\beta \cdot (\nabla^{LC}_\alpha u)^\beta \Rightarrow A_\alpha = -\frac{1}{2} \Gamma^\gamma_{\beta\alpha} \sigma_\gamma \sigma_\delta g^{\beta\delta}_{\text{Minkowski}}$ #

Final Lecture : Gauge inv functionals and characteristic classes

Gauge field $A \in \Lambda^1(\mathfrak{g})$ $\mathfrak{g} = \mathrm{Lie}\,G$, $\langle\, , \rangle$ Killing form , assume non-degenerate

$L(A)$ Lagrangian inv under Gauge transf. ie. intrinsically :

$$\begin{tikzcd} E \arrow[d] \\ S \end{tikzcd}$$

G bundle

eg. $L(A) = |F|^2 = \langle F^{\mu\nu}, F_{\mu\nu} \rangle = g^{\mu\alpha} g^{\nu\beta} \langle F_{\alpha\beta}, F_{\mu\nu} \rangle$

Thm (Yang - Mills) The extremal for $\int_S |F|^2 \, d\sigma =: S[A]$

satisfies $d_A^* F = \nabla_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} - [A_\mu, F^{\mu\nu}] = 0$. (sign problem ?)

$\nabla : \Lambda^0(E) \to \Lambda^1(E)$

a G - connection

$L = L(\nabla)$.

$F = dA + \underset{\substack{\uparrow \\ \text{matrix mult}}}{A \wedge A}$.

pf : $\frac{d}{d\varepsilon} S[A + \varepsilon\eta]\Big|_{\varepsilon = 0} = 2 \int_S \langle F, d\eta + \eta \wedge A + A \wedge \eta \rangle \, d\sigma$

$d\eta \overset{\shortparallel}{+} [A, \eta] = d_A \eta$

$=: 2 \int_S \langle d_A^* F, \eta \rangle \, d\sigma$.

Explicitly, $\int_S \langle F^{\mu\nu}, \partial_\mu \eta_\nu - \partial_\nu \eta_\mu + [A_\mu, \eta_\nu] + [\eta_\mu, A_\nu] \rangle \, d\sigma$

Stokes' $= \int_S \langle \partial_\mu F^{\mu\nu}, \eta_\nu \rangle - \underline{\langle \partial_\nu F^{\mu\nu}, \eta_\mu \rangle} - \langle [A_\mu, F^{\mu\nu}], \eta_\nu \rangle + \underline{\langle [A_\nu, F^{\mu\nu}], \eta_\mu \rangle}$

$\mu \leftrightarrow \nu$ switch

$= 2 \int_S \langle \partial_\mu F^{\mu\nu} - [A_\mu, F^{\mu\nu}], \eta_\nu \rangle \, d\sigma$

*

Rmk : This is the non-abelian generalization of Maxwell eq^n , eg. $G = SU(2)$.

Def^n : Gauge inv. closed form $\omega(A)$ with $\delta\omega(A)$ exact , hence $\delta \int_S \omega \equiv 0$ for S a space of dim = deg ω , is called a diff. geom. char. class.

① $c_i(A) := \mathrm{tr}\, F^i = \mathrm{tr}(F \wedge \cdots \wedge F) \in \Lambda^{2i}$ $dF = dA \wedge A - A \wedge dA = [dA, A] = [F, A]$

$dc_i(A) = \mathrm{tr}(dF \wedge \cdots F) + \mathrm{tr}(F \wedge dF \wedge \cdots) + \cdots$ $\delta F = d\delta A + \delta A \wedge A + A \wedge \delta A$

$= i\, \mathrm{tr}([dA, A] \wedge F^{i-1}) = i\, \mathrm{tr}([F, A] \wedge F^{i-1})$ $= d(\delta A) + \underset{\substack{\uparrow \\ \mathbb{Z}_2\text{-bracket}}}{[\delta A, A]}$

$= i\, \mathrm{tr}(F A F^{i-1} - A F^i) = i\, \mathrm{tr}(A F^i - A F^i) = 0$

#

$\delta c_i(A) = i\, \mathrm{tr}\big((d(\delta A) + [\delta A, A]) \wedge F^{i-1} \big) = i\, \mathrm{tr}(d(\delta A) \wedge F^{i-1}) + i\, \mathrm{tr}([\delta A, A] \wedge F^{i-1})$

for $i = 1$ get $\delta c_1(A) = d\, \mathrm{tr}(\delta A)$ exact , done.   For $G = U(n)$, $i = 1, \dots, n$

Ex. Show that $\delta c_i(A)$ is exact for all i . $\Rightarrow$ Chern classes $c_i(A) \in \Lambda^{2i}$

eg. may do $c(A) := \det\left(1 + \frac{\sqrt{-1}}{2\pi} F\right)$ directly. $\longleftrightarrow$ (up to some const $\left(\frac{\sqrt{-1}}{2\pi}\right)^i$)

② For $G = SO(m)$, $F \in \mathfrak{g} = \mathfrak{so}(m) \Rightarrow F^T = -F \Rightarrow (F^i)^T = (-1)^i F^i \Rightarrow c_{2k+1} \equiv 0$

get (only) Pontryagin classes $p_i := c_{2i} \in \Lambda^{4i}$

③ For $G = SO(2n)$, there is one more class $X_n := \varepsilon^{i_1 \cdots i_{2n}} F_{i_1 i_2} \wedge \cdots \wedge F_{i_{2n-1} i_{2n}} \in \Lambda^{2n}$

the Euler class . eg. for $SO(2)$, $X_1 = \varepsilon^{ij} F_{ij} = 2K \, d\sigma$.

Ex. Show that X_n is a characteristic class, also all products with p_i's . END.