Stable Maps Moduli Spaces 2004 11/17 at CTS p.

$\underline{M_g \text{ Riemann}}$: $\dim M_g = 3g-3$ $(g \geq 2)$, $M_0 = pt$, M_1 

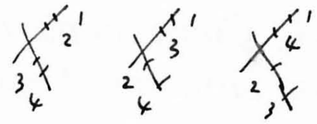
$\underline{\overline{M}_{g,n} \text{ Deligne - Mumford's}}$ stable curves

C = nodal curve $= \bigcup C_i$  $C_i \simeq \mathbb{P}^1 \ni \geq 3$ special pts

$C_i \simeq T \ni 1$ special pt

n marked pt (sm pts) arith genus $Pa(C) = g$

example : $M_{0,3} = pt$, $M_{04} = \mathbb{P}^1 \setminus 3 pts$

$\Rightarrow \overline{M}_{04} = \mathbb{P}^1 = M_{04} \cup \{$  $\}$

construction : $(C, \omega_C(p_1 + \cdots + p_n))$

$\exists f \in \mathbb{N}$, $\mathcal{L}^{\otimes f}$ v.a. $h^1 = 0$, "$\mathcal{L}$ is ample

$h^0(\mathcal{L}^{\otimes f}) = \chi(\mathcal{L}^{\otimes f}) = f(2g-2+n) + (1-g)$
$= (2f-1)(g-1) + fn =: N$

embedding $\Phi : C \hookrightarrow \mathbb{P}(H^0(\mathcal{L}^{\otimes f})) = \mathbb{P}^{N-1}$

$\underline{\text{The case without}}$ $\mathcal{L}^{\otimes f} = \Phi^* \mathcal{O}_{\mathbb{P}^{N-1}}(1)$ $\overset{\cup\!\!|}{\mathbb{C}^N} \equiv W$

$\underline{\text{marked pt}}$: consider $H = \mathrm{Hilb}_{\mathbb{P}^{N-1}}(P) \circlearrowleft G = PGL(N)$

$P(m) = mf(2g-2+n) + 1 - g$

$\underline{\text{The case with marked pts}}$:

each p_i det a pt in $\mathbb{P}(W)$, so look at

$Z_1 \hookrightarrow Z \hookrightarrow H \times \mathbb{P}(W)^n$ incidence subscheme

thm of inv II. finally $\overline{M}_{g,n} = Z /\!/ G$. (GIT) proper by

$\mathcal{O}_{\mathbb{P}(W)}(1)|_C$ and $\omega_C(\Sigma p_i)^{\otimes f}$ have same multi-degree

projective scheme. (stable red thm.)

$\underline{\overline{M}_{g,n}(X, \beta)}$ Kontsevich's space of stable maps.

$\ni (C, \{p_i\}_{i=1}^n, \mu)$

$Pa(C) = g$ nodal $C = \bigcup C_i$

$\mu(C_i) = pt$, then $g(C_i) = 0 \ni 3$ sp pts ; $g(C_i) = 1 \ni$ one sp pt.

stability $\Leftrightarrow$ finite automorphism

$\Leftrightarrow \mathcal{L} := \omega_C(p_1 + \cdots + p_n) \otimes \mu^* \mathcal{O}_{\mathbb{P}^r}(3)$ is ample.

when $X = \mathbb{P}^r$ then similar construction holds.

$r = 0$: $\overline{M}_{g,n}(\mathbb{P}^0, d) = \overline{M}_{g,n}$ no such

$d = 0$: $\overline{M}_{g,n}(\mathbb{P}^r, 0) = \overline{M}_{g,n} \times \mathbb{P}^r$

$\overline{M}_{0,0}(\mathbb{P}^1, 1) = pt$

Will construct $\overline{M}_{g,n}(\mathbb{P}^r, d)$ for all other (g, n, r, d).

<u>Thm 1</u>. $\exists$ coarse moduli space $\overline{M}_{g,n}(X, \beta)$ as a scheme

ie. $\exists$ nat'l transf of functors $\phi : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathrm{Hom}_{sch}(\cdot, \overline{M}_{g,n}(X,\beta))$

st $\phi(\mathrm{Spec}\, \mathbb{C})$ is a bijection

Any $\psi : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathrm{Hom}_{sch}(\cdot, Z)$ factors thr $\overline{M}_{g,n}(X,\beta)$

where $\overline{\mathcal{M}}_{g,n}(X,\beta)(S) = \{$ isom class of $\pi : C \to S$ flat proj
$\{p_i\}_{i=1}^n$ sections
$\mu : C \to X$ family of stable maps $\}$

<u>Thm 2</u>. X sm proj, convex

i.e. $H^1(C, f^* T_X) = 0 \quad \forall (C, f)$, $C = \mathbb{P}^1$

then $\overline{M}_{0,n}(X,\beta)$ is normal of pure $\dim = c_1(X).\beta + \dim X + n - 3$

which a orbifold (i.e. locally $\mathbb{C}^N/G$, $|G| < \infty$)

Moreover, $\overline{M}^*_{0,n}(X,\beta) :=$ auto free stable maps

is a fine moduli space (with univ. family).

<u>Thm 3</u>. X sm, proj, convex, the boundary of $\overline{M}_{0,n}(X,\beta)$

(i.e. with reducible domain curves) is a NCD up to a finite quotient.

<u>Example</u>: why coarse moduli is also useful : eg

evaluation maps $e_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathrm{Hom}(*, X)$

induces unique morphism $e_i : \overline{M}_{g,n}(X,\beta) \to X$.

Idea of pfs: (about Thm 1)

let $\mathbb{P}^r = \mathbb{P}(V)$; $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) = V^*$, $\bar{t} = (t_0, \cdots, t_r)$ as basis of V^*.

A $\bar{t}$-rigid stable family of (g, d, n) stable maps to $\mathbb{P}^r$ is a collection of data

$$\left(\pi : C \to S,\ \{p_i\}_{i=1}^{n},\ \{q_{ij}\}_{i=0\ j=1}^{r\ \ d},\ \mu \right) \text{ st}$$

(i) $(\pi : C \to S, \{p_i\}_{i=1}^{n}, \mu)$ is a stable family of maps to $\mathbb{P}^r$.

(ii) $(\pi : C \to S, \{p_i\}, \{q_{ij}\})$ is a family of Deligne–Mumford stable curves with $n + d(r+1)$ pts, $\{p_i\}, \{q_{ij}\}$ are sections

(iii) for $0 \le i \le r$, $\mu^* t_i = q_{i1} + \cdots + q_{id}$ as Cartier divisors

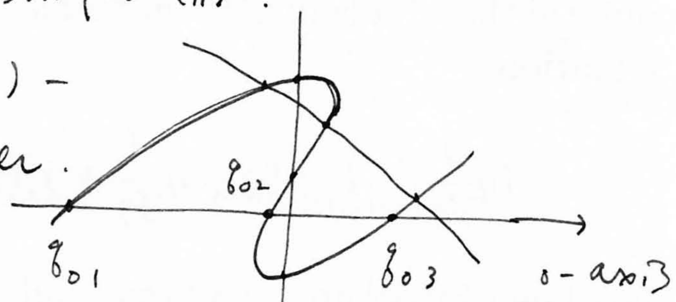
ie. each map int hyp planes $(t_i) \subset \mathbb{P}^r$ transversely (iii) and along unmarked sm pts (ii).

The point is, the new (sub)-functor $\overline{M}_{g,n}(\mathbb{P}^r, d, \bar{t})$ is simpler.

$g=0 \exists$ representable with univ family.

'fine moduli space !

as a quasi-proj var.



$\overline{U}_{g,m} \xrightarrow{\pi} $ univ. curve (for $g=0$)

$\mathcal{H}_i = \mathcal{O}_{\overline{U}_{0,m}}(q_{i1} + \cdots + q_{id})$

Reason : $\begin{matrix} C \\ \downarrow \\ S \end{matrix} \Rightarrow S \to \overline{M}_{g,m}$ $\quad m = n + d(r+1)$

Zariski open if $g=0$

let image $\Rightarrow B$, locally closed subscheme

Thm of whe II on equality of $\mathcal{H}_i$'s rel. to π

$(S \to B,\ \mu^* t_i)$ determines $C \to S$
$i=0,\dots,r$ up to diagonal action

the map μ is determined only by $\mathrm{div}(\mu^* t_i)$ $\forall i$

the $\bar{t}$-rigid moduli space = total space of these r distinct $\mathbb{C}^\times$ bundle over B.

$'' (r+1) - 1$

When $g=0$, $\overline{M}_{0,m}$ is fine and B is Zariski open. for $g>0$, B can be characterized by universal conditions ($\mathcal{H}$ balanced) etc.

$\overline{M}_{g,n}(\mathbb{P}^r, d)$ by gluing $\overline{M}_{g,n}(\mathbb{P}^r, d, \bar{t})$ for various $\bar{t}$ and take quotients. (Bertini thm)

$\overline{M}_{g,n}(X, \beta) \hookrightarrow \overline{M}_{g,n}(\mathbb{P}^r, d)$ if $X \xhookrightarrow{i} \mathbb{P}^r$, $i_* \beta = d\ell$

via $\overline{M}(\bar{t}, \bar{t}')/S_d^{r+1}$ $\triangle\ \overline{M}(\bar{t}, \bar{t})/S_d^{r+1}$.

* defined by $I_X(k) = H^0(\mathbb{P}^r, \mathcal{I}_X(k))$ s.b.g.s.

as sections of "v.b." $\pi_{M*} \mu^* \mathcal{O}_{\mathbb{P}^r}(k)$ on $\overline{M}_{0,n}(\mathbb{P}^r, d, \bar{t})$ with $\mathcal{U} \xrightarrow{\pi_M} \overline{M}_{0,n}(\mathbb{P}^r, d, \bar{t})$, then glue over $\bar{t}$

idea of pf on Thm 2.

Deformation theory ; $g=0$ case ($g \geq 1$ is similar)

cf. Kollar or Hartshorne

A. Zariski tangent space $T_{[\mu]} \mathrm{Hom}_\beta(C, X) = H^0(C, \mu^* T_X)$

$\mathrm{Def}(\mu) \equiv T_{[\mu]} M_{0,n}(X,\beta) = \mathrm{Def}_R(\mu) / H^0(C, T_C)$ ← auto of C.
i.e. with C fixed — 3-dim

and $0 \to \mathrm{Ker} \to \mathrm{Def}_R(\mu) \to T_{[\mu]} \mathrm{Hom}_\beta(C, X) \to 0$
deformation of markings, n-dim'l

B. Since X is convex ⇒

$\dim \mathrm{Def}(\mu) = \mu^* c_1(X).C + \dim X (1-g) + n - 3$

$= c_1(X).\beta + \dim X + (n-3)$.

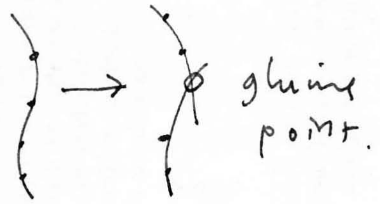
Also $H^1(C, \mu^* T_X) = 0$ ⇒ no obstructions ⇒ sm local moduli

So the global moduli is sm up to finite quotient.

Rmk : Care needs to be taken on boundary of $\overline{M}_{0n}(X,\beta)$,
namely A & B : e.g. $H^1(C, \mu^* T_X)$ still $= 0$.
can only do this for $g=0$

idea of pf of Thm 3.

boundary of $\overline{M}_{0,n}$ is clearly $\overline{M}_{0,A\cup\{\cdot\}} \times \overline{M}_{0,B\cup\{\cdot\}}$
with $A \cup B = \{1, \dots n\}$. For $\overline{M}_{0,n}(X,\beta)$,
similarly $D(A, B; \beta_1, \beta_2)$



$= \overline{M}_{0,A\cup\{\cdot\}}(X, \beta_1) \times_X \overline{M}_{0,B\cup\{\cdot\}}(X, \beta_2)$

$= (e_A \times e_B)^{-1}(\Delta)$ where e_A, e_B the evaluation of last pt.

~~Key point~~ : $D(A,B)$'s form NCD of $\overline{M}_{0,n}$, hence
by the construction of $\overline{M}_{0,n}(X, \beta, \bar{t})$, which is
locally Zariski open in $\overline{M}_{0,m} \times (\mathbb{C}^\times)^r$, hence
also share the same boundary behavior, ⇒ NCD.
$\overline{M}_{0,n}(X,\beta)$ = gluing / quotient ⇒ NCD up to finite quotient.

Gromov-Witten inv. $g=0$ case

$X = G/P$ homog var

$\Rightarrow T_X$ ggs $\Rightarrow h^1(C, f^*T_X) = 0$
ie. X is convex.

$H^{2i}(X) = A^i(X)$ Schubert cycles

$M_{0,n}(X,\beta) \xrightarrow{e_i} X$

$\langle \gamma_1, \cdots, \gamma_n \rangle_\beta := \int_{M_{0,n}(X,\beta)} e_1^*\gamma_1 \cup \cdots \cup e_n^*\gamma_n$

non-zero only if $\sum \deg \gamma_i = c_1(X).\beta + \dim X + (n-3)$

Fact: $\langle \gamma_1, \cdots, \gamma_n \rangle_\beta = \# \, e_1^{-1}(g_1\Gamma_1) \cap \cdots \cap e_n^{-1}(g_n\Gamma_n)$
for generic $g_i \in G$.

ie. # of maps $f: \mathbb{P}^1 \to X$ st $f_*[\mathbb{P}^1] = \beta$
$f(p_i) \in g_i\Gamma_i \ \forall i$.
in $M^*_{0n}(X,\beta)$!

Fact: I. $\langle \gamma_1, \cdots, \gamma_n \rangle_0 = 0$ unless $n=3$: $\langle \gamma_1 \gamma_2 \gamma_3 \rangle_0$
$= \int_X \gamma_1 \cup \gamma_2 \cup \gamma_3$

since $M_{0n}(X,0) = M_{0n} \times X$
$\searrow$ dim > 0 for $n > 3$.

II. $\langle 1, \gamma_2, \cdots, \gamma_n \rangle_{\beta \neq 0} = \int_{M_{0,n}(X,\beta)} e_2^*\gamma_2 \cup \cdots \cup e_n^*\gamma_n$
rel dim $= 1 > 0$
$\equiv 0$. $\quad M_{0,n-1}(X,\beta)$

< $\beta = 0$ then only $n=3$ get as in I. >

III. Divisor axiom: $\gamma_1 \in A^1(X)$

$\langle \gamma_1, \cdots, \gamma_n \rangle_\beta = (\gamma_1.\beta) \langle \gamma_2, \cdots, \gamma_n \rangle_\beta$

Pf : $= \int_{\psi_* M_{0,n}(X,\beta)} \gamma_1 \times (e_2^*\gamma_2 \cup \cdots \cup e_n^*\gamma_n)$

$\psi_* M_{0,n}(X,\beta) = \beta_1 \times M_{0,n-1}(X,\beta) + \alpha$ Kunneth

$\psi: M_{0,n}(X,\beta) \to X \times M_{0,n-1}(X,\beta)$, $\quad \Rightarrow \beta_1 = \beta$.

Quantum Cohomology

basis of coh : $T_0 = 1, T_1, \cdots, T_p \in A^1X$; $T_{p+1}, \cdots, T_m$

$g_{ij} = \int_X T_i \cup T_j$ dual basis $T^i = \sum_j g^{ij} T_j$.

$(g^{ij}) = (g_{ij})^{-1}$ then $\Delta \subset X \times X$ has $[\Delta] = \sum_{ef} g^{ef}\, T_e \otimes T_f$ ...

$T_i \cup T_j = \sum_k \langle T_i, T_j, T_k \rangle_0 T^k$

→ small q-coh prod : $T_i *_s T_j = \sum_{\beta \in A_1 X} \sum_k \langle T_i, T_j, T_k \rangle_\beta T^k q^\beta$

Define pre-potential. $\Phi(\gamma) = \sum_n \sum_\beta \frac{\langle \gamma^n \rangle_\beta}{n!} q^\beta$ ← not needed for $X = G/P$

Fact : $\forall n$, only finite # of β st. $\langle \gamma^n \rangle_\beta \neq 0$

since $\beta = \sum_{i=1}^p n_i \beta_i$, $n_i \geq 0$. $\int_{\beta_i} c_1(T_X) > 0$ (≥ 2)

so $\int_\beta c_1(x) + \dim X + (n-3) \leq n \dim X \Rightarrow \int_\beta c_1(x) \leq C$.

$T_i * T_j = \sum_k \Phi_{ijk} T^k$; where $\gamma = \sum t_i T_i$

$$\Phi_{ijk} := \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_k} = \sum_{n=0}^{\infty} \sum_{\beta \in A_1 X} \frac{1}{n!} \langle T_i, T_j, T_k, \gamma^n \rangle_\beta T^k q^\beta$$

The big Quantum product.

Notice. $T_0 * T_j = \langle T_0 T_j T_k \rangle_0 T^k = g_{jk} T^k = T_j$. ie $T_0 = id_*$.

- t_i, $i = 0 \ldots m$ are deformation parameters.

Theorem : $A^*X \otimes \mathbb{Q}[[t]] =: QH^*(x)$ is a comm ring under $*$.

Pf : $(T_i * T_j) * T_k = \sum_{e,f} \Phi_{ije} g^{ef} T_f * T_k$

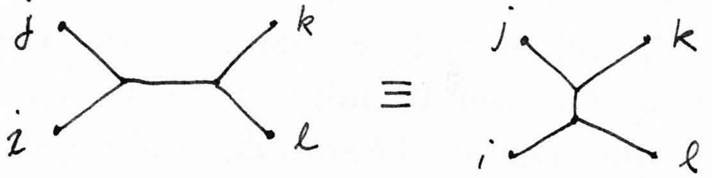
$= \sum_{e,f} \sum_c \Phi_{ije} g^{ef} \Phi_{fkc} T^c$

$T_i * (T_j * T_k) = \sum_{e,f} T_i * (\Phi_{jke} g^{ef} T_f)$

$= \sum_{e,f} \sum_c \Phi_{jke} g^{ef} \Phi_{ifc} T^c$

So associativity is equiv to the $\underline{\text{WDVV Eq}^{n}}$. Dijkgraaf

$$\sum_{e,f} \Phi_{ije}\, g^{ef}\, \Phi_{fkl} = \sum_{e,f} \Phi_{jke}\, g^{ef}\, \Phi_{ifl}$$



consider $D(A,B;\beta_1,\beta_2) = M_{0,A\bullet}(X,\beta_1)\times_X M_{0,B\bullet}(X,\beta_2)$

$A, B \neq \emptyset$

$$\begin{tikzcd} D(A,B;\beta_1,\beta_2) \arrow[d, "\alpha"] \arrow[r, hook, "\iota"'] & M_{0,A\bullet}(X,\beta_1)\times M_{0,B\bullet}(X,\beta_2) \\ M_{0,n}(X,\beta) & \end{tikzcd}$$

$A\cup B = \{1,\cdots,n\}$, $\beta_1+\beta_2=\beta$

<u>Lemma (Splitting Axiom)</u>:

$$\iota_*\alpha^*(e_1^*\gamma_1\cup\cdots\cup e_n^*\gamma_n) = \sum_{e,f} g^{ef}\Big(\prod_{a\in A} e_a^*\gamma_a \cup \underline{e_\bullet^* T_e}\Big)\times\Big(\prod_{b\in B} e_b^*\gamma_b\cup \underline{e_\bullet^* T_f}\Big)$$

pf:

$$\begin{tikzcd} M \arrow[d, "\rho"] & D \arrow[l, "\alpha"'] \arrow[r, hook, "\iota"] \arrow[d, "\eta"] \arrow[dr, phantom, "\Delta"] & M_A\times M_B \arrow[d, "\rho'"] \\ X^n & X^{n+1} \arrow[l, "p"] \arrow[r, "\delta"] & X^{n+2} \end{tikzcd}$$

δ diagonal embedding repeating the last factor

$$LHS = \iota_*\underline{\alpha^*\rho^*}(\gamma_1\times\cdots\times\gamma_n) = \iota_*\underline{\eta^* p^*}(\gamma_1\times\cdots\times\gamma_n)$$

$$= \iota_*\eta^*(\gamma_1\times\cdots\times\gamma_n\times[X]) = \rho'^*\delta_*(\gamma_1\times\cdots\times\gamma_n\times[X])$$

since Δ is a base change

$$= \rho'^*(\gamma_1\times\cdots\times\gamma_n\times[\Delta])$$

$$= \sum_{e,f} g^{ef}\rho'^*(\gamma_1\times\cdots\times\gamma_n\times T_e\times T_f) = RHS. \,//$$

Now $\sum_{e,f}\Phi_{ije}\, g^{ef}\,\Phi_{fkl} = \sum \frac{1}{n_1!\,n_2!}\langle T_i T_j T_e \gamma^{n_1}\rangle_{\beta_1}\, g^{ef}\,\langle T_k T_l T_f \gamma^{n_2}\rangle_{\beta_2}$

Let $G(qr|st) := \sum g^{ef}\langle \prod_{a\in A\cup\{q,r\}}\gamma_a, T_e\rangle_{\beta_1}\langle\prod_{b\in B\cup\{s,t\}}\gamma_b, T_f\rangle_{\beta_2}$

$q\,r\,s\,t \neq$ in $\{1,\dots,n\}$

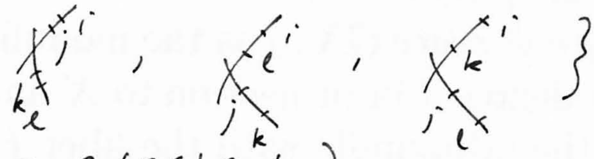
then it $= \sum\int_{D(A,B;\beta_1,\beta_2)} e_1^*\gamma_1\cup\cdots\cup e_n^*\gamma_n$

But the divisor $D(ij|kl) := \sum_{ij\in A,\ kl\in B} D(A,B;\beta_1,\beta_2)$

has $\underline{D(ij|kl)\sim D(il|jk)}$ on $M_{0,n}(X,\beta)$:

The pf is easy (and important):

$$\overline{M}_{0,n}(X,\beta) \longrightarrow \overline{M}_{0,n} \longrightarrow \underline{\overline{M}_{0,\{i,j,k,l\}} \simeq \mathbb{P}^1}$$

and $\mathbb{P}^1 \ni \{$  $\}$ //.

Thus $G(qr|st) = G(rs|qt)$.

for $\{q\,r\,s\,t\} = \{1,2,3,4\}$

$$\gamma_1 = T_i,\ \gamma_2 = T_j,\ \gamma_3 = T_k,\ \gamma_4 = T_l\ ;\ \gamma_i = \gamma \text{ other } i$$

get
$$G(qr|st) = \sum C^{n-4}_{n_1-2}\, g^{ef} \langle T_i T_j T_e \gamma^{n_1-2}\rangle_{\beta_1} \langle T_k T_l T_f \gamma^{n_2-2}\rangle_{\beta_2}$$
$$= (n-4)! \sum_{e,f} \Phi_{ije}\, g^{ef}\, \Phi_{fkl} \quad \text{done } \square$$

Applications to $\mathbb{P}^2$:

$T_0 = 1$, $T_1 =$ line, $T_2 =$ pt $\qquad T_i * T_j = \Phi_{ij0} T_2 + \Phi_{ij1} T_1 + \Phi_{ij2} T_0$

$$(T_1 * T_1) * T_2 = T_1 * (T_1 * T_2) \Rightarrow \Phi_{222} = \Phi_{112}^2 - \Phi_{111}\Phi_{122}$$

$$\Phi(\gamma) = \Phi(t_1 l + t_2\, pt) = \sum_{n \geq 0} \sum_{d \geq 0} \frac{\langle (t_1 l + t_2\, pt)^n \rangle_{dl}}{n!}$$
$$= \sum_{n \geq 0} \sum_{\substack{d \geq 0 \\ n_1 + n_2 = n}} \frac{1}{n_1!\, n_2!}\, t_1^{n_1} d^{n_1} t_2^{n_2} \langle pt^{n_2} \rangle_{dl}$$
$$= \sum_{d \geq 0} e^{d t_1}\, N_d\, \frac{t_2^{3d-1}}{(3d-1)!}$$
$$N_d = \langle \underbrace{pt, pt, \cdots, pt}_{3d-1} \rangle_{dl}$$

Compute directly get for $d \geq 2$:

$$\underline{\underline{N_d}} = \sum_{\substack{d_1 + d_2 = d \\ d_i > 0}} N_{d_1} N_{d_2} \left(d_1^2 d_2^2\, C^{3d-4}_{3d_1-2} - d_1^3 d_2\, C^{3d-4}_{3d_1-1} \right)$$

since $N_1 = \langle pt, pt \rangle_1 = 1$, get all N_d.

$N_2 = 1$, $N_3 = 12$, $N_4 = 620$, $N_5 = 87304 \ldots$

End.

Virtual cycles and Axioms.
4/24 at NTU, 2009
Dragon

$M/\mathbb{C}$ scheme.

$E^\bullet = [E^{-1} \to E^0]$ cpx of v.b./M

$\downarrow \phi$ morphism in $D^-_{coh}(M)$

$L^\bullet_M$ cotangent cpx of M

st. (a) $h^0(\phi) =$ isom
(b) $h^{-1}(\phi) =$ surj } a perfect obstruction theory

Thm (Li-Tian, Behrend-Fantechi)

$\exists\ [M]^{vir} \in A_d(M)$, $d = rk\, E^0 - rk\, E^{-1}$ expected dim.

construction: Say $M \hookrightarrow Y$, nonsingular scheme with $I = I_M$ ideal sh.

2 term cut-off $L^{\geq -1}_M = [I/I^2 \to \Omega_Y]$

May assume actual morphism of cpx's.

$$\begin{array}{ccccc} E^\bullet & & E^{-1} & \longrightarrow & E^0 \\ \downarrow \phi & & \downarrow & & \downarrow \\ L^\bullet & & I/I^2 & \longrightarrow & \Omega_Y \otimes \mathcal{O}_M \end{array}$$

Ex. M has enough locally free's $\Rightarrow$ can do this in $D^-_{coh}(M)$.

Mapping cone: exact by (a) + (b)

$$\begin{tikzcd} E^{-1} \arrow[r] \arrow[dr] & E^0 \oplus I/I^2 \arrow[r] & \Omega_Y|_M \arrow[r] & 0 \\ 0 \arrow[r] & Q \arrow[u] \arrow[r] & 0 & \end{tikzcd}$$

Recall: S coh sh on M, $C(S) := Spec_M\, Sym(S)$ abelian cone

S loc. free $\Rightarrow C(S) = S^*$.

$\Rightarrow \quad 0 \to T_Y|_M \to C(I/I^2) \times_M E_0 \to C(Q) \to 0$

where $E_i := E^{i*}$

Normal cone $C_{M/Y} = Spec_M \bigoplus_{k \geq 0} I^k/I^{k+1} \hookrightarrow C(I/I^2)$

- $\dim C_{M/Y} =$ pure dim $= \dim Y$
- $C_{M/Y} \times_M E_0 \hookrightarrow C(I/I^2) \times_M E_0$ is T_Y-inv.

$$D := C_{M/Y} \times_M E_0 / T_Y \hookrightarrow C(Q) \hookrightarrow E_1 \quad \text{sub cone}$$

$\searrow$ quotient cone $/_M$, $\dim = \dim E_0$

Finally, $[M]^{vir} := 0^! [D] \in A_{\dim E_0 - \dim E_1}(M, Q)$

the refinement int. product, $0 : M \hookrightarrow E_1$ the zero section.

The case of $\overline{M} = \overline{M}_{g,n}(X, \beta)$

(1) Let $M = Mor(C, X)$, $C, X /_{\mathbb{C}}$ proj scheme

$$\begin{array}{ccc} C \times M & \xrightarrow{f} & X \\ {\scriptstyle \pi}\downarrow & & \\ M & & \end{array} \quad \text{univ. morphism}$$

$$e : f^* L^\bullet_X \to L^\bullet_{C\times M} \to L^\bullet_{C\times M/C} \cong \pi^* L^\bullet_M$$

assume that ω_C (dualizing cpx) exists.

$$e \otimes \omega_C : f^* L^\bullet_X \overset{L}{\otimes} \omega_C \to \pi^* L^\bullet_M \overset{L}{\otimes} \omega_C = \pi^! L^\bullet_M$$

adjunction $\Rightarrow$

$$\begin{tikzcd} R\pi_*(f^* L^\bullet_X \overset{L}{\otimes} \omega_C) \arrow[r] \arrow[d, equal, "\text{Grothendieck dual}"'] & L^\bullet_M \\ (R\pi_* f^* T^\bullet_X)^\vee \arrow[ur, dashed, "\phi"'] & \end{tikzcd}$$

Prop (Behrend - Fantechi)

C Gorenstein curve, X sm $\Rightarrow$ ϕ perfect obstruction th.

(2) Global resolution : T v.b. on $\mathcal{C}$ (e.g $f^* T_X$)

$$\begin{array}{ccc} (\mathcal{C}, x_1, \dots, x_n) & \xrightarrow{f} & X \\ {\scriptstyle \pi}\downarrow & \text{stable map} / M & \\ M & & \end{array} \qquad L := \omega_{\mathcal{C}/M}\left(\sum_1^n x_i\right) \otimes f^* A^{\otimes 3}$$

is π- ample

$\exists$ N large st. $\begin{cases} \pi^* \pi_*(T \otimes L^N) \twoheadrightarrow T \otimes L^N \\ R^1 \pi_* (T \otimes L^N) = 0 \end{cases}$

Let $0 \to K \to F := \pi^* \pi_* (T \otimes L^N) \otimes L^{-N} \to T \to 0$

K = ker. exact sequence of v.b.'s $/M$

$\forall\ t \in M$,

$$H^0(C_t, F) = H^0(C_t, \pi_*(T\otimes L^N)_t \otimes L_t^{-N})$$
$$= \pi_*(T\otimes L^N)_t \otimes H^0(C_t, L_t^{-N}) = 0$$

$\Rightarrow H^0(C_t, K) = 0 \quad \forall t \in M.$

$\Rightarrow \pi_* F = 0, \quad \pi_* K = 0 \Rightarrow R^1\pi_* F, \ R^1\pi_* K$ loc. free

$$0 \to R^0\pi_* T \longrightarrow R^1\pi_* K \longrightarrow R^1\pi_* F \longrightarrow R^1\pi_* T \to 0$$

i.e. $R\pi_* T \simeq [R^1\pi_* K \to R^1\pi_* F]$ in $D^-_{coh}(M)$.

(3) Consider
$$\begin{array}{ccc} \overline{M} & \xleftarrow{\pi} & U \\ \tau\downarrow & & \\ \mathfrak{M} & & \end{array}$$
U univ. curve, $\mathfrak{M}$ Artin stack of quasi-stable curve

[Deformation theory for $f: C \to X$ (C fixed)
is known: Tangent $= H^0(C, f^*T_X)$
obstruction $= H^1(C, f^*T_X)$]

Get canonical relative perfect obst th: (1) $\Rightarrow$

$$E_\tau^\bullet := [R^\bullet\pi_* f^*T_X]^\vee \longrightarrow L_\tau^\bullet$$

Then
$$\begin{tikzcd} & E^\bullet \arrow[r, dashed] \arrow[d, dashed, "\phi"'] & E_\tau^\bullet \arrow[r, "\Delta"] \arrow[d] & \tau^* L^\bullet_{\mathfrak{M}}[1] \arrow[d] \\ \tau^* L^\bullet_{\mathfrak{M}} \arrow[r] & L^\bullet_{\overline{M}} \arrow[r] & L^\bullet_\tau \arrow[r] & \tau^* L^\bullet_{\mathfrak{M}}[1] \end{tikzcd}$$

this defines $L_\tau^\bullet$

where $\tau^* L^\bullet_{\mathfrak{M}} \cong [R\mathcal{H}om_{\mathcal{O}_{\overline{M}}}(-, \mathcal{O}_U)(\Omega_\pi(\Sigma x_i))]^\vee[-1]$

This gives ϕ, and in particular, at $(E^\bullet_{[f]})^\vee$:

$$0 \to Ext^0(\Omega_C(\Sigma x_i), \mathcal{O}_C) \to H^0(C, f^*T_X) \to Tan(f)$$
$$\to Ext^1(\Omega_C(\Sigma x_i), \mathcal{O}_C) \to H^1(C, f^*T_X) \to Obs(f) \to 0$$

Exercise: $H^1(C, f^*T_X) = 0 \Rightarrow [f] \in \overline{M}_{g,n}(X,\beta)$ is a smooth pt.

Axioms for (gravitational) descendent inv.

- Point mapping : $\overline{M}_{g,n}(X,0) = X \times \overline{M}_{g,n}$

$$\mathcal{E} = R^1\pi_* f^* T_X = T_X \boxtimes R^1\tilde{\pi}_* \mathcal{O}_U = \pi_1^* T_X \otimes (\pi_2^* \mathcal{H}_g)^\vee$$

Let $\lambda_i = c_i(\mathcal{H}_g)$: Mumford $\int_{\overline{M}_{1,1}} \lambda_1 = \frac{1}{24}$

$$[\overline{M}_{g,n}(X,0)]^{vir} = e(\mathcal{E}) \cap [X \times \overline{M}_{g,n}]$$

e.g. For $(g,n) = (1,1)$, $e(\mathcal{E}) = c_d(X) - c_{d-1}(X).\lambda_1$

$$\langle \alpha \rangle_{1,0} = -(c_{d-1}(X).\alpha) \int_{\overline{M}_{1,1}} \lambda_1 = -\frac{1}{24}(c_{d-1}(X).\alpha).$$

$$\langle \tau_1 \rangle_{1,0} = \frac{1}{24} \int_X c_d(X) = \frac{1}{24}\chi(X).$$

- Fund. class : $\pi_n : \overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n-1}(X,\beta)$

$\beta \neq 0$ or if $n + 2g \geq 4$

$$\psi_i = \pi_n^* \psi_i' + \tilde{D}(\underset{\uparrow}{i,n} \mid \underset{\uparrow}{\text{others}})$$

$C_1 \cup C_2$ with $f|_{C_1} = \text{constant}$

$$\langle \tau_{d_1}\gamma_1, \cdots, 1 \rangle_{g,\beta} = \sum_{i=1}^{n-1} \langle \tau_{d_1}\gamma_1 \cdots, \underline{\tau_{d_i - 1}\gamma_i}, \cdots \rangle_{g,\beta}.$$

- Divisor :

Virtual Localization :

$\mathbb{C}^\times$

$$\begin{tikzcd} & V \arrow[d] \\ X \arrow[r, hook] & Y \arrow[u, bend left, "v"] \end{tikzcd} \qquad \begin{tikzcd} E^\bullet = [\,V^\vee \arrow[r] \arrow[d, dashed] & \Omega_Y\,] \arrow[d, "\wr"] \\ L_X^\bullet = [\,I/I^2 \arrow[r] & \Omega_Y\,] \end{tikzcd}$$

$e_{ref}(V)$

Chow homology class.

$\Rightarrow \ [X]^{vir} =$ refined Euler class of V.

v $\mathbb{C}^\times$-inv section :

$$\begin{tikzcd} V_i^f \arrow[d] \\ Y_i \arrow[u, bend left, "v_i"] \end{tikzcd}$$

V_i^f = zero weight bundle

vanishes on $X_i = X \cap Y_i$

Y_i → fixed comp.

$[(V_i^f)^\vee \to \Omega_{Y_i}]$ = perf obst. th. of X_i

$N^{vir} \underset{\text{by def}}{=} [T_Y \to V]^m$

$T_Y^m = N_{Y_i/Y}$

$\Rightarrow \quad N_i^{vir} = [N_{Y_i/Y} \longrightarrow V_i^m]$

Want to show :

$e(N_i^{vir}) = e(N_{Y_i/Y})/e(V_i^m)$

$$\begin{array}{c} [X]^{vir} \overset{?}{=} i_* \sum \dfrac{[X_i]^{vir}}{e(N_i^{vir})} = i_* \dfrac{e_{ref}(V_i^f) \cap e(V_i^m)}{e(N_{Y_i/Y})} \\ \| \\ e_{ref}(V) \end{array}$$

On the other hand,

$$[Y] = i_* \sum \frac{[Y_i]}{e(N_{Y_i/Y})} \Rightarrow e_{ref}(V) = i_* \sum \text{"}\frac{e_{ref}(V) \cap [Y_i]}{e(N_{Y_i/Y})}\text{"}$$

e_{ref} commutes with pull back

$= e_{ref}(V_i) = e_{ref}(V_i^f) \cap e(V_i^m)$

since v_i in V_i^f

Example : Multiple cover formula. & GW on $\mathbb{P}^r$.

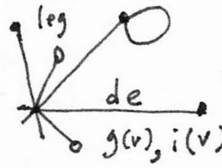
connected component.

$$\left(1+m_1\psi_1+m_1^2\psi_1^2+\cdots\right)\left(1+m_2\psi_2+m_2^2\psi_2^2+\cdots\right)\cdots\left(1+m_\ell\psi_\ell+\cdots\right)$$

Lemma: $$m_1^{k_1}\cdots m_\ell^{k_\ell}\int_{\overline{M}_{0,\ell m+2}}\psi_1^{k_1}\psi_2^{k_2}\cdots\psi_\ell^{k_\ell} = \binom{\ell m-1}{k_1\cdots k_\ell}$$

$k_1+\cdots+k_\ell = \ell m-1$ require.

marked pt = leg



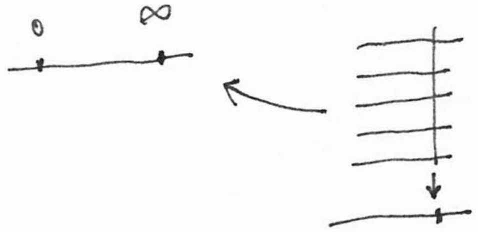
edge = non-contr. ⇓ nat'l.

$$\overline{M}_\Gamma := \prod_v \overline{M}_{g(v),\mathrm{val}(v)}$$

vertex = conn comp of $f^{-1}(\{p_1,\cdots,p_r\})$

$(x_0 : x_1) \mapsto (x_0 : t^{-1}x_1)$

$\mathcal{O}(-1)$



m_i

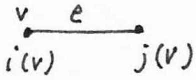
$$0\to\prod_e \mathbb{Z}/d_e \to A \to \mathrm{Aut}\,\Gamma \to 0$$

$$\overline{M}_\Gamma/A \hookrightarrow \overline{M}_{g,n}(\mathbb{P}^r, d)$$

non-singular !

$$w_F := \frac{\lambda_{i(F)}-\lambda_{j(F)}}{d_e}$$

flag $= \{(e,v)\}$



$\mathbb{P}^r$ case :

x_{-1}

$\mathbb{C}^\times \curvearrowright V=\mathbb{C}^{r+1}$, $p_i\in\mathbb{P}(V)$ fixed pt. $T_{p_i}V$ has wt. $\lambda_i-\lambda_j$; $j\neq i$

generic weights $-\lambda_0,\cdots,-\lambda_r$ $\quad = \mathrm{Hom}(\mathbb{C}p_i, V/\mathbb{C}p_i)$

perf. obst. th. on $\overline{M}_\Gamma/A$: dual $\to$ $0\to T^1\to E_{0,\Gamma}\to E_{1,\Gamma}\to T^2\to 0$

Also: $$0\to \underset{1}{\mathrm{Ext}^0(\Omega_C(D),\mathcal{O}_C)}\to \underset{2}{H^0(C,f^*T_X)}\to T^1\to \underset{4}{\mathrm{Ext}^1(\Omega_C(D),\mathcal{O}_C)}\to \underset{5}{H^1(C,f^*T_X)}\to T^2\to 0$$

$$e(N^{vir}) = \frac{e(B_2^m)\,e(B_4^m)}{e(B_1^m)\,e(B_5^m)}$$

Virtual fund. class

$\bar{M}_{g,n}(X,\beta)$ has exp. dim $= (c_1(X),\beta) + (\dim X - 3)(1-g) + n$

eg. CY 3 fold. v. dim $= 0$ $\forall g, \beta$ $(n=0)$

if $Ob(\Sigma, p_i, X) = 0$ eg. $g=0$, X convex then $[\bar{M}_{g,n}(X,\beta)]^{vir}$ in $H_{2\cdot vdim}(X,\mathbb{Q})$ $= \bar{M}_{g,n}(X,\beta)$

- Special case 1: $\bar{M}_{g,n}(X,\beta)$ is non-singular

$$[\bar{M}_{g,n}(X,\beta)]^{vir} = e(\underset{\text{v.b.}}{ob}) \cap [\bar{M}_{g,n}(X,\beta)]$$

- Special case 2: $g=0$, $X = (\ell) \subset \mathbb{P}^m$. defined by $(s=0)$

$i: \bar{M}_{0,n}(X,d) \hookrightarrow \bar{M}_{0,n}(\mathbb{P}^m, d)$ ("d" = all $\beta \in H_2(X,\mathbb{Z}) \mapsto d \cdot$ line)

convex, dim $= (m+1)d + m - 3 + n$

v. dim $= (m+1-\ell)d + (m-1) - 3 + n$

$= \dim RHS - (d\ell + 1)$

$$\Sigma = \bar{M}_{0,n+1}(\mathbb{P}^m, d) \xrightarrow{ev_{n+1}} \mathbb{P}^m$$
$$\downarrow \pi$$

$\dim H^0(\Sigma, f^* \mathcal{O}_{\mathbb{P}^m}(\ell)) = d\ell + 1$

$\forall$ genus 0 nodal Σ

$\Rightarrow$ Get v.b. $E_d := \pi_* ev_{n+1}^* \mathcal{O}(\ell) \longrightarrow \bar{M}_{0,n}(\mathbb{P}^m, d)$

HW. $\pi_* ev_{n+1}^* s$ is a section of E_d, vanishes on $i(\bar{M}_{0n}(X,d))$ exactly.

Thm: $i_*[\bar{M}_{0n}(X,d)]^{vir} = e(E_d) \cap \bar{M}_{0n}(\mathbb{P}^m, d)$.

For both cases,

$$\int_{[\bar{M}_{0,n}(X,\beta)]^{vir}} \phi = \int_{\bar{M}_{0,n}(X,\beta)} e(ob).\phi$$

How to calculate such integrals?

Answer: Localization.

$T = (\mathbb{C}^\times)^{m+1}$ acts on $V = \mathbb{C}^{m+1}$

$(t_0, \cdots, t_m)$ $\qquad (x_0, \cdots, x_m)$

$H^*_T \equiv H^*_T(pt) := H^*(BT) = H^*((\mathbb{C}P^\infty)^{m+1}) = \mathbb{Q}[\alpha_0, \cdots, \alpha_m]$

with $\alpha_i = c_1(\mathcal{L}_i)$, $\mathcal{L}_i$ = the i-th $\mathcal{O}_{P^\infty}(1)$

T acts on $X \ni H^*_T(X) := H^*(X \times_T BT)$ is a H^*_T-module

$$\begin{array}{cc} \| & \| \\ \mathbb{P}(V) \simeq \mathbb{P}^m & \mathbb{Q}[\alpha_0, \cdots, \alpha_m][H] / \prod_{i=0}^m (H - \alpha_i) \end{array}$$

diff action (linearization) on $\mathcal{O}_{\mathbb{P}^m}(-1)$:

e.g. $(x_0, \cdots, x_m) \mapsto t_0^{-1}(t_0 x_0, \cdots, t_m x_m)$ get $\mathcal{O}(-H + \alpha_0)$

Let $p_0, \cdots, p_m$ be T-fixed pt of $\mathbb{P}^m$

$\phi_i := H^{2m}_T(\mathbb{P}^m)$ the equivariant class of $p_i \xhookrightarrow{\iota_i} \mathbb{P}^m$

pairing: $a, b \in H^*_T(\mathbb{P}^m)$, $(a, b) := \int_{\mathbb{P}^m} a \cup b \in H^*_T$.

- Facts: (1) $T_{\mathbb{P}^m}|_{p_i}$ has weights $\alpha_i - \alpha_j$ $(j \neq i)$, by def of Hom.

(2). $H^0(\mathbb{P}^1, T\mathbb{P}^1)$ has wts $\alpha_0 - \alpha_1, 0, \alpha_1 - \alpha_0$

from $0 \to \mathcal{O} \to \mathcal{O}(1) \otimes V \to T\mathbb{P}^1 \to 0$ take H^0.

wt $= 0$. $\quad$ global sect x_0, x_1

wt $\{\alpha_0, \alpha_1\} \times \{-\alpha_0, -\alpha_1\}$.

- Facts: (3) $(f(H, \alpha), \phi_i) = \iota_i^* f(H, \alpha) = f(\alpha_i, \alpha)$

(4) $\phi_i = \prod_{j \neq i}(H - \alpha_j)$ (equiv. ext. of H^m)

(5) $a = b \iff (a, \phi_i) = (b, \phi_i)$ $\forall i$.

Atiyah-Bott localization theorem:

for any $\phi \in H^*_T(X)$, $\phi = \sum_F \frac{\iota_{F*} \iota_F^* \phi}{e(N_{F/X})}$ $\qquad \iota_F : F \hookrightarrow X$

i.e. $\int_X \phi = \sum_F \int_F \frac{i^* \phi}{e(N_{F/X})}$ $\quad$ fixed manifolds.

e.g. $X = \mathbb{P}^m$, $\int_{\mathbb{P}^m} f(H, \alpha) = \sum_{i=0}^m \operatorname{Res}_{H = \alpha_i} \frac{f(H, \alpha)}{\prod_j (H - \alpha_j)}$ (by (1)).

- Facts (6). For $f : \Sigma \to \mathbb{P}^1$ $d:1$ cover branched only at $0, \infty$ then $\Sigma \simeq \mathbb{P}^1$ and $(z_0, z_1) \mapsto (z_0^d, z_1^d) = (x_0, x_1)$.

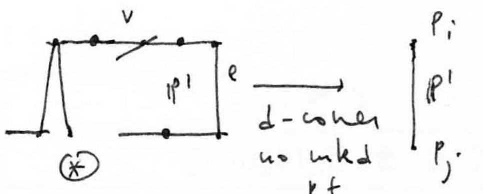
(7). On $\mathcal{O}_{\mathbb{P}^1}(1) \simeq \mathcal{O}(H)$, $H^0(\Sigma, f^*\mathcal{O}(1))$ has wts $\frac{1}{d}(i\alpha_0 + (d-i)\alpha_1)$.

Prop: For $\mathcal{O}(1) \simeq \mathcal{O}(H - \alpha_1)$, $H^0(\Sigma, f^*\mathcal{O}(1))$ wts $i(\alpha_0 - \alpha_1)/d \longrightarrow i = 0, \cdots, d$.

R.R. + Serre : $H^1(\Sigma, f^*\mathcal{O}(-1))$ $\qquad i(\alpha_0 - \alpha_1)/d : 1 \le i \le d-1$.

T-localization on $\overline{M}_{g,n}(\mathbb{P}^m, d)$

$f:(\Sigma, x_1,\dots,x_n) \to \mathbb{P}^m$ is T-fixed $\Leftrightarrow$



$\longleftrightarrow$ graph Γ : edge $\leftrightarrow$ non-const. comp e with $\deg(e) = d_e$

vertex $\leftrightarrow$ conn comp of $f^{-1}\{p_1,\dots,p_m\}$

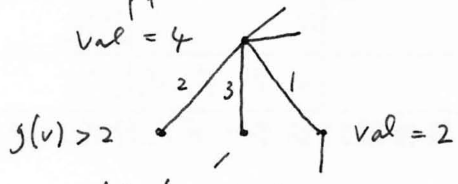
v with $p_v = f(v)$

$g(v)$

tail $\leftrightarrow$ marked points

$\mathrm{val}(v) = \#$ tails + edges

e.g.



$\gamma : \overline{M}_\Gamma : \prod_v \overline{M}_{g(v),\mathrm{val}(v)} \longrightarrow \overline{M}_{g,n}(\mathbb{P}^m, d)$; $\sum_e d_e = d$

automorphism on $\overline{M}_\Gamma$: $1 \to \prod_e \mathbb{Z}/d(e) \to A_\Gamma \to \mathrm{Aut}(\Gamma) \to 1$
semi-direct.

Flag : $F = (e,v)$ "$v \in e$" $\quad \omega_F := \dfrac{\alpha_{\mu(v)} - \alpha_{\mu(v')}}{d_e}$ wt on $T_{\cdot}\mathbb{P}^1$

point p_F

For $g=0$ (hence $g(v)=0\ \forall v$), since $h^1(\Sigma, f^*T\mathbb{P}^m) = 0$

$0 \to \mathrm{aut}(\Sigma, x_i) \to \mathrm{def}(f) \to \mathrm{def}(\Sigma, x_i, f)$ on $\overline{M}_\Gamma / A_\Gamma$

$\to \mathrm{def}(\Sigma, x_i) \to 0$

what we want is the moving part "ie. wt $\neq 0$".

$$e(N_\Gamma) = \underset{③}{e(\mathrm{def}(f)^{mov})} \cdot \underset{②}{e(\mathrm{def}(\Sigma,x_i)^{mov})} \Big/ \underset{①}{e(\mathrm{aut}(\Sigma,x_i)^{mov})}$$

① : $\prod_{\mathrm{val}(v)=1} \omega_F$ (e.g. in ⊛, the pt p_F is NOT special. the auto fixes $\mu(v)$ & $\mu(v')$ has wt 0)

② : boundary lemma : $i : \overline{M}_{g_1, A\cup p} \times \overline{M}_{g_2, B\cup q} \longrightarrow \overline{M}_{g,n}$

has $N_i = (L_p \boxtimes L_q)^*$ ie. $\otimes$ of T_p & T_q

def of const. comp $\to$ wt 0. so consider smoothing of node

$$\prod_F (\omega_F - \psi_F) \cdot \prod_{\mathrm{val}(v)=2} (\omega_{F_{v_1}} + \omega_{F_{v_2}})$$

ω_F

$\psi_F = \psi_{p_F}$

that has $\mathrm{val}(F) \geq 3$

V node joins 2 non-const. comp, no marked pts.

③ : On $\mathrm{def}(f) = H^0(\Sigma, f^* T\mathbb{P}^m)$:

Partial Normalization : $0 \to \mathcal{O}_\Sigma \to \bigoplus_v \mathcal{O}_{\Sigma_v} \oplus \bigoplus_e \mathcal{O}_{\Sigma_e} \longrightarrow \bigoplus_F \mathcal{O}_{p_F} \to 0$

$h^1 = 0$

$\Rightarrow H^0(\Sigma, f^*T\mathbb{P}^m) = \bigoplus_v T_{p_v}\mathbb{P}^m + \bigoplus_e H^0(\Sigma_e, f^*T\mathbb{P}^m) - \bigoplus_F T_{p_F}\mathbb{P}^m$

as T-representations.

The interesting part "$H^0(\Sigma_e, f^* T\mathbb{P}^m)$" $\to \overline{M}_\Gamma$ is a trivial bundle

$$0 \to \mathcal{O}_{\mathbb{P}^m} \to \mathcal{O}_{\mathbb{P}^m}(1) \otimes V \to T\mathbb{P}^m \to 0 \qquad \text{where } \mathbb{P}^m = \mathbb{P}(V)$$

$$\Rightarrow \quad 0 \to \mathbb{C} \to H^0(\Sigma_e, \mathcal{O}_{\Sigma_e}(d_e)) \otimes V \to H^0(\Sigma_e, f^* T\mathbb{P}^m) \to 0$$

$$\text{wt} = 0 \qquad \text{wt} = \frac{1}{d_e}(a\alpha_i + b\alpha_j) - \alpha_k \qquad \alpha = 0, \dots, m$$

$$k = j \qquad k = i \qquad a + b = d_e$$

the case $a = 0$ or $b = 0$ gives 2 wt 0.

another 0 wt to be canceled is in $\mathrm{aut}(\Sigma, x_i)$ in ①.

Thm:
$$\frac{1}{e(N_\Gamma)} = \underline{\prod_{\mathrm{val}(v)=1} w_F} \cdot \underline{\prod_F \frac{1}{w_F - \psi_F}} \cdot \underline{\prod_{\mathrm{val}(v)=2} \frac{1}{w_{F_{v_1}} + w_{F_{v_2}}}}$$

vertex
$$\underline{\prod_v \prod_{j \neq p_v} \frac{1}{\alpha_{p_v} - \alpha_j}} \cdot \underline{\prod_F \prod_{j \neq p_F} (\alpha_{p_F} - \alpha_j)}$$

$$\prod_{\text{edges}} \frac{(-1)^{d_e} (d_e)^{2d_e}}{(d_e!)^2 (\alpha_i - \alpha_j)^{2d_e}} \;\text{flags}\; \prod_{\substack{a+b=d_e \\ k \neq i, j}} \frac{1}{\frac{1}{d_e}(a\alpha_i + b\alpha_j) - \alpha_k}.$$

Cor. For $\overline{\Gamma} = \underset{p_0}{\bullet} \overset{d}{\text{———}} \underset{p_1}{\bullet}$; $\dfrac{1}{e(N_{\overline{\Gamma}})} = \dfrac{(-1)^{d-1} d^{2d-2}}{(d!)^2 (\alpha_0 - \alpha_1)^{2d-2}}$. (H.W.)

Aspinwall-Morrison formula:

X generic quintic $\subset \mathbb{P}^4$, consider $\mathbb{P}^1 = C \hookrightarrow X$ rigid, i.e. $N_{C/X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$

\# deg 1 curve $= 2875$

\# deg 2 curve $=$ \# rigid conics + "double covers to C"

$\overline{M}_0(X, 2)$ is not equi-dimensional!

Thm: The $g = 0$, degree d covers has contr. $1/d^3$ in $d[C] \in H_2(X, \mathbb{Z})$, where $f : \Sigma \longrightarrow C \hookrightarrow X$.

Pf: The Ob. bundle on $\overline{M}_0(\mathbb{P}^1, d)$ is "$H^1(\Sigma, f^* N_{C/X})$".

Will show $\displaystyle\int_{\overline{M}_0(\mathbb{P}^1, d)} e\big(H^1(\Sigma, f^*\mathcal{O}(-1)) \oplus H^1(\Sigma, f^*\mathcal{O}(-1))\big) = \frac{1}{d^3}$.

key idea: Choose action wt on both factors as $-H + \alpha_0$, $-H + \alpha_1$.

claim: If $\Gamma \neq \overline{\Gamma}$, then contr. $= 0$. If $\Gamma \neq \overline{\Gamma}$, it has nodes b_i.

$$0 \to f^*\mathcal{O}(-1) \to \oplus f|_{\Sigma_i}^* \mathcal{O}(-1) \to \oplus f|_{b_i}^* \mathcal{O}(-1) \to 0. \qquad \tilde{\Sigma} = \cup \Sigma_i \to \Sigma \to C$$

normalization

$$\Rightarrow \quad 0 \to \oplus H^0(b_i, f^*\mathcal{O}(-1)) \hookrightarrow H^1(\Sigma, f^*\mathcal{O}(-1))$$

$\overset{\shortparallel}{\mathbb{C}}$ if $f(b_i) = p_0$ get sub line bundle of trivial wt on $\mathcal{O}(-1) \simeq \mathcal{O}(-H + \alpha_0)$ at p_0.

Similarly if $f(b_i) = p_1 \Rightarrow e_T(\cdots) = 0$. Finally, notice $1/d$ from $A_{\overline{\Gamma}}$. done. $\square$

Descendent inv. $\psi_i = c_1(L_i)$, $L_i = x_i^* \omega_\pi$ in $(*)$

$$\left\langle \prod_{i=1}^n \tau_{a_i} \gamma_i \right\rangle_{g,\beta}^X = \int_{[\bar{M}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^n ev_i^* \gamma_i \cdot \psi_i^{a_i}$$

- String Eq'n: $\langle 1, \tau_{a_1}\gamma_1 \cdots \tau_{a_n}\gamma_n \rangle_{g,\beta} = \sum_i \langle \cdots \tau_{a_i - 1} \gamma_i \cdots \rangle_{g,\beta}$
- Dilaton Eq'n: $\langle \tau_1, \tau_{a_1}\gamma_1 \cdots \tau_{a_n}\gamma_n \rangle_{g,\beta} = (2g-2+n) \langle \prod \tau_{a_i}\gamma_i \rangle_{g,\beta}$
- Divisor Eq'n: $\langle D, \prod \tau_{a_i}\gamma_i \rangle_{g,\beta} = (D.\beta) \langle \prod \tau_{a_i}\gamma_i \rangle_{g,\beta} + \sum_i \langle \cdots \tau_{a_i - 1}(\gamma_i D) \cdots \rangle_{g,\beta}$

Proofs dep on Comparison lemma:

$(*)$ $\pi : \bar{M}_{g,n+1}(X,\beta) \to \bar{M}_{g,n}(X,\beta)$ then $\psi_i = \pi^*\psi_i + D(\{i, n+1\}, 0 \mid \{\ldots \hat{i} \ldots n\}, \beta)$ virtually

For $g=0$, $n \geq 3 \Rightarrow \psi_1 = D_{1|23}$

"pf:" $\nu : \bar{M}_{0n}(X,\beta) \to \bar{M}_{0,3} = pt \Rightarrow \psi_1 - \nu^* \overset{\overset{0}{\shortparallel}}{\psi_1} = bdr$.

- Top. recursion rel: $\langle \tau_{a_1}\gamma_1, \tau_{a_2}\gamma_2, \tau_{a_3}\gamma_3, \prod_{i \in S} \tau_{d_i}\delta_i \rangle_\beta$
  $(g=0)$
  $$= \sum \langle \tau_{a_1 - 1}\gamma_1, \prod_{i \in S_1} \tau_{d_i}\delta_i, T_e \rangle_{\beta_1} \langle T^e, \tau_{a_2}\gamma_2, \tau_{a_3}\gamma_3, \prod_{i \in S_2} \cdots \rangle_{\beta_2}$$

The SE, DE & TRR has their origin for $X = pt$.

$$F_g := \sum_{n \geq 0} \frac{\langle \tau^n \rangle_g}{n!} = \sum_{n \geq 0} \frac{1}{n!} \sum_{k_1, \cdots, k_n} t_{k_1} \cdots t_{k_n} \langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g$$

where $\tau = \sum_{i=0}^\infty t_i \tau_i$

$$F = \sum_{g \geq 0} \lambda^{2g-2} F_g \quad \& \quad Z = e^F$$

Then $L_{-1} Z = 0 \iff$ SE & $\langle \tau_0^3 \rangle_0 = 1$; $L_{-1} = -\frac{\partial}{\partial t_0} + \frac{\lambda^{-2}}{2} t_0^2 + \sum_{i=0}^\infty t_{i+1} \frac{\partial}{\partial t_i}$

$L_0 Z = 0 \iff$ DE & $\langle \tau_1 \rangle_1 = \frac{1}{24}$; $L_0 = -\frac{3}{2}\frac{\partial}{\partial t_1} + \sum_{i=0}^\infty \frac{2i+1}{2} t_i \frac{\partial}{\partial t_i} + \frac{1}{16}$

$\exists$ diff op's L_n $(n \geq -1)$ st. $[L_n, L_m] = (n-m) L_{n+m}$

Virasoro alg. gen by

Witten's conj. $L_n Z = 0$ $\forall n \geq -1$.

$L_n = -z^{n+1} \frac{d}{dz}$

solved by Kontsevich (matrix models).

Virasoro conj: (Eguchi-Hori-Xiong & S.Katz) also for general X.
but with op's dep on b_k^μ $T_\mu \psi^k$.

Quantum Diff eq'n & its fund. solution

$\gamma = \sum t_i T_i$, $T_0 = 1$, $\Phi(t) = \sum_{n,\beta} \frac{\langle t^n \rangle_\beta}{n!} \beta$, $t \in H = H^*(X)$

Dubrovin connection on TH :

$\nabla^{\hbar}_{T_i} = \hbar \frac{\partial}{\partial t_i} - T_i *$ is flat ($\Leftrightarrow$ WDVV eq'n)

QDE means flat solutions $\hbar \frac{\partial F}{\partial t_i} = T_i * F$

Def'n: J-function $J(t) = 1 + \frac{t}{\hbar} + \sum_{(n,\beta)\neq(0,0)} \frac{1}{n!} \cdot \left\langle \frac{T_b}{\hbar(\hbar-\psi)}, t^n \right\rangle_{\beta, n+1} T^b$.

consider $\hbar \frac{\partial}{\partial t_a} J_b =: \Phi_{ab} = g_{ab} + \sum \frac{1}{n!} \langle T_a, \frac{T_b}{\hbar-\psi}, t^n \rangle_\beta$

$= g_{ab} + \sum \sum_{k \geq 0} \frac{\hbar^{-k-1}}{n!} \langle T_a, \tau_k(T_b), t^n \rangle_\beta$

Prop: Φ_{ab} is a fund. sol. matrix for QDE :

i.e. $\hbar \frac{\partial}{\partial t_i} \left(\Phi_{ab} T^a \right) = T_i * \left(\Phi_{ab} T^a \right)$.

Pf: $LHS = \sum_{k \geq 0} \frac{\hbar^{-k}}{n!} \langle T_a, \tau_k(T_b), T_i, t^n \rangle_\beta T^a$ change index $a \mapsto s$

$RHS = T_i * T_b + \sum_{\substack{k \geq 0 \\ n_i, \beta_i}} \frac{\hbar^{-k-1}}{n_1! \cdot n_2!} \langle T_a, \tau_k(T_b), t^{n_1} \rangle_{\beta_1} g^{aj}$.

$\langle T_i, T_j, T_s, t^{n_2} \rangle_{\beta_2} T^s$

Apply TRR to $\underline{\tau_k(T_b)}, T_s, T_i$ to reduce k to $k-1$.

of LHS

$k=0$ gives $T_i * T_b$

$k \geq 1 \longmapsto k-1 \geq 0$ gives the remaining of RHS. $\square$

Rmk: This is equiv. to $\hbar \partial_i \partial_j J = C_{ij}^k \partial_k J$.

- Now, Mirror Sym Suggests A-model$/X$ $\simeq$ B-model$/X'$

for $t \in H^0(X) + H^2(X)$:
$t_0 + D$

Dubrovin conn. ? Picard-Fuchs eq'n

fund. sol. $\overset{?}{\longleftrightarrow}$ fund. sol.

(hyper geom. fcn)

string + div eq'n $\Rightarrow$ $J(t) = \sum_\beta e^{\frac{t_0+D}{\hbar} + (D.\beta)} \left\langle \frac{T_b}{\hbar(\hbar-\psi)} \right\rangle_\beta T^b$ $I(t)$.

- Idea: Prove J satisfies similar "recursive relations" as in I and prove some uniqueness.

$X \subset \mathbb{P}^m$ hyp. surf $\deg = \ell$, $0 \to E_d' \to E_d \to ev_2^* \mathcal{O}(\ell) \to 0$

J-side I-side

$\star$ $\underline{S(t, t_1)} = \underline{\sum_{d \geq 0} e^{\left(\frac{H}{t_1} + d\right)t} \, ev_2{}_* \left(\frac{e(E_d)}{t_1 - \psi_2}\right)}$ $\mapsto$ c.f. $\underline{S^*}$ next page.

$\langle S/\ell H, \phi_i \rangle = e^{\alpha_i t/t_1} \cdot \sum_{d \geq 0} e^{dt} \int_{\overline{M}_{0\,2}(\mathbb{P}^m, d)} \frac{e(E_d')}{t_1 - \psi_2} \, ev_2^*(\phi_i)$

$\dim \overline{M}_{0\,2}(\mathbb{P}^m, d) = (m+1)d + m - 1$ $\qquad$ " $Z_i(e^t, t_1)$

$rk(E_d') = \ell d$, " $\deg \phi_i = m$ " $\Rightarrow$ ψ_2-deg $\geq (m+1-\ell)d - 1$ $=: \delta \geq 0$

So $Z_i(e^t, t_1) = 1 + \sum_{d > 0} \left(\frac{e^t}{t_1^{m+1-\ell}}\right)^d \int \frac{\psi_2^\delta \cdot e(E_d')}{1 - \psi_2/t_1} \, ev_2^* \phi_i$ $\quad$ if $\ell \leq m$

$\underline{G_d = G_d^{i*} \cup G_d^{i0} \cup G_d^{i1}}$ $\qquad \sum_{\Gamma \in G_d} \int_{\overline{M}_\Gamma} \frac{*}{e(N_\Gamma)}$

w.r.t. i $\quad$ $x_2 \not\mapsto p_i$ $\quad$ $\Sigma_v \to p_i$, $\downarrow x_2$ $\quad$ $\mu(v) = i$, $val(v) = 2$. i.e. Σv is not contracted.

and focus on $x_2 \in \Sigma_v$. $\quad$ x_2 i.e. $\mu(v) = i$, $val(v) \geq 3$

Denote $G^{i0} = \bigcup_{d > 0} G_d^{i0}$ etc. $G_d^i = G_d^{i0} \cup G_d^{i1}$.

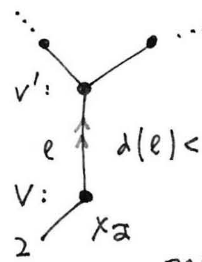
- $\Gamma \in G_d^{i*}$: clearly $ev_2^* \phi_i = 0$
- $\Gamma \in G_d^{i0}$: $\psi_2|_{\overline{M}_\Gamma}$ has trivial T-action $\Rightarrow$ $\psi_2^{\dim(v)+1} = 0 \in H_T^*(\overline{M}_\Gamma)$ $\qquad$ $\dim(v) = val(v) - 3$ $(\dim \overline{M}_{0, val(v)})$

  Γ has at most d edges, 2 tails, no loops $\Rightarrow$ $val(v) \leq d+2$

  So nilpotency of $\psi_2 \leq d$.
- $\Gamma \in G_d^{i1}$: let $e = \overline{vv'}$ & $j = \mu(v')$.

  if $d(e) < d$ let $\Gamma_j = \Gamma$ contr. e

  Then $\Gamma_j \in G^j_{d - d(e)}$, $|\mathrm{Aut}\,\Gamma| = |\mathrm{Aut}\,\Gamma_j|$.



one pt itself is a conn. comp.

$\underline{\text{lemma.}}$ Write $Z_i(e^t, t_1) = 1 + \sum_{d > 0} e^{dt} \, \zeta_{id}(\alpha, t_1)$

then ζ_{id} is a rat'l fcn, regular at $t_1 = \frac{\alpha_i - \alpha_j}{h}$ $\forall\, i \neq j, h \geq 1$.

pf: $\zeta_{id} \overset{*}{=} \sum_{\Gamma \in G_d^{i0}} \sum_{k=0}^{d-1} \frac{P_{\Gamma,k}(\alpha)}{t_1^{k+1}} + \sum_{\Gamma \in G_d^{i1}} \frac{P_\Gamma(\alpha)}{t_1 + \frac{\alpha_i - \alpha_j}{d(e)}}$ $\quad$ — Since $\psi_2|_{\overline{M}_\Gamma}$ is top-trivial of wt. $\frac{\alpha_j - \alpha_i}{d(e)}$.

with $P_{\Gamma,k}(\alpha), P_\Gamma(\alpha) \in \mathbb{Q}(\alpha)$.

※

if $l \le m$. denote $z_i(Q,\hbar) = Z_i(Q\hbar^{m+1-l}, \hbar)$

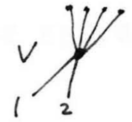
lemma. the contribution of G_d^{i0} to $z_i(Q,\hbar)$ is 0 if $l<m$,

$l=m$: $C_i(Q,\hbar) = -1 + \exp\left(-m!Q + \frac{(m\alpha_i)^m}{\prod_{j\neq i}(\alpha_i-\alpha_j)} Q\right)$.

pf: $l<m$, $d>0$ $\exists$ $\delta = (m+1-l)d - 1 > d$, the integrand

ψ_2^{δ} now restr to $\overline{M}_\Gamma = 0$ (nilp. condi).

$l=m$: Only one case $\int_{\overline{M}_\Gamma} \psi_2^{d-1} \frac{e(E_d')\, ev_2^*(\phi_i)}{e(N_\Gamma)}$ survives.

$\Gamma = v$ 

val$(v)=d+2$
d edges, each $d(e)=1$ this is done directly (HW.)

Hint: notice $d-1=(d+2)-3$: $\int_{\overline{M}_{0,n}} \psi_1^{\beta_1}\cdots\psi_n^{\beta_n} = \binom{n-3}{\beta_1,\dots,\beta_n}$ ※

if $\sum\beta_i = n-3$.

Now let $\Gamma\in G_d^{i1}$. If $d(e)=d$, get

contr. $Q^d\cdot c_i^j(d,\hbar)$: $c_i^j(d,\hbar) = \frac{1}{\frac{\alpha_i-\alpha_j}{\hbar}+d}\cdot \frac{\prod_{r=1}^{ld}\left(\frac{l\alpha_i}{(\alpha_j-\alpha_i)/d}+r\right)}{\prod_{k=0}^{m}\prod_{r=1}^{d} \left(\frac{\alpha_j-\alpha_k}{(\alpha_j-\alpha_i)/d}+r\right)}$

$(k,r)\neq(j,d)$

lemma. (key point #1)

If $d(e)<d$, then contraction $\Gamma\mapsto\Gamma_j$

$\Rightarrow$ $\mathrm{Cont}_\Gamma\, z_i(Q,\hbar) = Q^{d(e)}\cdot c_i^j(d(e),\hbar)\cdot \mathrm{Cont}_{\Gamma_j}\, z_j\left(Q, \frac{\alpha_j-\alpha_i}{d(e)}\right)$

pf. here flag (v',e) in $\Gamma \longrightarrow$ node of Σ

M_Γ has a line bundle quoti $\longleftrightarrow$ deformation of node

in $N_{\Gamma'}$ this deform disappears, but $\psi_2 \mapsto \psi_2 - \frac{\alpha_j-\alpha_i}{d(e)}$. ※

Prop (linear recursion & uniqueness):

(R): $z_i(Q,\hbar) = 1 + C_i(Q,\hbar) + \sum_{j\neq i,\, d>0} Q^d c_i^j(d,\hbar)\, z_j\left(Q,\frac{\alpha_j-\alpha_i}{d}\right)$

and this det. z_i uniquely.

- Now consider $Z_i^*(e^t,\hbar) = \sum_{d=0}^{\infty} e^{dt}\frac{\prod_{r=1}^{ld}(l\alpha_i+r\hbar)}{\prod_{j=0}^{m}\prod_{r=1}^{d}(\alpha_i-\alpha_j+r\hbar)}$

which is $\langle S^*(t,\hbar),\phi_i\rangle = e^{\frac{\alpha_i t}{\hbar}}\cdot l\alpha_i\cdot Z_i^*$; $S^* := \sum_{d\ge 0} e^{(\frac{H}{\hbar}+d)t}\frac{\prod_{r=0}^{ld}(lH+r\hbar)}{\prod_{j=0}^{m}\prod_{r=1}^{d}(H-\alpha_j+r\hbar)}$

For $l\le m$, $z_i^*(Q,\hbar) := Z_i^*(Q\hbar^{m+1-l},\hbar)$.

z_i^* $(l<m)$ or $e^{-m!Q}z_i^*$ $(l=m)$ satisfies (R), hence

$S=S^*$ or $S=e^{-m!e^t/\hbar}S^*$ done. ※