KONTSEVICH'S PROOF OF WITTEN CONJECTURE

YOU-CHENG CHOU

0.1. **nKdV-hierarchy.** We consider the operator

$$L = \partial_x^{n+1} + a_1(x)\partial_x^{n-1} + \cdots + a_n(x),$$

For any pair

$$(\alpha, p), \quad \alpha = 1, \ldots, n, \quad p = 0, 1, \ldots$$

we consider the following system of PDEs for functions $a_1(x, \underline{t}), \ldots, a_n(x, \underline{t})$:

$$\partial_{t^{\alpha,p}} L = [L, (L^{\frac{\alpha}{n+1}+p})_+],$$

where $(L^{\frac{\alpha}{n+1}+p})_+$ denotes the differential part of the pseudodifferential operator $L^{\frac{\alpha}{n+1}+p}$.

To construct pseudodifferential operator $L^{\frac{\alpha}{n+1}+p}$, it suffice to construct

$$L^{\frac{1}{n+1}} = \partial_x + l_1(x)\partial_x^{-1} + l_2(x)\partial_x^{-2} + \cdots,$$

where ∂^{-1} is the inverse operator to ∂ and its commutator with function $f(x)$ is as follows:

$$[\partial_x^{-1}, f(x)] = -f_x\partial_x^{-1} + f_{xx}\partial_x^{-2} - \ldots.$$

The last rule and equation $(L^{\frac{1}{n+1}})^{n+1} = L$ allow us to solve $l_i(x)$ recursively.

0.2. **Constructing Solutions to the KdV Hierarchy from the Sato Grassmanian.**

Definition 0.1. *A Sato space is an infinite dimensional vector subspace* $W \subset \mathbb{C}((z))$ *such that*

$$W = < f_1, f_2, \cdots >$$

for some $f_j(z) = z^{-j} + \alpha z^{-j+1} + \cdots = z^{-j}(1 + \circ(1))$.

Let $T_1, T_2, \ldots$ be an infinite sequence of formal variables, and denote by $M(z; T_1, T_2, \ldots)$ the function

$$M(z; T_1, T_2, \ldots) = e^{T_1 z^{-1} + T_2 z^{-2} + \cdots}.$$

For a Sato space $W = < f_1, f_2, \cdots >$, we define the τ-function associated it as the fraction

$$\tau_W(T_1, T_2, \ldots) = \frac{(\cdots \wedge Mf_3 \wedge Mf_2 \wedge Mf_1) \wedge z^0 \wedge z^1 \wedge z^2 \ldots}{\cdots \wedge z^{-3} \wedge z^{-2} \wedge z^{-1} \wedge z^0 \wedge z^1 \wedge z^2 \wedge \ldots}.$$

It depends on the space W itself, and not on the specific choice of the basis $f_1, f_2, \ldots$.

Proposition 0.2. *Let W be a Sato space such that $z^{-2}W \subset W$. Then we have*
(1) $\tau_W(T_1, T_2, \dots)$ does not depend on T_{2i} for $i > 0$;
(2) the second order differential operators

$$L = \partial_x^2 + 2\partial_{T_1}^2 \log\tau_W(-x + T_1, T_3, T_5, \dots)$$

satisfyies the KdV hierarchy

$$\partial_{T_{2k+1}} L = [L, (L^{\frac{2k+1}{2}})_+].$$

Proposition 0.3. *Let W be a Sato space generated by $f_i = z^{-i}(1 + \circ(1))$. Then for any $N \geq 0$,*

$$\frac{\det(f_i(z_j))}{\det z_j^{-i}} = \tau_W(T_1(z_*), T_2(z_*), \dots),$$

where $T_k(z_*) := \frac{1}{k}\sum_{i=1}^N z_i^k$.

0.3. **Witten's Conjecture.**
Let $\mathcal{M}_{g,n}$ ($\overline{\mathcal{M}}_{g,n}$) be the moduli space of smooth (nodal) genus g, n-pointed stable curves, $\mathcal{L}_i$ be line bundle on $\overline{\mathcal{M}}_{g,n}$ whose fiber at the moduli point $(C; x_1, \dots, c_n)$ is $T^*_{x_i}C$ and let $\psi_i := c_1(\mathcal{L}_i)$.

The intersection number is defined by

$$< \tau_{d_1} \cdots \tau_{d_n} > := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n},$$

where $g = \frac{\sum d_i + 3 - n}{3}$. We consider the formal power series of intersection number

$$F(t_0, t_1, \dots) = \sum_{n \geq 0} \sum_{d_1 \geq 0, \dots, d_n \geq 0} \frac{1}{n!} < \tau_{d_1} \cdots \tau_{d_n} > t_{d_1} \cdots t_{d_n}.$$

Theorem 0.4 (Witten's Conjecture)**.** *e^F is the τ-function for KdV with respect to variables $T_{2i+1} = \frac{t_i}{(2i+1)!!}$*

Remark 0.5. $F(t_0, t_1, \dots)$ is completely determined by $F(t_0, 0, 0, \dots) = \frac{1}{6}t_0^3$, KdV-hierarchy, string equation and dilaton equation.

0.4. **Combinatorial Model.** Let X be a compact Riemann surface and ρ be a meromorphic quadratic differential on X. Locally, $\rho = \phi(z)(dz)^2$, where $\phi(z)$ is a meromorphic function. In our case, we assume ϕ has only simple or double poles.

We define the horizontal line field as

$$\{v \in TX \mid \phi(z)(dz(v))^2 > 0\}.$$

Its integral curve is called horizontal trajectory.

Remark 0.6. For a generic quadratic differential, a generic horizontal trajectory is nonclosed.

Here we define a special kind of quadratic differential:

Definition 0.7. *A Jenkins-Strebel differential is a quadratic differential with only finite nonclosed horizontal trajectory.*

By a local analysis, we can see that if z_0 zero of order d of ρ, then there are $d+2$ horizontal trajectories issuing from z_0. If z_0 is a simple pole, then there is a unique horizontal trajectory issuing from z_0. Finally, if z_0 is a double pole with negative residue, then z_0 is surrounded by closed horizontal trajectories. We further list some properties of Jenkins-Strebel differential that we will need later.

Proposition 0.8. *Let X be a Riemann surface of finite type and $\rho(z) = \phi(z)dz^2$ be a Jenkin-Strebel differential on X. Then*

- *the connected component of $X \setminus \{$graph of nonclosed horizontal trajectory$\}$ is either open annulus or open disk;*
- *all closed horizontal trajectory in the same connected component have the same length. (We use the metric $dl^2 = |\phi(z)||dz|^2$.)*

Theorem 0.9. *(Strebel) For any $2n+1$-tuples $(X; x_1, \ldots, x_n; p_1, \ldots, p_n)$, where X is a Riemann surface of finite type, x_i are distinct points of X, $p_i > 0$, and $n > \chi(X)$, there exists a unique Jenkins-Strebel differential such that*

- *it has double pole at x_i and no other poles;*
- *connected components of $X \setminus \{$ graph of nonclosed horizontal trajectory $\}$ are open disks;*
- *the length of horizontal trajectory associated to x_i is p_i.*

We call the unique Jenkins-Strebel differential defined above the canonical Jenkins-Strebel differential.

Conversely, given an embedded graph (a graph in oriented topological surface X) with

- each valencies of vertex ≥ 3,
- face marked by $x_1, \ldots, x_n$
- fixed lengths of its edges,
- complement of embedded graph is a disjoint, union of open disks,

there exists unique complex structure such that its corresponding canonical Jenkins-Strebel differential determines the given embedded graph.

Now, we define $M_{g,n}^{comb} := \{$the moduli space of genus g connected embedded graphs with each valencies of vertex ≥ 3, n-marked faces, fixed lengths of its edges, and complement being a disjoint union of open disks $\}$.

Theorem 0.10. *$\mathcal{M}_{g,n} \times \mathbb{R}_+^n \cong \mathcal{M}_{g,n}^{comb}$ as real orbifolds.*

We can further generalize the above discussion to stable curve.

Definition 0.11. *For a stable curve C with given perimeter on its marked points, the canonical Jenkins-Strebel differential on C is a quadratic differential ρ such that:*

- $\rho \equiv 0$ *on the unmarked components;*
- ρ *is the canonical Jenkins-Strebel differential on the puntured marked components.*

$\overline{\mathcal{M}}_{g,n}^{comb} :=$ { the space of stable genus g embedded graphs with vertices of valencies ≥ 3 on smooth point, at most one valency on nodal points, n-marked faces, complement being disjoint union of open disks, and fixed length of edges on marked component and no graph on unmarked components.}

To determine the relation between $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}^{comb}$, we introduce the equivalence relation as follows: Let C be a stable curve with genus g and n marked points. We can canonically decompose C as the union of two curves $C = C^+ \cup C^0$, where C^+ is the union of all the components of C containing marked points, and C^0 is the union of those containing no marked points. Let $\xi_1, \ldots, \xi_u$ be the points that C^+ has in common with C^0. We say that $[(C; x_1, \ldots, x_n)]$ is equivalent to $[(C', x'_1, \ldots, x'_n)]$ if there is a family of nodal curves $\{C_s^0\}_{s\in S}$ over a connected base S, together with sections of smooth points $\tau_1, \ldots, \tau_u$, with the property that $(C; x_1, \ldots, x_n)$ (resp.,$(C', x'_1, \ldots, x'_n)$) can be obtained from C^+ and C_s^0 (resp.,$C_{s'}^0$) by identifying ξ_i with $\tau_i(s)$ (resp.,$\tau_i(s')$) for $i = 1, \ldots, u$.

It is easy to check that what we just defined is an equivalence relation. We let

$$Q : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}'_{g,n}$$

denote the projection via the equivalence relation. Now we can state the similar identifications for stable curves:

Theorem 0.12. $H : \overline{\mathcal{M}}'_{g,n} \times \mathbb{R}_+^n \to \overline{\mathcal{M}}_{g,n}^{comb}$ *is a homeomorphism.*

0.5. **Matrix Integral Model.** We recall some facts about matrix integral.

Let B be a $n \times n$ positive definite symmetric matrix. We consider the integral

$$c \int_{\mathbb{R}^n} e^{-\frac{1}{2}(Bx,x)} \prod_{i=1}^{n} dx_i,$$

where c is chosen such that the integral equals 1. With this normalization we have

$$< x_i x_j >:= c \int_{\mathbb{R}^n} x_i x_j e^{-\frac{1}{2}(Bx,x)} \prod_{i=1}^{n} dx_i = (B^{-1})_{ij}.$$

We can further generalized this computation.

Theorem 0.13 (Wick's formula)**.** *Let $f_1, \ldots, f_{2k}$ be linear functions of $x_1, \ldots, x_n$. Then*

$$< f_1 f_2 \cdots f_{2k} >= \sum_{\substack{p_1 < \cdots < p_k \\ q_1 < \cdots < q_k}} < f_{p_1} f_{q_1} > \cdots < f_{p_k} f_{q_k} > .$$

Let $\Lambda = (\Lambda_i)_{1\leq i\leq N}$ be a diagonal matrix with $\mathrm{Re}(\Lambda_i) > 0$ and $H = (h_{ij}) = (x_{ij} + \sqrt{-1}y_{ij})$ be a Hermitian matrix. We consider the following measure on the space of Hermitian matrices

$$d\mu_\Lambda(H) = C_{\Lambda,N} e^{-\frac{1}{2}\mathrm{tr} H^2\Lambda} \prod_{i=1}^{N} dx_{ii} \prod_{i<j} dx_{ij} dy_{ij},$$

where $C_{\Lambda,N}$ is chosen such that

$$\int_{\mathcal{H}_N} d\mu_\Lambda(H) = 1.$$

By direct computation, we have $C_{\Lambda,N} = (2\pi)^{-\frac{N^2}{2}} \prod_{i=1}^{N} \Lambda_i^{\frac{1}{2}} \prod_{i<j}(\Lambda_i + \Lambda_j)$.

In coordinated x_{ii}, x_{ij}, y_{ij}, we can write $\mathrm{tr}(H^2\Lambda) = (Bx, x)$, where

$$B = \begin{pmatrix} \Lambda_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \Lambda_N & & & & & & \\ & & & \Lambda_1+\Lambda_2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & \Lambda_{N-1}+\Lambda_N & & & \\ & & & & & & \Lambda_1+\Lambda_2 & & \\ & & & & & & & \ddots & \\ & & & & & & & & \Lambda_{N-1}+\Lambda_N \end{pmatrix}.$$

Hence we have

$$< x_{ii}^2 >_{\Lambda,N} := \int_{\mathcal{H}_N} x_{ii}^2 \, d\mu_\Lambda(H) = \frac{1}{\Lambda_1}, \quad < x_{ij}^2 >_{\Lambda,N} = < y_{ij}^2 >_{\Lambda,N} = \frac{1}{\Lambda_i + \Lambda_j},$$

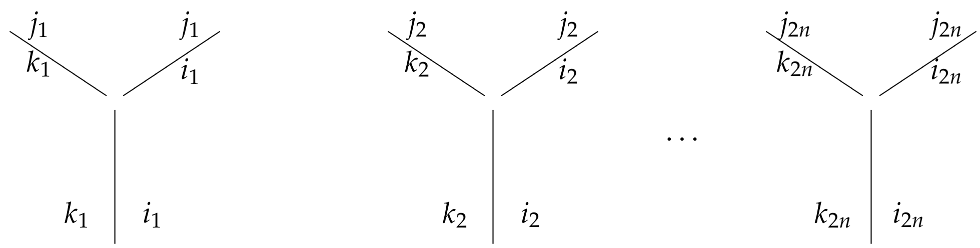
and similarly

$$< h_{ij}h_{ji} >_{\Lambda,N} = \frac{2}{\Lambda_i + \Lambda_j}, \; < h_{ij}h_{kl} >_{\Lambda,N} = 0 \text{ if } (i,j) \neq (l,k).$$

Now we compute

$$< e^{\frac{\sqrt{-1}}{6}tr(H^3)} >_{\Lambda,N} = < (1 - \frac{1}{2!}\frac{1}{6^2}(tr(H^3))^2 + \frac{1}{4!}\frac{1}{6^4}(tr(H^3))^4) - \cdots >_{\Lambda,N} .$$

By Wick's formula, the right hand side can be presented as the monomial of $< h_{i_n j_n} h_{i_m k_m} >$. Notice that each term can correspond to the gluing of 3-stars.



In this case an edge of the gluing corresponds to a pair $< h_{i_n j_n} h_{i_m k_m} >$. We define the weight of an gluing the product

$$\prod \frac{2}{\Lambda_i + \Lambda_j}$$

taken over all edges of the gluing.

Now we can rewrite the Kontsevich's model into graph sum:

$$< e^{\frac{\sqrt{-1}}{6} tr(H^3)} >_{\Lambda,N} \sim \sum_{G \in \mathcal{G}^{3,N}} \frac{(\frac{\sqrt{-1}}{2})^{|V(G)|}}{|\text{Aut}G|} \prod_{e \in E(G)} \frac{2}{\Lambda_e + \Lambda'_e},$$

where $\mathcal{G}^{3,N}$ is set of equivalent class with 3-valent graphs and with N possible colors $\Lambda_1, \cdots, \Lambda_N$ drawing on the face.

It can be observed by the following proposition:

Proposition 0.14.

$$< (trH)^{\alpha_1} \cdots (tr(H^k))^{\alpha_k} >_{\Lambda,N} = \alpha_1! \cdots \alpha_k! 2^{\alpha_2} \cdots k^{\alpha_k} \sum_{G \in \mathcal{G}^N} \frac{1}{|\text{Aut}G|} \prod_{e \in E(G)} \frac{2}{\Lambda_e + \Lambda'_e}.$$

0.6. **Proof of Witten conjecture.** The first step of the proof is to give a combinatorial formula for ψ_i. Let $\overline{\pi}_i : S^1(\mathcal{L}_i^{comb}) \to \overline{\mathcal{M}}_{g,n}^{comb}$. We want to find a closed 2-form w_i on $\overline{\mathcal{M}}_{g,n}^{comb}$ such that $\overline{\pi}_i^*(w_i) = d\phi$ and $\int_{S^1} \phi|_{\text{fiber}} = 1$.

Fix $p = (p_1, \dots, p_n) \in \mathbb{R}_+^n$, we have the commutative diagram:

$$\begin{array}{ccc} & \overline{\mathcal{M}}_{g,n} & \\ {\scriptstyle Q} \swarrow & & \searrow {\scriptstyle f} \\ \overline{\mathcal{M}}'_{g,n} & \xrightarrow[\sim]{h} & \overline{\mathcal{M}}_{g,n}^{comb}(p) \end{array}$$

By the way Q was constructed, the line bundle $\mathcal{L}_i$ restricts to a trivial line bundle on the fibers of Q and therefore drops to a well-defined line bundle $\mathcal{L}'_i$ on $\overline{\mathcal{M}}'_{g,n}$ with $Q^*(\mathcal{L}'_i) = \mathcal{L}_i$. Let $\mathcal{L}_i^{comb}$ be the pullback of $\mathcal{L}'_i$ via h^{-1}, so that

$$h^*(\mathcal{L}_i^{comb}) = \mathcal{L}'_i,\ f^*(\mathcal{L}_i^{comb}) = \mathcal{L}_i.$$

Now our goal becomes giving the combinatorial expression for its first Chern class.

Let $|a|/\Gamma_a$ be an orbisimplex of $\overline{\mathcal{M}}_{g,n}^{comb}(p)$, where a corresponds to an embedded graph $(G_a; x_1, \dots, x_n)$ whose i-th perimeter is equal to p_i and $\Gamma_a = \text{Aut}((G_a; x_1, \dots, x_n))$. The coordinates relative to the simplex $|a|$ are the lengths

$$\{l_e\}_{e \in E(G_a)}$$

of the edges of G_a. At each point x_i, we consider a cyclically ordered set of oriented edges of G_a

$$(e_1, \dots, e_v)$$

with possible repetitions. A repetition happens when the edge bounds the same component of G_a. We set

$$(w_i)_{|a|} = \sum_{1\le s<t\le v-1} d\Big(\frac{l_{e_s}}{p_i}\Big)\wedge d\Big(\frac{l_{e_t}}{p_i}\Big)$$

Lemma 0.15. *For each x_i and $p \in \mathbb{R}^n_+$,*

$$[w_i] = c_1(\mathcal{L}_i^{comb}) \in H^2(\overline{\mathcal{M}}^{comb}_{g,n}(p)).$$

In particular,

$$[f^*(w_i)] = c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}).$$

Now we can rewrite the intersection number

$$< \tau_{d_1},\dots,\tau_{d_n} >= \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n} = \int_{\overline{\mathcal{M}}^{comb}_{g,n}(p)} w_1^{d_1}\cdots w_n^{d_n}$$
$$= \int_{\mathcal{M}^{comb}_{g,n}(p)} w_1^{d_1}\cdots w_n^{d_n}$$

The last equality is true since the boundary is measure zero.

Let $\Omega = \sum_{i=1}^n p_i^2 w_i$.

$$\int_{\mathbb{R}^n_{\ge 0}} e^{-\sum\lambda_i p_i}\Big(\int_{\mathcal{M}^{comb}_{g,n}(p)} \frac{\Omega^d}{d!}\Big) dp_1\cdots dp_n$$
$$= \sum_{d_1+\cdots d_n=d} < \tau_{d_1}\cdots\tau_{d_n} > \prod_{i=1}^n \frac{2d_i!}{d_i!}\lambda_i^{-2(d_i+1)} = (1),$$

where Re$(\lambda_i)>0$ and $d = 3g-3+n$.

We use the combinatorial theorem due to Kontsevich.

Theorem 0.16. $\frac{\Omega^d}{d!} dp_1\wedge\cdots\wedge dp_n = 2^{2n+5g-5} dl_{e_1}\wedge\cdots\wedge dl_{e_{6g-6+3n}}$.

We have

$$(1) = \int_{\mathbb{R}^n_{\ge 0}} e^{-\sum\lambda_i p_i}\Big(\int_{\mathcal{M}^{comb}_{g,n}(p)} 2^{2n+5g-5} dl_{e_1}\wedge\cdots\wedge dl_{e_{6g-6+3n}}\Big)$$
$$= \sum_{G\in\mathcal{G}^{3,c}_{g,n}} \frac{2^{2n+5g-5}}{|\mathrm{Aut}G|}\int_{|a(G)|} e^{-\sum\lambda_i p_i} dl_{e_1}\wedge\cdots\wedge dl_{e_{6g-6+3n}},$$

where $\mathcal{G}^{3,c}_{g,n}$ is the isomorphism class of connected 3-valent embedded graph with genus g and n-marked points. We further do some change of variables.

$$\sum_{i=1}^n \lambda_i p_i = \sum_{e\in E(G)} (\lambda_e + \lambda'_e) l_e,$$

where λ_e and λ'_e are the perimeter of the two faces adjacent to the edge e.

Now we have the relation $(*)$

$$\sum_{\mathcal{G}^{3,c}_{g,n}} \frac{2^{-|V(G)|}}{|\mathrm{Aut}(G)|}\prod_{e\in E(G)}\frac{2}{\lambda_e+\lambda'_e} = \sum_{d_1+\cdots+d_n=d} < \tau_{d_1}\cdots\tau_{d_n} > \prod_{i=1}^n \frac{(2d_i-1)!!}{\lambda_i^{2d_i+1}}.$$

Theorem 0.17. *Let* $F(t_0, t_1, \cdots) = \sum_{n\geq 0} \sum_{d_1,\dots,d_n\geq 0} \frac{1}{n!} < \tau_{d_1} \cdots \tau_{d_n} > t_{d_1} \cdots t_{d_n}$. *Set* $\Lambda = \mathrm{diag}(\Lambda_1, \dots \Lambda_N)$ *with* $\mathrm{Re}(\Lambda_i) > 0$ *and* $t_i(\Lambda) = -(2i-1)!!\mathrm{Tr}(\Lambda^{-2i-1})$. *Then*

$$F(t_0(\Lambda), t_1(\Lambda), \dots) = \sum_{G\in\mathcal{G}^{3,c,N}} \frac{(\frac{\sqrt{-1}}{2})^{|V(G)|}}{|\mathrm{Aut}(G)|} \prod_{e\in E(G)} \frac{2}{\Lambda_e + \Lambda_e'},$$

where $\mathcal{G}^{3,c,N}$ *is the isomorphism class of connected* 3*-valent embedded graph with* N*-possible colors* $\Lambda_1, \cdots, \Lambda_N$ *drawing on the face.*

Corollary 0.18. $F(t_0(\Lambda), t_1(\Lambda), \cdots)$ *is the asymptotic expansion of* $\log < e^{\frac{\sqrt{-1}}{6}\mathrm{tr}H^3} >_{\Lambda,N}$ *as* $\Lambda^{-1} \to 0$.

Here we recall some properties of Airy function. We first study its asymptotic behaviour by stationary phase method.

$$\begin{aligned} a(y) &= \int_{-\infty}^{\infty} e^{\sqrt{-1}(x^3/3-xy)}dx \\ &\sim \int_{U(y^{1/2})} e^{\sqrt{-1}(x^3/3-xy)}dx + \int_{U(-y^{1/2})} e^{\sqrt{-1}(x^3/3-xy)}dx \\ &\sim \mathrm{const}\cdot \sum_{\pm y^{1/2}} y^{\frac{-3}{4}} e^{-\frac{2\sqrt{-1}}{3}y^{\frac{3}{2}}} f_1(y^{-\frac{1}{2}}), \end{aligned}$$

and similarly we have

$$\begin{aligned} a^{j-1}(y) &= \int_{-\infty}^{\infty} (-\sqrt{-1}x)^{j-1} e^{\sqrt{-1}(x^3/3-xy)}dx \\ &\sim \mathrm{const}\cdot \sum_{\pm y^{1/2}} y^{\frac{-3}{4}} e^{-\frac{2\sqrt{-1}}{3}y^{\frac{3}{2}}} f_j(y^{-\frac{1}{2}}), \end{aligned}$$

where $f_j(y) = y^{-j}(1 + \circ(1))$.

We have similar expression for matrix Airy function:

$$\begin{aligned} A(Y) &= \int_{\mathcal{H}_N} e^{\sqrt{-1}\mathrm{tr}(X^3/3-XY)}dX \\ &\sim \sum_{Y^{1/2}} \int_{U(Y^{\frac{1}{2}})} e^{\sqrt{-1}\mathrm{tr}(X^3/3-XY)}dX \\ &= \sum_{Y^{1/2}} \int_{U(0)} e^{\sqrt{-1}\mathrm{tr}((X+Y^{1/2})^3/3-(X+Y^{1/2})Y)}dX \\ &= \sum_{Y^{1/2}} e^{-\frac{2\sqrt{-1}}{3}\mathrm{tr}Y^{3/2}} \int_{U(0)} e^{\sqrt{-1}\mathrm{tr}(X^3/3-X^2Y^{1/2})}dX \\ &\sim \mathrm{const}\cdot \sum_{Y^{1/2}} e^{-\frac{2\sqrt{-1}}{3}\mathrm{tr}Y^{3/2}} Y_i^{-1/4} \prod_{i<j} (Y_i^{1/2} + Y_j^{1/2})^{-1/2} e^{F(\tilde{t_0}(Y^{1/2}),\cdots)} - (1), \end{aligned}$$

where $\tilde{t_i}(Y^{1/2}) = 2^{-(2i+1)/3}(2i-1)!!\mathrm{tr}(Y^{-i-1/2})$. We can express matrix Airy function in another form.

Lemma 0.19. *If Φ is a conjugacy invariant function on $\mathcal{H}_N$, then for any diagonal real matrix Y,*

$$\int_{\mathcal{H}_N} \Phi(X) e^{-\sqrt{-1}\mathrm{tr}XY} dX$$

$$= \frac{(-2\pi\sqrt{-1})^{N(N-1)/2}}{\det Y_i^{j-1}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi(D) e^{-\sqrt{-1}\mathrm{tr}DY} \det D_i^{j-1} dD_1 \cdots dD_N$$

Now we have

$$A(Y) = \frac{(-2\pi\sqrt{-1})^{N(N-1)/2}}{\det Y_i^{j-1}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_i e^{\sqrt{-1}\mathrm{tr}(D_i^3/3 - D_i Y_i)} \det D_i^{j-1} dD_1 \cdots dD_N$$

$$\sim \mathrm{const}\cdot \sum_{Y^{1/2}} e^{-\frac{2\sqrt{-1}}{3}\mathrm{tr}Y^{3/2}} \prod_{i=1}^{N} Y_i^{-\frac{3}{4}} \frac{\det f_j(Y_i^{-1/2})}{\det Y_i^{j-1}} - (2).$$

We can compare (1) and (2). Then we get

$$e^{F(t_0(\Lambda),\dots)} \sim \frac{\det(f_j(\Lambda_i))}{\det(\Lambda_i^{-j})}.$$

We conclude that $e^{F(t_0(\Lambda),\dots)}$ safisties KdV hierarchy with respect to variables:

$$T_{2i+1} = \frac{1}{2i+1}\mathrm{Tr}\Lambda^{-2i-1} = \frac{(-1)^{2i+1} t_i}{(2i+1)!!}.$$

Finally, notice that $T_i \to c^i T_i$ also satisfies KdV hierarchy. This proves Witten conjecture.

References

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