B04?01026 [illegible]

1. (Ex 1.1)

We know $(\mathcal{L}_E \eta)_{\alpha\beta} = \underbrace{E \eta_{\alpha\beta}}_{0\ (\because \eta_{\alpha\beta}:\ \text{constant})} + q^\rho_\alpha \eta_{\rho\beta} + q^\rho_\beta \eta_{\rho\alpha} = (d_F - d_1)\eta_{\alpha\beta}$.

Set $\begin{cases} Q = (q^\alpha_\beta) \\ G = (\eta_{\alpha\beta}) \end{cases}$. Then we have $Q^T G + G^T Q = (d_F - d_1) G$ — (1)

Also, since $\mathcal{L}_E \underset{(\overset{\parallel}{e_1})}{e} = -d_1 e$, we have $Qe_1 = d_1 e_1$

Now set $\underset{\acute{e}}{e_1}, e_2, \dots, e_n$ be the eigenvectors of Q with eigenvalues $\underbrace{d_1, \dots, d_n}_{\text{distinct}}$.

Then (1) $\Rightarrow$ "$\left[\left(\forall i,j,\ \langle e_i, e_j\rangle = e_i^T G e_j \neq 0\right) \Rightarrow \left(d_i + d_j = d_F - d_1\right)\right]$" ~ (2)

- In our case, $\eta_{11} = \langle e_1, e_1\rangle \neq 0$, so (2) implies $d_F = 3d_1$.
- Note that (2) implies that $[\langle e_i, e_j\rangle \neq 0 \Rightarrow \langle e_i, e_k\rangle = 0$ for $k \neq j]$
  since eigenvalues are distinct

So $\langle e_1, e_j\rangle = 0$ for $2 \le j \le n$.

- Also, we have $\langle e_j, e_j\rangle = 0$ for $2 \le j \le n$: otherwise $d_j + d_j = d_F - d_1 = 2d_1 \leadsto d_j = d_1$ ✗

Now since G is nondegenerate, we can inductively rearrange $e_1, \dots, e_n$ such that in the basis $e_1, \dots, e_n$,

$$\begin{cases} [G] = \left(\begin{array}{c|c} c & \\ \hline & \begin{matrix} 0 & & 1 \\ & ⋰ & \\ 1 & & 0 \end{matrix} \end{array}\right) = (\langle e_i, e_j\rangle)_{ij} \\ [Q] = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} \end{cases}$$ with)

$\therefore d_\alpha + d_{n-\alpha+2} = d_F - d_1 = 2d_1$, $\alpha = 2, \dots, n$

$\therefore F = \frac{c}{6}(t^1)^3 + \frac{1}{2} t^1 \sum_{\alpha=1}^{n-1} t^{\alpha+1} t^{\alpha-n+1} + f(t^2, \dots, t^n)$.

2. (Ex 1.2) (Another approach but a priori assume the existence of unity)

- Note that if we can find $e_1, \dots, e_n$ such that $e_i e_j = \delta_{ij} e_i$, $i,j = 1, \dots, n$, then the condition on the inner product holds trivially: Since for $i \neq j$,

$\langle e_i, e_j\rangle = \langle e_i, e_j \cdot e_j\rangle = \langle e_i e_j, e_j\rangle = \langle \delta_{ij} e_i, e_j\rangle = \delta_{ij}\langle e_i, e_j\rangle = 0$ for $i \neq j$.

- Now we use the facts in algebra:
  Given R : a finite dimensional algebra over a field k.
  Then R is semisimple (i.e. Jacobson radical of $R = 0$)
  $\Leftrightarrow R$ is a cartesian product of simple algebras
  $\Leftrightarrow$ (Wedderburn's theorem) $R = \prod_{i=1}^{n} M_{n_i}(D_i)$, i.e. R is a product of matrix algebras over D_i : division-k-algebras.

In our case, since the Frobenius algebra $\hat{R}$ is artinian and commutative,

so we have $\mathrm{Jac}(R) = \sqrt{\langle 0 \rangle} \underset{\substack{\uparrow \\ \text{assumption (no nilpotents)}}}{=} 0$

So by Wedderburn's theorem, we may write $R = \prod_{i=1}^{n} M_{n_i}(D_i)$

∙∙ $\underline{n_i = 1}$: for if not, then the matrix algebra $M_{n_i}(D_i)$ must have nilpotent elements

∙∙ $\underline{D_i \cong \mathbb{C}}$: Since $\mathbb{C}$ is algebraically closed and D_i are f.d. commutative associative division $\mathbb{C}$-algebra, we have D_i is a finite field extension over $\mathbb{C}$, i.e. $D_i \cong \mathbb{C}$.

So we have $R \underset{\substack{\uparrow \\ \text{(as ring)}}}{\cong} \mathbb{C}^n$, i.e. $e_i e_j = \delta_{ij} e_i$. #

3. (Ex 1.3) $\hat{V} := \nabla E - \frac{1}{2}(2-d)\,\mathrm{id}$.

Then $\langle \hat{V} x, y \rangle = \langle \nabla_x E, y \rangle - \frac{1}{2}(2-d)\langle x, y \rangle$

$= \langle \nabla_E x - \underbrace{[E, x]}_{\mathrm{Lie}_E x}, y \rangle - \frac{1}{2}(2-d)\langle x, y \rangle$

$= \left(\cancel{E\langle x, y \rangle}_{0} - \langle x, \nabla_E y \rangle \right) - \left(\cancel{\mathrm{Lie}_E \langle x, y \rangle}_{0} - \langle x, \mathrm{Lie}_E y \rangle - (2-d)\langle x, y \rangle \right) - \frac{1}{2}(2-d)\langle x, y \rangle$

$= - \langle x, \nabla_E y - \mathrm{Lie}_E y - \frac{1}{2}(2-d) y \rangle = - \langle x, \hat{V} y \rangle$. #

4. (Ex C1)

∙ Set $y(\tau) = a_0 + \sum_{n=1}^{\infty} a_n q^n$

Then
$$y'''(\tau) = \sum_{n=1}^{\infty} (2\pi i n)^3 a_n q^n$$

$$y''(\tau)\, y(\tau) = \left(\sum_{n=1}^{\infty} (2\pi i n)^2 a_n q^n \right) \left(a_0 + \sum_{n=1}^{\infty} a_n q^n \right)$$
$$= a_0 \left(\sum_{n=1}^{\infty} (2\pi i n)^2 a_n q^n \right) + \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} (2\pi i k)^2 a_k a_{n-k} q^n$$

$$(y'(\tau))^2 = \left(\sum_{n=1}^{\infty} (2\pi i n) a_n q^n \right)^2$$
$$= \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} (2\pi i)^2 k(n-k)\, a_k a_{n-k} q^n$$

Then since $y''' - 6yy'' + 9(y')^2 = 0$, we have

$$\begin{cases} (2\pi i)^3 a_1 - 6\cdot(2\pi i)^2 a_0 a_1 = 0 & \sim (*) \\ (2\pi i n)^3 a_n - 6(2\pi i n)^2 a_0 a_n - 6\sum_{k=1}^{n-1}(2\pi i k)^2 a_k a_{n-k} + 9\sum_{k=1}^{n-1}(2\pi i)^2 k(n-k)\, a_k a_{n-k} = 0 \\ (n\geq 2) \sim (**) \end{cases}$$

Now if $a_1 = 0$, then the recursive equation $(**)$ shows that $a_n = 0$ for $n \geq 2$, and hence y is a constant.

So we may assume $a_1 \neq 0$. Then $(*) \Rightarrow a_0 = \frac{\pi i}{3}$.

Also, modulo the ambiguity C4, $\tau \longmapsto \tau + \tau_0$, $a_n \longmapsto a_n e^{2\pi i \tau_0}$ $(n\geq 1)$, we may assume $a_1 = -8\pi i$.

Then by solving $(**)$, we know the nonconstant solution to C5 is unique (upto C4) to be

$$y(\tau) = \frac{\pi i}{3}(1 - 24q - 72q^2 - 96q^3 - 168q^4 - \cdots], \quad q = e^{2\pi i \tau}.$$

5. (Ex C6)

By (C.72d), we know the solution to $y''' - 6yy'' + 9(y')^2 = 0$ is $\frac{\pi i}{3} E_2(\tau)$, where

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\sigma(n) q^n.$$

(Remark: a direct way to show (C.72d) different from the text is to use the well-known Ramanujan-identities, which states that

$$\begin{cases} \frac{1}{2\pi i} E_2'(\tau) = \frac{1}{12}(E_2^2 - E_4) \\ \frac{1}{2\pi i} E_4'(\tau) = \frac{1}{3}(E_2E_4 - E_6) \\ \frac{1}{2\pi i} E_6'(\tau) = \frac{1}{2}(E_2E_6 - E_4^2). \end{cases}$$

Compute directly, we can get $\frac{\pi i}{3} E_2$ is a solution to the chazy equation.)

Now since $\frac{\pi i}{3} E_2$ is a solution, the recursive equation $(**)$ gives

$$(n^3 - n^2)\sigma(n) - 12\sum_{k=1}^{n-1}\left(3k(n-k) - 2k^2\right)\sigma(k)\sigma(n-k) = 0$$

$$\rightarrow \sigma(n) = \frac{12}{n^2(n-1)}\sum_{k=1}^{n-1} k(3n-5k)\,\sigma(k)\sigma(n-k).$$

6. (Ex 4.1)

$$V(u) := \frac{1}{2(n+1)}\left[\zeta_0^2 + \cdots + \zeta_n^2\right]\Big|_{\zeta_0+\cdots+\zeta_n=0}$$

$$= \frac{1}{2(n+1)}\left((\zeta_1+\cdots+\zeta_n)^2 + \zeta_1^2 + \cdots + \zeta_n^2\right)$$

$$\to \partial_i V(u) = \frac{1}{2(n+1)}\left(2(\zeta_1+\cdots+\zeta_n)(\partial_i\zeta_1+\cdots+\partial_i\zeta_n) + 2\zeta_1\partial_i\zeta_1 + \cdots + 2\zeta_n\partial_i\zeta_n\right) \sim (***)$$

Note that by (4.55), we have $\partial_i\zeta_a = \dfrac{-1}{(\zeta_a - q^i)\lambda'(q^i)}$

Also, since $\lambda(p) = (p+\zeta_1+\cdots+\zeta_n)\prod_{a=1}^{n}(p-\zeta_a)$, we have

$$\frac{\lambda'(p)}{\lambda(p)} = \frac{1}{p+(\zeta_1+\cdots+\zeta_n)} + \sum_{a=1}^{n}\frac{1}{p-\zeta_a} \sim (****)$$

$$\text{So } (***) = \frac{1}{n+1}\left((\zeta_1+\cdots+\zeta_n)\left(\sum_{a=1}^{n}\frac{-1}{(\zeta_a-q^i)\lambda'(q^i)}\right) + \sum_{a=1}^{n}\frac{-\zeta_a}{(\zeta_a-q^i)\lambda'(q^i)}\right)$$

$$\underset{\uparrow\,(****)}{=} \frac{1}{n+1}\cdot\frac{-1}{\lambda'(q^i)}\left[\underbrace{(\zeta_1+\cdots+\zeta_n)\Bigg(\underbrace{-\frac{\lambda'(q^i)}{\lambda(q^i)}}_{=0} + \frac{1}{q^i+(\zeta_1+\cdots+\zeta_n)}\Bigg)}_{\downarrow\; 1-\left(\frac{q^i}{\zeta_1+\cdots+\zeta_n+q^i}\right)} + \sum_{a=1}^{n}\left(1+\frac{q^i}{(\zeta_a-q^i)}\right)\right]$$

$$= \frac{1}{n+1}\cdot\frac{-1}{\lambda'(q^i)}\Bigg((n+1) + \underbrace{\sum_{a=1}^{n}\frac{q^i}{\zeta_a-q^i} - \frac{q^i}{\zeta_1+\cdots+\zeta_n+q^i}}_{=\,-q^i\cdot\frac{\lambda'(q^i)}{\lambda(q^i)}\,=\,0}\Bigg) = \frac{-1}{\lambda'(q^i)} = \eta_{ii}(u)$$

#

7. (Ex 4.2)

We have $p = p(k) = k + \frac{1}{n+1}\left(\frac{t^n}{k} + \frac{t^{n-1}}{k^2} + \cdots + \frac{t^1}{k^n}\right) + O\left(\frac{1}{k^{n+1}}\right)$

Also, $\lambda(p(k)) = \lambda^{\frac{1}{n+1}}$.

$$\text{So } \operatorname*{res}_{p=\infty}\left(\lambda(p)^{\frac{n-\alpha+1}{n+1}}\,dp\right) = \operatorname*{res}_{k=\infty}\left(k^{n-\alpha+1}\,p'(k)\,dk\right)$$

$$= \operatorname*{res}_{k=\infty}\left(k^{n-\alpha+1}\left[1+\frac{1}{n+1}\left(\sum_{\alpha=1}^{n} -(n-\alpha+1)\frac{t^\alpha}{k^{n-\alpha+2}} + O\left(\frac{1}{k^{n+2}}\right)\right)\right]\right) = \frac{-(n-\alpha+1)}{n+1}t^\alpha$$

#

8. (Ex E1 (2))

By computation, we have

- $$\frac{d\Omega_i}{dt_3} = \frac{2w_i'(w_j-w_i)(w_i-w_k) - w_i\left[(w_j'-w_i')(w_i-w_k) + (w_j-w_i)(w_i'-w_k')\right]}{4\left(\sqrt{(w_j-w_i)(w_i-w_k)}\right)^3}$$

$$= \frac{\cancel{(w_j-w_i)(w_i-w_k)}\left[2w_i' + 2w_iw_k + 2w_iw_j\right]}{4\left(\sqrt{(w_j-w_i)(w_i-w_k)}\right)^3}$$

(by Ex C3) $w_j'-w_i' = 2w_k(w_i-w_j)$

$$= \frac{2w_jw_k}{4\sqrt{(w_j-w_i)(w_i-w_k)}}$$

(by Ex C3)

- $$\frac{ds}{dt_3} = \frac{(w_3'-w_1')(w_2-w_1) - (w_2'-w_1')(w_3-w_1)}{(w_2-w_1)^2}$$

$$= \frac{2(w_3^2 + w_1w_2 - w_3w_1 - w_3w_2)}{(w_2-w_1)} = \frac{2(w_3-w_1)(w_3-w_2)}{(w_2-w_1)}$$

by (Ex C3) $w_j'-w_i' = 2w_k(w_i-w_j)$

- $$\frac{d\Omega_i}{ds} = \frac{d\Omega_i}{dt_3}\cdot\left(\frac{ds}{dt_3}\right)^{-1} = \frac{w_jw_k}{2\sqrt{(w_j-w_i)(w_i-w_k)}}\cdot\frac{w_2-w_1}{2(w_3-w_1)(w_3-w_2)}$$

$$\Omega_j\Omega_k = \frac{w_jw_k}{4\sqrt{(w_i-w_k)(w_k-w_j)}\sqrt{(w_i-w_j)(w_j-w_k)}}$$

$$\to \frac{\frac{d\Omega_i}{ds}}{\Omega_j\Omega_k} = \begin{cases} i=1: & \dfrac{(w_2-w_1)}{(w_3-w_1)(w_3-w_2)}\cdot\left(\dfrac{\sqrt{(w_3-w_2)(w_2-w_1)}\sqrt{(w_1-w_3)(w_3-w_2)}}{\sqrt{(w_2-w_1)(w_1-w_3)}}\right)^{\text{need } w_3-w_2} \\ i=2: & \dfrac{(w_2-w_1)}{(w_3-w_1)(w_3-w_2)}\cdot\left(\dfrac{\sqrt{(w_2-w_1)(w_1-w_3)}\sqrt{(w_1-w_3)(w_3-w_2)}}{\sqrt{(w_3-w_2)(w_2-w_1)}}\right)^{\text{need } w_1-w_3} \\ i=3: & \dfrac{(w_2-w_1)}{(w_3-w_1)(w_3-w_2)}\left(\dfrac{\sqrt{(w_2-w_1)(w_1-w_3)}\sqrt{(w_3-w_2)(w_2-w_1)}}{\sqrt{(w_1-w_3)(w_3-w_2)}}\right)^{\text{need } w_2-w_1} \end{cases}$$

So if we can select a brach of $\sqrt{\ }$ such that the three "need" holds, then we are done.

9、(HW on I)

Two ODE with rational poles are locally gauge invariant iff their Sph are the same.

Pf) One direction is clear, so it suffices to prove ($\Leftarrow$)

Now let $\begin{cases} \frac{d\Psi}{d\lambda} = A(\lambda)\Psi & \sim ① \\ \frac{d\Phi}{d\lambda} = B(\lambda)\Phi & \sim ② \end{cases}$ $\lambda_0 \in \mathbb{CP}^1$ is an irregular singularity of rank r, and

$$Sph_{\lambda_0}(A) = Sph_{\lambda_0}(B) = \{\Lambda_{-r}, \Lambda_{-r+1}, \cdots, \Lambda_0, \overbrace{S_1, \cdots, S_{2r}}^{\text{Stokes matrices}}\}$$

We denote $\begin{cases} \{\Psi_n(\lambda)\}_{n=1,\cdots,2r+1} \\ \{\Phi_n(\lambda)\}_{n=1,\cdots,2r+1} \end{cases}$ the canonical solutions of ① and ② in the same Stokes sector Ω_n.

· Set $G(\lambda) = \Phi_1(\lambda)\Psi_1(\lambda)^{-1}$, $\lambda \in \Omega_1$. Since $\Psi_1, \Phi_1 \in GL(n, \mathcal{O}(\Omega_1))$, we know $G(\lambda)$ is holo. invertible in Ω_1.

· Also, for $\lambda \in \Omega_n$, the analytic continuation of G to Ω_n is given by

$$G(\lambda) = \left(\Phi_n(\lambda) S_{n-1}^{-1} S_{n-2}^{-1} \cdots S_1^{-1}\right)\left(\Psi_n(\lambda) S_{n-1}^{-1} \cdots S_1^{-1}\right)^{-1} = \Phi_n(\lambda)\Psi_n(\lambda)^{-1}$$

· Also, for $\lambda \in \Omega_{2r+1}$, $G(\lambda) = \Phi_{2r+1}(\lambda)\Psi_{2r+1}(\lambda)^{-1} = \left(\Phi_1(\lambda) e^{2\pi i \Lambda_0}\right)\left(\Psi_1(\lambda) e^{2\pi i \Lambda_0}\right)^{-1}$

$$= \Phi_1(\lambda)\Psi_1(\lambda)^{-1}.$$

This shows that $G(\lambda)$ admits an analytic continuation to the small puncture disk $B(\lambda_0, r) \setminus \{\lambda_0\}$.

· Also using the asymptotic of the solution Φ, Ψ as $\lambda \to \lambda_0$, we get

$$G(\lambda) = \Phi_n(\lambda)\Psi_n(\lambda)^{-1} \sim (\underset{\substack{\downarrow \\ \text{constant matrix}}}{P}\ \underset{\substack{\downarrow \\ \text{formal powers series with constant term } I}}{\hat{\Phi}(\lambda)}\ e^{\Lambda})(Q\hat{\Psi}(\lambda)e^{\Lambda})^{-1} = O(1) \text{ as } \begin{matrix}\lambda \to \lambda_0 \\ \lambda \in \Omega_n\end{matrix}$$

$$\Lambda = \sum_{k=1}^{r} \frac{\Lambda_{-k}}{-k}(x-\lambda_0)^{-k} + \Lambda_0 \ln(x-\lambda_0)$$

So λ_0 is a removable singularity of G.

Hence Φ and Ψ are locally gauge invariant by G.

Exercise 1.(b)
Note

$$\begin{aligned}(X\circ L_Y c(Z,W)+Y\circ L_X c(Z,W)-L_{X\circ Y}c(Z,W))^i &= c^i_{jk}X^jZ^mW^n(\underline{Y^l(\partial_l c^k_{mn})}-(\partial_l Y^k)c^l_{mn}+(\partial_m Y^l)c^k_{ln}+(\partial_n Y^l)c^k_{ml})\\ &\quad+c^i_{pq}Y^pZ^sW^t(\underline{X^r(\partial_r c^q_{st})}-(\partial_r X^q)c^r_{st}+(\partial_s X^r)c^q_{rt}+(\partial_t X^r)c^q_{sr})\\ &\quad\underline{-Z^mW^n(c^a_{bc}X^bY^c\partial_a c^i_{mn}-\partial_a(c^i_{bc}X^bY^c)c^a_{mn}+\partial_m(c^a_{bc}X^bY^c)c^i_{an}+\partial_n(c^a_{bc}X^bY^c)c^i_{ma})}.\end{aligned}$$

Then by seeing the terms underlined, we have

$L_{X\circ Y}c = X\circ L_Y c + L_X c\circ Y$ for all X,Y $\Leftrightarrow c^i_{jk}\partial_l c^k_{mn}+c^i_{lk}\partial_j c^k_{mn}-c^k_{jl}\partial_k c^i_{mn}+c^k_{mn}\partial_k c^i_{jl}-c^i_{kn}\partial_m c^k_{jl}-c^i_{mk}\partial_n c^k_{jl}=0$ and $c^a_{bc}c^d_{ef}=c^a_{bf}c^d_{ec}$.

Textbook C.5
We have

$$p(z) = \frac{1}{z^2} + \frac{1}{20}g_2z^2 + \frac{1}{28}g_3z^4 + \frac{1}{1200}g_2^2z^6 + O(z^7) \text{ and } \zeta(z) = \frac{1}{z} - \frac{1}{60}g_2z^3 - \frac{1}{140}g_3z^5 - \frac{1}{8400}g_2^2z^7 + o(z^8).$$

Then by (C.63), which can be checked by seeing their poles, we have

$$\begin{aligned} \eta\frac{\partial p}{\partial w} + \eta'\frac{\partial p}{\partial w'} + \zeta\frac{\partial p}{\partial z} &= -2p^2 + \frac{1}{3}g_2 \\ &= \frac{-3}{10}g_3z^2 - \frac{1}{84}g_2^2z^4 + O(z^5) \end{aligned}$$

while the left hand side is

$$\eta(\frac{1}{20}\frac{\partial g_2}{\partial w}z^2 + \frac{1}{28}\frac{\partial g_3}{\partial w} + O(z^5)) + \eta'(\frac{1}{20}\frac{\partial g_2}{\partial w'}z^2 + \frac{1}{28}\frac{\partial g_3}{\partial w'} + O(z^5)).$$

Then compare their coefficients we have

$$\eta\frac{\partial g_2}{\partial w} + \eta'\frac{\partial g_2}{\partial w'} = \frac{-3}{10}g_3 \times 20 = -6g_3$$

and

$$\eta\frac{\partial g_3}{\partial w} + \eta'\frac{\partial g_3}{\partial w'} = \frac{-1}{84}g_2^2 \times 28 = \frac{-1}{3}g_2^2.$$

Textbook 3.5
By the Proposition 3.5, where the case for t_1 was done in class, we have

$$\mu_\alpha \psi_{i\alpha} = \sum_k V_{ik}\psi_{k\alpha} \text{ and } \eta_{\alpha\beta} = \sum_i \psi_{i\alpha}\psi_{i\beta}.$$

Then the original problem

$$V_{ij} = \sum_{\alpha,\beta} \eta^{\alpha\beta}\mu_\alpha \psi_{i\alpha}\psi_{j\beta}$$

is equivalent to

$$V_{ij} = \sum_{\alpha,\beta,k} \eta^{\alpha\beta} V_{ik}\psi_{k\alpha}\psi_{j\beta}.$$

Thus it suffices to show

$$\sum_{\alpha,\beta} \eta^{\alpha\beta}\psi_{k\alpha}\psi_{j\beta} = \delta_{kj}.$$

Write $E := (\eta_{ij})_{i,j}$ and $P := (\psi_{ij})_{i,j}$. Then

$$\sum_{\alpha,\beta} \eta^{\alpha\beta}\psi_{k\alpha}\psi_{j\beta} = \delta_{kj} \Leftrightarrow PE^{-1}P^t = I \Leftrightarrow E = P^t P \Leftrightarrow \eta_{\alpha\beta} = \sum_i \psi_{i\alpha}\psi_{i\beta}$$

which is given above.

Textbook 4.1

$$\begin{aligned}
\frac{\partial V}{\partial u^i} &= \sum_a \frac{\partial V}{\partial \xi_a} \frac{\partial \xi_a}{\partial u^i} \\
&= \sum_a \frac{-1}{2(n+1)} (2(\xi_1 + \cdots + \xi_n) + 2\xi_a) \cdot \frac{-1}{(\xi_a - q^i)\lambda''(q^i)} \\
&= \frac{1}{(n+1)\lambda''(q^i)} \sum_a \frac{\xi_a - \xi_0}{\xi_a - q^i} \\
&= \frac{1}{(n+1)\lambda''(q^i)} \cdot (-\mathrm{res}_{p=\infty} \frac{(p-\xi_0)\lambda'(p)}{(p-q^i)\lambda(p)}) \\
&= \frac{1}{(n+1)\lambda''(q^i)} \cdot (n+1) \\
&= \frac{1}{\lambda''(q^i)} \\
&= -\eta_{ii}.
\end{aligned}$$

Hence it should be that

$$\frac{\partial V}{\partial u^i} = -\eta_{ii}.$$

Exercise B1:

Recall that under type I Transf. $\Rightarrow \partial_\alpha = \partial_\kappa \cdot \hat{\partial}_\alpha$

So, $\langle \hat{\partial}_\alpha, \hat{\partial}_\beta \rangle_\kappa = \langle \partial_\kappa \cdot \partial_\kappa, \hat{\partial}_\alpha \cdot \hat{\partial}_\beta \rangle = \langle \partial_\kappa \cdot \hat{\partial}_\alpha, \partial_\kappa \cdot \hat{\partial}_\beta \rangle = \langle \partial_\alpha, \partial_\beta \rangle = \eta_{\alpha\beta}$

$\Rightarrow \hat{t}^\alpha$ is flat coord. w.r.t. $\langle , \rangle_\kappa$.

$\langle \hat{\partial}_\alpha \cdot \hat{\partial}_\beta, \hat{\partial}_\gamma \rangle_\kappa = \langle \partial_\kappa \cdot \partial_\kappa, (\hat{\partial}_\alpha \cdot \hat{\partial}_\beta) \cdot \hat{\partial}_\gamma \rangle = \langle \partial_\kappa \cdot \hat{\partial}_\gamma, \partial_\kappa \cdot (\hat{\partial}_\alpha \cdot \hat{\partial}_\beta) \rangle$

$= \langle \partial_\gamma, \partial_\kappa \cdot (\hat{\partial}_\alpha \cdot \hat{\partial}_\beta) \rangle \qquad \hat{\partial}_\alpha \cdot \hat{\partial}_\beta = \hat{C}_{\alpha\beta}^{\varepsilon} \hat{\partial}_\varepsilon \qquad \partial_\kappa \cdot (\hat{\partial}_\alpha \cdot \hat{\partial}_\beta) = \partial_\kappa \cdot (\hat{C}_{\alpha\beta}^{\varepsilon} \hat{\partial}_\varepsilon)$

$= \hat{C}_{\alpha\beta}^{\varepsilon} \cdot (\partial_\kappa \cdot \hat{\partial}_\varepsilon) = \hat{C}_{\alpha\beta}^{\varepsilon} \partial_\varepsilon$

$\langle \hat{\partial}_\alpha \cdot \hat{\partial}_\beta, \hat{\partial}_\gamma \rangle_\kappa = \hat{C}_{\alpha\beta}^{\varepsilon} \langle \partial_\gamma, \partial_\varepsilon \rangle = \hat{C}_{\alpha\beta}^{\varepsilon} \eta_{\varepsilon\gamma} = \hat{C}_{\alpha\beta}^{\varepsilon} \hat{\eta}_{\varepsilon\gamma} = \hat{C}_{\alpha\beta\gamma}$

$= \hat{\partial}_\alpha \hat{\partial}_\beta \hat{\partial}_\gamma \hat{F}(\hat{t}) \quad \square$

Exercise B.2:

Recall type II transf.: $\hat{t}^1 = \frac{1}{2} \frac{t_\sigma t^\sigma}{t^n} \quad \hat{t}^\alpha = \frac{t^\alpha}{t^n}, \ (\alpha \neq 1, n) \quad \hat{t}^n = \frac{-1}{t^n}$

$\hat{F}(\hat{t}) = (\hat{t}^n)^2 F + \frac{1}{2} \hat{t}^1 \hat{t}_\sigma \hat{t}^\sigma$

$\hat{\eta}_{\alpha\beta} = \eta_{\alpha\beta}$

Now, for $n=2$, $(\eta_{\alpha\beta}) = (\hat{\eta}_{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{t}^1 = \frac{1}{2} \frac{t_\sigma t^\sigma}{t^2} = \frac{\eta_{12} t^2 t^1 + \eta_{21} t^1 t^2}{2t^2} = t^1$

$\hat{t}^2 = \frac{-1}{t^2} \quad \Rightarrow \quad t^2 = \frac{-1}{\hat{t}^2}$

1.24 b: $F(t^1, t^2) = \frac{1}{2}(t^1)^2 t^2 + (t^2)^2 \log(t^2)$, $d = -1$

Then $F(t^1, t^2) = G(\hat{t}^1, \hat{t}^2) = \frac{-1}{2} (\hat{t}^1)^2 \left(\frac{1}{\hat{t}^2}\right) + \frac{1}{(\hat{t}^2)^2} \log\left(\frac{-1}{\hat{t}^2}\right) = \frac{-(\hat{t}^1)^2}{2\hat{t}^2} - \frac{1}{(\hat{t}^2)^2} \log(-\hat{t}^2)$

$\hat{F}(\hat{t}) = (\hat{t}^2)^2 \left(\frac{-1}{2} (\hat{t}^1)^2 \left(\frac{1}{\hat{t}^2}\right) - \frac{1}{(\hat{t}^2)^2} \log(-\hat{t}^2) \right) + \frac{1}{2} \hat{t}^1 (2 \hat{t}^1 \hat{t}^2)$

$= \frac{1}{2} (\hat{t}^1)^2 \hat{t}^2 - \log(-\hat{t}^2)$

$\hat{d} = 2 - d = 3$

Exercise C12:

Chazy eqn: $\gamma''' = 6\gamma\gamma'' - 9\gamma'^2$

The cubic eqn: $w^3 + \frac{3}{2}\gamma(\tau)w^2 + \frac{3}{4}\gamma'(\tau)w + \frac{1}{4}\gamma''(\tau) = 0$ — $(*)$

$w_1(\tau), w_2(\tau), w_3(\tau)$: roots of $(*)$

Then $\begin{cases} w_1 + w_2 + w_3 = \frac{-3}{2}\gamma \\ w_1w_2 + w_2w_3 + w_1w_3 = \frac{3}{4}\gamma' \\ w_1w_2w_3 = \frac{-1}{4}\gamma'' \end{cases}$ $\Rightarrow$ $\begin{aligned} \gamma &= \frac{-2}{3}(w_1+w_2+w_3) \\ \gamma' &= \frac{4}{3}(w_1w_2+w_2w_3+w_3w_1) \\ \gamma'' &= -4w_1w_2w_3 \end{aligned}$

$$\begin{cases} \dot w_1 + \dot w_2 + \dot w_3 = \frac{-3}{2}\gamma' = -2(w_1w_2 + w_2w_3 + w_1w_3) \\ \dot w_1 w_2 + w_1 \dot w_2 + \dot w_2 w_3 + w_2\dot w_3 + \dot w_1 w_3 + w_1 \dot w_3 = \frac{3}{4}\gamma'' = -3w_1w_2w_3 \\ \qquad \overset{\|}{(w_2+w_3)\dot w_1 + (w_1+w_3)\dot w_2 + (w_1+w_2)\dot w_3} \\ \dot w_1 w_2 w_3 + w_1\dot w_2 w_3 + w_1w_2\dot w_3 = \frac{-1}{4}\gamma''' = -\frac{3}{2}\gamma\gamma'' + \frac{9}{4}\gamma'^2 = -4(w_1+w_2+w_3)w_1w_2w_3 \\ \qquad + 4(w_1w_2+w_2w_3+w_3w_1)^2 \end{cases}$$

$$= w_1^2w_2^2 + w_2^2w_3^2 + w_3^2w_1^2 + 2w_1w_2^2w_3 + 2w_1w_2w_3^2 + 2w_1^2w_2w_3 - 4w_1^2w_2w_3 - 4w_1w_2^2w_3 - 4w_1w_2w_3^2$$

$$= w_1^2w_2^2 + w_2^2w_3^2 + w_1^2w_3^2 - 2w_1w_2^2w_3 - 2w_1w_2w_3^2 - 2w_1^2w_2w_3$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ w_2+w_3 & w_1+w_3 & w_1+w_2 \\ w_2w_3 & w_1w_3 & w_1w_2 \end{pmatrix} \begin{pmatrix} \dot w_1 \\ \dot w_2 \\ \dot w_3 \end{pmatrix} = \begin{pmatrix} -(w_1w_2+w_2w_3+w_1w_3) \\ -6w_1w_2w_3 \\ (w_1w_2+w_2w_3+w_3w_1)^2 - 4(w_1+w_2+w_3)w_1w_2w_3 \end{pmatrix}$$

We can verify that:

$$\begin{cases} \dot w_1 = -w_1(w_2+w_3) + w_2w_3 \\ \dot w_2 = -w_2(w_1+w_3) + w_1w_3 \\ \dot w_3 = -w_3(w_1+w_2) + w_1w_2 \end{cases}$$ is sol'n of above linear eqn □

Ex 3.1:

(a)

$\partial_\alpha \xi_\beta = z\, C^\gamma_{\alpha\beta}(t)\, \xi_\gamma$ — (*)

Show that any sol'n of the system must be of the form $\xi_\alpha = \partial_\alpha \tilde{t}$:

pf: $\partial_\alpha \xi_\beta = \partial_\beta \xi_\alpha = z\, C^\gamma_{\alpha\beta}(t)\, \xi_\gamma \Rightarrow \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha = 0$

Therefore, consider $\xi := \xi_\alpha dt^\alpha$. Then $d\xi = \partial_\beta \xi_\alpha dt^\beta \wedge dt^\alpha = 0$

Locally, ξ admits a primitive $\tilde{t}$ s.t. $d\tilde{t} = \xi \Rightarrow \partial_\alpha \tilde{t}\, dt^\alpha = \xi_\alpha dt^\alpha \Rightarrow \xi_\alpha = \partial_\alpha \tilde{t}$

(b) $\xi^1_\alpha, \ldots\ldots, \xi^n_\alpha$: fundamental system of sol'n of the sysytem (*) for a given z

The corresponding function $\tilde{t}^1, \ldots, \tilde{t}^n$ are flat coord. for the deformed connection $\tilde{\nabla}(z)$.

pf: By (a), we know that $\xi^i = \begin{pmatrix} \xi^i_1 \\ \vdots \\ \xi^i_n \end{pmatrix} = \mathrm{grad}(\tilde{t}^i) = \begin{pmatrix} \partial_1 \tilde{t}^i \\ \vdots \\ \partial_n \tilde{t}^i \end{pmatrix}$ is linearly indep.

$\Rightarrow$ The Jacobian matrix $J(t, \tilde{t}; z) = \left(\frac{\partial \tilde{t}^i}{\partial t_j}\right) = (\xi^i_j)$ is non-singular $\Rightarrow$ $(\tilde{t}^1, \ldots, \tilde{t}^n)$ indeed forms a system of local coord.

Denote $\partial_\alpha := \frac{\partial}{\partial t^\alpha}$ $\tilde{\partial}_\alpha := \frac{\partial}{\partial \tilde{t}^\alpha}$ Then $\partial_\alpha = \frac{\partial \tilde{t}^\beta}{\partial t^\alpha} \cdot \tilde{\partial}_\beta = \xi^\beta_\alpha \tilde{\partial}_\beta$

$$\tilde{\nabla}_{\partial_i} \partial_j = \tilde{\nabla}_{\partial_i}(\xi^k_j \tilde{\partial}_k) = \partial_i \xi^k_j \tilde{\partial}_k + \xi^k_j \tilde{\nabla}_{\partial_i} \tilde{\partial}_k = z C^l_{ij} \xi^k_l \tilde{\partial}_k + \xi^k_j \tilde{\nabla}_{\partial_i} \tilde{\partial}_k$$

$$\| \qquad\qquad = z C^l_{ij} \xi^k_l \tilde{\partial}_k + \xi^k_j \xi^s_i \tilde{\nabla}_{\tilde{\partial}_s} \tilde{\partial}_k$$

$$\nabla_{\partial_i} \partial_j + z \cdot \partial_i \cdot \partial_j = z \cdot C^l_{ij} \partial_l = z \cdot C^l_{ij} \xi^k_l \tilde{\partial}_k \qquad = z \cdot C^l_{ij} \xi^k_l \tilde{\partial}_k + \xi^r_j \xi^s_i \tilde{\Gamma}^k_{sr} \tilde{\partial}_k$$

$\Rightarrow \xi^r_j \xi^s_i \tilde{\Gamma}^k_{sr} \tilde{\partial}_k = 0 \Rightarrow \forall i, j, k,\ \xi^r_j \xi^s_i \tilde{\Gamma}^k_{sr} = 0 \quad \because (\xi^j_i)$ is invertible

$\therefore \tilde{\Gamma}^k_{sr} = 0. \Rightarrow (\tilde{t}^1, \ldots, \tilde{t}^n)$ is the flat coord. in Dubrovin connection $\tilde{\nabla}$ 12

Ex 3.2: $n = 3$, $d = 1$. $F(t) = \frac{1}{2} t_1^2 t_3 + \frac{1}{2} t_1 t_2^2 \underbrace{- \frac{t_2^4}{16} \gamma(t_3)}_{+ f(t_2, t_3)}$ $\quad e_i \cdot e_j = C^k_{ij} e_k$

$C^k_{ij} = \eta^{kl} C_{lij}$, $\eta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$C^1_{11} = 1 \quad C^2_{11} = C^3_{11} = 0$

$C^1_{12} = C^3_{12} = 0$, $C^2_{12} = 1$. $C^1_{13} = C^2_{13} = 0$ $C^3_{13} = 1$. $C^1_{22} = C_{223} = -\frac{3}{4} t_2^2 \gamma'$

$C^2_{22} = C_{222} = -\frac{3}{2} t_2 \gamma$ $C^3_{22} = C_{122} = 1$ $C^1_{33} = C_{333} = \frac{-t_2^4}{16} \gamma'''$

$C^1_{32} = C_{233} = -\frac{t_2^3}{4} \gamma''$ $C^2_{32} = C_{322} = \frac{-3}{4} t_2^2 \gamma'$ $C^2_{33} = C_{332} = \frac{-t_2^3}{4} \gamma''$ $C^3_{33} = C_{133} = 0$

$C^3_{32} = C_{132} = 0$

(Recall that the associativity condition : $(e_2 e_2) e_3 = e_2 (e_2 e_3)$

$C^2_{223} : C_{333} + C_{222} C_{233} \Rightarrow \frac{9}{16} t_2^4 \gamma'^2 = \frac{-t_2^4}{16} \gamma''' + \frac{3}{8} t_2^4 \gamma \gamma'' \Rightarrow \gamma''' - 6\gamma\gamma'' + 9\gamma'^2 = 0$ Chazy eqn)

$$\vec{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \quad \partial_\alpha \vec{\xi} = \begin{pmatrix} \partial_\alpha \xi_1 \\ \partial_\alpha \xi_2 \\ \partial_\alpha \xi_3 \end{pmatrix} = z \begin{pmatrix} C^\beta_{\alpha 1} \xi_\beta \\ C^\beta_{\alpha 2} \xi_\beta \\ C^\beta_{\alpha 3} \xi_\beta \end{pmatrix} = z \begin{pmatrix} C^1_{\alpha 1} & C^2_{\alpha 1} & C^3_{\alpha 1} \\ C^1_{\alpha 2} & C^2_{\alpha 2} & C^3_{\alpha 2} \\ C^1_{\alpha 3} & C^2_{\alpha 3} & C^3_{\alpha 3} \end{pmatrix} \vec{\xi}.$$

$$\partial_1 \vec{\xi} = z \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{\xi}$$

$$\partial_2 \vec{\xi} = z \begin{pmatrix} 0 & 1 & 0 \\ -\frac{3}{4} t_2^2 \gamma' & -\frac{3}{2} t_2 \gamma & 1 \\ -\frac{t_2^3}{4} \gamma'' & -\frac{3}{4} t_2^2 \gamma' & 0 \end{pmatrix} \vec{\xi} \qquad \partial_3 \vec{\xi} = z \begin{pmatrix} 0 & 0 & 1 \\ -\frac{t_2^3}{4} \gamma'' & -\frac{3}{4} t_2^2 \gamma' & 0 \\ -\frac{t_2^4}{16} \gamma''' & -\frac{t_2^3}{4} \gamma'' & 0 \end{pmatrix} \vec{\xi}$$

$$\Rightarrow \Bigg(\partial_2 + z \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ \frac{3}{4} t_2^2 \gamma' & \frac{3}{2} t_2 \gamma & -1 \\ \frac{t_2^3}{4} \gamma'' & \frac{3}{4} t_2^2 \gamma' & 0 \end{pmatrix}}_{=U} \Bigg) \vec{\xi} = 0 \qquad \Bigg(\partial_3 + z \underbrace{\begin{pmatrix} 0 & 0 & -1 \\ \frac{t_2^3}{4} \gamma'' & \frac{3}{4} t_2^2 \gamma' & 0 \\ \frac{t_2^4}{16} \gamma''' & \frac{1}{4} t_2^3 \gamma'' & 0 \end{pmatrix}}_{=V} \Bigg) \vec{\xi} = 0$$

The system $\partial_\alpha \xi_\beta = z C^\gamma_{\alpha\beta}(t) \xi_\gamma$ — $(*)$ becomes

$$\begin{cases} (\partial_1 - z I) \vec{\xi} = 0 \\ (\partial_2 + z U) \vec{\xi} = 0 \\ (\partial_3 + z V) \vec{\xi} = 0 \end{cases} \quad \text{— } (*)'$$

If $\vec{\xi}$ is a sol'n of $(*)'$, then $(\partial_2 + zU)\vec{\xi} = 0$, $(\partial_3 + zV)\vec{\xi} = 0$

$$\Rightarrow [\partial_2 + zU, \partial_3 + zV] \vec{\xi} = 0 \rightsquigarrow [\partial_2 + zU, \partial_3 + zV] = 0$$

Also, note that $[\partial_2 + U, \partial_3 + V] = (\partial_2 + U)(\partial_3 + V) - (\partial_3 + V)(\partial_2 + U)$

$$= \partial_2 \partial_3 + \partial_2 V + U \partial_3 + UV - \partial_3 \partial_2 - \partial_3 U - V \partial_2 - VU \quad \text{— } (**)$$

$$\partial_2 V = \begin{pmatrix} 0 & 0 & 0 \\ \frac{3}{4} t_2^2 \gamma'' & \frac{3}{2} t_2 \gamma' & 0 \\ \frac{t_2^3}{4} \gamma''' & \frac{t_2}{2} \gamma'' & 0 \end{pmatrix} \qquad \partial_3 U = \begin{pmatrix} 0 & 0 & 0 \\ \frac{3}{4} t_2^2 \gamma'' & \frac{3}{2} t_2 \gamma' & 0 \\ \frac{t_2^3}{4} \gamma''' & \frac{3}{2} t_2^2 \gamma' & 0 \end{pmatrix}$$

$$UV = \begin{pmatrix} -\frac{t_2^3}{4} \gamma'' & -\frac{3}{4} t_2^2 \gamma' & 0 \\ \frac{t_2^4}{16}(6\gamma\gamma'' - \gamma''') & \frac{9 t_2^3}{8}(\gamma\gamma' - 2\gamma'') & -\frac{3}{4} t_2^2 \gamma' \\ \frac{3}{16} t_2^5 \gamma\gamma'' & \frac{9}{16} t_2^4 \gamma'^2 & -\frac{t_2^3}{4} \gamma'' \end{pmatrix} \qquad UV - VU = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{t_2^4}{16}(\gamma''' - 6\gamma\gamma'' + 9\gamma'^2) & 0 & 0 \\ 0 & \frac{t_2^4}{16}(\gamma''' - 6\gamma\gamma'' + 9\gamma'^2) & 0 \end{pmatrix}$$

$$VU = \begin{pmatrix} -\frac{t_2^3}{4} \gamma'' & -\frac{3}{4} t_2^2 \gamma & 0 \\ \frac{9}{16} t_2^4 \gamma'^2 & \frac{9}{8} t_2^3 (\gamma\gamma' - 2\gamma'') & -\frac{3}{4} t_2^2 \gamma' \\ \frac{3}{16} t_2^5 \gamma\gamma'' & \frac{t_2^4}{16}(\gamma''' - 6\gamma\gamma'') & -\frac{1}{4} t_2^3 \gamma'' \end{pmatrix}$$

Plug $\vec{\xi}$ into (**), $\cancel{\partial_z\partial_3\xi} + \underbrace{(\partial_2 V - \partial_3 U)}_{=0}\xi + \underbrace{U\partial_3\xi - V\partial_2\xi}_{=-zUV\xi + zVU\xi} + [U, V]\xi - \cancel{\partial_3\partial_2\xi}$

$= (-z+1)[U, V]\xi = 0 \Rightarrow [U, V] = 0$ and $[U, V] = 0$ gives Chazy Eqn.

Ex 3.3: For $\omega_1, \omega_2 \in \Omega^1(M)$, define $(\omega_1, \omega_2)^* = L_E(\omega_1 \cdot \omega_2)$

Then for $u, v \in \Gamma(TM)$, $(E \cdot u, v) = \langle u, v \rangle$

pf: $\partial_\alpha := \frac{\partial}{\partial t^\alpha}$ $\quad \partial_\alpha \cdot \partial_\beta = C_{\alpha\beta}^\gamma(t)\partial_\gamma \rightsquigarrow dt^\alpha \cdot dt^\beta := C_\gamma^{\alpha\beta} dt^\gamma$

Then $C_\gamma^{\alpha\beta} = \eta^{\beta\varepsilon} C_{\varepsilon\gamma}^\alpha$

Therefore, $g^{\alpha\beta} = (dt^\alpha, dt^\beta)^* = L_E(dt^\alpha \cdot dt^\beta) = E^\gamma (dt^\alpha \cdot dt^\beta)(\partial_\gamma)$

$= E^\gamma C_\gamma^{\alpha\beta} = E^\gamma \eta^{\beta\varepsilon} C_{\varepsilon\gamma}^\alpha$

Suppose $g^{\alpha\beta}$ is invertible, $\delta_\tau^\beta = g_{\tau\alpha} g^{\alpha\beta} = g_{\tau\alpha} E^\gamma \eta^{\beta\varepsilon} C_{\varepsilon\gamma}^\alpha$

$\Rightarrow \underbrace{\eta_{\theta\beta}\delta_\tau^\beta}_{=\eta_{\theta\tau} = \langle \partial_\theta, \partial_\tau \rangle} = \eta_{\theta\beta} g_{\tau\alpha} E^\gamma \eta^{\beta\varepsilon} C_{\varepsilon\gamma}^\alpha = g_{\tau\alpha}\delta_\theta^\varepsilon E^\gamma C_{\varepsilon\gamma}^\alpha = E^\gamma C_{\theta\gamma}^\alpha g_{\tau\alpha}$

$\Rightarrow (E \cdot \partial_\theta, \partial_\tau) = (E^\gamma \partial_\gamma \cdot \partial_\theta, \partial_\tau) = (E^\gamma C_{\gamma\theta}^\alpha \partial_\alpha, \partial_\tau) = E^\gamma C_{\gamma\theta}^\alpha g_{\alpha\tau}$

$\Rightarrow \langle \partial_\theta, \partial_\tau \rangle = (E \cdot \partial_\theta, \partial_\tau)$

Denote #: dual via $\langle , \rangle$, *: dual via $(,)$:

Then for $w \in \Gamma(TM)$, $\langle (dt^\beta)^\#, w \rangle = dt^\beta(w)$ $\quad (dt^\beta)^\# = A^\tau \partial_\tau$

$\to \langle (dt^\beta)^\#, \partial_\gamma \rangle = A^\tau \eta_{\tau\gamma} = \delta_\gamma^\beta \Rightarrow \underbrace{A^\tau \eta_{\tau\gamma}\eta^{\gamma\varepsilon}}_{=A^\tau\delta_\tau^\varepsilon = A^\varepsilon} = \delta_\gamma^\beta \eta^{\gamma\varepsilon} = \eta^{\beta\varepsilon}$

$\Rightarrow (dt^\beta)^\# = \eta^{\beta\varepsilon}\partial_\varepsilon$

Also, $\underbrace{(dt^\alpha \cdot dt^\beta)(\partial_\gamma)}_{=C_\gamma^{\alpha\beta}} = dt^\alpha((dt^\beta)^\# \cdot \partial_\gamma) = \delta_\tau^\alpha C_{\varepsilon\gamma}^\tau \eta^{\beta\varepsilon} = C_{\varepsilon\gamma}^\alpha \eta^{\beta\varepsilon}$

On the other hand, for $\omega \in \Omega^1(M)$, $u \in \Gamma(TM)$, $(u^*, \omega)^* = \omega(u)$

Write $\partial_\alpha^* = B_\tau dt^\tau$ $\quad (\partial_\alpha^*, dt^\beta)^* = B_\tau g^{\tau\beta} = \delta_\alpha^\beta$

$\Rightarrow \underbrace{B_\tau g^{\tau\beta} g_{\beta\varepsilon}}_{=B_\tau\delta_\varepsilon^\tau = B_\varepsilon} = \delta_\alpha^\beta g_{\beta\varepsilon} = g_{\alpha\varepsilon} \quad \Rightarrow \quad \partial_\alpha^* = g_{\alpha\varepsilon} dt^\varepsilon$

$(\partial_\alpha, \partial_\beta) = (\partial_\alpha^*, \partial_\beta^*)^* = g_{\alpha\gamma} g_{\beta\tau} (dt^\gamma, dt^\tau)^* = g_{\alpha\gamma} g_{\beta\tau} g^{\gamma\tau} = \delta_\alpha^\tau g_{\beta\tau} = g_{\beta\alpha} = g_{\alpha\beta}$

Claim: $\langle\,,\,\rangle^* = \mathcal{L}_e (\,,\,)^*$ (Proof of (3.39) in Dubrovin's book)

Proved in the flat coord. (t^α) $\quad e = \partial_1 \quad E = E^\alpha \partial_\alpha \qquad \langle dt^\alpha, dt^\beta\rangle = \eta^{\alpha\beta}$

$g^{\alpha\beta} = (dt^\alpha, dt^\beta)^* = i_E(dt^\alpha \cdot dt^\beta) = E^\gamma C^{\alpha\beta}_\gamma = E^\gamma \eta^{\beta\varepsilon} C^\alpha_{\varepsilon\gamma}$

$(\mathcal{L}_e g)(dt^\alpha, dt^\beta) = \mathcal{L}_e(g(dt^\alpha, dt^\beta)) - g(\mathcal{L}_e dt^\alpha, dt^\beta) - g(dt^\alpha, \mathcal{L}_e dt^\beta)$

$\mathcal{L}_e dt^\alpha = d(i_e dt^\alpha) = d(\delta^\alpha_1) = 0$

$\Rightarrow (\mathcal{L}_e g)(dt^\alpha, dt^\beta) = \mathcal{L}_e i_E(dt^\alpha \cdot dt^\beta) = e((dt^\alpha \cdot dt^\beta)(E))$

$= \partial_1(E^\gamma C^{\alpha\beta}_\gamma) = \partial_1(E^\gamma) C^{\alpha\beta}_\gamma + E^\gamma \partial_1(C^{\alpha\beta}_\gamma) = d_1 C^{\alpha\beta}_1 + E^\gamma \partial_1(C^{\alpha\beta}_\gamma)$

$= d_1 \eta^{\beta\varepsilon} C^\alpha_{1\varepsilon} + E^\gamma \partial_1(C^{\alpha\beta}_\gamma) = d_1 \eta^{\alpha\beta} + E^\gamma \partial_1(C^{\alpha\beta}_\gamma)$

$E^\gamma \partial_1(C^{\alpha\beta}_\gamma) = E^\gamma \partial_1(\eta^{\beta\varepsilon} C^\alpha_{\varepsilon\gamma}) = E^\gamma \eta^{\beta\varepsilon} \partial_1(C^\alpha_{\varepsilon\gamma}) = E^\gamma \eta^{\beta\varepsilon} \partial_1(\eta^{\alpha\tau} C_{\tau\varepsilon\gamma})$

$= E^\gamma \eta^{\beta\varepsilon} \eta^{\alpha\tau} \partial_1(C_{\tau\varepsilon\gamma}) = E^\gamma \eta^{\beta\varepsilon} \eta^{\alpha\tau} \partial_\tau(C_{1\varepsilon\gamma}) = E^\gamma \eta^{\beta\varepsilon} \eta^{\alpha\tau} \partial_\tau(\eta_{\varepsilon\gamma}) = 0$

Set $d_1 = 1$, $\quad (\mathcal{L}_e g)(dt^\alpha, dt^\beta) = \eta^{\alpha\beta}$

Exercise 4.3:

$F = \frac{1}{2} t_1^2 t_3 + \frac{1}{2} t_1 t_2^2 - \frac{1}{24} t_2^4 + t_2 e^{t_3} \qquad E := t_1 \partial_1 + \frac{1}{2} t_2 \partial_2 + \frac{3}{2} \partial_3$

$f(t_2, t_3) := \frac{-1}{24} t_2^4 + t_2 e^{t_3}$

When $n=3$, WDVV $\Leftrightarrow$ $f_{223}^2 = f_{333} + f_{222} f_{233}$

$f_{223} = 0 \qquad f_{333} = t_2 e^{t_3} \qquad f_{222} = -t_2 \qquad f_{233} = e^{t_3}$

$\rightsquigarrow$ F satisfies WDVV

$\partial_1 F = t_1 t_3 + \frac{1}{2} t_2^2 \quad \partial_2 F = t_1 t_2 - \frac{1}{6} t_2^3 + e^{t_3} \quad \partial_3 f = \frac{1}{2} t_1^2 + t_2 e^{t_3}$

$\mathcal{L}_E F = t_1 \partial_1 F + \frac{1}{2} t_2 \partial_2 F + \frac{3}{2} \partial_3 F = 2F + \frac{3}{4} t_1^2$

$\rightsquigarrow$ F is indeed a sol'n of WDVV with Euler v.f. E

$c_{111} = 0 = c_{112} \quad c_{113} = 1 \quad c_{122} = 1 \quad c_{123} = c_{133} = 0 \quad c_{113} = 1$

$c_{223} = 0 \quad c_{233} = e^{t_3} \quad c_{222} = -t_2 \quad c_{333} = t_2 e^{t_3}$

$$\eta_{ij} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \eta^{ij} = (\eta_{ij})^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$C_{11}^1 = C_{113} = 1 \quad C_{11}^2 = C_{11}^3 = 0 \quad C_{12}^1 = C_{12}^3 = 0 = C_{13}^2 = C_{13}^1$

$C_{23}^1 = C_{223} = e^{t_3} \quad C_{23}^2 = C_{232} = 0 \quad C_{23}^3 = C_{123} = 0 \qquad (C_{ij}^k = \eta^{kl} C_{ijl})$

$C_{22}^1 = C_{223} = 0 \quad C_{22}^2 = C_{222} = -t_2 \quad C_{22}^3 = C_{221} = 1$

$C_{33}^1 = C_{333} = t_2 e^{t_3} \quad C_{33}^2 = C_{233} = e^{t_3} \quad C_{33}^3 = C_{133} = 0$

$C_k^{ij} = C_{lk}^i \eta^{lj} = C_{rlk} \eta^{lj} \eta^{ri}$

$C_1^{11} = C_{331} = 0 \quad C_2^{11} = C_{332} = e^{t_3} \quad C_3^{11} = C_{333} = t_2 e^{t_3}$

$C_1^{12} = C_{321} = 0 \quad C_2^{12} = C_{322} = 0 \quad C_3^{12} = C_{323} = e^{t_3}$

$C_1^{13} = C_{311} = 1 \quad C_2^{13} = C_{312} = 0 \quad C_3^{13} = C_{313} = 0$

$C_1^{23} = C_{121} = 0 \quad C_2^{23} = C_{221} = 1 \quad C_3^{23} = C_{123} = 0$

$C_1^{22} = C_{122} = 1 \quad C_2^{22} = C_{222} = -t_2 \quad C_3^{22} = C_{223} = 0$

$C_1^{33} = C_{111} = 0 \quad C_2^{33} = C_{211} = 0 \quad C_3^{33} = C_{113} = 1$

$g^{11} = E^i C_i^{11} = \frac{1}{2} t_2 e^{t_3} + \frac{3}{2} t_2 e^{t_3} = 2 t_2 e^{t_3} \quad g^{12} = E^i C_i^{12} = \frac{3}{2} e^{t_3}$

$g^{13} = E^i C_i^{13} = t_1 \quad g^{22} = E^i C_i^{22} = t_1 - \frac{1}{2} t_2^2 \quad g^{23} = E^i C_i^{23} = \frac{1}{2} t_2$

$g^{33} = \frac{3}{2}$

$\Rightarrow g = 2 t_2 e^{t_3} dt_1^2 + 3 e^{t_3} dt_1 dt_2 + 2 t_1 dt_1 dt_3 + (t_1 - \frac{1}{2} t_2^2) dt_2^2 + t_2 dt_2 dt_3 + \frac{3}{2} dt_3^2$

or

$$g^{\alpha\beta} = \begin{pmatrix} 2t_2 e^{t_3} & \frac{3}{2}e^{t_3} & t_1 \\ \frac{3}{2}e^{t_3} & t_1 - \frac{1}{2}t_2^2 & \frac{1}{2}t_2 \\ t_1 & \frac{1}{2}t_2 & \frac{3}{2} \end{pmatrix}$$

$$t_1 = 2^{\frac{-1}{3}} e^{\frac{2}{3}z} \left(e^{x+y} + e^{-x} + e^{-y}\right)$$

$$t_2 = 2^{\frac{-2}{3}} e^{\frac{1}{3}z} \left(e^{-(x+y)} + e^{x} + e^{y}\right)$$

$$t_3 = z$$

$$dt_1 = \Big(t_1 - \overbrace{2^{\frac{-1}{3}} e^{\frac{2}{3}z}\left(2e^{-x} + e^{-y}\right)}^{A_1}\Big)dx + \Big(t_1 - \overbrace{2^{\frac{-1}{3}} e^{\frac{2}{3}z}\left(2e^{-y} + e^{-x}\right)}^{A_2}\Big)dy + \frac{2}{3}t_1\,dz$$

$$dt_2 = \Big(t_2 - \underbrace{2^{\frac{-2}{3}} e^{\frac{1}{3}z}\left(e^{y} + 2e^{-(x+y)}\right)}_{B_1}\Big)dx + \Big(t_2 - \underbrace{2^{\frac{-2}{3}} e^{\frac{2}{3}z}\left(e^{x} + 2e^{-(x+y)}\right)}_{B_2}\Big)dy + \frac{1}{3}t_2\,dz$$

$$dt_3 = dz$$

$$2t_2 e^{t_3} \cdot \frac{4}{9}t_1^2 + 3e^{t_3} \cdot \frac{2}{9}t_1 t_2 + 2t_1 \cdot \frac{2}{3}t_1 + \left(t_1 - \frac{1}{2}t_2^2\right) \cdot \frac{1}{9}t_2^2 + \frac{1}{3}t_2^2 + \frac{3}{2}$$

$$= \frac{8}{9}t_1^2 t_2 e^{t_3} + \frac{2}{3}t_1 t_2 e^{t_3} + \frac{4}{3}t_1^2 + \frac{1}{9}t_2^2 t_1 - \frac{1}{18}t_2^4 + \frac{1}{3}t_2^2 + \frac{3}{2}$$

<u>Ref:</u> Dubrovin, Zhang - Extended Affine Weyl Groups and Frobenius Manifolds (1998) arxiv: hep-th/9611200

Φ: irred, reduced root system in $V := \mathbb{E}^n$ (Euclidean space) $\quad r_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$

i) $|\Phi| < \infty$ and $0 \notin \Phi$

ii) $\alpha \in \Phi$, $r_\alpha(\Phi) = \Phi$

iii) $\forall \alpha, \beta \in \Phi$, $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$

iv) $\alpha \in \Phi$, $c\alpha \in \Phi \Rightarrow c = \pm 1$

v) Φ cannot decompose into orthogonal proper subset

Fix a simple roots of Φ: $\alpha_1, \ldots\ldots, \alpha_n \in \Phi$ (i.e. $\forall \beta \in \Phi$, $\beta = \sum_{i=1}^{n} k_i \alpha_i$, where k_i are all ≤ 0 or all ≥ 0)

$\alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i,\alpha_i)}$: coroot $\quad i = 1, \ldots, n$

$\rightarrow A_{ij} = (\alpha_i, \alpha_j^\vee) \in \mathbb{Z}$ (i.e. The entries of Cartan matrix)

$W(\Phi)$: The Weyl gp associated Φ i.e.

$$W = \langle r_{\alpha_1^\vee}, \ldots\ldots, r_{\alpha_n^\vee} \rangle$$

Then affine Weyl gp $W_a(\Phi) \curvearrowright V$ by: $x \longmapsto w(x) + \sum_{i=1}^{n} m_i \alpha_i^\vee$, $w \in W(\Phi)$, $m_i \in \mathbb{Z}$

Introduce fundamental weight $\omega_1, \ldots, \omega_n \in V$ s.t. $(\omega_i, \alpha_j^\vee) = \delta_{ij}$

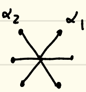
So, pick ω_k for $k \in \{1, \ldots, l\}$,

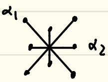
we define extended affine Weyl gp $\widetilde{W} := \widetilde{W}_a^{(k)}(\Phi) \curvearrowright \widetilde{V} = V \oplus \mathbb{R}$

to be the gp generated by:

(1) $\vec{x} = (x, x_{l+1}) \longmapsto (w(x) + \sum_{i=1}^{m} m_i \alpha_i^\vee, x_{l+1}) \qquad w \in W(\Phi), m_j \in \mathbb{Z}$, $\quad x \in \mathbb{R}^l$

and (2) $\vec{x} = (x, x_{l+1}) \longmapsto (x + \omega_k, x_{l+1} - 1)$

A_2:  Dynkin diagram $\bullet\!\!-\!\!\bullet$ $\qquad$ Weyl gp $= S_3$

B_2:  Dynkin diagram $\bullet\!\Rightarrow\!\bullet$

Exercise 4.4: $F = \frac{1}{2} t_1^2 t_3 + \frac{1}{2} t_1 t_2^2 - \frac{1}{48} t_2^4 + \frac{1}{4} t_2^2 e^{t_3} + \frac{1}{32} e^{2t_3}$

with Euler v.f. $E = t_1 \partial_1 + \frac{1}{2} t_2 \partial_2 + \partial_3$.

$f(t_2, t_3) := -\frac{1}{48} t_2^4 + \frac{1}{4} t_2^2 e^{t_3} + \frac{1}{32} e^{2t_3}$ $\quad f_{223} = \frac{1}{2} e^{t_3}$ $\quad f_{223}^2 = \frac{1}{4} e^{2t_3}$

$f_{333} = \frac{1}{4} e^{2t_3} + \frac{1}{4} t_2^2 e^{t_3}$ $\quad f_{222} = -\frac{1}{2} t_2$ $\quad f_{233} = \frac{1}{2} t_2 e^{t_3}$

$\rightsquigarrow$ F satisfies WDVV

$\partial_1 F = t_1 t_3 + \frac{1}{2} t_2^2$ $\quad \partial_2 F = t_1 t_2 - \frac{1}{12} t_2^3 + \frac{1}{2} t_2 e^{t_3}$ $\quad \partial_3 F = \frac{1}{2} t_1^2 + \frac{1}{4} t_2^2 e^{t_3} + \frac{1}{16} e^{2t_3}$

$$\mathcal{L}_E F = t_1 \partial_1 F + \frac{1}{2} t_2 \partial_2 F + \partial_3 F = t_1^2 t_3 + \frac{1}{2} t_1 t_2^2 + \frac{1}{2} t_1 t_2^2 - \frac{1}{24} t_2^4 - \frac{1}{4} t_2^2 e^{t_3} + \frac{1}{2} t_1^2 + \frac{1}{4} t_2^2 e^{t_3} + \frac{1}{16} e^{2t_3}$$

$$= 2F + \frac{1}{2} t_1^2$$

$\Rightarrow$ F is sol'n of WDVV with Euler vf. E

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Exercises

<u>Ex 2 part (b)</u>. $M :=$ space of $\{P(z) = z^5 + a_3 z^3 + a_2 z^2 + a_1 z + a_0 \mid a_i \in \mathbb{R}\}$

For $x \in M$, $T_x M \simeq \mathbb{C}[z]/P'(z)$ so we can shift alg. str. on $\mathbb{C}[z]/P'(z)$ to $T_x M$. Define

$$g_x(p(z), q(z)) := -\mathrm{Res}_{z=\infty} \frac{p(z)\cdot q(z)}{P'(z)} dz$$
$$= \mathrm{Res}_{w=0} \frac{p(\frac{1}{w})\cdot q(\frac{1}{w})}{P'(\frac{1}{w})} \cdot \frac{1}{w^2} dw$$

(cf. Sabbah, Isomonodromy Deformations and Frob. Mfds, p. 246). We show that under this definition,

$$ds^2 = \frac{1}{5} da_0 da_3 + \frac{1}{5} da_1 da_2 - \frac{3}{25} a_3 da_2 da_3 - \frac{1}{25} a_2 da_3^2.$$

Indeed, observe that $\dfrac{1}{P'(\frac{1}{w})} = \dfrac{1}{5\prod_{i=1}^4 (\frac{1}{w} - \alpha_i)} = \frac{1}{5} \prod_{i=1}^4 (w + \alpha_i w^2 + \alpha_i^2 w^3 + \alpha_i^3 w^4 + \cdots)$. $\therefore$

- $g_x(\frac{\partial}{\partial a_0}, \frac{\partial}{\partial a_3}) = \mathrm{Res}_{w=0} \frac{1}{w^5} \cdot \frac{1}{5} \prod_{i=1}^4 (w + \alpha_i w^2 + \alpha_i^2 w^3 + \cdots) dw = \frac{1}{5}$
- Similarly,
- $g_x(\frac{\partial}{\partial a_2}, \frac{\partial}{\partial a_3}) = \mathrm{Res}_{w=0} \frac{1}{w^7} \cdot \frac{1}{5} \prod_{i=1}^4 (w + \alpha_i w^2 + \alpha_i^2 w^3 + \cdots) dw = \frac{1}{5}\left(\sum_{i \neq j} \alpha_i \alpha_j + \sum_i \alpha_i^2\right) = -\frac{3}{25} a_3$

  use $0 = (\alpha_1 + \cdots + \alpha_4)^2 = \sum \alpha_i^2 + 2 \sum_{i \neq j} \alpha_i \alpha_j$
- $g_x(\frac{\partial}{\partial a_3}, \frac{\partial}{\partial a_3}) = \mathrm{Res}_{w=0} \frac{1}{w^8} \cdot \frac{1}{5} \prod_{i=1}^4 (w + \alpha_i w^2 + \alpha_i^2 w^3 + \cdots) dw$

$$= \frac{1}{5}\left(\sum_{i \neq j \neq k} \alpha_i \alpha_j \alpha_k + \sum_{i \neq j} \alpha_i^2 \alpha_j + \sum_i \alpha_i^3\right) = -\frac{2}{25} a_2.$$

$$\sum_i \alpha_i^2 (\alpha_1 + \cdots + \alpha_4) = 0$$

Next we show the flat coor. is given by $\begin{cases} a_0 = t_0 + \frac{1}{5} t_2 t_3 \\ a_1 = t_1 - \frac{1}{5} t_3^2 \\ a_2 = t_2 \\ a_3 = t_3 \end{cases}$. Indeed, because

$\begin{cases} da_0 = \frac{1}{5} t_2 dt_3 + \frac{1}{5} t_3 dt_2 \\ da_1 = dt_1 + \frac{2}{5} dt_3 \\ da_2 = dt_2, \ da_3 = dt_3 \end{cases}$, we have $ds^2 = \frac{1}{5} dt_0 dt_3 + \frac{1}{25} t_2 dt_3^2 + \frac{1}{25} t_3 dt_2 dt_3 + \frac{1}{5} dt_1 dt_2 + \frac{2}{25} dt_2 dt_3 - \frac{3}{25} dt_2 dt_3 - \frac{1}{25} t_2 dt_3^2$

$$= \frac{1}{5} dt_0 dt_3 + \frac{1}{5} dt_1 dt_2.$$

<u>Ex 1 part (b)</u>. (Assuming $\circ$ is assoc.) <u>cf.</u> Hertling, thm 2.14 and thm 2.15.

First, observe that the assumption ∇C is a symm. $(0,4)$ tensor implies $\nabla \circ$ is a symm. $(1,3)$ tensor, because

$$(\nabla C)(X, Y, Z, W) = \nabla_X g(Y \circ Z, W) - g(Y \circ Z, W) - g(\nabla_X Y \circ Z, W) - g(Y \circ \nabla_X Z, W) - g(Y \circ Z, \nabla_X W)$$
$$= g(\nabla_X(Y \circ Z), W) - g(\nabla_X Y \circ Z, W) - g(Y \circ \nabla_X Z, W)$$
$$= g((\nabla \circ)(X, Y, Z), W).$$

Compute $(\mathrm{Lie}_{X \circ Y}(\circ) - X \circ \mathrm{Lie}_Y(\circ) - Y \circ \mathrm{Lie}_X(\circ))(Z, W)$

$= \nabla \circ (X \circ Y, Z, W) - X \circ \nabla \circ (Y, Z, W) - Y \circ \nabla \circ (X, Z, W) - \nabla \circ (Z \circ W, X, Y) + Z \circ \nabla \circ (W, X, Y) + W \circ \nabla \circ (Z, X, Y)$. Want: This is 0.

This will hold if the following claim is true:

<u>claim.</u> $\varphi : (X, Y, Z, W) \mapsto \nabla \circ (X \circ Y, Z, W) + Z \circ \nabla \circ (W, X, Y) + W \circ \nabla \circ (Z, X, Y)$ is symm. in X, Y, Z, W.

<u>pf of claim.</u> Observe $(X, Y, Z, W) \mapsto \nabla \circ (Z, X \circ Y, W) + W \circ \nabla \circ (Z, X, Y)$

$$= \nabla_Z (X \circ Y \circ W) - (\nabla_Z X \circ Y) \circ W - X \circ Y \circ \nabla_Z W + W \circ \nabla_Z (X \circ Y) - W \circ \nabla_Z X \circ Y - W \circ X \circ \nabla_Z Y$$

is symm. in X, Y, W. $\therefore \varphi$ is symm. in $(X, W), (Y, W), (Z, W)$. $\therefore$ ok.

Exercise 1.2. Choose $\{\tilde{e}_1, \dots, \tilde{e}_n\} =: \beta$, basis of A, s.t. $[\langle\cdot,\cdot\rangle]_\beta = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = I$.

$\forall i$, define $l_i : A \to A$, $x \mapsto e_i x$ linear, and $l_i \circ l_j = l_j \circ l_i$ $\forall i, j$. Observe:

- $\langle l_i b, c\rangle = \langle b, l_i c\rangle$ $\therefore l_i = l_i^*$ $\therefore \forall i$, l_i is diagonalizable.
- $\forall i$, l_i comm. w/. I. $\Rightarrow$ can find a basis that simultaneously diagonalizes $l_1, \dots, l_n$, $[\langle\cdot,\cdot\rangle]_\beta$.

(still denote this basis by $\tilde{\beta}$ and its elements by $\tilde{e}_i$.)

$\therefore l_i \cdot \tilde{e}_i = c_i \tilde{e}_i$ for some $c_i \neq 0$ ($\because \nexists$ nilpotent). Moreover, for $i \neq j$, $\tilde{e}_i \tilde{e}_j = c\tilde{e}_i = c'\tilde{e}_j$ for some c, c'.

$\therefore \tilde{e}_i \tilde{e}_j = 0$, for $i \neq j$. Finally, we just choose $e_i := \frac{1}{\sqrt{c_i}} \tilde{e}_i$. Then $e_i e_j = \delta_{ij} e_i$.

(Remark: Under this choice, $\langle e_i, e_i\rangle \neq 1$ in general!)

Exercise 1.3. Recall that $(\mathcal{L}_E \eta)_{\alpha\beta} = D\eta_{\alpha\beta}$ for some D. Want:

$$\langle (\nabla E - \tfrac{1}{2}D)X, Y\rangle + \langle X, (\nabla E - \tfrac{1}{2}D)Y\rangle = 0.$$

Since $\nabla_\cdot E = \nabla_E \cdot - \mathcal{L}_E \cdot$, we have

$$LHS = \underbrace{\langle \nabla_E X, Y\rangle + \langle X, \nabla_E Y\rangle}_{} - \langle \mathcal{L}_E X, Y\rangle - \langle X, \mathcal{L}_E Y\rangle - \underbrace{D\langle X, Y\rangle}_{=\,-\mathcal{L}_E\langle X, Y\rangle}$$

$$= 0$$

Exercise C.3.

Define $\Omega := \frac{d\gamma}{d\tau} - \frac{1}{2}\gamma^2$. We shall first explain why $\Omega d\tau^2$ is inv. under Möbius transform; this will give a hint to prove this exercise. Showing $\Omega d\tau^2$ is inv. under Möbius trans. is equv. to showing

$$\frac{\tilde{\gamma}' - \frac{1}{2}(\tilde{\gamma})^2}{(c\tau+d)^4} = \gamma' - \frac{1}{2}\gamma^2.$$

Compute

$$LHS = \frac{1}{(c\tau+d)^4}\Big[\underbrace{\big(2c(c\tau+d)\gamma + (c\tau+d)^2\gamma' + 2c^2\big)(c\tau+d)^2}_{=\,\tilde{\gamma}'} - \frac{1}{2}\underbrace{\big((c\tau+d)^2\gamma + 2c(c\tau+d)\big)^2}_{=\,\tilde{\gamma}^2}\Big] = \gamma' - \frac{1}{2}\gamma^2.$$

Through this computation, we see the γ' term in $\tilde{\gamma}'$ must not be cancelled in order to have invarince, while the less-differentiated terms in $\tilde{\gamma}'$, eg. $2c(c\tau+d)\gamma$, must be cancelled with terms in the Möbius trans. of other terms (here, $\tilde{\gamma}^2$).

Now we back to Exercise C.3. The fact $P d\tau^k$ is inv. is equiv. to $\frac{P(\tilde{\gamma}, \tilde{\gamma}', \tilde{\gamma}'', \dots)}{(c\tau+d)^{2k}} = P(\gamma, \gamma', \gamma'', \dots)$.

Use lexigraphical order: $\gamma < \gamma' < \gamma'' < \cdots$

__claim.__ Under lexigraphical order, $LT(P) = a\gamma^{(n), a_n}\gamma^{(n-1), a_{n-1}} \cdots (\gamma')^{a_1}\gamma^{a_0}$, w/. $a_0 = 0$.

(Then, $P - a(\nabla^{n-1}\Omega)^{a_n}(\nabla^{n-2}\Omega)^{a_{n-1}} \cdots (\Omega)^{a_1}$ has less degree and $P - a(\nabla^{n-1}\Omega^{a_n}) \cdots (\Omega)^{a_1} d\tau^k$ is still inv. Proceed in this way $\Rightarrow$ $P = Q(\Omega, \nabla\Omega, \nabla^2\Omega, \dots)$)

__pf of claim.__ Suppose $a_0 > 0$. Under Möbius trans, $a\gamma^{(n), a_n} \cdots (\gamma')^{a_1}\gamma^{a_0}$ becomes $a\tilde{\gamma}^{(n), a_n} \cdots (\tilde{\gamma}')^{a_1}\tilde{\gamma}^{a_0}$. It produces a $\gamma^{(n), a_n} \cdots (\gamma')^{a_1}$ term:

$$a\left(\prod_{i=1}^{n} (c\tau+d)^{2((n+1)a_n + \cdots + 2a_1)}\right) \cdot 2c^{a_0} (c\tau+d)^{a_0} \gamma^{(n), a_n} \cdots (\gamma')^{a_1} \text{ --- } (*).$$

This term won't be cancelled by any other term: for example, for any $b_0 < a_0$, the $\gamma^{(n), a_n} \cdots (\gamma')^{a_1}$ term in $\tilde{\gamma}^{(n), a_n} \cdots (\tilde{\gamma}')^{a_1}\tilde{\gamma}^{b_0}$ has fewer exponent of $c\tau+d$, since we considered the leading term. However, the invariance assumption says $(*)$ must be cancelled! $\divideontimes$ $\therefore a_0 = 0$. #claim

Equation H.3. Want: $g^{\alpha\beta}(t) - \lambda\eta^{\alpha\beta} = g^{\alpha\beta}(t^1-\lambda, t^2, \dots, t^n)$.

By Remark 3.2, $\underset{t_\eta}{\langle\ ,\ \rangle^*} = \mathcal{L}_e \underset{g}{(\ ,\ )^*}$. $\therefore \frac{\partial}{\partial\lambda} LHS = -(\mathcal{L}_e g)^{\alpha\beta}$

$$= -\frac{\partial}{\partial t^1} g^{\alpha\beta} + g(\underbrace{\mathcal{L}_e dt^\alpha}_{d\delta_1^\alpha = 0}, dt^\beta) + g(dt^\alpha, \underbrace{\mathcal{L}_e dt^\beta}_{d\delta_1^\beta = 0})$$

$$= -\frac{\partial}{\partial t^1} g^{\alpha\beta} = \frac{\partial}{\partial\lambda} RHS.$$

$\therefore$ $LHS = RHS + G$, where G is indep. of λ. Take $\lambda = 0 \Rightarrow$ find $G = 0$

Exercise E.1. Here, the potential function is $F = \frac{1}{2}t_1^2 t_3 + \frac{1}{2}t_1 t_2^2 - \frac{t_2^4}{16}\gamma(t_3)$. To prove the result, we use (3.54): u_i's are the indep. sol. of

$$\sum_\gamma \partial_\gamma u\, C^\gamma_{\alpha\beta}(t) = \partial_\alpha u\, \partial_\beta u \text{ —— } (*)$$

Compute $C_{111} = 0$, $C_{112} = 0$, $C_{113} = 1$,
$C_{121} = 0$, $C_{122} = 1$, $C_{123} = 0$,
$C_{222} = -\frac{3}{2}t_2\gamma$, $C_{223} = -\frac{3}{4}t_2^2\gamma'(t_3)$, $C_{332} = -\frac{1}{4}t_2^3\gamma''(t_3)$, $C_{333} = -\frac{1}{16}t_2^4\gamma'''(t_3)$. $\therefore$ $(*)$ becomes

$$\partial_1 u\, C_{3\alpha\beta} + \partial_2 u\, C_{2\alpha\beta} + \partial_3 u\, C_{1\alpha\beta} = \partial_\alpha u\, \partial_\beta u.$$

We check this holds $\forall \alpha,\beta$ for $u = u_1$. The computation for u_2 and u_3 is similar.
$\alpha = 1, \beta = 1$: $LHS = 1\cdot 1 + 0 + 0$, $RHS = 1\cdot 1$. ok!
$\alpha = 1, \beta = 2,3$: similarly ok!
$\alpha = 2, \beta = 2$: Want: $-\frac{3}{4}t_2^2\gamma'(t_3) + 0 + \frac{1}{2}t_2^2 w_1'(t_3) = t_2^2 w_1^2(t_3) \Leftrightarrow \frac{3}{2}\gamma'(t_3)w_1 - w_1 w_1' + 2w_1^3 = 0$.
Recall: Exercise C.2: $w_1' = -w_1(w_2+w_3) + w_2 w_3$. $\therefore -w_1 w_1' + 2w_1^3 = w_1^2(w_1+w_3) - w_1 w_2 w_3 + 2w_1^3$
$= -\frac{3}{2}\gamma(t_3)w_1^2 + \frac{1}{4}\gamma''(t_3) + w_1^3$. By (C.7) $\Rightarrow$ ok!
$\alpha = 2, \beta = 3$, and $\alpha = 3, \beta = 3$: The method is similar as above; one just uses (C.7) and (C.8).

Exercise 1.4. (perhaps this question is not doable!)
A natural choice for this 2 dim'l comm. gp of diffeo, preserving $\circ$ and η, is the flow genearated by e and E.
Recall: $\mathcal{L}_E e = -d_1 e$. $\therefore$ $d_1 = 0 \Rightarrow [E, e] = 0$, ie. flow generated by E commutes with flow generated by e.
We have $\mathcal{L}_E(\circ) = \circ$ and $\mathcal{L}_E \eta = D$ for some $D \in \mathbb{C}$, while $\mathcal{L}_e(\circ) = 0$. ... what does "preserving $\circ$ and η" exactly mean under this context?