Frobenius Manifolds

March, 2nd, 2018.

• History of Frobenius Manifolds:

~1980, Saito's singularity Theory

~1990, Vafa's Quantum Cohomology (A-model)

"Moduli" of C-Y 3folds (B-model)

(1) (1994) Dubrovin: Geometry of 2D TFT [textbook]

$\updownarrow$

PDE: WDVV eqns

(2) (1998) Manin: Frobenius Manifolds, Quantum Cohomology and Moduli Space (Gromov–Witten Theory)

(3) (2004) Hertling: Frobenius Manifold & Moduli Space for Singularity.

Main Question: Analytic Continuation of Frobenius Mfds?

Compact Example? (Compactification??)

• Definition of Frobenius Manifold:

M: C^∞-mfd $g: TM \times TM \longrightarrow \mathbb{R}$ "metric" (not necessarily positive def, only require non-deg.)

A commutative multiplication str: $\circ : TM \times TM \longrightarrow TM$.

In any local coord $(x^1, \dots, x^n)$, the basis $\{\partial_j := \frac{\partial}{\partial x_j}\}$ on $TM|_U$, we denote the str const, $\partial_j \circ \partial_k = C_{jk}^m \partial_m$. They satisfy:

(1) g is flat i.e. $\mathrm{Riem}(g) \equiv 0$. Therefore, $\exists$ flat coord. $(t^1, \dots, t^n)$ s.t. $g_{ab} := g(\partial_a, \partial_b)$ are constants

(2) $\exists\, e: M \longrightarrow TM$: identity/unity section. s.t. $e \circ X = X$, $\forall X \in TM$

global section

(3) $g(X \circ Y, Z) = g(X, Y \circ Z)$ Define $c(X, Y, Z) := g(X \circ Y, Z)$: totally symmetric (0,3)-tensor

$\rightsquigarrow \partial_i \circ \partial_j = g^{\ell k} C_{ij\ell}\, \partial_k$ $C_{ij\ell}$: totally symmetric w.r.t. i, j, k.

(4) ∇: L-C connection w.r.t. g $(\nabla_X C)(Y, Z, W) = (\nabla_Y C)(X, Z, W)$

Regard ∇C: (0,4)-tensor Then above identity: totally symmetric!

$\rightsquigarrow$ This is the associativity of $\circ$

(5) $\nabla e = 0$

Rmk: Later, we will introduce the Euler v.f. E

In flat coord. $(t^i)_{i=1}^n$ we may assume that $e = \frac{\partial}{\partial t_1}$

(4) becomes: $\boxed{\partial_a C_{bcd} = \partial_b C_{acd}}$ Calculus (!) implies:

$\exists$ (loc.) fcn $F(t^1, \dots, t^n)$ s.t. $C_{abc} = \dfrac{\partial^3 F}{\partial t^a \partial t^b \partial t^c}$

$\rightsquigarrow$ The function F determines everything on the Frob. mfd:

Metric: $g(\cdot,\cdot) = g(e\circ\cdot, \cdot) = c(e,\cdot,\cdot) \Rightarrow g_{ab} = \dfrac{\partial^3 F}{\partial t^1 \partial t^a \partial t^b}$

Str const. of multiplicative str: $C^k_{ij} = g^{k\ell} \dfrac{\partial^3 F}{\partial t^i \partial t^j \partial t^\ell} \rightsquigarrow C^k_{ij}$: function of F.

Associativity of $\circ$: $(\partial_i \circ \partial_j) \circ \partial_k = \partial_i \circ (\partial_j \circ \partial_k)$

$$\begin{array}{cc} (\partial_i \circ \partial_j) \circ \partial_k & \partial_i \circ (\partial_j \circ \partial_k) \\ \| & \| \\ (C^\ell_{ij} \partial_\ell \circ \partial_k) & \partial_i \circ (C^\ell_{jk} \partial_\ell) \\ \| & \| \\ \sum_{\ell,s} C^\ell_{ij} C^s_{\ell k} \partial_s & \sum_{\ell,s} C^s_{i\ell} C^\ell_{jk} \partial_s \end{array}$$

(Witten-Dijgraaf-Verlinde-Verlinde)

$$\boxed{\sum_\ell C^\ell_{ij} C^s_{\ell k} = \sum_\ell C^s_{i\ell} C^\ell_{jk}} \quad \text{— (1)}$$

WDVV eqn

Now, in flat coord.,
$$\boxed{\frac{\partial^3 F}{\partial t^i \partial t^j \partial t^m} g^{m\ell} \frac{\partial^3 F}{\partial t^\ell \partial t^k \partial t^n} = \frac{\partial^3 F}{\partial t^i \partial t^\ell \partial t^n} g^{\ell s} \frac{\partial^3 F}{\partial t^j \partial t^k \partial t^s}}$$

<u>Exer 0</u>: Axiom (4) $\Longleftrightarrow$ WDVV eqn

<u>Issue</u>: WDVV eqn only true in flat coord.

<u>Goal</u>: Find a coord.-free expression for WDVV eqn.

The existence of potential $F \Rightarrow \boxed{\dfrac{\partial C^a_{bc}}{\partial t^d} - \dfrac{\partial C^a_{bd}}{\partial t^c} = 0}$ — (2)

$$0 = \frac{\partial}{\partial t^b}\left(C^d_{pa} C^p_{ef} - C^d_{pf} C^p_{ae}\right) - \frac{\partial}{\partial t^f}\left(C^d_{pe} C^p_{ab} - C^d_{pb} C^p_{ae}\right) + C^p_{ae}\left(\frac{\partial C^d_{pf}}{\partial t^b} - \frac{\partial C^d_{pb}}{\partial t^f}\right)$$
$$+ C^p_{ef}\left(\frac{\partial C^d_{ab}}{\partial t^p} - \frac{\partial C^d_{ap}}{\partial t^b}\right) - C^p_{ab}\left(\frac{\partial C^d_{ef}}{\partial t^p} - \frac{\partial C^d_{ep}}{\partial t^f}\right) + C^d_{pb}\left(\frac{\partial C^p_{ef}}{\partial t^a} - \frac{\partial C^p_{ea}}{\partial t^f}\right) - C^d_{pf}\left(\frac{\partial C^p_{ab}}{\partial t^e} - \frac{\partial C^p_{ae}}{\partial t^b}\right)$$

$$\Rightarrow \left(C^d_{pa} \frac{\partial C^p_{ef}}{\partial t^b} + C^d_{pb} \frac{\partial C^p_{ef}}{\partial t^a}\right) - \left(C^d_{pe} \frac{\partial C^p_{ab}}{\partial t^f} + C^d_{pf} \frac{\partial C^p_{ab}}{\partial t^e}\right) + \left(C^p_{ef} \frac{\partial C^d_{ab}}{\partial t^p} - C^p_{ab} \frac{\partial C^d_{ef}}{\partial t^p}\right) = 0 \quad (3)$$

<u>Claim</u>: (3) holds in general coord. system (x^i)

<u>Exer 1</u>: (a) Verify the claim

(b) (Hertling - Manin): (3) is the coord. expression of the following:

$X, Y, Z \in \Gamma(TM)$, $h_{X \circ Y}(Z) = X \circ h_Y(Z) + h_X(Z) \circ Y$

Still another form of WDVV:

$\mathcal{L}: TM \longrightarrow TM$ (1,1)-tensor i.e. a vector-valued 1-form.

Torsion of $\mathcal{L}$: $T_{\mathcal{L}}(X,Y) := [\mathcal{L}X, \mathcal{L}Y] - \mathcal{L}([\mathcal{L}X, Y]) - \mathcal{L}([X, \mathcal{L}Y]) + \mathcal{L}^2[X,Y]$ — X,Y extends to v.f.

Hantjes tensor: $\mathcal{H}_{\mathcal{L}}(X,Y) := T_{\mathcal{L}}(\mathcal{L}X, \mathcal{L}Y) - \mathcal{L}T_{\mathcal{L}}(\mathcal{L}X, Y) - \mathcal{L}T_{\mathcal{L}}(X, \mathcal{L}Y) + \mathcal{L}^2 T_{\mathcal{L}}(X,Y)$

, for X, Y, vectors

Now, for $X \in TM$, $\mathcal{L}_X(Y) = X \circ Y \Rightarrow \mathcal{L}_X: TM \to TM$ X: v.f.

Exer 1 (c): $\mathcal{H}_{\mathcal{L}_X} = 0 \iff$ WDVV eqn.

Example from Singularity Theory (Saito):

Surface Singularity: $f(x,y,z) = x^2 + y^2 + p(z) = 0$ in $\mathbb{C}^3$

$$p(z) = z^5 + \underline{a_3} z^3 + \underline{a_2} z^2 + \underline{a_1} z + \underline{a_0}$$: deformation of A_4 sing.

moduli parameter: a_3 — A_2-sing.; a_2, a_1 — ord. double pt.; a_0 — C^∞

A_n sing: $x^2 + y^2 + z^{n+1} = 0$ iso. sing.

$A = \mathbb{C}[\![x,y,z]\!] / \underbrace{(f_x, f_y, f_z)}_{\text{complete intersection}} \simeq \mathbb{C}[z]/(p'(z))$, $\dim_{\mathbb{C}} A = 4$, $p'(z) = 5z^4 + 3a_3 z^2 + 2a_2 z + a_1$

$p(z), q(z) \in A$. $p(z) \circ q(z) = p(z) \cdot q(z) \pmod{p'(z)}$

The metric is given by Grothendieck's residue:

Rmk: 0-dim'l complete intersection $A = k[\![z_1, \dots, z_m]\!]/(f_1, \dots, f_m)$ finite-dim'l k-v.s. (say $k = \mathbb{C}$)

$I: A \longrightarrow k$ trace map

$\varphi \in A$, $I(\varphi) := \operatorname{Res}_{f_m=0} \operatorname{Res}_{f_{m-1}=0} \cdots\cdots \operatorname{Res}_{f_1=0} \left(\dfrac{\varphi\, df_1 \wedge \dots \wedge df_m}{f_1 \cdots\cdots f_m} \right) \in k$.

It is a trace i.e. $g(a,b) := I(ab)$ is non-deg.

Exer 2: (a) Show that $g(p(z), q(z)) = \sum_{\alpha_j} \dfrac{p(\alpha_j)\, q(\alpha_j)}{p''(\alpha_j)}$

(b) The metric is flat in $\mathbb{R}^4$ (take $k = \mathbb{R}$), with

$$ds^2 = da_0\, da_3 + da_1\, da_2 - \frac{3}{5} a_3\, da_2\, da_3 - \frac{1}{5} a_2\, da_3^2$$

The flat coord.
$$\begin{cases} t_0 = a_0 - \frac{1}{5} a_2 a_3 \\ t_1 = a_1 - \frac{1}{5} a_3^2 \\ t_2 = a_2 \\ t_3 = a_3 \end{cases}$$

Write down: $\dfrac{\partial}{\partial t^i} \circ \dfrac{\partial}{\partial t^j} = C_{ij}^k \dfrac{\partial}{\partial t^k}$

March, 9th, 2018.

- WDVV with Euler vector field:

$F(t) = F(t^1, \dots, t^n)$: potential $\quad C_{\alpha\beta\gamma} = \partial^3_{\alpha\beta\gamma} F \quad t = \sum_{\alpha=1}^{n} t^i e_i$

1) $\eta_{\alpha\beta} := C_{1\alpha\beta}$ is a constant, non-deg. matrix (metric)

$C^\gamma_{\alpha\beta} = \eta^{\gamma\varepsilon} C_{\varepsilon\alpha\beta}$: "str const."

2) Associativity : $e_\alpha \cdot e_\beta = C^\gamma_{\alpha\beta}(t) e_\gamma$ is a ring A_t, with unity e_1

Then $(e_\alpha \cdot e_\beta) \cdot e_\gamma = e_\alpha \cdot (e_\beta \cdot e_\gamma) \Rightarrow \sum_s C^s_{\alpha\beta} C^\ell_{s\gamma} = \sum_s C^\ell_{\alpha s} C^s_{\beta\gamma} \longrightarrow$ WDVV eqn

3) Additional "quasi-homogeneity property" :

require : $\deg(t) = 1$, $\deg(e_1) = 0$ (so, $\deg(t^1) = 1$)

$t = t^1 e_1 + t^2 e_2 + \dots + t^n e_n$

<u>Notation</u> : $d_\alpha := \deg(t^\alpha) \quad q_\alpha := \deg(e^\alpha) = 1 - d_\alpha$

(<u>Recall</u> : F: weighted homogeneous of $\deg F = d_F$ means :

$F(c^{d_1} t_1, c^{d_2} t_2, \dots, c^{d_n} t_n) = c^{d_F} F(t)$, $\forall c \in \mathbb{C}^*$

$\Big\downarrow \frac{d}{dc}\big|_{c=1}$

$\mathcal{L}_E F = d_F F$, $E(t) = \sum_\alpha d_\alpha t^\alpha \partial_\alpha$, $\partial_\alpha := \frac{\partial}{\partial t_\alpha}$)

[Since we only care about $\partial^3_{\alpha\beta\gamma} F$, we require only :

$$\mathcal{L}_E F = d_F F + A_{\alpha\beta} t^\alpha t^\beta + B_\alpha t^\alpha + C \quad \text{——} \quad (*)$$

, for some const. d_F, $A_{\alpha\beta}$, B_α, C, with Euler v.f.

$E(t) = \sum_{\beta,\alpha} (q^\alpha_\beta t^\beta + r^\alpha) \partial_\alpha$ s.t. $\mathcal{L}_E e = -d_1 e$

Quasi-homogeneity property

The case:

$e = \frac{\partial}{\partial t^1}$ $\quad E(t) = d_\alpha t^\alpha \partial_\alpha$

$\mathcal{L}_E e = [E, e]$

$= -d_1 e$

<u>Lemma 1</u> : $(\mathcal{L}_E \eta)_{\alpha\beta} = (d_F - d_1) \eta_{\alpha\beta}$ i.e. E is an infinitesimal conformal transf. w.r.t. the flat metric $\langle , \rangle = (\eta_{\alpha\beta})$

<u>Rmk</u> : conformal transf. in Euclidean space : $n=2$, conformal transf. iff holomorphic

$n \geq 3$: conformal is generated by : isometry, dilation, inversion. (Liouville, cf. Dubrovin, Fomenko, Novikov Vol. I)

pf : Apply $\partial_\beta \partial_\alpha \partial_1$ to $(*)$ and do the commutator with E.

$\partial_\beta \partial_\alpha \partial_1 E F = d_F \partial_\beta \partial_\alpha \partial_1 F = d_F \eta_{\alpha\beta}$

Observe : $\partial_1 E F = [\partial_1, E] F + E \partial_1 F = d_1 \partial_1 F + E \partial_1 F \Rightarrow \partial_\beta \partial_\alpha \partial_1 E F = d_1 \eta_{\alpha\beta} + \partial_\beta \partial_\alpha E \partial_1 F$

$\partial_\beta \partial_\alpha E \partial_1 F = \partial_\beta ([\partial_\alpha, E] + E \partial_\alpha) \partial_1 F = \partial_\beta (q^\rho_\alpha \partial_\rho \partial_1 F) + \partial_\beta E \partial_\alpha \partial_1 F$

$\Rightarrow d_F \eta_{\alpha\beta} = d_1 \eta_{\alpha\beta} + q^\rho_\alpha \eta_{\rho\beta} + q^\rho_\beta \eta_{\rho\alpha}$

Cor: If $\eta_{11} = \eta(e_1, e_1) = 0$, and $Q = (q^\alpha_\beta)$ has simple eigenvalues, then by a linear change of coord., we may assume: $(\eta_{\alpha\beta}) = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$, and then

$$F(t) = \frac{1}{2}(t^1)^2 t^n + \frac{1}{2} t^1 \sum_{\alpha \neq 1, n} t^\alpha t^{(n+1)-\alpha} + f(t_2, \dots, t_n)$$

Also, for $d_1 = 1$, $q_1 = 0$, $q_n = d$, $d_F = 2 + d_n = 1 + d_\alpha + d_{n+1-\alpha} = 3 - d$

$\rightsquigarrow q_\alpha + q_{(n+1)-\alpha} = 2 - (d_\alpha + d_{(n+1)-\alpha}) = 2 - (2-d) = d.$

pf: $\langle e_1, e_1 \rangle = 0 \Rightarrow$ Can chose eigenvector e_n of Q s.t. $\langle e_1, e_n \rangle = 1$

Now, on $\operatorname{span}\{e_1, e_n\}^\perp$, can use Q-eigenvector to get $(\star)$. (Check!

Rmk: In general, if Q is diagonalizable, then $E(t)$ can be transf. (by a linear transf.) to:

$$E(t) = \sum_\alpha d_\alpha t^\alpha \partial_\alpha + \sum_{\{\alpha \mid d_\alpha = 0\}} r^\alpha \partial_\alpha$$

• Example:

1. $n=2$, WDVV is empty. Scaling conditions $\Rightarrow$

   i) $F(t_1, t_2) = \frac{1}{2} t_1^2 t_2 + t_2^k$, $\quad k = \frac{3-d}{1-d}$ $\quad d \neq 1, 2, 3$, $d = \deg(e_2)$

   ii) $F(t_1, t_2) = \frac{1}{2} t_1^2 t_2 + t_2^2 \log t_2$, $\quad d = -1$

   iii) $F(t_1, t_2) = \frac{1}{2} t_1^2 t_2 + \log t_2$, $\quad d = 3$

   iv) $F(t_1, t_2) = \frac{1}{2} t_1^2 t_2 + e^{\frac{2}{r} t_2}$, $\quad d = 1, r \neq 0$

   v) $F(t_1, t_2) = \frac{1}{2} t_1^2 t_2$, $\quad d = 1, r = 0$

2. $n=3$, $F(t) = \frac{1}{2} t_1^2 t_3 + \frac{1}{2} t_1 t_2^2 + f(t_2, t_3)$

WDVV $\Rightarrow f(x,y)$ satisfies $f_{xxy}^2 = f_{yyy} + f_{xxx} f_{xyy}$

pf: multiplicative table: $e_1 = e$, e_2, e_3 $\quad (\eta_{\alpha\beta}) = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$

$$e_2^2 = F_{221} \eta^{13} e_3 + F_{222} \eta^{22} e_2 + F_{223} \eta^{31} e_1 = e_3 + f_{xxx} e_2 + f_{xxy} e_1$$

$$e_2 \cdot e_3 = F_{231} e_3 + F_{232} e_2 + F_{233} e_1 = f_{xxy} e_2 + f_{xyy} e_1$$

$$e_3^2 = F_{331} e_3 + F_{332} e_2 + F_{333} e_1 = f_{xyy} e_2 + f_{yyy} e_1$$

Associativity (WDVV) $: (e_2 \cdot e_2) \cdot e_3 = e_2 (e_2 \cdot e_3)$

$$\Rightarrow e_3^2 + f_{xxx} e_2 \cdot e_3 + f_{xxy} e_3 = f_{xxy} e_2^2 + f_{xyy} e_2$$

$$\Rightarrow \cancel{f_{xxy}} e_3 + \cancel{f_{xyy}} e_2 + f_{yyy} e_1 + \cancel{f_{xxx} f_{xxy}} e_2 + f_{xxx} f_{xyy} e_1 = f_{\cancel{xxy}} e_3 + (\cancel{f_{xxy}} f_{xxx} + \cancel{f_{xyy}}) e_2 + f_{xxy}^2 e_1$$

(Check)

Scaling conditions: (a) $(1-\frac{d}{2})xf_x+(1-d)yf_y=(3-d)f \quad d\neq 1,2,3.$

(b) $d=1$: $\frac{1}{2}xf_x+rf_y=2f$

(c) $d=2$: $rf_x-yf_y=f$

(d) $d=3$: $\frac{1}{2}xf_x+2yf_y=\text{const.}$

<u>idea</u>: Use the first integral of $E(t)$, will do case (d):

<u>Question</u>: Find a good. coord. system.

v.f. on the (x,y)-plane. $(\frac{x}{2}, 2y)$ i.e. $\begin{cases} x'=\frac{1}{2}x \\ y'=2y \end{cases} \Rightarrow \begin{cases} x=c_1e^{\frac{t}{2}} \\ y=c_2e^{2t} \end{cases}$

Define $s=y/x^4$ is a const. along any integral curve.

We set $s=yx^{-4}$ $\quad t=x$ $\quad \begin{cases} x=t \\ y=st^4 \end{cases}$

Change of Variable: $\frac{\partial}{\partial t}f(x,y)=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}=f_x+4t^3sf_y$

$=\frac{2}{t}(\frac{1}{2}x\partial_x+2y\partial_y)f \Rightarrow \frac{t}{2}\frac{\partial}{\partial t}f=c \Rightarrow f_t=\frac{2c}{t} \Rightarrow f=2c\log t+\phi(s)$

$\Rightarrow f(x,y)=2c\log x+\phi(\frac{y}{x^4})$ — (**)

<u>Thm</u>: $f(x,y)$ satisfies $f_{xxy}^2=f_{yyy}+f_{xxx}f_{xyy}$ can be transformed into ODE of $\phi=\phi(z)$ via (**)

$$\phi'''=400\phi'^2+32c\phi''+1120z\phi'\phi''+784z^2\phi''^2+16cz\phi'''+160z^2\phi'\phi'''+192z^3\phi''\phi'''$$

This is a special case of Painlevé VI (as well as all other $n=3$ cases)

- Coord.-free form of Euler v.f.: $E\in\Gamma(TM)$,
- $\nabla(\nabla E)=0$ (This make sense since $\eta_{\alpha\beta}$ is flat)
- $Q=\nabla E$ is a covariant const. operator $\Rightarrow$ Eigenvalues of Q are const. fcns on M. s.t.

$$\begin{cases} \nabla_\gamma\nabla_\beta E=0 \\ \mathcal{L}_E C_{\alpha\beta}^{\gamma}=C_{\alpha\beta}^{\gamma} \\ \mathcal{L}_E e=-e \\ \mathcal{L}_E\eta_{\alpha\beta}=D\eta_{\alpha\beta} \text{ for some const. } D. \end{cases}$$

March, 16th 2018.

- Symmetry of WDVV:

Type I: Legendre-type Transformation $S_\varkappa$ $(\varkappa = 1, \dots, n)$: Coord. change

$t^\alpha \longmapsto \hat{t}^\alpha$ s.t. (a) $\hat{t}_\gamma = \partial_\gamma \partial_\varkappa F(t)$

(b) $\hat{F}_{\hat{\mu}\hat{\nu}}(\hat{t}) = F_{\mu\nu}(t)$

(c) $\hat{\eta}_{\hat{\alpha}\hat{\beta}} = \eta_{\alpha\beta}$

(1) $\partial_\alpha = \frac{\partial}{\partial t^\alpha} = \frac{\partial \hat{t}^\beta}{\partial t^\alpha} \frac{\partial}{\partial \hat{t}^\beta} = \eta^{\beta\gamma} \partial_\alpha \partial_\varkappa \partial_\gamma F(t) \frac{\partial}{\partial \hat{t}^\beta} = \eta^{\beta\gamma} F_{\alpha\varkappa\gamma} \hat{\partial}_\beta = F^\beta_{\varkappa\alpha} \hat{\partial}_\beta$

Cor: If $\partial_\varkappa \cdot$ is invertible, then $\partial_\alpha = \partial_\varkappa \cdot \hat{\partial}_\alpha$. In particular, $\partial_\varkappa = \partial_\varkappa \cdot \hat{\partial}_\varkappa \Rightarrow \partial_\varkappa = e$.

pf: $\partial_\alpha \cdot \partial_\varkappa = F^\beta_{\varkappa\alpha} \hat{\partial}_\beta \cdot \partial_\varkappa$. Let $G_\varkappa$ be the inverse matrix for $\partial_\varkappa \cdot$:

$$G^\alpha_{\varkappa\alpha_1} F^\lambda_{\varkappa\alpha} \partial_\lambda = \partial_{\alpha_1}$$

$$\parallel$$

$$G^\alpha_{\varkappa\alpha_1} F^\beta_{\varkappa\alpha} \hat{\partial}_\beta \cdot \partial_\varkappa = \hat{\partial}_{\alpha_1} \cdot \partial_\varkappa.$$

Rmk: $S_\varkappa$'s commute for different $\varkappa$'s. S_1 = identity. So, at the end, we need to renumbering the indices to switch $\varkappa \longleftrightarrow 1$.

(2) $\underset{\substack{\parallel \\ F_{\alpha\mu\nu}}}{\partial_\alpha F_{\mu\nu}} = \eta^{\beta\gamma} F_{\alpha\varkappa\gamma} \hat{\partial}_\beta \hat{F}_{\hat{\mu}\hat{\nu}} = \eta^{\beta\gamma} F_{\alpha\varkappa\gamma} \hat{F}_{\hat{\beta}\hat{\mu}\hat{\nu}} = F^\beta_{\varkappa\alpha} \hat{F}_{\hat{\beta}\hat{\mu}\hat{\nu}}$

$\Rightarrow$ Under $t^\alpha \longmapsto \hat{t}^\alpha$, $F_{\alpha\mu\nu} = F^\beta_{\varkappa\alpha} \hat{F}_{\hat{\beta}\hat{\mu}\hat{\nu}}$

(3) Preserving WDVV eqn:

$$\underbrace{\hat{F}_{\hat{\alpha}_1\hat{\beta}\hat{\lambda}}}_{\cdot\, F^{\alpha_1}_{\varkappa\alpha}} \hat{\eta}^{\hat{\lambda}\hat{\mu}} \underbrace{\hat{F}_{\hat{\mu}\hat{\gamma}_1\hat{\delta}}}_{\cdot\, F^{\gamma_1}_{\varkappa\gamma}} = \underbrace{\hat{F}_{\hat{\gamma}_1\hat{\beta}\hat{\lambda}}}_{\cdot\, F^{\gamma_1}_{\varkappa\gamma}} \hat{\eta}^{\lambda\mu} \underbrace{\hat{F}_{\hat{\mu}\hat{\alpha}_1\hat{\delta}}}_{\cdot\, F^{\alpha_1}_{\varkappa\alpha}} \Rightarrow \text{WDVV in original coord. system.}$$

Example: $(n=2, d=1, r=2)$ $F = \frac{1}{2}(t^1)^2 t^2 + e^{t^2}$ $\eta_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\varkappa = 2$, S_2: $\hat{t}^1 = \hat{t}_2 = F_{22} = e^{t^2}$ Check that: $\hat{F}(\hat{t}^1, \hat{t}^2) = \frac{1}{2}\hat{t}^1(\hat{t}^2)^2 + \frac{1}{2}(\hat{t}^1)^2 (\log \hat{t}^1 - \frac{3}{2})$

$\hat{t}^2 = \hat{t}_1 = F_{12} = t^1$

, then switch $1 \leftrightarrow 2$. $\hat{F}(\hat{t}^1, \hat{t}^2) = \frac{1}{2}\hat{t}^2(\hat{t}^1)^2 + \frac{1}{2}(\hat{t}^2)^2(\log \hat{t}^2 - 3/2)$

$\rightsquigarrow$ The case $d = -1$.

Rmk: If $q_{\varkappa_1} = \cdots = q_{\varkappa_s}$, then $\forall c = (c^1, \dots, c^s)$, can consider

S_c: $\hat{t}_\alpha = \sum_{i=1}^s c^i \partial_\alpha \partial_{\varkappa_i} F(t) \Rightarrow S_c$ is a transf. if $c^i \partial_{\varkappa_i} \cdot$ is invertible.

Type 2: The Inversion I

$$\begin{cases} \hat{t}^1 = \frac{1}{2}\frac{t_\alpha t^\alpha}{t^n} \qquad \hat{t}^\alpha = \frac{t^\alpha}{t^n} \text{ for } \alpha \neq 1, n \qquad \hat{t}^n = \frac{-1}{t^n} \\ \hat{F}(\hat{t}) = \frac{1}{(t^n)^2}\left(F(t) - \frac{1}{2}t^1 t_\alpha t^\alpha\right) = (\hat{t}^n)^2 F + \frac{1}{2}\hat{t}^1 \hat{t}_\alpha \hat{t}^\alpha \\ \hat{\eta}_{\alpha\beta} = \eta_{\alpha\beta} \end{cases}$$

It is conformal: $\eta_{\alpha\beta}\, d\hat{t}^\alpha d\hat{t}^\beta = \frac{1}{(t^n)^2}\eta_{\alpha\beta}\, dt^\alpha dt^\beta$

Effect on Euler v.f. (Dubrovin Lemma B.1).

<u>Exercise</u>: $SL(2;\mathbb{C})$ acts on sol'n of WDVV eqns with $d=1$.

$t^1 \longmapsto t^1 + \frac{1}{2}\frac{c}{ct^n+d}\sum_{\alpha\neq 1} t_\alpha t^\alpha \qquad t^\alpha \longmapsto t^\alpha/(ct^n+d) \quad \alpha \neq 1, n$

$t^n \longmapsto (at^n+b)/(ct^n+d)$

- 1-dim'l affine connection: real/cpx

<u>Def</u>: This is given by a function $\gamma(\tau) = \Gamma_{11}^1(\tau)$ s.t. on k-form $\Omega^{\otimes k}$:

$f\,d\tau^k \in \Omega^{\otimes k}$, $\quad \nabla f\, d\tau^{k+1} := \left(\frac{df}{d\tau} - k\gamma(\tau) f\right) d\tau^{k+1}$ —— $(*)$

In $\tilde{\tau}$: $f(\tau)d\tau^k = \left(f(\tau)\,(d\tau/d\tilde{\tau})^k\right) d\tilde{\tau}^k =: \tilde{f}(\tilde{\tau})\, d\tilde{\tau}^k$

$$\Rightarrow \left(\frac{d\tilde{f}}{d\tilde{\tau}} - k\tilde{\gamma}(\tilde{\tau})\tilde{f}\right) d\tilde{\tau}^{k+1} = \left(\frac{d\tilde{\tau}}{d\tau}\frac{d}{d\tilde{\tau}}\left(\tilde{f}(\tilde{\tau})\left(\frac{d\tilde{\tau}}{d\tau}\right)^k\right) - k\gamma(\tau)\tilde{f}(\tilde{\tau})\left(\frac{d\tilde{\tau}}{d\tau}\right)^k\right)$$

$$= \left(\frac{d\tilde{f}(\tilde{\tau})}{d\tilde{\tau}} + \tilde{f}(\tilde{\tau})\cdot k\frac{d^2\tilde{\tau}}{d\tau^2}\left(\frac{d\tilde{\tau}}{d\tau}\right)^{k-1}\left(\frac{d\tau}{d\tilde{\tau}}\right)^{k+1} - \frac{k\gamma(\tau)\tilde{f}(\tilde{\tau})}{d\tilde{\tau}/d\tau}\right) d\tilde{\tau}^{k+1} \qquad \left(\frac{d\tau}{d\tilde{\tau}}\right)^{k+1}(d\tilde{\tau})^{k+1}$$

$\Rightarrow$ Transformation rule: $\boxed{\tilde{\gamma}(\tilde{\tau}) = \frac{\gamma(\tau)}{d\tilde{\tau}/d\tau} - \frac{d^2\tilde{\tau}/d\tau^2}{(d\tilde{\tau}/d\tau)^2}}$ —— $(**)$

e.g. $\tilde{\tau} = \overset{\text{Mö}}{\frac{a\tau+b}{c\tau+d}} \qquad \frac{d\tilde{\tau}}{d\tau} = \frac{ad-bc}{(c\tau+d)^2} = \frac{1}{(c\tau+d)^2} \qquad \frac{d^2\tilde{\tau}}{d\tau^2} = \frac{-2c}{(c\tau+d)^3}$

$\Rightarrow \tilde{\gamma}(\tilde{\tau}) = (c\tau+d)^2\gamma(\tau) + 2c(c\tau+d)$

No local inv.: Locally, may solve: $\omega = \phi\, d\tau$ s.t. $\nabla\omega = 0$ i.e. $\phi' - \gamma\phi = 0$

Then we set: $\omega = \phi\, d\tau =: dx \qquad x$: flat parameter

Notice that $(*)$ reads as: $\nabla f\, d\tau^{k+1} = \phi^k \frac{d}{d\tau}(f\phi^{-k})\, d\tau^{k+1}$

Def: (Projective Structure)
1-dim'l affine connection w/ Möbius transformation as symmetries is called a projective str.

Prop: (1) $\Omega d\tau^2$: quadratic differential, $\Omega = \frac{d\gamma}{d\tau} - \frac{1}{2}\gamma^2$ is an inv. under Möbius transf. (Check)

(2) γ can be reduced to 0 by Möbius transf. $\Leftrightarrow \Omega = 0$

pf of (2): ($\Rightarrow$) trivial ($\Leftarrow$) $\Omega = 0 \Rightarrow \frac{d\gamma}{d\tau} - \frac{1}{2}\gamma^2 = 0$

$$2\frac{d\gamma}{\gamma^2} = d\tau \Rightarrow 2\int \frac{d\gamma}{\gamma^2} = \tau - \tau_0 \Rightarrow -2\frac{1}{\gamma} = \tau - \tau_0 \Rightarrow \gamma = \frac{-2}{\tau - \tau_0}$$

Now, set $\tilde{\tau} := \frac{-1}{\tau - \tau_0} \Rightarrow \tilde{\gamma}(\tilde{\tau}) = -2(\tau - \tau_0) + 2(\tau - \tau_0) = 0$ $\circ$

Rmk: Recall that moduli of flat connection $\longleftrightarrow$ $\{$rep. of $\pi_1(X, x)\}$
$\Rightarrow$ No local inv.

Example: $L = -\frac{d^2}{dx^2} + u(x)$ on $D = S^1$ or $\mathbb{CP}^1$
$Ly_1 = Ly_2 = 0$ (loc.) Set $\tau = \frac{y_2(x)}{y_1(x)}$ as our new local coord. $\longrightarrow$ projective str.!

The non-trivial affine connection is defined s.t. x = flat coord.
$\Rightarrow \Omega d\tau^2 = 2u\,dx$. (Check)

Exercise: $p(\gamma, \gamma', \ldots)$: poly. s.t. $p\,d\tau^k$ (fixed k) is inv. under Möbius transf. for any affine γ, then $p = Q(\Omega, \nabla\Omega, \nabla^2\Omega, \ldots)$, where Q is a grd homogeneous, $\nabla^l\Omega$ has degree $l+2$.

Chazy Eqn: 3d Frobenius chart with $d = q_3 = 1$, $r = 0$. $E = t^1\partial_1 + \frac{1}{2}t^2\partial_2$
Ask for sol'n of WDVV, periodic in t^3, period $= 1$, analytic $(t^1, t^2, t^3) = (0, 0, i\infty)$

$$F(t^1, t^2, t^3) = \frac{1}{2}(t^1)^2 t^3 + \frac{1}{2}(t^1)(t^2)^2 - \frac{(t^2)^4}{16}\gamma(t^3), \quad \gamma(\tau) = \sum_{n \geq 0} a_n q^n, \quad q = e^{2\pi i \tau}$$

WDVV eqn becomes: $\gamma''' = 6\gamma\gamma' - 9\gamma'^2$ (cf. 1.13 (b))
Exer: $\gamma(\tau) = \frac{\pi i}{3}(1 - 24q - 72q^2 - 96q^3 - \cdots)$ → coeff. in $\mathbb{Z}$.

$d=1$, $SL(2;\mathbb{C})$ acts on the ODE ! $\Rightarrow$ Fits into the theory of affine connections

Eqns from $Q(\Omega, \nabla\Omega, \dots, \nabla^k\Omega) = 0$ $\Omega = d\gamma/d\tau - \frac{1}{2}\gamma^2$ Set $u = \frac{1}{2}\frac{\Omega d\tau^2}{w^2}$, where $w = dx$, $\nabla w = 0$

$\Omega = d\gamma/d\tau - \frac{1}{2}\gamma^2$

Set $u = \frac{1}{2}\frac{\Omega d\tau^2}{w^2}$, where $w = dx$, $\nabla w = 0$

$L = -\frac{d^2}{dx^2} + u(x) \to y_1(x), y_2(x)$ normalize y_1, y_2 by $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 1$

$\tau = y_2/y_1 \rightsquigarrow \gamma = \frac{d(y_1^2)}{dx} = \left(\frac{1}{\tau'}\right)'$ since $\frac{d\tau}{dx} = \frac{y_2' y_1 - y_2 y_1'}{y_1^2} = \frac{1}{y_1^2}$

$k=0$, $\Omega = 0$

$k=1$, $\nabla\Omega = 0 \Rightarrow \gamma'' - 3\gamma\gamma' + \gamma^3 = 0$

$k=2$, $\nabla^2\Omega + c\Omega^2 = 0$ for some c

$\Updownarrow$

$\gamma''' - 6\gamma\gamma'' + 9\gamma'^2 + (c-12)(\gamma' - \frac{1}{2}\gamma'^2)^2 = 0$ $c=12$ gives Chazy eqn !

In terms of u, the eqn is $u'' + 2cu^2 = 0$

<u>Observation</u>: $\wp'^2 = 4\wp^3 - g_2\wp - g_3 \Rightarrow 2\wp''\wp' = 12\wp^2\wp' - g_2\wp'$

$\Rightarrow \wp'' = 6\wp^2 - g_2/2$

Thus, $g_2 = 0$, $\underbrace{g_3 = 1}_{\text{normalize}}$ $u = \frac{-3}{c}\wp_0(x)$ $\wp_0$: equi-an-harmonic elliptic function.

<u>ODE</u> Lamé eqn $y'' + (A\wp + B)\wp = 0$

$y'' + \frac{3}{c}\wp_0(x)y = 0$ Let $t = 1 - \wp_0^3(x)$, we get:

$$t(t-1)\frac{d^2y}{dt} + \left(\frac{7}{6}t - \frac{1}{2}\right)\frac{dy}{dt} + \frac{y}{12c} = 0$$

March, 23rd, 2018.

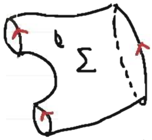
2D Topological field Theory (Atiyah's axioms)

1. Space of local physical states A i.e. $\dim_{\mathbb{C}} A = n < \infty$

2. $S=(\Sigma, \partial\Sigma) \longmapsto v_\Sigma \in A_{(\Sigma,\partial\Sigma)} := \begin{cases} \mathbb{C}, & \text{if } \partial\Sigma=\phi,\ \Sigma: \text{connected} \\ \bigotimes_{i=1}^{k} A_i, \quad A_i = \begin{cases} A, & \text{if } C_i \text{ has orientation induced from } \Sigma. \\ A^*, & \text{if not.} \end{cases} & \text{if } \partial\Sigma = C_1 \amalg \dots \amalg C_k,\ C_i \simeq S^1 \end{cases}$

(Σ: cpt, oriented surface)

e.g.



$v_{(\Sigma,\partial\Sigma)} \in A_{(\Sigma,\partial\Sigma)} = A^* \otimes A^* \otimes A = \mathrm{Hom}(A\otimes A, A)$

Also, we require the assignment is a top. inv. i.e. They are the same under homeo. s.t.

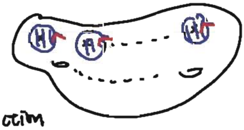
1) normalization :  $\xmapsto{v}$ $id \in A^* \otimes A$

2) multiplication : $v_{S_1 \amalg S_2} = v_{S_1} \otimes v_{S_2}$

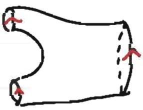
3) factorization : S'  $\xrightarrow{\text{glueing}}$  S

$v_{S'} \in A \otimes A^* \otimes \cdots$ Then v_S = contraction of $v_{S'}$ along the cut.

Def: • Genus g, s-pt correlator :

$v_{g,s} :=$ v of  genus $= g$

$v_{g,s} \in \otimes^s A^*$ symmetric function

• $c := v$ of  $\in A^* \otimes A^* \otimes A = \mathrm{Hom}(A\otimes A, A)$ gives the alge. str.

genus 0

• $\eta := v$ of  $\in A^* \otimes A^*$ i.e. $\eta = v_{0,2}$ symmetric pairing

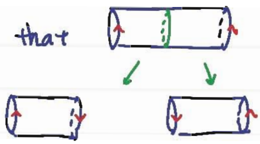
• $e := v$ of a cap (disk)  $\in A$

Thm: (c, η, e) gives A a Frobenius alge. str (primary chiral fields of the theory) pf: Associativity: $(\alpha*\beta)*\gamma = \alpha*(\beta*\gamma)$ $\quad \alpha,\beta,\gamma\in A$.

$g=0$

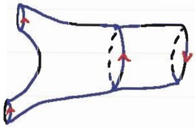
Unity: cut and glue a disk = do nothing

Non-degeneracy of η:

Let $\tilde{\eta} := v$ of  Then observe that 

$\Rightarrow \eta\tilde{\eta} = \mathrm{id}$

$\Rightarrow \eta$: invertible

Frobenius Property: $\eta(\alpha*\beta,\gamma) = v_{0,3}(\alpha,\beta,\gamma)$ via  symm. in α,β,γ.

Rmk: This use only the genus 0 part!

Actual Physics consideration:

QFT on D-dim'l mfd Σ

Data: • A family of local fields $\phi_\alpha(x)$ $x \in \Sigma$

• A Lagrangian $\mathcal{L} = \mathcal{L}(\phi, \phi', \phi'', \dots)$

$\rightarrow$ $S[\phi] = \int_\Sigma \mathcal{L}(\phi, \phi', \dots)$: action ← involves metric $g_{ij}(x)$

• Quantization via Feymann's path integrals:

partition function: $Z_\Sigma = \int_{\mathcal{A}} [d\phi]\, e^{-S[\phi]}$

$\mathcal{A} := \left(\begin{matrix} \text{The space of} \\ \text{all fields} \end{matrix} \right)$ "measure"

Correlation functions: $\langle \phi_\alpha(x), \phi_\beta(y), \dots \rangle_\Sigma := \int_{\mathcal{A}} [d\phi]\, \phi_\alpha(x)\, \phi_\beta(y) \dots e^{-S[\phi]}$

The classical theory is conformal if $\delta S = 0$, for any $\delta g_{ij} = \varepsilon g_{ij}$

topological if $\delta S = 0$ $\forall\, \delta g_{ij}$

TQFT: The correlation function depends only on the topo. on Σ, not on $x, y, \dots$

$\langle \phi_\alpha, \phi_\beta, \dots \rangle_g$ ← genus

Now, for a TFT arising from TQFT (or a TCFT, if integrable over all conf. classes of Σ)

May consider deformations preserving the top. inv. $L \mapsto L + \sum t^\alpha L_\alpha$

$\Rightarrow$ Moduli space (local)

Method: QFT with nilpotent symmetry $\underset{\text{charge}}{Q} : \mathcal{H} \longrightarrow \mathcal{H}$ ← some Hilbert space of states, $Q^2 = 0$

Observables := operator on $\mathcal{H}$ which commutes with Q i.e. $\underbrace{\{Q, \psi\}}_{\mathbb{Z}_2\text{-bracket}} = 0$

Q-cohomology on all operators:

By $\mathbb{Z}_2$-graded Jacobi identity: $\{Q, \{Q, \phi\}\} \pm \{Q, \{\phi, Q\}\} \pm \{\phi, \{Q, \cancel{Q}\}\} = 0$

$\Rightarrow \{Q, \{Q, \phi\}\} = 0$

Define: $A = \ker Q / \operatorname{im} Q$ (The primary states fields)

$\because$ Q is a symmetry $\Rightarrow \langle \{Q, \psi\}, \phi_1, \phi_2, \dots \rangle = 0$

(*) If for any primary field, $\phi_\alpha = \phi_\alpha(x)$, we may solve:

$$d\phi_\alpha(x) = \{Q, \underset{\text{1-form}}{\phi_\alpha^{(1)}(x)}\} \qquad d\phi_\alpha^{(1)}(x) = \{Q, \underset{\text{2-form}}{\phi_\alpha^{(2)}(x)}\}$$

Then $d_x \langle \phi_\alpha(x), \phi_\beta(y) \dots\dots \rangle = \langle \{Q, \phi_\alpha(x)\}, \phi_\beta(y) \dots\dots \rangle$

$\Rightarrow$ All correlators are topo-inv.

<u>Fact</u>: We can do (*) for 2d N=2 supersymmetry QFT by "twisting"

$$\{Q, \underbrace{\int_{\underset{\text{1-cycle}}{c}} \phi_\alpha^{(1)}}_{\text{observable}}\} = \int_c \{Q, \phi_\alpha^{(1)}\} = \int_c d\phi_\alpha = 0 \quad \Rightarrow$$

$$\{Q, \underbrace{\int_\Sigma \phi_\alpha^{(2)}}_{\text{observable}}\} = \int_\Sigma d\phi_\alpha^{(1)} = 0 \quad (\text{Assume } \partial\Sigma = \phi)$$

<u>Thm</u> (Dijkgraaf - Verlinde - Verlinde, 1991)

1) $L \longmapsto \tilde{L}(t) := L - \sum_{\alpha=1}^{n} t^\alpha \phi_\alpha^{(2)}$ preserve top-inv.

2) The family of primary chiral algebra A_t satisfies WDVV eqn

<u>Rmk</u>: This is the origin of WDVV.

A new axiom to TFT (Dubrovin):

The above "canonical moduli space" of a TCFT is a Frobenius mfd

<u>Rmk</u>: Cecotti - Vafa 1991: Top - anti top fusion: tt^* eqn as additional str

<u>Example</u>:
1. σ-model A (Gromov-Witten)
2. σ-model B (Calabi-Yau Moduli)
3. Landau-Ginzburg model (Singularity Theory)

Chazy Eqn: WDVV for $n=3$, $d=1$. analytic at $(0,0,i\infty)$

Geometric Realization: $\rightsquigarrow$ Universal torus

$$\mathcal{L} = \{ \text{The lattice } \mathbb{Z}^2 \simeq L \subset \mathbb{C} \}$$

$$L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \mathbb{Z}\,2\omega + \mathbb{Z}\,2\omega' \Rightarrow \dim_{\mathbb{C}} \mathcal{L} = 2.$$

$$\begin{tikzcd} \mathbb{C}/L \arrow[r, hook] \arrow[dr] & M \arrow[d] \\ & L \in \mathcal{L} \end{tikzcd} \qquad \dim_{\mathbb{C}} M = 3.$$

Invariant elliptic function on M: $\underset{(\omega,\omega')\in\mathcal{L}}{\overset{z\in\mathbb{C}/L}{f(z;\omega,\omega')}} = f(z+2n\omega+2m\omega';\omega,\omega')$

$= f(z; c\omega'+d\omega, a\omega'+b\omega)$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{Z})$

Weierstrass $\wp$-fcn:

e.g. $\wp \equiv \wp(z;\omega,\omega') = \dfrac{1}{z^2} + \displaystyle\sum_{a\in L^\times} \left(\frac{1}{(z-a)^2} - \frac{1}{a^2} \right)$ $\quad a = 2m\omega + 2n\omega'$

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3 \quad g_2(\omega,\omega') = 60\sum_{a\in L^\times}\frac{1}{a^4} \quad g_3(\omega,\omega') = 140\sum_{a\in L^\times}\frac{1}{a^6}$$

Frobenius-Stickelberger:

Euler v.f.: $\omega\dfrac{\partial}{\partial\omega} + \omega'\dfrac{\partial}{\partial\omega'} + z\dfrac{\partial}{\partial z}$ i.e. f: inv. elliptic $\Rightarrow$ Ef inv. elliptic

(tautological one)

Weierstrass ζ-fcn: $\zeta(z;\omega,\omega') = \dfrac{1}{z} + \displaystyle\sum_{a\in L^\times}\left(\frac{1}{z-a} + \frac{1}{a} + \frac{z}{a^2}\right)$ is not elliptic

$\zeta' = -\wp$

$$\zeta(z+2m\omega+2n\omega';\omega,\omega') = \zeta(z;\omega,\omega') + 2m\eta + 2n\eta'$$

qsi-period $\quad \eta = \zeta(\omega) \quad \eta' = \zeta(\omega')$

behaves well under $SL(2;\mathbb{Z})$

The 2nd Euler v.f:

$$\eta\frac{\partial}{\partial\omega} + \eta'\frac{\partial}{\partial\omega'} + \zeta\frac{\partial}{\partial z} \quad (\text{o.k.})$$

• Possible Topic for Report:
Y. Manin: Quantum Cohomology / Gromov-Witten Theory
C. Hertling: Saito's Singularity Theory
S. Barannikov, M. Kontsevich: $\frac{1}{2}\infty$ - Variation of Hodge Str / Calabi-Yau Moduli
Sabbah: Isomonodromic Deformation of ODE

• Isomonodromic Deformation as Frobenius Mfds:
"moduli space of ODE over $\mathbb{P}^1$" Riemann-Hilbert Problem
For our purpose, we consider only the simplest case with one reg and one irreg. sing. on $z \in \mathbb{P}^1$ → Birkhoff normal form

$$\Lambda \psi = 0, \text{ where } \Lambda = \frac{d}{dz} - U - \frac{1}{z} V, \quad U, V \in M_n(\mathbb{C})$$

$\rightsquigarrow$ reg. at $z=0$, irreg. at $w = \frac{1}{z} = 0$
Assume that U has distinct eigenvalues $U \sim \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ $\lambda_i \neq \lambda_j$

(In fact, may assume that U is "regular" i.e. (min-poly = char. poly) $\underset{\text{equiv.}}{\equiv}$
Only one Jordan block for each eigenvalues)
V: skew-symmetry
The general thm of isomonodromic deformation
$\Rightarrow$ The space of all isomonodromic deformation has coord. $(u_1, \dots, u_n) = \operatorname{diag} U$, called the space $M(\Lambda)$ $\rightsquigarrow$ $V = V(u)$

<u>Thm</u>: A Frob. mfd str on $M(\Lambda)$ is defined by: $\partial_i := \frac{\partial}{\partial u_i}$
1) $\partial_i \cdot \partial_j = \delta_{ij} \partial_j$ i.e. ∂_i: idempotent
2) $\langle , \rangle := \sum_{i=1}^n \psi_i^2(u) (du^i)^2$, where $\psi = \begin{pmatrix} \psi_1(u) \\ \vdots \\ \psi_n(u) \end{pmatrix}$ is an eigenvector of $V(u)$
3) $e = \sum_{i=1}^n \partial_i$
4) $E := \sum_{i=1}^n u^i \partial_i$

$\rightsquigarrow$ This gives $M(\Lambda)$ a semi-simple Frob. mfd str.
Different eigenvectors gives Legendre-type transf.

Two gradients associated to Frob. mfd:

- Dubrovin connection: $z \in \mathbb{C}$, $\tilde{\nabla}^z_u v := \nabla_u v + z u \cdot v$ (D1)

Lemma: $\tilde{\nabla}^z$ is flat for all z $\Longleftrightarrow$ • is associativite and $\exists$ potential

pf: flat iff $[\tilde{\nabla}^z_\alpha, \tilde{\nabla}^z_\beta] = 0$ $\quad \tilde{\nabla}^z_\alpha \tilde{\nabla}^z_\beta \partial_\gamma = \tilde{\nabla}^z_\alpha (\nabla_\beta \gamma + z C^\varepsilon_{\beta\gamma} \partial_\varepsilon)$

$= \tilde{\nabla}^z_\alpha (z \cdot C^\varepsilon_{\beta\gamma} \partial_\varepsilon) = z (\partial_\alpha C^\varepsilon_{\beta\gamma}) \partial_\varepsilon + z^2 C^\delta_{\alpha\varepsilon} C^\varepsilon_{\beta\gamma} \partial_\delta$

$\Rightarrow [\tilde{\nabla}^z_\alpha, \tilde{\nabla}^z_\beta](\partial_\gamma) = 0$ iff $z (\partial_\alpha C^\varepsilon_{\beta\gamma} - \partial_\beta C^\varepsilon_{\alpha\gamma}) \partial_\varepsilon + z^2 (C^\delta_{\alpha\varepsilon} C^\varepsilon_{\beta\gamma} - C^\delta_{\beta\varepsilon} C^\varepsilon_{\alpha\gamma}) \partial_\gamma = 0$

iff $\begin{cases} \partial_\alpha C^\varepsilon_{\beta\gamma} - \partial_\beta C^\varepsilon_{\alpha\gamma} = 0 \rightsquigarrow \text{potential} \\ C^\delta_{\alpha\varepsilon} C^\varepsilon_{\beta\gamma} - C^\delta_{\beta\delta} C^\varepsilon_{\alpha\gamma} = 0 \rightsquigarrow \text{WDVV} \end{cases}$ $\square$

We need to ask for the differentiation in z-direction $\quad M \times \mathbb{P}^1$, t^i, $\overset{\Psi}{z}$

(D2) $z \partial_z \xi_\alpha = z E^\gamma(t) C^\beta_{\gamma\alpha}(t) \xi_\beta + Q^\gamma_\alpha \xi_\gamma \longrightarrow$ ODE! (fixed t)

flatness of (D1) $\Leftrightarrow$ compatibility of the linear ODE: $\partial_\alpha \xi_\beta = z C^\gamma_{\alpha\beta}(t) \xi_\gamma$

$\Leftrightarrow \exists$ n linear-indep sol'n.

New Euler v.f.: $\mathcal{E} = z \frac{\partial}{\partial z} - E$

So, $z \partial_z \xi = \mathcal{L}_E \xi = \iota_E d\xi + d\iota_E \xi = z E^\gamma(t) C^\beta_{\gamma\alpha}(t) \xi_\beta + Q^\gamma_\alpha \xi_\alpha$, $\quad Q = \nabla E$

(D2) $\Rightarrow \partial_z \xi_\alpha = \underbrace{E^\gamma(t) C^\beta_{\gamma\alpha}(t)}_{U} \xi_\beta + \frac{1}{z} \underbrace{Q^\gamma_\alpha}_{V} \xi_\gamma$

- Intersection form on T^*M (Saito)

First, $\cdot : TM \otimes TM \longrightarrow TM$ induces multiplication str on T^*M.

$(\omega_1, \omega_2)^* := \iota_E (\omega_1 \cdot \omega_2)$

Exer: $(E \cdot u, v) = \langle u, v \rangle$ on TM $\rightsquigarrow$ non-deg. near t^1-axis when t small

original flat metric

Lemma: In flat coord. define $\Gamma^{ij}_k := (dx^i, \nabla_k dx^j)^* = -g^{is} \Gamma^j_{sk}$ on TM

Then $\Gamma^{\alpha\beta}_\gamma = (\frac{d+1}{2} - q_\beta) C^{\alpha\beta}_\gamma$, $\quad C^{\gamma\delta}_\alpha := C^\gamma_{\alpha\beta} \eta^{\beta\delta}$

pf: (Calculation).

Prp: ① $(,)^* \rightsquigarrow g^{ij}$, ② $\langle , \rangle^* \rightsquigarrow \eta^{ij}$ forms a flat pencil i.e.

(1) $h^{ij} := g^{ij} + \lambda \eta^{ij}$ is flat $\forall \lambda$

(2) $\Gamma^{ij}_k = \Gamma^{ij}_{① k} + \lambda \Gamma^{ij}_{② k}$

Now, let M be s.s. Frobenius mfd i.e. T_tM is s.s. (no nilpotent element) for a generic pt. $t \in M$ ($\Rightarrow \exists e_i$ s.t. $e_i \cdot e_j = \delta_{ij} e_j$)

Main lemma (Canonical coord.):

In a nbd of s.s. point $t \in M$, $\exists u^1, \dots, u^n$ s.t. $\partial_i \cdot \partial_j = \delta_{ij} \partial_j$, $\partial_i := \frac{\partial}{\partial u^i}$

pf: $\exists$ v.f. $V_1, \dots, V_n$ (smooth) in a nbd of t s.t. $V_i \cdot V_j = \delta_{ij} V_i$

Claim: $[V_i, V_j] = 0 \quad \forall i, j$

pf: Let $[V_i, V_j] := \sum_k f_{ij}^k V_k$. Now, $R \equiv 0$ (curvature op.) for the Dubrovin connection $\tilde{\nabla}^z$

Write $\tilde{\nabla}^z_{V_i} V_j = \sum_k A_{ij}^k V_k \quad \tilde{\nabla}^z_{V_i} V_j = \nabla_{V_i} V_j + z V_i \cdot V_j = \Gamma_{ij}^k V_k + z \delta_{ij} V_j$

$\Rightarrow \quad 0 = R(V_i, V_j) V_k = [\tilde{\nabla}^z_{V_i}, \tilde{\nabla}^z_{V_j}] V_k - \tilde{\nabla}_{[V_i, V_j]} V_k$

z^1-coeff: $0 = \sum_l (\Gamma_{jk}^l \delta_{il} + \Gamma_{ik}^l \delta_{jk} - \Gamma_{ik}^l \delta_{jl} - \Gamma_{jk}^l \delta_{ik} - f_{ij}^l \delta_{lk}) V_l$

Take $l = k$. $\Rightarrow f_{ij}^k = 0$ □

Prop: Near a semi-simple pt. $t \in M$, all roots of

(*) $\det(g^{\alpha\beta}(t) - u\eta^{\alpha\beta}) = 0$ are simple and gives canonical coord. (u^i)

(g: $(\,,)^*$, η: $\langle\,,\rangle^*$)

Conversely, if (*) has simple roots near $t \in M$, then t is a s.s. pt. and the roots (u^i) are canonical coord.

pf: ($\Rightarrow$) In canonical coord. given in the main lemma, $\Rightarrow$

$du^i \cdot du^j = \eta_{ii}^{-1} du^i \delta_{ij} \quad (\langle \partial_i, \partial_j \rangle = \langle e, \partial_i \cdot \partial_j \rangle = \langle e, \delta_{ij} e_i \rangle = \eta_{ii}(u) \delta_{ij})$

$\because \; e = \sum_i \partial_i$

$E = \sum_i u^i \partial_i$ (After a shift) Exer: Show that $E = \sum (u^i + c^i) \partial_i$ using the axiom: $\mathcal{L}_E \cdot = \cdot$ (1.50b) or (1.9) $\mathcal{L}_E F = d_F F + (\text{quadratic})$

$\Rightarrow g^{ij}(u) = u^i \eta_{ii}^{-1} \delta_{ij}$

$(g^{ij} = (du^i, du^j)^* = i_E(du^i \cdot du^j) = i_E(\eta_{ii}^{-1} du^i \delta_{ij}) = u^i \eta_{ii}^{-1} \delta_{ij})$

Then (*) becomes $\prod_{i=1}^n (u^i - u) = 0 \Rightarrow$ Roots $u = u^i \quad i = 1, \dots, n$.

($\Leftarrow$) Consider $U = (U^\alpha_\beta(t))$: matrix of $E\cdot$ on T_tM

By exercise $(E \cdot v, w) = \langle v, w \rangle \Rightarrow U^\alpha_\beta = g^{\alpha\varepsilon} \eta_{\varepsilon\beta}$

The char. poly. of U coincide with $(*)$.

If $E_\cdot$ has simple eigenvalues ($\Rightarrow E_\cdot$ is s.s. i.e. diagonalizable)

$\Rightarrow$ Any other $V_\cdot$ also has diagonalization $\Rightarrow V_\cdot$ is s.s. but it may not have simple roots $\Rightarrow T_eM$ semi-simples. $\square$

Recall: s.s. frob. mfd $\rightsquigarrow$ canonical coord. $u^1, \ldots, u^n$

canonical coord. are orthogonal $\Rightarrow$ $ds_1^2 = \sum_{i=1}^{n} \frac{\eta_{ii}^{(u)}(u)}{u_i^2} du_i \otimes du_i$ $ds_0^2 = \sum_{i=1}^{n} \eta_{ii}^{(u)}(u) du^i \otimes du^i$

ds_1^2: intersection form ds_0^2: original flat metric

$e = \sum_{i=1}^{n} \partial_i$, where $\partial_i := \frac{\partial}{\partial u^i}$

$$\eta_{ii}^{(u)}(u) = \langle \partial_i, \partial_i \rangle = \langle e \cdot \partial_i, \partial_i \rangle = \langle e, \partial_i \cdot \partial_i \rangle = \langle \frac{\partial}{\partial t^1}, \partial_i \rangle \overset{(*)}{=} dt_1(\partial_i) = \partial_i(t_1)$$

$(*)$: $dt_1 = \eta_{ij}^{t} dt^j$ $\langle \frac{\partial}{\partial t^1}, v \rangle = \eta_{ij}^{(t)} v^j = (dt_1)(v)$

$\rightsquigarrow$ We find that the the metric $\eta_{ii}^{(u)}$ is derivative of the function t_1

Therefore, t_1 is a local function, serving as the "metric potential"

We denote t_1 by ρ.

Flatness of ds_0^2 in canonical coord := Darboux – Egoroff System

$$\Gamma_{ij}^{k} = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \qquad g^{ii} = \frac{1}{\rho_{ij}}$$

For $i \neq j$, $\Gamma_{ij}^{i} = \frac{\rho_{ij}}{2\rho_i}$ $\Gamma_{ij}^{j} = \frac{\rho_{ij}}{2\rho_i}$

$i = j$, $\Gamma_{ii}^{i} = \frac{\rho_{ii}}{2\rho_i}$ $\Gamma_{ii}^{k} = \frac{-\rho_{ik}}{2\rho_i}$ $(k \neq i)$

Def: Rotation coeff. $\gamma_{ij} = \frac{\partial_j \sqrt{\rho_i}}{\sqrt{\rho_j}} = \frac{\rho_{ij}}{2\sqrt{\rho_i \rho_j}}$: symmetry in i, j

Prop: 1) $\partial_k \gamma_{ij} = \gamma_{ik} \gamma_{kj}$ for i, j, k distinct

2) $\sum_{k=1}^{n} \partial_k \gamma_{ij} = 0$

$\Big\}$ flatness of ds_0^2

3) $\sum_{k=1}^{n} u^k \partial_k (\gamma_{ij}) = -\gamma_{ij}$ $\}$ flatness of e

pf: $0 = R_{ijk} \Leftrightarrow \nabla_i \nabla_j \partial_k = \nabla_j \nabla_i \partial_k$ $(i \neq j)$

For $k \neq i, j$, get: $\nabla_i \left(\frac{1}{2} \frac{\rho_{jk}}{\rho_j} \partial_j + \frac{1}{2} \frac{\rho_{jk}}{\rho_k} \partial_k \right) - \nabla_j \left(\frac{1}{2} \frac{\rho_{ik}}{\rho_i} \partial_i + \frac{1}{2} \frac{\rho_{ik}}{\rho_k} \partial_k \right)$

Claim: Expand this, we get $(*)$: $\frac{1}{2} g_{ijk} = \frac{1}{4} \left(\frac{\rho_{ij}\rho_{ik}}{\rho_i} + \frac{\rho_{ji}\rho_{jk}}{\rho_j} - \frac{\rho_{ki}\rho_{kj}}{\rho_k} \right)$

In fact, (1) are the same as $(*)$.

(2) comes from the case $k = i \neq j$

See the pic.

There is an easy to get (2):

<u>Lemma</u>: $e\rho_i = \sum_{k=1}^{n} \rho_{ik} = 0 \iff e$ is flat

pf: $\nabla_i e = \nabla_i(\partial_i + \sum_{j\neq i} \partial_j) = \sum_{j=1}^{n} \Gamma_{ij}^{i} \partial_i = \sum_{j=1}^{n} \frac{\rho_{ij}}{2\rho_i} \partial_i = \sum_{j=1}^{n} \frac{e\rho_i}{2\rho_i} \partial_i = 0$

$\iff e\rho_i = 0$ $\square$

$\rightsquigarrow \sum_{k=1}^{n} \rho_{ijk} = 0 \longrightarrow$ direct to get (2).

<u>Rmk</u>: $\begin{cases} \partial_k \gamma_{ij} = \gamma_{ik}\gamma_{kj} & i,j,k = \text{distinct} \\ \sum_{k=1}^{n} \partial_k \gamma_{ij} = 0 \end{cases}$ Darboux-Egoroff System

does NOT need the flatness of e.

- Scaling structure:

Recall: Euler v.f. E is a v.f. s.t. $\mathcal{L}_E g = Dg$, $D \in \mathbb{C}$ ($D = 2-d$) $(d_1 = 1)$, $\mathcal{L}_E \circ = 0$

$\Rightarrow \deg \eta_{ii} = -d$, $\deg(\gamma_{ij}) = -1$, $\deg(t_1) = -d+1 = \deg(\rho)$

In particular, (c) $\sum_{k=1}^{n} u^k \partial_k \gamma_{ij} = -\gamma_{ij}$ $\qquad \gamma_{ij} = \frac{\partial_j \sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}}$

<u>Cor</u>: Darboux system is the compatibility eqns of the linear system on $\psi = \begin{pmatrix} \psi_1(u) \\ \vdots \\ \psi_n(u) \end{pmatrix}$ Linear system $\begin{cases} \partial_k \psi_i = \gamma_{ik}\psi_k & (k \neq i) \\ \sum_{k=1}^{n} \partial_k \psi_i = 0 \end{cases} \longleftrightarrow \partial_k \psi = [\Gamma, E_k]\psi$

$\Gamma = (\gamma_{ij})$ $\quad E_k = E_{kk} = \begin{array}{c} \begin{array}{ccc} & k & \end{array} \\ \begin{pmatrix} 0 & \vdots & 0 \\ \cdots & 1 & \cdots \\ 0 & \vdots & 0 \end{pmatrix} k \end{array}$

Regard $[\Gamma, E_k]$ as connection form, compatible eqn $\longleftrightarrow$ connection is flat

(2) $\Rightarrow \partial_i \gamma_{ij} + \partial_j \gamma_{ij} = -\sum_{k\neq i,j} \partial_k \gamma_{ij} \underset{(1)}{=} -\sum_{k\neq i,j} \gamma_{ik}\gamma_{jk}$

(3) $u^i \partial_i \gamma_{ij} + u^j \partial_j \gamma_{ij} = -\gamma_{ij} - \sum_{k\neq i,j} u^k \gamma_{ik}\gamma_{kj}$

Cramers' rule $\Rightarrow \partial_i \gamma_{j} = \frac{1}{u^j - u^i}\left(\sum_{k\neq i,j} (u^j - u^k)\gamma_{ik}\gamma_{kj} - \gamma_{ij} \right)$

Write $V(u) := [\Gamma(u), U]$, where $U = \begin{pmatrix} u^1 & & \\ & \ddots & \\ & & u^n \end{pmatrix} = \sum_k u^k E_k$

$V_{ij} = (u^j - u^i)\gamma_{ij} \rightsquigarrow$ skew symm.

Lemma: Darboux System $\Leftrightarrow$ $(*)$: $\partial_k V = [[\Gamma, E_k], V]$ (Lax Pair)
$k = 1, \dots, n.$

Lax Pair: $\frac{dV}{dt} = [P, V] \Rightarrow$ Eigenvalue of V are const. in t.

Then $\partial_k V = [[\Gamma, E_k], V] \rightsquigarrow V$: parallel

Cor: 1) V acts on sol'n space of the linear system
2) Eigenvalues of V are indep. of u.
3) A sol'n ψ with $V\psi = \mu\psi \Leftrightarrow \psi(cu) = c^{\mu}\psi(u)$.

pf: 1) $\partial_k(V\psi) = ([\Gamma, E_k]V - V[\Gamma, E_k])\psi + V[E, \Gamma_k]\psi = [\Gamma, E_k](V\psi)$
2) done ($\because$ V is parallel under this connection)
3) $\sum_{k=1}^{n} u^k \partial_k \psi = \sum_{k=1}^{n} u^k [\Gamma, E_k]\psi = [\Gamma, U]\psi = V\psi$

Now, $V\psi = E(\psi) = \mu\psi \Rightarrow \psi$ is homogeneous of weight μ. $\square$

Exercise: Supply the detail pfs of prop. 3.5 and 3.6.

Cor: (s.s. Frobenius mfd modulo generalized Legendre type transf)

$\updownarrow$

(sol'n of $(*)$ with diagonalizable $V(u)$)

Adding the spectral parameter z, let $\Lambda = \frac{\partial}{\partial z} - U - \frac{1}{z}V(u)$

Prop: Darboux system (Equivalently, $(*)$) is the compatibility eqn for

$(\mathrm{I}^u)\begin{cases} \partial_k \psi_i = \gamma_{ik}\psi_k \\ \sum_{k=1}^{n} \partial_k \psi_i = z\psi_i \end{cases}$ and (II^u) $\Lambda\psi = 0$, where now $\psi = \begin{pmatrix} \psi_1(u, z) \\ \vdots \\ \psi_n(u, z) \end{pmatrix}$

proof is straightforward.

Rmk: These are Dubrovin connection I, II via $\psi_i^{\alpha}(u, z) = z^{\frac{d}{2}-1} \dfrac{\partial_i \tilde{t}^{\alpha}(t(u), z)}{\sqrt{\eta_{ii}(u)}}$

$\tilde{t}^{\alpha}$: flat coord. for Dubrovin connection (c.f. (3.3), ex. 3.1)

<u>Recall</u>: $\Lambda\Psi = 0$, $\Lambda = \frac{\partial}{\partial z} - U(u) - \frac{1}{z}V(u)$ Birkhoff Normal Form

ODE on $\mathbb{P}^1$ $U(u) = \begin{pmatrix} u^1 & & \\ & \ddots & \\ & & u^n \end{pmatrix}$ $u^i \neq u^j$ $V(u) = [\Gamma(u), U(u)]$ $\Gamma(u) = (\gamma_{ij})$

analytic continuation

Φ_0: fund. sol'n matrix

When $z = 0 \Rightarrow$ reg. singular pt.

Monodromy matrix M_0, eigenvalue $M_0 = (\mu_1, \dots, \mu_n)$

$\mapsto$ determine the local sol of ODE (e.g. if $\mu_i \neq \mu_j \pmod{\mathbb{Z}}$)

const. coord. transf. $\parallel$ monodromy matrix

$z = \infty$ $z = \frac{1}{w}$ $\frac{\partial}{\partial z} = \frac{\partial w}{\partial z}\cdot\frac{\partial}{\partial w} = \frac{-1}{z^2}\frac{\partial}{\partial w} = -w^2\frac{\partial}{\partial w}$

$\Lambda \sim \frac{\partial}{\partial w} + \frac{1}{w^2}U(u) + \frac{1}{w}V(u)$ Poincaré rank $r := h-1$

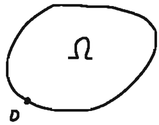
h

$h \geq 2$: $y' = \frac{dy}{dw} = \frac{a(w)}{w^h}y$ $a(w)$: holo. in w or asymptotic expansion of some holo. fcn f in w.

$y = \vec{y}$ in $\mathbb{C}^n$

f: holo. on Ω

When $n = 1$, $y = c\exp\left(\int \frac{a(w)}{h}dw\right)$



$\Omega \subset \mathbb{C}$

$a(w) = a_0 + a_1 w + \dots + a_{h-1}w^{h-1} + r(w)$

$y = c\exp\left(\sum_{i=0}^{h-2}\frac{a_i}{w^{(h-1)-i}}\cdot\frac{-1}{(h-1)-i} + a_{h-1}\log w\right)\cdot\exp\left(\int r(w)dw\right)$

$= c(w)\, w^{a_{n-1}}\exp\left(\underbrace{\sum_{i=0}^{h-2}\frac{a_i}{w^{(h-1)-i}}}_{\text{irreg. part}}\right)$

$c\exp\left(\int r(w)dw\right)$

$a(w) = a_0 + a_1 w + a_2 w^2 + \cdots$ formal power series

a is asymp. to f if $\lim_{\substack{w\to 0 \\ \text{in } \Omega}} |w|^{-k}\left(f - \sum_{i=0}^{k} a_i w^i\right) = 0$ $\forall k = 0, 1, 2, \dots$

$\rightarrow$ We can inductively solve a_i, for all i.

How to generalize this to n-dim'l ?

$\rightsquigarrow$ diagonalize

- Block-diagonalization Process :

Set $y = pZ$ $p: n\times n$, $\det p(0) \neq 0$ Gauge transf.

$y' = p'Z + pZ' = \frac{a(w)}{w^h}pZ \Rightarrow Z' = \frac{b(w)}{w^h}Z$, where $pb := ap - w^h p'$

Expect: $\exists p$ s.t. b is diagonal

Set $p = \sum_{j=0}^{\infty} p_j w^j$, $b = \sum_{j=0}^{\infty} b_j w^j$ (Let the given $a(w) = \sum_{j=0}^{\infty} a_j w^j$)

Then $p_0 b_0 - a_0 p_0 = 0$

$(*)$ $p_k b_0 - a_0 p_k = \sum_{j=0}^{k-1}(a_{k-j}p_j - p_j b_{k-j}) - (k-h+1)p_{k-h+1}$, $\forall h \geq 1$

Assume that a_0 has 2 sets of eigenvalues $\{\lambda_1, \dots, \lambda_p\} \ni \lambda_i$, $\{\lambda_{p+1}, \dots, \lambda_n\} \ni \lambda_j^*$ $\forall i, j$

May assume
$$a_0 = \begin{array}{c} \scriptstyle p \\ \left(\begin{array}{c|c} a_0^{11} & 0 \\ \hline 0 & a_0^{22} \end{array}\right) \\ \scriptstyle n-p \end{array}$$
(via a linear change P_0)

Now, we may set $p_0 = Id.$, then $b_0 = a_0$

$\rightsquigarrow (*)' : [P_k, a_0] = -b_k + H_k \quad (k \geq 1)$

$\because h > 1$, H_k deps only on p_j, b_j with $j < k$.

<u>Ansatz</u>: Set $b_k = \left(\begin{array}{c|c} b_k^{11} & 0 \\ \hline 0 & b_k^{22} \end{array}\right)$ $\quad p_k = \left(\begin{array}{c|c} 0 & p_k^{12} \\ \hline p_k^{21} & 0 \end{array}\right)$ $(k \geq 1)$ $\quad p_0 = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right)$

Then $(*)'$ becomes:
$$\left(\begin{array}{c|c} 0 & p_k^{12} a_0^{22} - a_0^{11} p_k^{22} \\ \hline p_k^{21} a_0^{11} - a_0^{22} p_k^{21} & 0 \end{array}\right) = \left(\begin{array}{c|c} -b_k^{11} + H_k^{11} & H_k^{12} \\ \hline H_k^{21} & -b_k^{22} + H_k^{22} \end{array}\right)$$

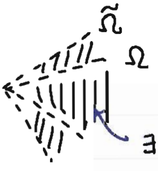
$\because a_0^{11}, a_0^{22}$ has no common eigenvalues

$\Rightarrow$ p_k^{12} and p_k^{21} are uniquely solved (Ex.)

$b_k^{11} = H_k^{11} \qquad b_k^{22} = H_k^{22}$

<u>Thm</u>: The formal fund. sol'n matrix $Y_f = P_0 C \exp(\Lambda(w))$, is given by:
$$P_0 \underbrace{\left(\sum_{j=0}^{\infty} \psi_j w^j\right)}_{C(w)} \exp\left(-\sum_{k=1}^{h-1} \frac{b_{(h-1)-k}}{k} \cdot \frac{1}{w^k} + b_{h-1} \log w\right)$$

diagonal matrix



holo.

$\exists$ fund. sol'n matrix y in Ω

<u>Question</u>: How large could Ω be s.t. $Y_f \sim y$ in Ω?
(In the sense after taking away the factor $e^{\Lambda(w)}$)

Now, for another $\tilde{y}$ in $\tilde{\Omega}$, $\zeta := y^{-1}\tilde{y}$ is a const. in $\Omega \cap \tilde{\Omega}$

$\Rightarrow \zeta = \lim_{w\to 0} \exp(-\Lambda(w))\,(I + O(w))\,\exp(\Lambda(w))$ is const.

$\Rightarrow \zeta_{ij} = \lim_{w\to 0} \exp(\Lambda(w)_{jj} - \Lambda(w)_{ii})\,(\delta_{ij} + O(w)) \Rightarrow S_{ii} = 1 \quad \forall i$

To get $S = \mathrm{id.}$, we must have that: $\forall i < j$,

$\Omega \supset \ell := \{ w \mid \mathrm{Re}(\lambda_i - \lambda_j) w^{-(n-1)} = 0 \}$ $\quad r := n-1 \quad \lambda_i, \lambda_j$: eigenvalue of a_0

$(*)_1$

$(*)_2$: For ε small, $\mathrm{Re}(e^{i\varepsilon}(\lambda_i - \lambda_j) w^{-(n-1)} > 0)$

$\Rightarrow S_{ij} = 0$ (Let $w \to 0$ in $e^{i\varepsilon}(\lambda_i - \lambda_j) w^{-(n-1)}$ or $e^{-i\varepsilon}(\lambda_i - \lambda_j) w^{-(n-1)}$

$\exists\ 2r$ rays ℓ's for any fixed $i<j$ with $(*)_1$, $(*)_2$

$\rightarrow$ Stokes rays $\ell_{ij}^{(1)}, \dots, \ell_{ij}^{(2r)}$

<u>Def</u>: Ω is a Stokes sector if Ω contains exactly one Stokes ray for each $i<j$

$\overset{\shortparallel}{\{ w \in \mathbb{C} \mid \arg(w) \in (\theta_1, \theta_2) \}}$

<u>Exercise</u>: $\Omega = \angle(\theta - \delta, \theta + \frac{2\pi}{2r})$ with δ small is a Stokes sector,

and others are $\Omega_j = \exp\left(\frac{(j-1)\pi i}{r}\right)\Omega \quad j = 1, \dots, 2r \quad \Omega_{2r+1} := \Omega_1$

$\Omega_1, \dots, \Omega_{2r}$: open cover of nbd of 0

$\Omega_i \cap \Omega_j$ contains no rays $\Rightarrow S_j := y_j^{-1} y_{j+1}$ is triangular after reordering of indices

$(y_j$: fund. sol'n on $\Omega_j)$

<u>Def</u>: S_j $(j = 1, \dots, 2r)$ are called the Stokes matrices $T = S_1 \cdots S_{2r}$: (usual) monodromy

(local) Stokes phenomenon $Sph = \{ b_0, \dots, b_r, S_0, \dots, S_{2r} \}$

<u>Exercise</u> Two (local) ODE are equiv. iff Sph are the same.

(Hint: $y' = Ay$, $\tilde{y}' = \tilde{A}\tilde{y}$ Let $g(w) = \tilde{y}_1 y_1^{-1}$ on Ω_1, analytic conti g to $\Omega_2, \dots, \Omega_{2r}$, and prove boundedness on $B_0^{\times}$)

Stokes Str in our Special case:

$$L = \frac{d}{dz} - A(z), \quad A(z) = U + O(1/z) \qquad (*): A^t(-z) = A(z)$$

(In our case, $A = U(u) + \frac{1}{z}V(u)$ $\quad (*) \Leftrightarrow V$ is skew-symme).

Exercise (Birkhoff) Any L with $(*)$-condition can be transformed to normal form $\Lambda = \frac{d}{dz} - (U(u) + \frac{1}{z}V(u))$ loc. near ∞, via gauge transf. $g(z) = I + O(1/z)$ with $g(z)g^t(-z) = Id$.

Lemma: $(*) \Leftrightarrow$ constancy of inner product of in z.

$\psi' = A\psi \quad \varphi' = A\varphi \quad \langle \psi, \varphi \rangle := \psi^T(-z)\varphi(z)$

Then $\langle \psi, \varphi \rangle' = (\psi^t(-z)\varphi(z))' = -\psi^t(-z)A^t(-z)\varphi(z) + \psi^t(-z)A(z)\varphi(z) = 0$

$\Leftrightarrow (*)$

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Now, let $r=1$, for the $z=\infty$ sing. pt. $\forall i \neq j$, has two rays

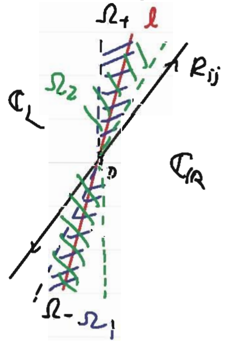
$\mathrm{Re}(z(u^i - u^j)) = 0$

$\mathrm{Re}(e^{i\varepsilon} z(u^i - u^j)) > 0$ for ε small

Stokes sector $\Omega_1 \cap \Omega_2 = \Omega_+ \sqcup \Omega_-$

$$\mathbb{C} \setminus \ell = \underset{\underset{\Psi_R}{\uparrow}}{\mathbb{C}_R} \sqcup \underset{\underset{\Psi_L}{\uparrow}}{\mathbb{C}_L}$$

fund. sol'n matrix



S_+ on Ω_+ $\quad S_-$ on Ω_-

$S := S_+ = S_1$; $S_- = S_2^{-1}$

$$\begin{cases} \Psi_R S_1 = \Psi_L & \text{on } \Omega_+ \\ \Psi_L S_2 = \Psi_R & \text{on } \Omega_- \end{cases} \quad \Updownarrow \quad \Psi_R S_- = \Psi_L$$

Prop: $(*) \Leftrightarrow \overset{S_2}{\overset{\|}{S_-^{-1}}} = S_+^t = S^t$

→ We call S the Stokes matrix for Λ.

Pf: $(\Leftarrow)$ Consider

$$\left.\begin{array}{lll} \Psi_L(z)\Psi_L^t(-z) \text{ on } \Omega_2 & \overset{z\in\Omega_1}{=} & \Psi_R(z) S_+ \Psi_L^t(-z) \\ \Psi_R(z)\Psi_R^t(-z) \text{ on } \Omega_1 & \underset{z\in\Omega_2}{=} & \Psi_R(z) S_-^{-t} \Psi_L^t(-z) \end{array}\right\} \Rightarrow \text{glued to a fcn } F \text{ on } \mathbb{C}^\times$$

At $z=\infty$, $F \sim (1 + O(1/z))e^{zU}e^{-zU^t}(1 + O(1/z))$ → bdd at $z=\infty$

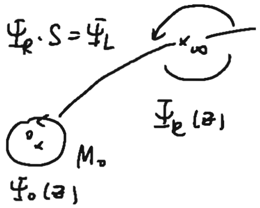
Also, check $z=0$, F is bdd

Then by Liouville thm $\Rightarrow \Psi_L(z)\Psi_L^t(-z) = \text{const.}$ (By $F \sim id$ at $z=\infty$)

$\Psi_R(z)\Psi_R^t(-z) = \text{const.}$

$\Rightarrow \Psi_L(z)\Psi_L^t(-z) = Id. \Rightarrow \Psi_L^t(-z)\Psi_L(z) = Id. \Rightarrow \langle \Psi_L, \Psi_L \rangle = \text{const.} \Rightarrow (*)$

Converse is just reverse the process.



Monodromy data for $\Lambda\Psi = 0$.

$(S, M_0, C, \mu_1, \dots, \mu_n)$

$\uparrow$ Stokes matrix; $\downarrow$ monodromy; $\searrow$ connection matrix defined by $\Psi_0(z) = \Psi_R(z) C$; $\searrow$ eigen M_0

$$\Psi_0(z\cdot e^{2\pi i}) = \Psi_R(z\cdot e^{2\pi i}) C = \Psi_R(z) S_2^{-1} S_1^{-1} C = \Psi_0(z) \underbrace{C^{-1} S^t S^{-1} C}_{M_0}$$

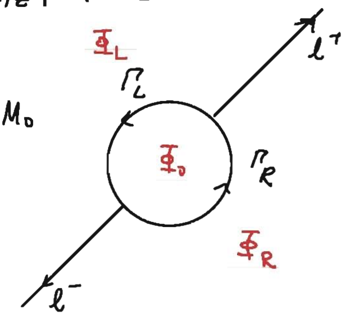
Q: Construct Λ from the Stokes data

Riemann-Hilbert Boundary Value Problem:

To determine Λ or $V(u)$, it is equiv. to determine its sol'n.

Consider $\Psi = \Phi \exp(zU)$ with $\Phi(z) = Id + O(1/z)$ for $z \to \infty$

$\uparrow$ holo. in any angular sector.



$$(*) \begin{cases} z^{L_0}\Phi_0(z) = \Phi_R(z)\cdot e^{zU}\cdot C & \text{on } \Gamma_R \\ z^{L_0}\Phi_0(z) = \Phi_L(z)\cdot e^{zU} S^{-1} C & \text{on } \Gamma_L \\ \Phi_L(z) e^{zU} = \Phi_R(z)\cdot e^{zU} S & \text{on } l_+ \\ \Phi_L(z) e^{zU} = \Phi_R(z)\cdot e^{zU} S^t & \text{on } l_- \end{cases}$$

$e^{2\pi i L_0} = M_0$

$\rightarrow$ Jimbo-Miwa-Ueno (1982)

JMU: Consider
$$\Psi(u,z) := \begin{cases} \Phi_R(u,z) e^{zU} & z \in \mathbb{C}_R, |z|>1 \\ \Phi_L(u,z) e^{zU} & z \in \mathbb{C}_L, |z|>1 \\ \Phi_0(u,z) z^{L_0} & |z|<1 \end{cases}$$

$$\Theta := \underline{\partial_i} \Psi \cdot \Psi^{-1}$$

$\uparrow$ $\partial/\partial u_i$

$$U = \begin{pmatrix} u_1 & & \\ & \ddots & \\ & & u_n \end{pmatrix}$$

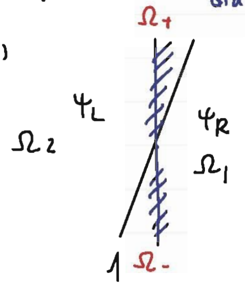
Recall: (1) $L = \frac{d}{dz} - A(z)$, $A(z) = U + \underbrace{O(1/z)}_{\text{exact order } 1}$ $\longleftrightarrow$ $\Lambda = \frac{d}{dz} - (U + \frac{1}{z} V)$

$(*)$-condition: $A^t(-z) = A(z)$ $\longleftrightarrow$ $V^t = -V$

The gauge transf. $L \mapsto g^{-1}(z) L g(z)$ with $g(z) = \mathrm{Id} + o(1/z)$ and $g(z) g^t(-z) = id$ preserves $(*)$

$\frac{b_0}{w^2} + \frac{b_1}{w}$ ← skew-symm, diag. $\Rightarrow b_1 = 0$

(2)



$\Psi_L = \Psi_R S_+$ on Ω_+ $(S := S_+)$

$\Psi_L = \Psi_R S_-$ on Ω_-

$(*) \Rightarrow S_- = S_+^t$

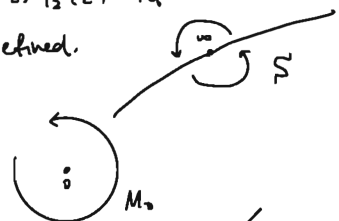
pf: $\Psi_L(z) \Psi_L^t(-z)$ on Ω_2

$\Psi_R(z) \Psi_R^t(-z)$ on Ω_1

$S_- = S_+^t \iff \Psi_L(z)\Psi_L^t(-z)$ and $\Psi_R(z)\Psi_R^t(-z)$ are the same on Ω_+, Ω_-

So, $(*) \Rightarrow (\Psi_1^t(-z)\Psi_2(z))' = 0$ May assume $\Psi_1^t(-z)\Psi_2(z) = id$

$\Rightarrow \Psi_L(z)\Psi_L^t(-z)$ and $\Psi_R(z)\Psi_R^t(-z)$ are well-defined.



- Monodromy data $(S, M_0, C, \underbrace{\mu_1, \dots, \mu_n}_{\text{eigen}(M_0)})$

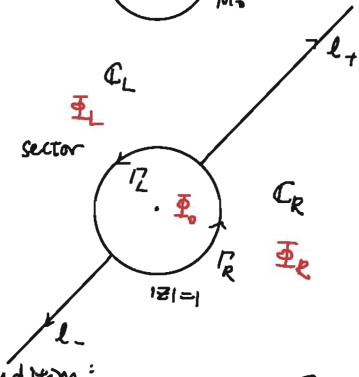
$\Psi_0(z) = \Psi_R(z) \cdot C$

$\Rightarrow M_0 = C^{-1} S^t S^{-1} C$

- Riemann-Hilbert ∂-valued Problem:

Consider $\Psi = \Phi e^{zU}$ $(b_1 = 0)$

$\Phi \sim \mathrm{Id} + O(1/z)$ for $z \to \infty$ in a Stokes' sector



(Hard thm (Wasow):

Ψ_{formal} :

Ψ_{holo} :

Take away e^{zU}, then $\Phi_f \sim \Phi_{hol}$)

Now, we have the following ∂-valued condition:

$$\begin{cases} z^{L_0} \Phi_0(z) = \Phi_R(z) e^{zU} \cdot C & \text{on } \Gamma_R \\ z^{L_0} \Phi_0(z) = \Phi_L(z) e^{zU} S^{-1} C & \text{on } \Gamma_L \\ \Phi_L(z) e^{zU} = \Phi_R(z) e^{zU} \cdot S & \text{on } \ell_+ \\ \Phi_L(z) \cdot e^{zU} = \Phi_R(z) \cdot e^{zU} \cdot S^t & \text{on } \ell_- \end{cases}$$

Ex: $\Phi_{R/L}(z) = \mathrm{Id} + \frac{\Gamma}{z} + O(1/z^2)$ as $z \to \infty$

$V = [\Gamma, U] = \Phi_0(0) L_0 \Phi_0(0)^{-1}$

Thm (Jimbo-Miwa-Ueno, Malgrange)
$\exists$ meromorphic $V(\tilde{u}, \xi, M_0, C, \mu_1, \dots, \mu_n)$, where $\tilde{u} \in \widetilde{\mathbb{C}^n \setminus \Delta}$ s.t.
$\Lambda = d/dz - (U + \frac{1}{z}V)$ has the monodromy data.

Prop: For a given mondromy data indep. of u $\Leftrightarrow$ $\partial_k V(u) = [[\Gamma, E_k], V(u)]$, where $V = [\Gamma, U]$. Also, $\Gamma(cu) = c^{-1}\Gamma(u)$ $\quad k = 1, \dots, n.$

pf: $(\Rightarrow)$ Consider the piecewise analytic function

$$\Psi(u,z) := \begin{cases} \Phi_R(u,z)\, e^{zU}, & z \in \mathbb{C}_R, \ |z|>1 \\ \Phi_L(u,z)\, e^{zU}, & z \in \mathbb{C}_L, \ |z|>1 \\ \Phi_0(u,z)\, z^{L_0}, & |z|<1 \end{cases}$$

JMU
$\Rightarrow$ $\partial_i \Psi \cdot \Psi^{-1}$ has no jumps on $l_+ \cup \Gamma_L \cup l_- \cup \Gamma_R$
$\partial_i = \frac{\partial}{\partial u^i}$ no effect on monodromy

$$\partial_i \Psi \cdot \Psi^{-1} = (1 + \frac{\Gamma}{z} + \cdots)\, z E_i\, e^{zU} e^{-zU} (1 - \frac{\Gamma}{z} + \cdots)$$
$$+ (\frac{\partial_i \Gamma}{z} + \cdots)\, e^{-zU} (1 - \frac{\Gamma}{z} + \cdots)$$
$$= zE_i + [\Gamma, E_i] + O(1/z)$$

Note that $\partial_i \Psi \cdot \Psi^{-1}$ is analytic at $z=0$
$(\partial_i \Phi_0 z^{L_0} \cdot z^{-L_0} \Phi_0^{-1} = (\partial_i \Phi_0)\Phi_0^{-1}$ is analytic)

By Liouville's thm $\Rightarrow$

$$(**) \begin{cases} \partial_i \Psi = (zE_i + [\Gamma, E_i])\Psi & i = 1, \dots, n. \\ (z\frac{\partial}{\partial z} - \sum u^i \partial_i)\Psi = 0 \end{cases}$$

Compatibility $\Rightarrow$ $\partial_i \partial_j \Psi = \partial_j \partial_i \Psi$ $\Rightarrow$
$[\partial_i \Gamma, E_i] - [\partial_i \Gamma, E_j] - [[\Gamma, E_i], [\Gamma, E_j]] = 0$
This is equiv. to RHS.

$(\Leftarrow)$ If $\Gamma(u)$ satisfies RHS $\Leftrightarrow$ (a), (b), (c) for $\Gamma = (\gamma_{ij})$, then $(**)$ is compatible.

Now, we consider $\Lambda = \frac{d}{dz} - U - \frac{1}{z}V$, $V = [\Gamma, U]$

For any Ψ solving the RH ∂-valued problem, $\Psi = \Psi(u,z)$ at the pt. u, monodromy data might depend on u,

Ex: $\Lambda\Psi = 0$

<u>Claim</u>: $\frac{\partial}{\partial u^i} - zE_i - [\Gamma, E_i]\Psi$ and $\left(z\frac{\partial}{\partial z} - \sum u^i \frac{\partial}{\partial u^i}\right)\Psi$ are sol'n of $\Lambda(\cdot)=0$

pf: $\left(\frac{\partial}{\partial z} - U - \frac{1}{z}V\right)\left(\frac{\partial}{\partial u^i} - zE_i - [\Gamma, E_i]\right)\Psi = (-E_i + E_i)\Psi = 0$

only crossing term

$$\left(\frac{\partial}{\partial z} - U - \frac{1}{z}V\right)\left(z\frac{\partial}{\partial z} - \sum u^i \frac{\partial}{\partial u^i}\right)\Psi = \left(\frac{\partial}{\partial z} - U - \frac{1}{z}V\right)\Psi = 0$$

$\Rightarrow$ (i) $(\partial_i - zE_i - [\Gamma, E_i])\Psi = \Psi \cdot T_i(u)$

(ii) $\left(z\frac{\partial}{\partial z} - \sum u^i \partial_i\right)\Psi = \Psi\, T(u)$ indep of z

(i) $\Rightarrow$ $[(\partial_i - zE_i - [\Gamma, E_i]\Psi)\Psi^{-1} = O(1/z)$

By asymp. at $z=\infty$

Similarly for (ii).

Therefore, we go back to $(**)$ $\Rightarrow$ ∂_i has no effect on $M_0, S, C, \mu_1, \dots, \mu_n$ along the jumping curves. □

$\leftrightarrow$ Appendix E

<u>Rmk</u>: (1) This isomonodromy space is isomorphic the one for the regular singular system $\lambda \in \mathbb{C}$: $\left(\frac{d}{d\lambda} + \sum_{i=1}^{n} \frac{A_i}{\lambda - u_i}\right)\phi = 0$ on $\mathbb{P}^1$, $\sum_{i=1}^{n} A_i \neq 0$ via Fourier-Laplace transf. ($\Rightarrow$ reg. at $\lambda = \infty$)

(Sabbah = max./universal integrable deformation)

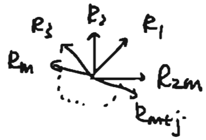
Indeed, $A_i = E_i V$ in our case. ($\mathrm{rk}(E_i V) = 1$)

The general case for $n=3$ (4 reg. sing. pts) $\Leftrightarrow$ Painlevé VI eqn

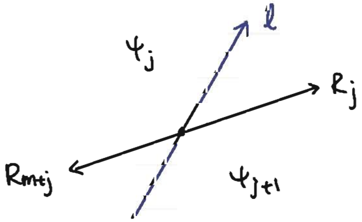
(2) Analytic Continuation of WDVV and Braids gps B_n

Appendix F

Given ℓ & $\{$Stokes rays $R_1, \dots, R_{2m}\}$



$m = \frac{n(n-1)}{2}$



$\Psi_{j+1} = \Psi_j \cdot K_{R_j}$

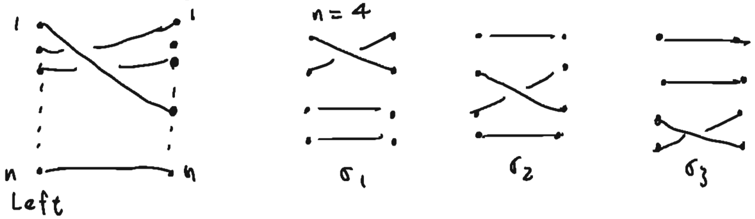
Stokes' factors K_R, $R = R_j$.

$K_{ii} = 1$, $K_{ij} \neq 0 \Rightarrow R_{ji} = R$

$\Rightarrow S = K_{R_1} \cdots K_{R_m}$ $\quad (K_{-R}^{-1} = K_R^t)$

Braid group $B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i \cdot \sigma_{i+1} \cdot \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, & 1 \le i \le n-2 \\ \sigma_i \sigma_j = \sigma_j \sigma_i & |i-j| \ge 2 \end{array} \right\rangle$

Braid : (Isotopy Class of homeo. on $\mathbb{R}^3$ s.t.)



2018.6.4

- Monodromy Group of a Frobenius mfd M:

Assume M is analytic, and $C_{\alpha\beta}^{\gamma}(t)$ is analytic in $t\in M$.

$(\alpha,\beta)^* := \iota_E(\alpha\circ\beta)$ on T^*M $\Leftrightarrow$ $(E\cdot u, v) = \langle u, v\rangle$ on TM

$g^{\alpha\beta} := (dt^\alpha, dt^\beta)^*$ $\quad \Sigma = \{ t \mid \Delta(t) := \det(g^{\alpha\beta}(t)) = 0\} \underset{\uparrow}{\subset} M$

On $M\setminus\Sigma$, $g_{\alpha\beta}$ exists. and it is flat $\qquad$ analytic subset

Isometry $\Phi: \underset{\uparrow}{\Omega} \longrightarrow \widetilde{M\setminus\Sigma}$ $\leftarrow$ simply connected, flat

an extended domain in Euclidean space E^n ($\mathbb{R}^n$ or $\mathbb{C}^n$)

$\rightsquigarrow$ $\mu: \pi_1(M\setminus\Sigma) \longrightarrow \mathrm{Iso}(E^n)$ = The gp of rigid motion

$W(M) := \mathrm{im}(\mu)$: monodromy gp of M

Explicit Construction of Φ:

flat coord. for $(\,,\,)^*$: $x = x(t^1,\ldots,t^n)$ can be solved via:

$$g^{\alpha\varepsilon}\partial_\varepsilon\partial_\beta x + \Gamma_\beta^{\alpha\varepsilon}(t)\partial_\varepsilon x = 0 \qquad \alpha,\beta = 1,\ldots,n$$

$\rightsquigarrow$ $\exists$ $\underbrace{x^1(t),\ldots\ldots, x^n(t)}_{\text{analytic.}}$ s.t. $g^{ab} = \dfrac{\partial x^a}{\partial t^\alpha}\cdot\dfrac{\partial x^b}{\partial t^\beta} g^{\alpha\beta}(t) = \text{const.}$

Since $\partial_\alpha\partial_\beta x - \Gamma_{\alpha\beta}^\gamma(t)\partial_\gamma x = 0$ $\quad \Gamma_{\alpha\beta}^{\gamma} = -g_{\alpha\varepsilon}\Gamma_\gamma^{\varepsilon\beta}$ has poles on Σ.

sol'n are analytic outside Σ. This gives Φ and μ:

γ: closed path on $M\setminus\Sigma$ gives $\tilde{x}^a(t) = \sum \underbrace{A_b^a(\gamma)}_{\text{in } O(n)} x^b(t) + B^a(\gamma)$

Ex.: All $x^a(t)$ are weighted homogeneous in t of weight $\frac{1}{2}(1-d)$, if $d\neq 1$
$\Rightarrow$ $B^a = 0$.

Nilpotent Loci: $\Sigma_{nil} := \{ t\in M \mid (T_tM, \cdot) \text{ is not s.s.}\}$

$$\Delta_{nil} := \mathrm{disc}_\lambda(\det(g^{\alpha\beta}(t) - \lambda\eta^{\alpha\beta}))$$
$$= \mathrm{disc}_\lambda(\det(g^{\alpha\beta}(t^1-\lambda, t^2,\ldots, t^n)) = \mathrm{disc}_\lambda \Delta(t^1-\lambda, t^2,\ldots,t^n)$$

Then $\Sigma_{nil} \subset$ zero loci of Δ_{nil}.

<u>Claim</u>: $ds^2 = g_{\alpha\beta}(t)\,dt^\alpha dt^\beta$ on $M\setminus\Sigma$ extends analytically on the double cover $\pi: \hat{M} \longrightarrow M$, where $\hat{M} := \{(w,t)\in\mathbb{C}\times M \mid w^2 = \Delta(t)\}$ outside $\Sigma\cap\Sigma_{nil}$

<u>Rmk</u>: $M\setminus\Sigma_{nil} \neq \phi \Leftrightarrow M$ is generically s.s.

Define I by $i \in I$ iff $u^i(t_0)=0$ $\quad u^1, \dots, u^n$: can. coord. at t_0.

$$\pi^* ds^2 = \pi^*\left(\sum_{i=1}^{n} \frac{\rho_i}{u^i}(du^i)^2\right) = \sum_{i\in I} \rho_i \, d(2\sqrt{u^i})^2 + \sum_{i \notin I} \frac{\rho_i}{u^i}(du^i)^2$$

<u>Lemma</u>: In flat coord. $x^a(t)$, every component of $\Sigma \setminus \Sigma_{nil}$ is a hyperplane in E^n

<u>Cor</u>: If $d \neq 1$, then the local monodromy is a reflection ($A^2 = Id$, $B=0$)

<u>pf of lemma</u>: Say the component is given by $u^n = 0$.

Enough to show: $b_{ij} \equiv 0$ on it. (in u^i-coord.)

First, pick $u_0^n \neq 0$, on the slice $u^n = u_0^n$,

normal vector $N = \dfrac{\partial_n}{|\partial_n|}$

$$1 \le i,j \le n-1$$

The second fundamental form: $b_{ij} = (\nabla_{\partial_i}\partial_j, N) = \Gamma_{ij}^n \, \partial_n \cdot \partial_n \dfrac{1}{|\partial_n|}$

$$= |\partial_n| \Gamma_{ij}^n = |\partial_n| \frac{1}{2} g^{ns}(\partial_i g_{sj} + \partial_j g_{si} - \partial_s g_{ij})$$

$$= \frac{-1}{2}\frac{1}{|\partial_n|}\delta_{ij}\,\partial_n\left(\frac{\rho_i}{u^i}\right) \qquad \frac{1}{|\partial_n|} = \sqrt{\frac{u_0^n}{\rho_n}} \longrightarrow 0 \text{ as } u_0^n \to 0$$

analytic func. $\qquad \rho_n = \eta_{nn} \neq 0$

$\Rightarrow (u^n = 0)$ is a hyperplane in flat Euclid. coord. $x^a(t)$ $\quad \square$

• Generalized Hypergeometric Eqn associated to a Frob. mfd and its monodromy:
Euler v.f. E leads to (1) Dubrovin connection on $\underset{(t,z)}{M \times \mathbb{P}^1}$, flat coord. $\tilde{t}^\alpha(t,z)$

(2) Flat pencil of metrics $g^{\alpha\beta}(t) - \lambda\eta^{\alpha\beta} = g^{\alpha\beta}(t^1-\lambda, t^2, \dots, t^n)$, flat coord.
$\tilde{x}^\alpha(t,\lambda) = x^\alpha(t^1-\lambda, t^2, \dots, t^n)$.

Def'n / Prop: $\xi_\varepsilon := \partial_\varepsilon x(t,\lambda)$ (No upper index means one of the coord. $\tilde{x}^a$)
satisfies the generalized hypergeometric eqn in $\lambda \in \mathbb{P}^1$

$$(*) \quad (g^{\alpha\varepsilon} - \lambda\eta^{\alpha\varepsilon}) \frac{d}{d\lambda} \xi_\varepsilon = \mu^{\alpha\varepsilon}\left(\frac{1}{2} - \mu_\varepsilon\right)\xi_\varepsilon \qquad \mu_\varepsilon := q_\varepsilon - \frac{d}{2}$$

pf: $\Gamma_1^{\alpha\varepsilon} = \left(\frac{d+1}{2} - q_\varepsilon\right) C_1^{\alpha\varepsilon} = \left(\frac{1}{2} - \mu_\varepsilon\right)\eta^{\alpha\varepsilon}$ in the defining eqn of $\tilde{x}^\cdot(t,\lambda)$
for $\beta=1$

$$(g^{\alpha\varepsilon} - \lambda\eta^{\alpha\varepsilon})\partial_\beta\partial_\varepsilon x + \underset{\substack{\uparrow \\ \text{w.r.t. the pencil}}}{\Gamma_\beta^{\alpha\varepsilon}} \partial_\varepsilon x = 0$$

Now, take $\beta = 1$,

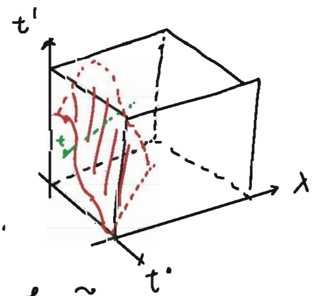
$$(\star) \quad (g^{\alpha\varepsilon} - \lambda\eta^{\alpha\varepsilon})\partial_1\partial_\varepsilon x + \left(\frac{1}{2} - \mu_\varepsilon\right)\eta^{\alpha\varepsilon}\partial_\varepsilon x = 0$$

Note that $\tilde{x}^a(t,\lambda) = x^a(t^1-\lambda, t^2, \dots, t^n)$

$$\partial_1 = \frac{\partial}{\partial t^1} = -\frac{d}{d\lambda}$$

$$\Rightarrow (g^{\alpha\varepsilon} - \lambda\eta^{\alpha\varepsilon})\frac{d}{d\lambda}\xi_\varepsilon = \mu^{\alpha\varepsilon}\left(\frac{1}{2} - \mu_\varepsilon\right)\xi_\varepsilon. \quad \square$$

The singular pts are at $\Sigma_\lambda := \{t \in M \mid \Delta(t^1-\lambda, t^2, \dots, t^n) = 0\}$ or rather Σ_t



Two observations
(1) Monodromy of $(*)$ = monodromy of $M \setminus \Sigma$
(This is seen from the defining eqn $\star$ and interpret in two ways)
(2) $(*)$ is a reg. sing. system when M is s.s.

Prop (H.1): If $\tilde{t}^\alpha(t,z)$ is normalized at $z\partial_z \tilde{t} = \mathcal{L}_E \tilde{t}$, then

$$\tilde{x}(t,\lambda) = \oint z^{\frac{d-3}{2}} e^{-\lambda z}\, \tilde{t}(t,z)\, dz$$

At s.s. pt. $t \in M$, using canonical coord. u^i, we have:

let $\phi_i^{\bullet}(u,\lambda) := \dfrac{\partial_i \tilde{X}^{\bullet}(t(u),\lambda)}{\sqrt{\rho_i}}$, $\phi^{\bullet} = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix}$

, then
$$\begin{cases} \partial_j \phi_i = \gamma_{ij}\phi_j & j \neq i \\ \sum_{k=1}^{n} (u^k - \lambda)\partial_k \phi_i = \frac{-1}{2}\phi_i \\ (\lambda I_n - U)\frac{d\phi}{d\lambda} = -\left(\frac{1}{2}I_n + V(u)\right)\phi \end{cases}$$

Compare to the earlier system on $\psi = \begin{pmatrix} \psi_1(u,z) \\ \vdots \\ \psi_n(u,z) \end{pmatrix}$
$$\begin{cases} \partial_i \psi_j = \gamma_{ij}\psi_j & j \neq i \\ \sum_{k=1}^{n} \partial_k \psi_i = z\psi_i \end{cases}$$

$$\phi(u,\lambda) = \oint e^{-\lambda z}\psi(u,z)\frac{dz}{\sqrt{z}}$$

5.11.2018

W: Coexter := gp generated by reflections in $V/\mathbb{R}$

(We assume $W \curvearrowright V$ irred, $\dim_{\mathbb{R}} V = n$, coord. $x^1, \dots, x^n$, with Euclid. metric)

$|W| < \infty \Rightarrow \exists!\, a \in V$ s.t. a is W-inv. pt. (Then take $a = 0$)

a Chamber $\mathcal{C} = (\mathcal{C}, (H_i)_{i=1}^n)$ is a simplicial cone at $a = 0$

∂-faces

Let S_H = reflection in the hyperplane H

Coexter element (transformation)

$C := S_{H_1} \cdots\cdots S_{H_n} \Longrightarrow$ All are conjugate.

Thm

Ref: Bourbaki - Lie Groups and Lie Algebra Chapter 4-6

$h := h(W)$ = order of C : Coexter number

Char. poly of C: $p(T) = \prod_{j=1}^n \left(T - \exp\left(\frac{2\pi i}{h} m_j\right)\right)$

m_j: exponent of W

$0 < m_1 \leq m_2 \leq \cdots\cdots \leq m_n \leq h$

real poly. $\Rightarrow \begin{cases} m_j + m_{n+1-j} = h & \dagger \\ \sum m_j = \frac{1}{2} n h & \ddagger \end{cases}$

Thm (Bourbaki, Lie Group and Lie Algebra, V §6, Thm 1 + Cor)

(1) $m_1 = 1$, $m_n = h - 1$, simple root

(2) $\mathcal{H}$ = The set of hyperplanes $\# \mathcal{H} = \sum_{j=1}^n (p_j - 1) = \sum_j m_j$ i.e. $p_j - 1 = m_j$

Here, $p_j = \deg f^j(x)$, $f^1(x), \dots, f^n(x)$: homogeneous W-inv. poly.

Goal: $V \otimes \mathbb{C}$ $x^1, \dots, x^n$

$\downarrow$

"M" $= V_{\mathbb{C}}/W$ $\quad M = \mathrm{Spec}\left(\mathbb{C}[y^1, \dots, y^n]\right) \quad \mathbb{C}[y^1, \dots, y^n] = \mathbb{C}[x^1, \dots, x^n]^W$

$\downarrow$

Frobenius manifold.

(3) $|W| = \prod_{j=1}^n p_j$

Sketch of pf: $J(f)(1, 0, \dots, 0) \neq 0$ $\exists \sigma \in S_n$ s.t. $0 \neq \frac{\partial f^j}{\partial x^{\sigma(j)}}(1, 0, \dots, 0)$, $\forall j = 1, \dots, n$

$\Rightarrow f^j(x) = \underset{\substack{\#\\0}}{*} (x^1)^{p_j - 1} x^{\sigma(j)} + \cdots\cdots$

(Here, we retake $x^1, \dots, x^n$ to be eigenvector of C, $x^i \xrightarrow{C} e^{\frac{2\pi i}{h} m_i} x^i$)

Note that $m_1 = 1$

Acts by C: $\mapsto * e^{\frac{2\pi i}{h}\left(\underbrace{p_j - 1 + m_{\sigma(j)}}_{\equiv 0 \ (\mathrm{mod}\ h)}\right)} (x^1)^{p_j - 1} x^{\sigma(j)} + \cdots \Rightarrow p_j - 1 = m_{\sigma(j)} + \mu_j h$, $\mu_j \geq 0$

$$|\mathfrak{h}| \underset{\uparrow}{=} \sum_{j=1}^{n}(p_j - 1) = \sum_{j=1}^{n} m_{\sigma(j)} + h\sum \mu_j \Rightarrow \mu_j \equiv 0 \quad \forall j.$$

proved by a Poincaré series argument (Which proves (3) at the same time)

□

Now, we come back to Dubrovin's convention in lecture 4:

$$\begin{array}{ll} V_{\mathbb{C}} & (x^1, \ldots, x^n) \\ \downarrow & \\ M = V_{\mathbb{C}}/W & (y^1, \ldots, y^n) \end{array}$$

inv. poly. $y^1(x), \ldots, y^n(x)$, $\deg y^j = d_j \searrow$ $d_1 = h > d_2 \geq \cdots \geq d_n = 2$

$$\rightsquigarrow \begin{cases} d_i + d_{n+1-i} = h+2 \\ \sum_i d_i = n(h+\frac{1}{2}) \end{cases}$$

<u>Claim</u>: $\partial_1 = \frac{\partial}{\partial y_1} \rightsquigarrow$ identity (After change to flat coord. t')

intersection form

$$g^{ij}(y) := (dy^i, dy^j)^* = \sum_{a=1}^{n} \frac{\partial y^i}{\partial x^a}\frac{\partial y^j}{\partial x^a}$$

$$\sum_k \Gamma^{ij}_k(y)\, dy^k = \sum_{a,b} \frac{\partial y^i}{\partial x^a}\frac{\partial^2 y^j}{\partial x^a \partial x^b} dx^b + 0 \nwarrow \text{connection in } X\text{-coord.} \equiv 0$$

Check: W-inv.

<u>Thm</u> (Saito's): $\exists!$ Frob. str on $(M, (\,,)^*)$ with $E = \frac{1}{h}\sum d_i y^i \frac{\partial}{\partial y^i} = \frac{1}{h}\sum x^a \frac{\partial}{\partial x^a}$

$$e = \partial_1 = \frac{\partial}{\partial y^1}$$

<u>Observation</u>: $g^{ij}(y)$ and $\Gamma^{ij}_k(y)$ are linear in y'

$$\deg(g^{ij}(y)) = d_i + d_j - 2 \leq 2d_1 - 2$$

$$\deg \Gamma^{ij}_k(y) = d_i + d_j - 2 - d_k \leq 2d_1 - 2$$

Saito's metric (1974) : $\eta^{ij}(y) = \partial_1 g^{ij}(y) \Rightarrow$ indep. of y'

$\deg \eta^{ij} = d_i + d_j - 2 - h \Rightarrow \eta^{ij} = 0$ for $i+j > n+1$

$c_i := \eta^{i,n+1-i}$: const.

$$\rightsquigarrow \eta^{ij} = \begin{pmatrix} & * & & c_1 \\ & & \cdot^{\cdot^{\cdot}} & \\ & c_{n-1} & & 0 \\ c_n & & & \end{pmatrix}$$

Let $D(y) = \det(g^{ij}(y)) = C y_1^n + a_1 y_1^{n-1} + \cdots + a_n$ $\quad \deg a_k = kh$.

$\uparrow$ $\deg = n(h+2) - 2n = nh$

$c = \det(\eta_{ij}) = (-1)^{\frac{n(n-1)}{2}} c_1 \cdots c_n$

Claim: $c \neq 0$

pf: Let $C\vec{\gamma} = e^{\frac{2\pi i}{h}}\vec{\gamma}$ eigenvector of $m_1 = 1$

As inv. poly, $y^k(\vec{\gamma}) = y^k(C\vec{\gamma}) = e^{\frac{2\pi i d_k}{h}} y^k(\vec{\gamma})$

$\Rightarrow y^k(\vec{\gamma}) = 0$ for $k = 2, \dots, n$.

But $D(y(\vec{\gamma})) \neq 0 \Rightarrow c \neq 0$ $\square$

Cor: (1) $g^{ij} + \lambda \eta^{ij}$ forms a flat pencil of metrics and η^{ij} is globally, poly. defined on M.

$\eta_{ij}(y)$ is also poly. in y.

Denote η_{ij} by $\langle\, ,\rangle$ and its Christoffel symbol $\gamma_k^{ij} = \partial_1 \Gamma_k^{ij}$

(2) The flat coord. $t^1, \dots, t^n$ exists globally as homogeneous poly. of deg d_i.

pf: pf of (1) $\longleftrightarrow$ Appendix D

pf of (2): In $y^\cdot$ coord., $t^\cdot$ are solved from (for $y^\cdot$ small)

$$\partial_k \xi_l + \sum_s \underbrace{\gamma_{kl}^s}_{\parallel} \xi_s = 0 \Rightarrow \xi_l^\alpha(y) =: \partial_l t^\alpha(y)$$

$$\sum_i \eta_{il} \gamma_k^{is} \text{ poly in } y^\cdot \qquad \partial_l = \partial/\partial y^l$$

The system in $t^\cdot(y)$ is quasi-homogeneous i.e. $y^i \longmapsto c^{d_i} y^i$

$\Rightarrow t^\cdot(y)$ are quasi-homogeneous in y^i

$\because$ All weight > 0 $\Rightarrow$ $t^\cdot(y)$ are poly. in $y^\cdot$

$\Rightarrow t^\cdot(y(x))$: poly. in x.

$\because y^\cdot \longleftrightarrow t^\cdot$ is invertible, may choose $\deg t^\alpha = \deg y^\alpha = d_\alpha$ in $x^\cdot$ (induction from low degree)

$\Rightarrow$ The sol'n $t^\cdot$ works globally on M.

We fix (choose) $t^n = y^n = \frac{1}{2h}\sum (x^i)^2$

Then in the flat coord., $g^{n\alpha} = \sum_a \frac{\partial t^n}{\partial x^a}\frac{\partial t^\alpha}{\partial x^a} = \frac{1}{h}\sum x^a \frac{\partial t^\alpha}{\partial x^a} = \frac{d_\alpha}{h} t^\alpha$

Notation: $\left(\begin{array}{l} i,j,k: x^\cdot \\ a,b,c: y^\cdot \\ \alpha,\beta,\gamma: t^\cdot \end{array}\right.$

(*) $\Gamma_\beta^{n\alpha} dt^\beta = \sum_{a,b} \frac{\partial t^n}{\partial x^a}\frac{\partial^2 t^\alpha}{\partial x^a \partial x^b} = \frac{1}{h}\sum_a x^a d\left(\frac{\partial t^\alpha}{\partial x^a}\right)$

$$= \frac{1}{h}\Big(\underbrace{d(E t^\alpha)}_{d(d_\alpha t^\alpha)} - \underbrace{\sum_a dx^a \frac{\partial t^\alpha}{\partial x^a}}_{d(t^\alpha)}\Big) = \frac{d_\alpha - 1}{h} dt^\alpha$$

$$\Rightarrow \Gamma_\beta^{n\alpha} = \frac{d_\alpha - 1}{h}\delta_\beta^\alpha$$

<u>Main Lemma</u>: Let $\eta^{\alpha\beta} = \partial_1 (dt^\alpha, dt^\beta)^*$ be the anti-diagonal const. metric in $t^\cdot$, then $g^{\alpha\beta}(t) = \frac{d_\alpha + d_\beta - 2}{h} \eta^{\alpha\lambda}\eta^{\beta\mu}\partial_\lambda\partial_\mu F(t)$, where

$F(t)$ is quasi-homogeneous poly. of degree $2h+2$

$\rightsquigarrow F(t)$ determines a poly. Frob. str with $e = \partial_1$, $E = \frac{1}{h}\sum d_\alpha t^\alpha \frac{\partial}{\partial t^\alpha}$

<u>pf</u>: In Appendix D, flatness of pencil $g^{\alpha\beta} + \lambda\eta^{\alpha\beta} \Rightarrow$

$$\Gamma^{\alpha\beta}_\gamma = \partial_\gamma f^{\alpha\beta} = \eta^{\alpha\varepsilon}\partial_\varepsilon\partial_\gamma f^\beta \Rightarrow \deg(f^\beta) = d_\beta + h.$$

$\because$ $g^{\alpha\sigma}\Gamma^{\beta\gamma}_\sigma = g^{\beta\sigma}\Gamma^{\alpha\gamma}_\sigma$ for $\alpha = n \rightsquigarrow$ By $(*)$,

$$\frac{1}{\cancel{h}}\sum_{\varepsilon,\sigma} d_\sigma t^\sigma \eta^{\beta\varepsilon}\partial_\varepsilon\partial_\sigma f^\gamma = \frac{1}{\cancel{h}}(d_\gamma - 1) g^{\beta\gamma} \quad (\star)$$

$$\Rightarrow \sum_\varepsilon \eta^{\beta\varepsilon}(d_\gamma - d_\varepsilon + h)\partial_\varepsilon f^\gamma \underset{\varepsilon+\beta=n+1}{=} (d_\gamma + d_\beta - 2)\eta^{\beta,n+1-\beta}\partial_{n+1-\beta} f^\gamma$$

$$\Rightarrow \frac{\eta^{\beta,n+1-\beta}\partial_{n+1-\beta} f^\gamma}{d_\gamma - 1} = \frac{\eta^{\gamma,n+1-\gamma}\partial_{n+1-\gamma} f^\beta}{d_\gamma - 1}$$

Let $F^\gamma := \frac{h}{d_\gamma - 1} f^\gamma$, then $\eta^{\beta,n+1-\beta}\partial_{n+1-\beta} F^\gamma = \eta^{\gamma,n+1-\gamma}\partial_{n+1-\gamma} F^\beta$

$\Rightarrow \exists F(t)$ s.t. $F^\gamma(t) = \eta^{\gamma,n+1-\gamma}\partial_{n+1-\gamma} F(t)$ (loc.) $\rightarrow$ globally, poly.

Back to $(\star)$, get the formula for $g^{\beta\gamma}$. ▫

- The Associativity for $C^{\alpha\beta}_\gamma = \eta^{\alpha\lambda}\eta^{\beta\mu}\partial_\lambda\partial_\mu\partial_\gamma F$ follows from the flatness of the pencil. since $\Gamma^{\alpha\beta}_\gamma = \frac{d_\beta - 1}{h} C^{\alpha\beta}_\gamma$

For $\alpha = n$: $(*) \Rightarrow C^{n\alpha}_\beta = \delta^\alpha_\beta \Rightarrow C^\beta_{1\alpha} = \delta^\beta_\alpha \Rightarrow \partial_1$ is identity.

Also, $C_{1\beta\gamma} = \delta^\alpha_\beta \eta_{\alpha\gamma} = \eta_{\beta\gamma}$ ▫ unique

<u>Thm</u>: This Frobenius mfd is semisimple.

5.18.2018

Last time, we construct a semisimple Frobenius structure on $V_{\mathbb{C}}/W$, where W is a finite coxeter gp.

Suppose $W = A_n$ (i.e. S_{n+1}) acts on $\mathbb{R}^{n+1} = \{(\xi_0, \ldots\ldots, \xi_n) \mid \xi_i \in \mathbb{R}\}$ by permutations $i \longmapsto \sigma(i)$

$\xi_0 + \cdots\cdots + \xi_n = 0$ is preserved $\simeq \mathbb{R}^n$

inv. poly. in ξ_a's. $M = \mathbb{C}^n / A_n$

Recall: We had defined A_n-model

$$M = \{\lambda(p) \in \mathbb{C}[p] \mid \lambda(p) = p^{n+1} + a_n p^{n-1} + \cdots + a_1\} \qquad (1.65)$$

(1.66) (a) $A_\lambda = \mathbb{C}[p] / (\lambda'(p))$: alge. str.

(b) $\langle f, g \rangle_\lambda = \operatorname{res}_{p=\infty} (fg/\lambda')$

is a Frobenius manifold.

In fact, it is $\mathbb{C}^n / A_n$.

Rmk: Relation with (isolated) A_n singularity

$(x^2 + y^2 + \lambda(z) = 0) \subseteq \mathbb{R}^3$
surface.

Lemma: Given any $\partial', \partial'', \partial''', \ldots$ tangent vectors at λ,

1) $$\langle \partial', \partial'' \rangle_\lambda = -\sum_{p \neq \infty} \operatorname{res}_{d\lambda_p = 0} \frac{\partial'(\lambda dp)\, \partial''(\lambda dp)}{d\lambda} \qquad d\lambda = \lambda'(p)\, dp$$

$$c(\partial', \partial'', \partial''')_\lambda = -\sum_{p \neq \infty} \operatorname{res}_{d\lambda_p = 0} \frac{\partial'(\lambda dp)\, \partial''(\lambda dp)\, \partial'''(\lambda dp)}{dp\, d\lambda}$$

2) Let $q^1, \ldots, q^n$: critical pt. of $\lambda(p)$ i.e. $\lambda'(q^i) = 0$

$u^i = \lambda(q^i)$: critical values.

Then at $\lambda \in M$ s.t. $\lambda(p)$ has no multiple roots, we have:

$$\langle\, , \rangle_\lambda = \sum_{i=1}^n \eta_{ii}(u) (du^i)^2, \quad \eta_{ii}(u) = \frac{-1}{\lambda''(q_i)}$$

(We will show later u^i are canonical coord.)

3) Moreover, $$(\partial', \partial'')_\lambda = -\sum_{p \neq \infty} \operatorname{res}_{d\lambda_p = 0} \frac{\partial'(\log\lambda dp)\, \partial''(\log\lambda dp)}{d\log\lambda}$$

↑

The metric induced by the inv. Euclidean space.

pf: (1) Apply $\operatorname{res}_{p=\infty} \omega + \operatorname{res}_{p \neq \infty} \omega = 0$ for $\omega = \frac{\partial'(\lambda dp)\, \partial''(\lambda dp)}{d\lambda}$ mero. 1-form.

Note we have the identification: $\partial' \longleftrightarrow \partial'\lambda$

for tangent vectors $\sum c_i \frac{\partial}{\partial a_i} \longmapsto \sum_i c_i p^{i-1}$

Similarly, $\operatorname*{res}_{p=\infty} \dfrac{\partial'(\lambda dp)\,\partial''(\lambda dp)\,\partial'''(\lambda dp)}{dp\, d\lambda} = \operatorname*{res}_{p=\infty} \dfrac{r\cdot h\, dp}{\lambda'} + \underbrace{\operatorname*{res}_{p=\infty} q\cdot h\, dp}_{=0}$

$\partial'\lambda = f \qquad f(p)g(p) = q(p)\lambda'(p) + \underline{r(p)}$ (division)

$\partial''\lambda = g \qquad$ i.e. $f\cdot g = r$ in A_λ $\quad \deg_p < n$

$\partial'''\lambda = h$

$\| \qquad \langle q.h\rangle_\lambda = \langle f\cdot g\cdot h\rangle_\lambda = c(\partial',\partial'',\partial''')$

(2) Coord. systems : $a_1, \dots, a_n$ (NOT very useful)

ξ_a $(a=1,\dots,n)$ $\quad \lambda(p) = \prod_{a=1}^{n}(p-\xi_n) = (p+\xi_1+\dots+\xi_n)\prod_{a=1}^{n}(p-\xi_n)$

$u^i = \lambda(q^i)$, $i=1,\dots,n$ $\quad \lambda'(p) = \prod_{i=1}^{n}(p-q^i)$

We have, by def'n:

Lagrange Interpolation

In u-coord. system: $\partial_i\lambda\big|_{p=q^j} = \delta_{ij} \Rightarrow \partial_i\lambda(p) = \dfrac{1}{p-q^i}\dfrac{\lambda'(p)}{\lambda''(q^i)}$

To relate to ξ_a-coord, we have:

$$\partial_i\lambda(p) = (\partial_i\xi_1+\dots+\partial_i\xi_n)\prod_{b=1}^{n}(p-\xi_b) - \sum_{b=1}^{n}\frac{\lambda(p)}{p-\xi_b}\partial_i\xi_b$$

Put $p=\xi_a$, then $\dfrac{1}{\xi_a-q^i}\dfrac{\lambda'(\xi_a)}{\lambda''(q_i)} = -\lambda'(\xi_a)\cdot\partial_i\xi_a$

$$\Rightarrow \partial_i\xi_a = \frac{-1}{\xi_a-q^i}\frac{1}{\lambda''(q^i)} \quad \text{Jacobi matrix}$$

$$\langle\partial_i,\partial_j\rangle_\lambda = -\sum_{p\neq\infty}\operatorname*{res}_{\lambda'(p)=0}\left(\frac{\frac{1}{p-q^i}\frac{\lambda'(p)}{\lambda''(q^i)}\frac{1}{p-q^j}\frac{\lambda'(p)}{\lambda''(q^j)}dp}{\lambda'(p)}\right) = \begin{cases} 0, & \text{for } i\neq j \\ \dfrac{-1}{\lambda''(q_i)}, & \text{for } i=j\end{cases}$$

For $c(\partial_i,\partial_j,\partial_k)$, we have : $c(\partial_i,\partial_j,\partial_k)\neq 0$ only if $i=j=k$

and $c(\partial_i,\partial_i,\partial_i) = \operatorname*{res}_{p=q^i}\left(\dfrac{\lambda'(p)^2\, dp}{(p-q^i)^3\lambda''(q^i)^3}\right)$

$\| $

$\langle\partial_i\cdot\partial_i,\partial_i\rangle = \dfrac{-1}{\lambda''(q^i)}$ $\quad \Rightarrow \partial_i\cdot\partial_j = \delta_{ij}$ i.e. They are idempotents

$\|$

$\langle\partial_i,\partial_i\rangle = \dfrac{-1}{\lambda''(q^i)}$ $\qquad$ i.e. u^i's are canonical coord.

3) The metric $(\partial_i, \partial_j)_\lambda = -\sum_{\substack{p\neq\infty \\ d\lambda_p = 0}} \operatorname{res} \frac{\partial_i(\log \lambda_p)\partial_j(\log \lambda d p)}{d\log\lambda}$

$$= -\sum_{\substack{p\neq\infty \\ d\lambda_p=0}} \operatorname{res} \frac{(\partial_i\lambda)(\partial_j\lambda)}{\lambda\lambda'} dp$$

$$\Rightarrow g_{ij} := (\partial_i, \partial_j)_\lambda = -\frac{1}{u^i}\frac{1}{\lambda''(q^i)}\delta_{ij}$$

Now, we compute in the (ξ_a) - coord. system :

Exercise : On $\mathbb{R}^{n+1} \supset \underset{\overset{\shortparallel}{E}}{\{\xi_0 + \cdots + \xi_n = 0\}} \subseteq \mathbb{R}^n$

Then the induced Euclidean metric on E is given by :

$$g^{ab} = \delta_{ab} - \frac{1}{n+1}$$

$$(d\xi_a, d\xi_b) = \sum_{i=1}^n \frac{1}{g_{ii}(u)}\frac{\partial\xi_a}{\partial u^i}\cdot\frac{\partial\xi_b}{\partial u^i} = -\sum_{i=1}^n u^i \cancel{\lambda''(q^i)} \frac{1}{(\xi_a - q^i)\cancel{\lambda''(q^i)}}\cdot\frac{1}{(\xi_b - q^i)\lambda''(q^i)}$$

$$= -\sum_{i=1}^n \frac{u^i}{(\xi_a - q^i)(\xi_b - q^i)\lambda''(q^i)} \underset{\text{if } a\neq b}{\uparrow}{=} -\operatorname{res}_{\substack{d\lambda_p=0 \\ p\neq\infty}}\left(\frac{\lambda(p)}{(p-\xi_a)(p-\xi_b)\lambda'(p)}\right) = \operatorname{res}_{p=\infty}\left(\frac{\lambda(p)}{(p-\xi_a)(p-\xi_b)\lambda'(p)}\right)$$

$$= \frac{-1}{n+1}$$

If $a=b$, $(d\xi_a, d\xi_b) = -\operatorname{res}_{d\lambda_p=0}\left(\frac{\lambda(p)}{(p-\xi_a)^2\lambda'(p)}\right)$, we get one more term 1

$$\Rightarrow (d\xi_a, d\xi_b) = \delta_{ab} - \frac{1}{n+1} = g^{ab}.$$

Now, we would like to construct the flat coord. (t°) of $\langle , \rangle$

$\lambda = \lambda(p) = p^{n+1} + a_n p^{n-1} + \cdots + a_1$, fcn in p with parameter

$p = p(\underset{\overset{\shortparallel}{k}}{\lambda^{\frac{1}{n+1}}})$ when λ is large

Expand p into $p = k + \frac{1}{n+1}\left(\frac{t^n}{k} + \cdots + \frac{t^1}{k^n}\right) + O\left(\frac{1}{k^{n+1}}\right)$

$\rightsquigarrow$ def. of $t^1, \ldots, t^n$

Claim : In fact, $t^i = t^i(a_1, \ldots, a_n)$ forms a coord. system

Put into : $k^{n+1} = p(k)^{n+1} + a_n p(k)^{n-1} + \cdots + a_1$

Set $wt(p) = 1 \rightsquigarrow \deg(\lambda) = n+1$ $\deg(k) = 1$ $\Rightarrow \deg t^i = \deg a_i = n+2-i$

May solve $\begin{cases} a_i = -t^i + f_i(t^{i+1}, \dots, t^n) & i = 1, \dots, n \\ a_n = -t^n \\ \vdots \end{cases}$

<u>Exercise</u>: $t^\alpha = -\frac{n+1}{n-\alpha+1} \operatorname*{res}_{p=\infty} \left(\lambda^{\frac{n-\alpha+1}{n+1}}(p)\, dp \right)$

<u>Claim</u>: $\langle \partial_\alpha, \partial_\beta \rangle = \delta_{\alpha+\beta, n+1}$

<u>Lemma</u> (Thermaodynamical Identity)

$\lambda = \lambda(p, t^1, \dots, t^n)$ $p = p(\lambda, t^1, \dots, t^n)$: 1-varable inverse fcn to each other, with parameter $t^1, \dots, t^n$. $\partial_\alpha := \frac{\partial}{\partial t^\alpha}$

$\Rightarrow (\partial_\alpha \lambda)\Big|_{p=\text{const.}} dp = -(\partial_\alpha p)\Big|_{\lambda=\text{const.}} d\lambda$

6.25. 2018

Recall last time:

(1) $\partial_i \lambda = -\frac{1}{p-q^i} \frac{\lambda'(p)}{\lambda''(q^i)}$ $\quad\quad$ $\partial_i \xi_a = -\frac{1}{(\xi_a - q^i)\lambda''(q^i)}$

(2) Kodaira - Spencer map: $\partial \in T_\lambda M \longmapsto \partial(\lambda dp)$

Finish the proof of flat coord. $t^1, \dots, t^n$:

$$p = p(k) = k + \frac{1}{(n+1)}\left(\frac{t^n}{k} + \frac{t^{n-1}}{k^2} + \cdots + \frac{t^1}{k^n}\right) + O\left(\frac{1}{k^{n+1}}\right) \quad ;$$

pf:

$$\partial_\alpha = \frac{\partial}{\partial t^\alpha} \quad -\partial_\alpha(\lambda dp)|_{p=\text{const.}} = \partial_\alpha(p\, d\lambda)|_{\lambda=\text{const.}}$$

$$= \left(\frac{1}{n+1}\frac{1}{k^{n+1+\alpha}} + O\left(\frac{1}{k^{n+1}}\right)\right) \underbrace{d(k^{n+1})}_{\substack{\| \\ (n+1)k^n dk}} = k^{\alpha-1} dk + O\left(\frac{1}{k}\right) dk$$

$$\langle \partial_\alpha, \partial_\beta \rangle_\lambda = \operatorname*{res}_{p=\infty} \frac{k^{\alpha-1}dk \cdot k^{\beta-1}dk}{d(k^{n+1})} = \operatorname*{res}_{p=\infty} \frac{k^{\alpha+\beta-2}dk}{(n+1)k^n} = \frac{1}{n+1}\delta_{\alpha+\beta, n+1} \text{ const}$$

Now, generalize this picture to Hurwitz Spaces:

$\lambda^{-1}(\infty) \in C$: Riem. surface of genus $= g$ $\quad$ (The previous example can be regarded as $\mathbb{P}^1 \xrightarrow{\lambda} \mathbb{P}^1$ with $(n+1)$-covers)

$\downarrow \lambda$ $\quad$ Fix the infinity profile

$\infty \in \mathbb{P}^1$

$$\lambda^{-1}(\infty) = \sum_{i=0}^m (n_i + 1)\infty_i \quad (\text{divisor})$$

$\vec{n} = (n_0, \dots, n_m)$ $\quad$ n_i: ramification degree at $\infty_i \geq 0$

The space of λ with genus g with ramification data $\vec{n}$: $M_{g,\vec{n}}$

Let $N = \dim M_{g,\vec{n}}$

For a generic $\lambda \in M_{g,\vec{n}}$, the finite branch pts p_j are all double pts

N_f: The number of finite branch pt.

By Riemann - Hurwitz formula $\Rightarrow$

$$2-2g = \sum_{i=0}^m (n_i+1) - \sum_{i=0}^m n_i - N_f \Rightarrow N_f = 2g + 2m + \overbrace{\sum_{i=0}^m n_i}^{\substack{|\vec{n}| \\ \|}}$$

Riemann Existence Thm: Given the ram. data $\Rightarrow \exists \lambda$ up to finite choices

$\Rightarrow N_f = N$.

Rmk: Riemann's count : $\dim M_g = 3g - 3$ (for $g \geq 2$)

$\dim M_{g,n} = 3g - 3 + n$

This Hurwitz number (the finite # of choices) had been recently determined by Okonkov - Pandphardipande.

~~proof of Riemann count~~: Riemann's approach: By Riemann-Roch

$\Rightarrow h^0(C, \mathcal{O}(\lambda^{-1}(\infty))) = (|\vec{n}|+m+1)+1-g$ if $\deg|\vec{n}|$ large $(\geq 2g-1)$

$\Rightarrow \dim M_{g,m+1} = N_f - h^0(C, \mathcal{O}(\lambda^{-1}(\infty))) = 3g+m-2 = (3g-3)+(m+1)$

Modern approach (Kodaira - Spencer)

$$T_{[C]} M_g \cong H^1(C, T_C) = H^0(C, \underbrace{K_C \otimes T_C^\vee}_{\underset{K_C^{\otimes 2}}{\parallel}})^\vee$$

$K_C^{\otimes 2}$: quadratic differential

Use R-R to Calculate:

$h^0(C, K_C \otimes T_C^\vee) = (4g-4)+1-g = 3g-3.$

Canonical word. (Candidate) : $u^j := \lambda(p_j)$ $(j=1,\dots,N)$ where $d\lambda|_{p_j} = 0$ (p_j : finite critical pts)

We consider only the open part $\hat{M} \subset M_{g,\vec{n}}$ with $u^i \neq u^j$ (for $i \neq j$)

Define product str : $\partial_i \cdot \partial_j = \delta_{ij} \partial_i$ $\quad (\partial_i := \frac{\partial}{\partial u^i})$

$e = \sum_i \partial_i \qquad E = \sum_{i=1}^N u_i \partial_i$

Def'n of the metric η (candidate) :

Given an 1-form Ω on $\hat{M}$, $\langle a, b \rangle_\Omega := \Omega(a \cdot b)$

We call Ω admissible if η defines a Frob. mfd on $\hat{M}$.

Construction of Ω:

Given any Q : quadratic diff. on C. Given $[\lambda : C \to \mathbb{P}^1] \in \hat{M}$,

$$\Omega_Q := \sum_{i=1}^N du^i \operatorname{res}_{p_i} \left(\frac{Q}{d\lambda} \right)$$

We need to work on a covering $\tilde{M}$ of $\hat{M}$, consists of sets :

$$(C, \lambda, \underbrace{k_0, \dots, k_m}_{\text{branched data}}, \underbrace{a_1, \dots, a_g}_{\text{A-cycle}}, \underbrace{b_1, \dots, b_g}_{\text{B-cycle}})$$

in $H^1(C, \mathbb{Z})$

Torrelli marking: They form a symplect basis w.r.t. intersection

• ∞_i : $k_i^{n_i+1}(p) = \lambda(p)$ near $p = \infty_i$

local word. is $z_i = k_i^{-1}$.

$\Rightarrow \mathcal{Q}$ can also be multi-valued: $\gamma \in H_1(C, \mathbb{Z})$, $Q \longmapsto Q + q_\gamma d\lambda$

$\rightsquigarrow \Omega_{\mathcal{Q}} \mapsto \Omega_{Q}$.

In fact, we will take $Q = \phi^2$ for some (multi-valued) differential on C, called the primary differential.

eg. Type I: $\phi_{t^{i,\alpha}} = \frac{-1}{\alpha} dk_i^{\alpha} + (\text{reg.})$ near ∞_i $\quad i = 0 \dots m$

$= -k_i^{\alpha-1} dk_i \quad \alpha - 1 = 0, 1, \dots, n_i - 1 \quad \alpha = 1, \dots, n_i$

subject to normalization condition: $\oint_{a_j} \phi_{t^{i,\alpha}} = 0$

$\rightsquigarrow$ (single-valued) Abelian differential of 2nd kind (All residue $= 0$)

gives $|\vec{n}| = \sum_{i=0}^{m} n_i$

Type II: ϕ_{v^i} $(i = 1, \dots, m)$ $\rightsquigarrow$ $\alpha = n_i + 1$ case in type I

Type III: ϕ_{w^i} $(i = 1, \dots, m)$ Abelian diff. of the 3rd kind

pole with pole at $\underset{-1}{\underset{\downarrow}{\infty_0}}, \underset{1}{\underset{\downarrow}{\infty_i}}$ with residue (with only simple pole)

Type IV: multivalued diff. ϕ_{r^i} $(i = 1, \dots, g)$ $\phi_{r^i}(p + b_j) - \phi_{r^i}(p) = -\delta_{ij} d\lambda$

and $\int_{a_j} \phi_{r^i} = 0$

Type V: ϕ_{s^i} $(i = 1, \dots, g)$ $\sim H^0(C, K)$ basis $\int_{a_j} \phi_{s^i} = \delta_{ij}$

To understand this, the most important thing is the genernalized contour:

$\tilde{C}$: The simply connected domain

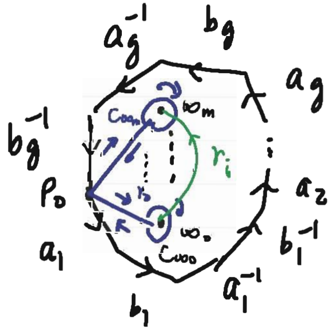
$\mathfrak{D}$ consits of : $a_1, \dots, a_g, b_1, \dots, b_g \rightarrow 2g$

$\lambda \cdot C_{\infty_i}$, γ_i $\quad i = 1, \dots, m \quad \rightarrow 2m$

$\uparrow$

connects ∞_0 to ∞_i

$\lambda^{\frac{k}{n_j+1}} C_{\infty_j}$ $\quad k = 1, \dots, n_j \quad j = 0, \dots, m$ $\sim |\vec{n}|$



Primes form: $E(P,Q)$ on $C \times C$ (Tata lecture on Theta I, II)
Ch.2

$\mathbb{P}^1$: $f = \dfrac{\prod(z_i - a_i)}{\prod(z_i - b_i)}$

T: $f = \dfrac{\prod \sigma(z_i - a_i)}{\prod \sigma(z_i - b_i)}$

S: Abel-Jacobi

$$E(P,Q) = \vartheta\begin{bmatrix}\delta' \\ \delta''\end{bmatrix}\left(\int_Q^P \vec{\omega}\right) / (\cdots\cdots)$$

$$\sim \frac{P-Q}{\sqrt{dp}\sqrt{dQ}}\left(1 + O(P-Q)\right)$$

$(P,Q) \in C \times C$, $B(P,Q) = \log E(P,Q)\, dP\, dQ$ Only pole is along $\Delta \subset C \times C$
i.e. $P = Q$
(double pole)

$C_\alpha \in \mathfrak{D}$, $\phi(p) \triangleq \int_{C_\alpha} B(P,Q)$ $\rightsquigarrow$ Type I – V is a basis
in Q-variable. of this construction.

Thm: Any ϕ constructed above with multiplication $\partial_i \cdot \partial_j = \delta_{ij}\partial_i$
$\eta: \langle \partial', \partial'' \rangle = \Omega_{\phi^2}(\partial' \cdot \partial'')$ $\quad \Omega_{\phi^2} = \sum_{i=1}^{N} du^i \operatorname{res}_{P_i}\left(\frac{\phi^2}{d\lambda}\right)$
gives the same Frobenius manifold str. with flat coord.

$$t^\alpha := \int_{C_\alpha \ni p} \phi(p) \longleftrightarrow \phi_\alpha := \int_{C_\alpha^*} B(P,Q)$$

$C_\alpha \in \mathfrak{D}$.

Sketch: How to prove the flatness $\partial_j = \frac{\partial}{\partial u_j}$

Lemma (5.1):
$\forall \omega^1, \omega^2$ on $\widetilde{M}$, with suitable meromorphic behavior (5.35) at $\infty_0, \ldots, \infty_m$

$$\frac{1}{2\pi i}\int_{\partial \tilde{C}} \omega^1(p) \int_{P_0}^{P} \partial_j \omega^2 \overset{(!)\ \text{imagination}}{=} -\operatorname{res}_{P_j}\left(\frac{\omega^1 \omega^2}{d\lambda}\right) \qquad \partial_j(\lambda\, dp) = \frac{1}{p - q_i}(\vdots)$$

has pole at $P = P_j$

$\|$

explicit countour calculation

$\partial_j \langle \omega^1, \omega^2 \rangle \longleftarrow$ pairing (5.36)

$\Rightarrow \langle \omega^1, \omega^2 \rangle$ is a symmetric pairing up to a const.

Modern approach: Rauch Variation formula (Arxiv: 1605.07644)

$$\frac{\partial}{\partial u_i} B(Q_1, Q_2) = \operatorname*{Res}_{P = P_i} \frac{B(P,Q_1)B(P,Q_2)}{d\lambda(P)}$$

η: $\eta_{jj} = \partial_j \underbrace{\langle \phi, \phi \rangle}_{}$ Need to check the rotation coeff.
$\rightarrow \rho$: metric potential satisfies Darboux eqn

$$\partial_k \gamma_{ij} = \gamma_{ik} \gamma_{kj}$$

$$\sum \partial_k \gamma_{ij} = 0$$

Plan:

- KdV hierachy:
- Witten Conjecture
- Combinatorial Model
- Matrix Integral Model
- Proof

- nkdV hierachy

$L = \partial_x^{n+1} + a_1(x)\partial_x^{n-1} + \cdots + a_n(x)$

$\partial_{t^{\alpha,p}} L = [L, (L^{\frac{\alpha}{n+1}+p})_+]$ $\quad \alpha = 1, \ldots, n$, $p = 0, 1, \ldots$

$a_i = a_i(x, \underline{t})$

$L^{1/n+1} = \partial_x + b_0(x) + b_1(x)\partial^{-1} + \cdots \rightsquigarrow (\partial_x + b_0 + b_1\partial^{-1} + \cdots)^n = L$

Note that $[\partial_x^{-1}, f] = -f_x\partial_x^{-1} + f_{xx}\partial_x^{-3} + \cdots$

Bi-Hamiltonian Str: ∂_x^{i-} coeff. of $\partial_{t^{\alpha,p}} L$

$\partial_{t^{\alpha,p}} \underline{a_i(x,t)} = \{a_i(x,t), H^{\alpha,p}\} = \{a_i(x,\underline{t}), H^{\alpha,p-1}\}_1$

$H^{\alpha,p} = H^{\alpha,p}[a_i, (a_i)_x, (a_i)_{xx}, \ldots\ldots]$ $\quad \{,\}, \{,\}_1$: Poisson brackets

Example $(n=1)$

KdV hierachy $L = \partial_x^2 + u$ $\quad \partial_{t_1} u(x,\underline{t}) = [L, (L^{\frac{2i+1}{2}})_+]$

$H_{1,1} = \frac{1}{2}u^2 \quad H_{1,2} = u^3 - \frac{1}{2}u_x^2$

$$\{F, G\} := \int \frac{\delta F}{\delta u}\frac{\partial}{\partial x}\left(\frac{\delta G}{\delta u}\right) dx$$

$$\{F, G\}_1 := \int \frac{\delta F}{\delta u}\left(\frac{-1}{2}\partial_x^3 + 2u\partial_x + u_x\right)\frac{\delta G}{\delta u} dx$$

- Semi-Classical Limit: $x \to \varepsilon x$ $\quad t_k \to \varepsilon t_k$ $\quad$ Take $\varepsilon \to 0$

Stable sol'n $\rightsquigarrow$ semisimple Frobenius manifold (c.f. Dubrovin Lecture 6)

- Construction of Sol'n of KdV hierachy: (Ref: Lando, Zvonkin - Graphs on Surfaces and their application)

Def: A Sato space is an infinite dim'l vector space $W \subseteq \mathbb{C}((z))$

s.t. $W = \langle f_1, f_2, \ldots\ldots \rangle$, where $f_i = z^{-i} + a_i z^{-i+1} + \cdots = z^{-i}(1 + o(1))$

$M(z, T_1, T_2, \ldots) := \exp(T_1 z^{-1} + T_2 z^{-2} + \cdots)$

$$\tau_W(T_1, T_2, \ldots) := \frac{\cdots \wedge Mf_2 \wedge Mf_1 \wedge z^0 \wedge z^1 \wedge \cdots}{\cdots \wedge z^{-2} \wedge z^{-1} \wedge z^0 \wedge z^1}$$

(In fact, indep. of choices of basis)

<u>Example</u>: $f_1 = z^{-1} + a_1 z^0 + a_2 z^{1} + \cdots \quad f_2 = z^{-2} + \frac{a_2}{a_1} z^{-1} \quad f_j = z^{-j} \ (j \geqq 3)$

$$Mf_1 = \cdots + \left(T_1 + a_1\left(\tfrac{1}{2}T_1^2 + T_2\right) + a_2\left(\tfrac{1}{6}T_1^3 + T_1T_2 + T_3\right)\right) z^{-2}$$
$$+ \left(1 + a_1 T_1 + a_2\left(\tfrac{1}{2}T_1^2 + T_2\right)\right) z^{-1}$$
$$Mf_2 = \cdots + \left(1 + a_2/a_1 \cdot T_1\right) z^{-2} + \frac{a_2}{a_1} z^{-1} + \cdots$$
$$\tau_W(T_1, T_2, \ldots) = 1 + a_1 T_1 + a_2 T_2^2 + \frac{a_2^2}{a_1}\left(\tfrac{1}{3}T_1^3 - T_3\right)$$

<u>Prop</u>: W: Sato space s.t. $z^{-2}W \subset W$

Then (1) $\tau_W(T_1, T_2, \ldots)$ does NOT depends on T_{2i}

(2) $L(x, T_1, T_3, \ldots) = \frac{\partial^2}{\partial x^2} + 2\frac{\partial^2}{\partial T_1^2}\log \tau_W(x + T_1, T_3, \ldots)$

satisfying KdV hierachy $\frac{\partial}{\partial T_{2k+1}}\left(2\frac{\partial^2}{\partial T_1^2}\log \tau_W\right) = [L, (L^{\frac{2k+1}{2}})_+]$

<u>Prop</u>: W: Sato space generated by $f_j = z^{-j}(1 + O(1))$

$$\frac{\det(f_i(z_j))}{\det(z_j^{-i})} = \tau_W(T_1(z_*) + T_2(z_*) + \cdots) \qquad T_k(z_*) = \frac{1}{k}\sum_{i=1}^{N} z_i^k$$

Witten Conjecture:

$M_{g,n}$ ($\overline{M_{g,n}}$) : moduli space of smooth (nodal) genus g, n-pointed stable curve

($\mathrm{Aut}(C; p_1, \ldots, p_n) < \infty$)

$\mathcal{L}_i$ $(i = 1, \ldots, n)$: line bundle on $\overline{M_{g,n}}$, whose fiber at $(C; x_1, \ldots, x_n)$ is $T^*_{x_i}C$, $\psi_i := c_1(\mathcal{L}_i)$

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle := \int_{\overline{M_{g,n}}} \psi_1^{d_1} \cdots \psi_n^{d_n} \quad \left(= 0 \text{ if } d_1 + \cdots + d_n \neq \dim_{\mathbb{C}} \overline{M_{g,n}} = 3g - 3 + n\right)$$

$$g = \frac{\sum d_i + 3 - n}{3}$$

$$F(t_0, \ldots) = \sum_{n \geqq 0} \sum_{\substack{d_1 \geqq 0 \\ \vdots \\ d_n \geqq 0}} \frac{1}{n!} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle t_{d_1} \cdots t_{d_n}$$

<u>Witten Conjecture</u>: e^F is the τ-function for the KdV hierarchy w.r.t.

$$T_{2i+1} = \frac{t_i}{(2i+1)!!}$$

<u>Rmk</u>: Witten conj. + String eqn + Dilaton eqn. $\left\{ \int_{\overline{M_{0,3}}} \psi_1^0 \psi_2^0 \psi_3^0 = \langle \tau_0 \tau_0 \tau_0 \rangle = 1 \quad \langle \tau_1 \rangle = \frac{1}{24} \right.$ initial datum

$\rightsquigarrow$ Solve all intersection number recursively.

Combinatorial Model:

X: cpt. Riemann surf. ρ: quad. differential Loc, $\rho = \varphi(z)(dz)^2$

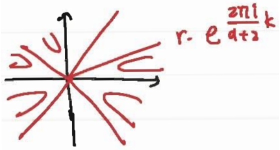
Assume $\varphi(z)$ has only simple or double pole.

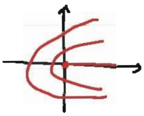
Horizontal line field $= \{ v \in TX \mid \varphi(z)(dz(v))^2 > 0 \}$

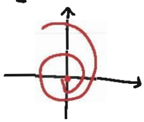
• No pole or No zero.

$\xrightarrow{\text{integral curve}}$ Horizontal trajectories

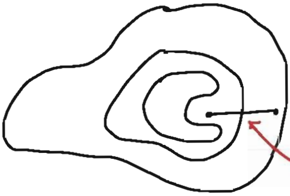
• $z^d (dz)^2 \quad d \geq 1$ • $\dfrac{(dz)^2}{z}$ • $c\dfrac{(dz)^2}{z^2} \quad c<0$







Rmk: For a generic quadratic diff. a generic hori traj. is non-closed (Not homeo to S^1)



Def: A Jenkin-Strebel differential is a quad. diff. with only finitely many non-closed hori. traj.

Prop: ρ: J-S diff. on X

• The connected component of $X \setminus \{\text{non-closed hori. traj.}\}$ is either open annulus or open disk.

• All closed hori. traj. in the same connected component have the same length

$(ds^2 = |\varphi(z)|^2 |dz|^2)$

Thm: (Strebel)

For any $(2n+1)$-type $(X, x_1, \dots, x_n; p_1, \dots, p_n)$, where X is a Riem. surf. of finite type, x_i: distinct pts on X $p_i > 0$ and $n > \chi(X)$, then $\exists!$ J-S diff s.t. • It has double pole at x_i and no other pole

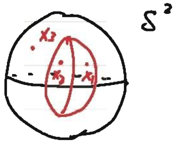
• Connected component of $X \setminus \{\text{non-closed hori. traj.}\}$ are open disks

- The length of hori traj. associated to x_i is p_i.

$\rightsquigarrow$ Such J-S diff is called "canonical J-S diff".

"Conversely", given an embedded graph in a 2-dim'l $\overset{\text{cpt}}{\text{topo.}}$ mfd with (genus $= g$)

(*)
- valencies of each vertex ≥ 3
- face marked by $\{x_1, \dots, x_n\}$
- connected component of $X \setminus \{\text{graph}\}$ are open disks
- fixed length of its edge.

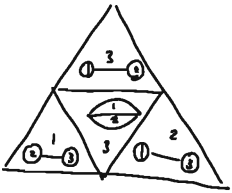


Then $\exists!$ cpx str on X s.t. Corresponding canonical J-S diff. determines the embedded graph.

$$\mathcal{M}_{g,n}^{\text{comb}} := \left\{ \text{The moduli space of genus } g \text{ connected embedded graph with } (*) \right\}$$

<u>Thm</u>: $M_{g,n} \times \mathbb{R}_+^n \cong \mathcal{M}_{g,n}^{\text{Comb.}}$ as orbifolds

<u>Examples</u>: $M_{0,3} \times \mathbb{R}_+^3 \cong \mathcal{M}_{0,3}^{\text{comb.}}$



<u>Rmk</u>: $V - E + n = 2 - 2g$

$3V = 2E$

$\Rightarrow E = \dim_{\mathbb{R}} \overline{\mathcal{M}_{g,n}} + n$