Notes on Isomonodromy Deformations of Linear ODE

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1. Preliminaries and Settings

We consider the ODE $\dfrac{dY}{dx} = A(x)Y$, $\begin{cases} A(x) \equiv A(x,t) = \sum_{\nu=1}^{M} \sum_{k=0}^{r_\nu} \dfrac{A_{-k}^{(\nu)}(t)}{(x-a_\nu(t))^{k+1}} - \sum_{k=1}^{r_\infty} A_{-k}^{(\infty)}(t) x^{k-1} \\ r_\nu \geq 0, \quad t = (t_1, \dots, t_q) \end{cases}$ $\sim (*)$

- We assume that

  ·· If $r_\nu = 0$, then a_ν has the following decomposition:

  we have $A_{-r_\nu}^{(\nu)} = G^{(\nu)} \Lambda_{-r_\nu}^{(\nu)} G^{(\nu)-1}$, where $\Lambda_{-r_\nu}^{(\nu)}$ is diagonal, $(\Lambda_{-r_\nu}^{(\nu)})_{ii} - (\Lambda_{-r_\nu}^{(\nu)})_{jj} \notin \mathbb{Z}$ for $i \neq j$. Then [Wason, Chap 2] shows that on a neighborhood of a_ν, the local solution $Y^{(\nu)}(x)$ can be uniquely written as $Y^{(\nu)}(x) = G^{(\nu)} \hat{Y}^{(\nu)}(x) \cdot (x-a_\nu)^{\Lambda_0^{(\nu)}}$, where

  $$\hat{Y}^{(\nu)}(x) = I + \sum_{j=1}^{\infty} Y_j^{\nu} (x-a_\nu)^j, \quad (x-a_\nu)^{\Lambda_0^{(\nu)}} := e^{\Lambda_0^{(\nu)} \log(x-a_\nu)}.$$

  ·· If $r_\nu > 0$, then a_ν has the following decomposition:

  we have $A_{-r_\nu}^{(\nu)} = G^{(\nu)} \Lambda_{-r_\nu}^{(\nu)} G^{(\nu)-1}$, where $\Lambda_{-r_\nu}^{(\nu)}$ is diagonal, $(\Lambda_{-r_\nu}^{(\nu)})_{ii} - (\Lambda_{-r_\nu}^{(\nu)})_{jj} \neq 0$ for $i \neq j$. Then [Wason Chap 4] shows that the canonical solutions $\overline{Y}_n^{(\nu)}(x)$ in the stokes sectors $S_n^{(\nu)}$, $\nu = 1, \dots, 2r_\nu+1$, have the asymptotic expansion

  $$\left(\begin{array}{l} \overline{Y}_n^{(\nu)}(x) \sim G^{(\nu)} \hat{Y}^{(\nu)}(x)\, e^{\Lambda^{(\nu)}(x)} \\ x \in S_n^{(\nu)}, \ x \to a_\nu, \ n = 1, \dots, 2r_\nu+1 \end{array} \right) \text{ where } \begin{cases} \hat{Y}^{(\nu)}(x) = I + \sum_{j=1}^{\infty} Y_j^{\nu} (x-a_\nu)^j \text{ and} \\ \Lambda^{(\nu)}(x) = \sum_{k=-r_\nu}^{-1} \dfrac{\Lambda_k^{(\nu)}}{k} (x-a_\nu)^k + \Lambda_0^{(\nu)} \ln(x-a_\nu). \end{cases} \sim (**)$$

  Here, $\Lambda_k^{(\nu)}$ are diagonal for all k. Also, $Y_{2r_\nu+1}^{(\nu)}(x) = Y_1^{(\nu)}(x) \cdot e^{2\pi i \Lambda_0^{(\nu)}}$

  (In the above setting, if $\nu = \infty$, $x - a_\nu := \dfrac{1}{x}$.)

  ·· Note that the analytic dependence of the solutions of complex linear ODE implies that we may assume the above sectors $\{S_n^{(\nu)}\}_{n=1}^{2r_\nu+1}$ can be chosen such that under the map $x \mapsto x - a_\nu(t)$, the sectors are the same in a small open set of t.

  ·· Also, [Wasow Chap 3] shows that for an analytic function in multivariables, the asymptotic expansion has analytic dependence on other variables, i.e. $f(x,t) \sim \sum_{j=0}^{\infty} f_j(t)(x-a(t))^j$. $[f(x,t) = \text{holo.} \Rightarrow f_j(t) = \text{holo}]$.
  So we may assume the asymptotic condition (**) holds uniformly in t.

  ·· By Gauge transform, we may assume $G^{(\infty)} \equiv I$.

The singularity data of (*) consists of $\{ a_\nu, A_k^{(\nu)}, \ldots, G^{(\nu)} \ (\nu = 1, \dots, M;\ k = 0, \dots, -r_\nu+1)$ and $A_k^{(\infty)} \ (k = -1, \dots, -r_\infty) \}$, denoted by $\mathcal{M}$.

it defines (*) / (1) Another coordinate of $\mathcal{M}$ is that $\{ a_\nu, G^{(\nu)}, (A_k^{(\nu)})_{\text{off diagonal}} \ (k \neq 0, \nu = 1, \dots, m, \infty), A_0^{(\nu)}, \Lambda_k^{(\nu)} \}$

The monodromy data of (*) consists of $\{ a_1, \dots, a_m, \infty, \ Sph^{(\nu)} \ (\nu = 1, \dots, m, \infty), \ \underbrace{C^{(1)}, \dots, C^{(M)}}_{\text{connection matrices}} \}$

Here $Sph^{(\nu)} := \begin{cases} \{\Lambda_0^{(\nu)}\} & \text{if } r_\nu = 0 \\ \{\Lambda_{-r_\nu}^{(\nu)}, \dots, \Lambda_0^{(\nu)}, \underbrace{S_1^{(\nu)}, \dots, S_{2r_\nu}^{(\nu)}}_{\text{stokes matrices}} \} & \text{if } r_\nu > 0 \end{cases}$

So we have $Y_{l+1}^{(\nu)}(x) = \bar{Y}_l^{(\nu)}(x) S_l^{(\nu)}$ and $\bar{Y}_1^{\infty}(x) = \bar{Y}_1^{(\nu)} C^{(\nu)}$

- **Def**: Under the assumption, an isomonodromy deformation means that the canonical solutions can be chosen such that $\Lambda_0^{(\nu)}$, $S_n^{(\nu)}$, $C^{(\nu)}$ are independent of t.

2. **Main Results**:

- Now set
$$\begin{cases} Y(x,t) \triangleq \bar{Y}_1^{(\infty)}(x,t) = Y_1^{(\infty)}(x,t) \\ Y_l^{(\nu)}(x,t) \triangleq G^{(\nu)}(t)^{-1} \bar{Y}_l^{(\nu)}(x,t) = G^{(\nu)-1} Y(x) C^{(\nu)-1} S_1^{(\nu)} \cdots S_{l-1}^{(\nu)} \end{cases}$$
, which is the solutions of $\frac{dY}{dx} = A^{(\nu)}(x,t) Y$, where $\begin{cases} A^{(\nu)} := G^{(\nu)-1} A G^{(\nu)} \\ A^{(\infty)} := A \end{cases}$

- In the following, the exterior differentiation d is wrt t.

Set
$$\begin{cases} B^{(\infty)}(x,t) \triangleq \hat{Y}^{(\infty)}(x,t)\, d'\Lambda^{(\infty)}(x,t)\, \hat{Y}^{(\infty)}(x,t)^{-1} = \sum_{k=-r_\infty}^{\infty} B_k^{(\infty)}(t)\, x^{-k} \\ B^{(\nu)}(x,t) \triangleq \hat{Y}^{(\nu)}(x,t)\, d'\Lambda^{(\nu)}(x,t)\, \hat{Y}^{(\nu)}(x,t)^{-1} = \sum_{k=-r_\nu-1}^{\infty} B_k^{(\nu)}(t)\, (x-a_\nu(t))^k \end{cases}$$
, where
$$\begin{cases} d'\Lambda^{(\infty)}(x,t) \triangleq d\Lambda^{(\infty)}(x,t) - d\Lambda_0^{(\infty)}(t) \log\left(\frac{1}{x}\right) \quad \left(= \sum_{k=1}^{r_\infty} d\Lambda_{-k}^{(\infty)}(t) \frac{x^k}{(-k)} \right) \\ d'\Lambda^{(\nu)}(x,t) \triangleq d\Lambda^{(\nu)}(x,t) - d\Lambda_0^{(\nu)}(t) \log\left(\frac{1}{x}\right) \quad \left(= \sum_{k=1}^{r_\nu} \frac{d\Lambda_{-k}^{(\nu)}}{-k(x-a_\nu)^k} + \sum_{k=0}^{r_\nu} \frac{-\Lambda_{-k}^{(\nu)}}{(x-a_\nu)^{k+1}} da_\nu \right) \end{cases}$$

Theorem 1: $(*)$ is isomonodromy
$$\overset{(a)}{\Longleftrightarrow} \begin{cases} dY_l^{(\mu)} = \Omega^{(\mu)} Y_l^{(\mu)} & (\mu = 1, \cdots, m, \infty,\ l = 1, \cdots, 2r_\nu) \quad (*1) \\ dG^{(\mu)} = \Theta^{(\mu)} G^{(\mu)} & (\mu = 1, \cdots, m) \end{cases} \sim (*2)$$

$$\overset{(b)}{\Longleftrightarrow} \begin{cases} dA^{(\mu)} = \frac{\partial \Omega^{(\mu)}}{\partial x} + [\Omega^{(\mu)}, A^{(\mu)}] & (\mu = 1, \cdots, m, \infty) \\ dG^{(\mu)} = \Theta^{(\mu)} G^{(\mu)} & (\mu = 1, \cdots, m) \end{cases} \sim (**)$$

(isomonodromy deformation equation)

Here,
$$\begin{cases} \Omega^{(\infty)}(x,t) \triangleq \sum_{\nu=1}^{m} G^{(\nu)} \left((B^{(\nu)})_{\text{sing}_x} \right) G^{(\nu)-1} + (B^{(\infty)})_{\text{holo}_x} \\ \Theta^{(\nu)}(x,t) \triangleq \left[\sum_{\mu \neq \nu} G^{(\mu)}(t) \left((B^{(\mu)})_{\text{sing}_x} \right) G^{(\mu)}(t)^{-1} + (B^{(\infty)})_{\text{holo}_x} \right]_{x = a_\nu(t)} \\ \qquad + G^{(\nu)}(t) \left(Y_1^{(\nu)}(t)\, da_\nu(t) - B_0^{(\nu)}(t) \right) G^{(\nu)}(t)^{-1} \\ \Omega^{(\nu)}(x,t) \triangleq G^{(\nu)}(t)^{-1} \left(\Omega^{(\infty)} - \Theta^{(\nu)} \right) G^{(\nu)}(t). \end{cases}$$

$(B^{(\nu)})_{\text{sing}_x}$: singular part in x; $(B^{(\infty)})_{\text{holo}_x}$: analytic part in x

Note: $Y_1^{(\nu)}$: the term in the formal power series $\hat{Y}^{(\nu)}$

Theorem 2 : ($\ast\ast$) is a complete integrable system in the sense of Frobenius.

Moreover, the parameter t can be chosen to be $\{a_1, \ldots, a_m, \Lambda^{(\nu)}_{-r_\nu}, \ldots, \Lambda^{(\nu)}_{-1}, \nu = 1, \ldots, m, \infty$

Theorem 3 : In theorem 2, the general solution of ($\ast\ast$), is meromorphic in t on the affine manifold with parameter $\{a_1, \ldots, a_m, \Lambda^{(\nu)}_{-r_\nu}, \ldots, \Lambda^{(\nu)}_{-1}, \nu = 1, \ldots, m, \infty\}$ outside $\{a_\nu = a_\mu$ for $\mu \neq \nu$, $(\Lambda^{(\nu)}_{-r_\nu})_{ii} = (\Lambda^{(\nu)}_{-r_\nu})_{jj}$ for $i \neq j\}$.

3. Proof of Theorem 1 :

ⓐ) • Note that $\begin{cases} (\ast 1), \mu = \infty \\ (\ast\ast 2) \end{cases}$ will imply $(\ast 1), \mu \neq \infty$:

$$\Omega^{(\mu)} \stackrel{?}{=} dY^{(\mu)}_\ell \cdot Y^{(\mu)-1}_\ell = -G^{(\mu)-1} dG^{(\mu)} + G^{(\mu)-1} dY^{(\infty)}_1 Y^{(\infty)-1}_1 G^{(\mu)}$$

$\left(Y^{(\mu)}_\ell = G^{(\mu)-1} Y^{(\infty)}_1 C^{(\mu)-1} S^{(\mu)}_1 \cdots S^{(\mu)}_{\ell-1}\right.$ and $C^{(\mu)}$, $S^{(\mu)}_k$ are constant$\left.\right)$

$$\| \qquad G^{(\mu)-1}(\Omega^{(\infty)} - \Theta^{(\nu)}) G^{(\mu)}$$

• Note ① $\because$ $S^{(\infty)}_k$ are constant $\therefore$ $dY^{(\infty)}_\ell \cdot Y^{(\infty)-1}_\ell$ is independent of ℓ

② [Wasow Chap3] shows that the partial derivative and the asymptotic expansion commutes, so $d\hat{Y}^{(\nu)}$ can be differentiated termwise.

So we get

$\cdot\cdot$ $$dY^{(\infty)}_\ell \cdot Y^{(\infty)-1}_\ell \sim d(\hat{Y}^{(\infty)} e^{\Lambda^{(\infty)}}) \cdot (\hat{Y}^{(\infty)} e^{\Lambda^{(\infty)}})^{-1}$$
$$= d\hat{Y}^{(\infty)} \cdot \hat{Y}^{(\infty)-1} + \hat{Y}^{(\infty)} \underbrace{d\Lambda^{(\infty)}}_{= d'\Lambda^{(\infty)}} \hat{Y}^{(\infty)-1} \qquad (x \to \infty)$$

since $d\Lambda^{(\infty)}_0 = 0$ by assumption.

↘ coincides with the analytic part of $\Omega^{(\infty)}$

$\cdot\cdot$ $$dY^{(\infty)}_\ell \cdot Y^{(\infty)-1}_\ell \sim d(G^{(\nu)} \cdot \hat{Y}^{(\nu)} \cdot e^{\Lambda^{(\nu)}} \cdot C^{(\nu)}) \cdot (G^{(\nu)} \cdot \hat{Y}^{(\nu)} \cdot e^{\Lambda^{(\nu)}} \cdot C^{(\nu)})^{-1}$$
$$= \underbrace{dG^{(\nu)} \cdot G^{(\nu)-1}}_{\text{constant in } x} + G^{(\nu)} \Big(\underbrace{d\hat{Y}^{(\nu)} \cdot \hat{Y}^{(\nu)-1}}_{\text{holomorphic in } x} + \hat{Y}^{(\nu)} \underbrace{d\Lambda^{(\nu)}}_{= d'\Lambda^{(\nu)}} \hat{Y}^{(\nu)-1} \Big) G^{(\nu)-1} \qquad (x \to a_\nu)$$

since $d\Lambda^{(\nu)}_0 = 0$

↘ coincides with the singular part at a_ν (of Ω^∞)

$\Rightarrow$ $dY^{(\infty)}_\ell \cdot Y^{(\infty)-1}_\ell = \Omega^{(\infty)}$

Compare the constant term of $\cdot\cdot$ above at $x = a_\nu$, we get

$$\Big[\sum_{\mu \neq \nu} G^{(\mu)-1} (B^{(\mu)})_{\text{sing}_x} G^{(\mu)} + (B^{(\infty)})_{\text{holo}_x}\Big]\Big|_{x = a_\nu} = dG \cdot G^{(\nu)-1} + G^{(\nu)}(-Y^{(\nu)}_1 da_\nu + B^{(\nu)}_0) G^{(\nu)-1}$$

This shows ($\ast\ast 2$). $\ast$

$\overset{(a)}{\Leftarrow}$ and $\overset{(b)}{\Rightarrow}$: Consider $\begin{cases} \dfrac{dY}{dx} = A^{(\mu)}(x,t)\,Y & \sim ① \\ dY = \Omega^{(\mu)}\,Y & \sim ② \end{cases}$, $\underline{d① - \frac{d}{dx}②}$ gives $dA^{(\mu)} = \dfrac{\partial \Omega^{(\mu)}}{\partial x} + [\Omega^{(\mu)}, A^{(\mu)}]$

- $\boxed{\text{Claim}}$: ($\ast\ast$) will gives the following : $d\Lambda_0^{(\mu)} = 0$ (This will be proved in the proof of theorem 2)
- $Y_\ell^{(\mu)}$ and $Y_{\ell+1}^{(\mu)}$ are both solutions to system ① and ②, so

$$dY_\ell^{(\mu)} \cdot Y_\ell^{(\mu)-1} = \Omega^{(\mu)} = dY_{\ell+1}^{(\mu)}\; Y_{\ell+1}^{(\mu)-1} = dY_\ell^{(\mu)-1}\, Y_\ell^{(\mu)} + Y_\ell^{(\mu)}\, dS_\ell^{(\mu)}\, S_\ell^{(\mu)-1}\, Y_\ell^{(\mu)-1}$$

$$\Rightarrow dS_\ell^{(\mu)} = 0$$

- Similarly, $dY_1^{(\mu)} \cdot Y_1^{(\mu)-1} = \Omega^{(\mu)} = G^{(\mu)-1}(\Omega^{(\infty)} - \Theta^{(\mu)})\,G^{(\mu)} = G^{(\mu)-1}(dY_1^{(\infty)} \cdot Y_1^{(\infty)-1} - dG^{(\mu)} \cdot G^{(\mu)-1})$

$$= G^{(\mu)-1} d(G^{(\mu)} Y_1^{(\mu)} C^{(\mu)})\,(G^{(\mu)} Y_1^{(\mu)} C^{(\mu)})^{-1} G^{(\mu)} - G^{(\mu)-1} dG^{(\mu)} = dY_1^{(\mu)} \cdot Y_1^{(\mu)-1} + Y_1^{(\mu)}\, dC^{(\mu)} \cdot C^{(\mu)-1}\, Y_1^{(\mu)-1}$$

$$\Rightarrow dC^{(\mu)} = 0.$$

$\overset{(b)}{\Leftarrow}$: It suffices to show ($\ast 1$).

Consider $dY_\ell^{(\mu)} - \Omega^{(\mu)} Y_\ell^{(\mu)} \sim ③$

$$\frac{\partial}{\partial x}\left(dY_\ell^{(\mu)} - \Omega^{(\mu)} Y_\ell^{(\mu)}\right) = d\left(A^{(\mu)} Y_\ell^{(\mu)}\right) - \left(\frac{\partial \Omega^{(\mu)}}{\partial x}\right) Y_\ell^{(\mu)} - \Omega^{(\mu)} \cdot A^{(\mu)}\, Y_\ell^{(\mu)}$$

$$= A^{(\mu)}\, dY_\ell^{(\mu)} + \underbrace{\left(dA^{(\mu)} - \frac{\partial \Omega^{(\mu)}}{\partial x} - \Omega^{(\mu)} \cdot A^{(\mu)}\right)}_{= -A^{(\mu)} \Omega^{(\mu)}} Y_\ell^{(\mu)} = A^{(\mu)}\left(dY_\ell^{(\mu)} - \Omega^{(\mu)} Y_\ell^{(\mu)}\right)$$

So ③ is also a solution to $\dfrac{dY}{dx} = A^{(\mu)}\,Y$. Hence, by uniqueness,

$dY_\ell^{(\mu)} - \Omega^{(\mu)} Y_\ell^{(\mu)} = Y_\ell^{(\mu)} \cdot K_\ell^{(\mu)}$ for some constant matrix $K_\ell^{(\mu)}$ in x.

- At a_μ, $Y_\ell^{(\mu)} \sim \hat{Y}^{(\mu)} e^{\Lambda^{(\mu)}}$. So $K_\ell^{(\mu)} = Y_\ell^{(\mu)-1} dY_\ell^{(\mu)} - Y_\ell^{(\mu)-1} \Omega^{(\mu)} Y_\ell^{(\mu)}$

$$= e^{-\Lambda^{(\mu)}}\Big(\hat{Y}_\ell^{(\mu)-1} d\hat{Y}_\ell^{(\mu)} + \underbrace{d\Lambda^{(\mu)}}_{= d'\Lambda^{(\mu)}} - \Omega^{(\mu)} \hat{Y}_\ell^{(\mu)}\Big) e^{\Lambda^{(\mu)}} = e^{-\Lambda^{(\mu)}}\Big(Y_1 da_\mu + (+(B^{(\mu)})_{\text{sing}} + Y_1 da_\mu - B_0^{(\mu)})\hat{Y}_1^{(\mu)} + \hat{Y}^{(\mu)-1}(B^{(\mu)})\hat{Y}^{(\mu)} + O(x - a_\mu)\Big) e^{\Lambda^{(\mu)}}$$

by claim, $d\Lambda_0^{(\mu)} = 0$.

- If a_μ is irregular, then approximate a_γ with $\mathrm{Re}\,(\Lambda^{(\mu)})_{ii} - (\Lambda^{(\mu)})_{jj} < 0$, we have exponential decay, so the off diagonal term of $K_\ell^{(\mu)} = 0$.
- If a_μ is regular, since we assume $(\Lambda_0^{(\mu)})_{ii} - (\Lambda_0^{(\mu)})_{jj} \notin \mathbb{Z}$, $(e^{\Lambda_0^{(\mu)}} K_\ell^{(\mu)} e^{-\Lambda_0^{(\mu)}})_{ij}$ $= (K_\ell^{(\mu)})_{\alpha\beta}\,(x - a_\gamma)^{(\Lambda_0^{(\mu)})_{\alpha\alpha} - (\Lambda_0^{(\mu)})_{\beta\beta}}$ can't has the asymptotic expansion $\sum_{j=1}^{\infty} f_j (x - a_\gamma)^{j}$ unless $K_\ell^{(\mu)} = 0$.
- For the diagonal, the above expression cancelled the constant term, so $K_\ell^{(\mu)} \equiv 0$.
- Similar for ∞. $\ast$

4. Proof of theorem 2:

Recall: We consider $\frac{dY}{dx} = A(x)Y$ (*), $A(x) = A(x,t) = \sum_{\nu=1}^{m}\sum_{k=0}^{r_\nu} \frac{A^\nu_{-k}}{(x-a_\nu)^{k+1}} - \sum_{k=1}^{r_\infty} A^\infty_{-k} x^{k-1}$, $r_\nu \geq 0$.

Here a_ν can be regular or irregular. In both case, we assume it is a "generic" case, and then the local solution near a_ν will be given by or asymptotic to $G^{(\nu)}\hat{Y}^{(\nu)}(x)\, e^{\Lambda^{(\nu)}(x)}$, where

$$\begin{cases} A^\nu_{-r_\nu} = G^{(\nu)}\Lambda^{(\nu)}_{-r_\nu} G^{(\nu)-1} \\ \hat{Y}^{(\nu)}(x) = I + \sum_{j=1}^{\infty} Y^\nu_j (x-a_\nu)^j \qquad\qquad (G^{(\infty)} := I) \\ \Lambda^{(\nu)}(x) = \sum_{k=-r_\nu}^{-1} \frac{\Lambda^{(\nu)}_k}{k}(x-a_\nu)^k + \Lambda^{(\nu)}_0 \ln(x-a_\nu). \end{cases}$$

- We define the normalization form: $A^{(\nu)}(x) := G^{(\nu)-1}A(x)\,G^{(\nu)}$, $A^{(\nu)}(x) = \sum_{k=0}^{r_\nu}\frac{A^{(\nu)}_{-k}}{(x-a_\nu)^{k+1}} + (\text{analytic part})$

  Then $A^{(\nu)}_{-r_\nu} = \Lambda^{(\nu)}_{-r_\nu}$ is diagonal. Also, the local solution of $\frac{dY}{dx} = A^{(\nu)}(x)Y$ at a_ν will be / be asymptotic to $\hat{Y}^{(\nu)}(x)\, e^{\Lambda^{(\nu)}(x)}$.

- The singularity data of (*), denoted by $\mathcal{R}$, consists of

  $\{a_1, \dots, a_m, G^{(1)}, \dots, G^{(m)}$, and $A^{(\nu)}_{-k}$ $(\nu=1,\dots,m,\ k=0,\dots,r_\nu)$ or $(\nu=\infty, k=1,\dots,r_\infty)\}$

  ↳ it determines (*).

- **Fact**: The singularity data can be replaced by

  $\{a_1, \dots, a_m, G^{(1)}, \dots, G^{(m)}$, and $\begin{cases}\Lambda^{(\nu)}_{-k} & (k=1,\dots,r_\nu) \\ (A^{(\nu)}_{-k})_{OD} & (k=1,\dots,r_\nu-1)\end{cases}$ $(\nu=1,\dots,m,\infty)$, $A^{(\nu)}_0$ $(\nu=1,\dots,m)\}$

  ↖ off diagonal part

证) (Recall): The formal power series $\hat{Y}^{(\nu)}(x)\, e^{\Lambda^{(\nu)}(x)}$ is defined to be the $\boxed{\text{unique}}$ solution of $\frac{d}{dx}\left(\hat{Y}^{(\nu)}(x)\, e^{\Lambda^{(\nu)}(x)}\right) = A^{(\nu)}(x)\left(\hat{Y}^{(\nu)}(x)\, e^{\Lambda^{(\nu)}(x)}\right)$

$$\Updownarrow$$

$$\frac{d}{dx}\hat{Y}^{(\nu)} = A^{(\nu)}\hat{Y}^{(\nu)} - \hat{Y}^{(\nu)}\frac{d}{dx}\Lambda^{(\nu)} \quad \sim (\Delta).$$

•• To determine the coefficients, note that $\hat{Y}^{(\nu)}(x)$ can be written uniquely as $F^{(\nu)}(x)\,D^{(\nu)}(x)$, where

$$\begin{cases} F^{(\nu)}(x) = I + \sum_{k=1}^{\infty} \underbrace{F^{(\nu)}_k}_{\text{off diagonal matrices}} (x-a_\nu)^k \\ D^{(\nu)}(x) = I + \sum_{k=1}^{\infty} \underbrace{D^{(\nu)}_k}_{\text{diagonal matrices}} (x-a_\nu)^k \end{cases}$$

•• Then (Δ) equation becomes $\frac{d}{dx}F^{(\nu)}(x) + F^{(\nu)}(x)\cdot\frac{d}{dx}\left(\log D^{(\nu)}(x) + \Lambda^{(\nu)}(x)\right) = A^{(\nu)}(x)\,F^{(\nu)}(x)$ $\sim(\Delta\Delta)$

•• Taking diagonal part of $(\Delta\Delta)$, we get $\frac{d}{dx}\left(\log D^{(\nu)}(x) + \Lambda^{(\nu)}(x)\right) = \left(A^{(\nu)}(x)\,F^{(\nu)}(x)\right)_D$ $\sim(\Delta\Delta\Delta)$

↑ diagonal part

•• Then $(\Delta\Delta)$ becomes $\frac{d}{dx}F^{(\nu)}(x) + F^{(\nu)}(x)\left(A^{(\nu)}(x)\,F^{(\nu)}(x)\right)_D = A^{(\nu)}(x)\,F^{(\nu)}(x)$. $\sim(\Delta\Delta\Delta\Delta)$

$\cdot\cdot$ Note that we can use ($\triangle\triangle\triangle\triangle$) to solve recursively the coefficients of $F^{(\nu)}(x)$. 6

$\cdots$ Consider the negative degree term of ($\triangle\triangle$), we get ↘ Plug in ($\triangle\triangle\triangle$), the others are determined

$$[F_j^{(\nu)}, A_{-r_\nu}^{(\nu)}] = \sum_{k=1}^{j} \left(A_{-r_\nu+k}^{(\nu)} \cdot F_{j-k}^{(\nu)} - F_{j-k}^{(\nu)} \Lambda_{-r_\nu+k}^{(\nu)}\right) \quad (0 \le j \le r_\nu).$$

eg. $0 = A_{-r_\nu}^{(\nu)} - \Lambda_{-r_\nu}^{(\nu)}$

$$\underbrace{[F_1^{(\nu)}, A_{-r_\nu}^{(\nu)}]}_{\text{off diagonal}} = A_{-r_\nu+1}^{(\nu)} - \underbrace{\Lambda_{-r_\nu+1}^{(\nu)}}_{\text{diagonal}}$$

$$\underbrace{[F_2^{(\nu)}, A_{-r_\nu}^{(\nu)}]}_{\text{off diagonal}} = A_{-r_\nu+1}^{(\nu)} F_1^{(\nu)} - F_1^{(\nu)} \Lambda_{-r_\nu+1}^{(\nu)} + A_{-r_\nu+2}^{(\nu)} - \underbrace{\Lambda_{-r_\nu+2}^{(\nu)}}_{\text{diagonal}}$$

$\vdots$

From (eg), we see the singularity data $(A_{-k}^{(\nu)})$ $(k=0,\dots,r_\nu)$ can be replaced by

$(A_{-k}^{(\nu)})_{OD}$ $(k=1,\dots,r_\nu)$, $\Lambda_{-k}^{(\nu)}$ $(k=1,\dots,r_\nu)$, $A_0^{(\nu)}$.

(Rmk: the generic assumption of $\Lambda_{-r_\nu}^{(\nu)}$ makes sure that the bracket in (eg) is solvable.)

Theorem 2:

Let N be the affine manifold generated by the singularity data.

Then the system

$$\begin{cases} dA^{(\mu)} = \dfrac{\partial \Omega^{(\mu)}}{\partial x} + [\Omega^{(\mu)}, A^{(\mu)}] & (\mu=1,\dots,m,\infty) \\ dG^{(\mu)} = \Theta^{(\mu)} G^{(\mu)} & (\mu=1,\dots,m) \end{cases} \sim (\star\star)$$

is complete integrable on $N - \{a_\nu = a_\mu \text{ for } \nu\neq\mu\} - \underbrace{\{(\Lambda_{-r_\nu}^{(\nu)})_{ii} = (\Lambda_{-r_\nu}^{(\nu)})_{jj} \text{ for } i\neq j\}}_{\text{critical variety}}$.

Moreover, the parameter t can be chosen to be

$\{a_1,\dots,a_m, \Lambda_{-k}^{(\nu)}\ (\nu=1,\dots,m,\infty,\ k=1,\dots,r_\nu)\}$.

Recall the definition: $B^{(\nu)} := \hat{Y}^{(\nu)} d'\Lambda^{(\nu)} \hat{Y}^{(\nu)-1}$, $d'\Lambda^{(\nu)} := d\Lambda^{(\nu)} - d\Lambda_0^{(\nu)} \cdot \log(x-a_\nu)$.

$$\Omega^{(\infty)} := \sum_{\nu=1}^{m} G^{(\nu)} \left((B^{(\nu)})_{\mathrm{sing}_x}\right) G^{(\nu)-1} + (B^{(\infty)})_{\mathrm{holo}_x}$$

$$\Theta^{(\nu)} := \left[\sum_{\mu\neq\nu} G^{(\mu)}\left((B^{(\mu)})_{\mathrm{sing}_x}\right) G^{(\mu)-1} + (B^{(\infty)})_{\mathrm{holo}_x}\right]\Bigg|_{x=a_\nu} + G^{(\nu)}\left(Y_1^{(\nu)} da_\nu - B_0^{(\nu)}\right) G^{(\nu)-1}$$

$$\Omega^{(\nu)} := G^{(\nu)-1} \left(\Omega^{(\infty)} - \Theta^{(\nu)}\right) G^{(\nu)}$$

Step 1 : Reduce (※※) to finitely many one forms.

Set
$$\begin{cases} Z^{(\nu)} := dA^{(\nu)} - \dfrac{\partial \Omega^{(\nu)}}{\partial x} - [\Omega^{(\nu)}, A^{(\nu)}] & , \nu = 1, \cdots, m, \infty . \\ H^{(\nu)} := dG^{(\nu)} - \Theta^{(\nu)} G^{(\nu)} & , \nu = 1, \cdots, m . \end{cases}$$

The idea is to consider $\Sigma^{(\nu)} := dF^{(\nu)} + F^{(\nu)} (\Omega^{(\nu)} F^{(\nu)})_D - \Omega^{(\nu)} F^{(\nu)}$

(This comes from $d(\hat{Y}^{(\nu)} e^{\Lambda^{(\nu)}}) = \Omega^{(\nu)} (\hat{Y}^{(\nu)} e^{\Lambda^{(\nu)}})$). off diagonal !!

- Lemma : $\Sigma^{(\nu)} \in O(x-a_\nu)$ (resp. $\Sigma^{(\infty)} \in O(\frac{1}{x})$)

pf.) Note
$$\begin{cases} dF^{(\nu)} = -F_1^{(\nu)} da_\nu + O(x-a_\nu) \\ \Omega^{(\nu)} = \underbrace{(B^{(\nu)})}_{\hat{Y}^{(\nu)} d'\Lambda^{(\nu)} \hat{Y}^{(\nu)-1} = F^{(\nu)} d'\Lambda^{(\nu)} F^{(\nu)-1}} - Y_1^\nu da_\nu + O(x-a_\nu) \sim (■) \end{cases}$$

So
$$\begin{aligned} \Sigma^{(\nu)} &\equiv -F_1^{(\nu)} da_\nu + F^{(\nu)} (F^{(\nu)} d'\Lambda^{(\nu)} - Y_1^\nu da_\nu F^{(\nu)})_D - F^{(\nu)} d'\Lambda^{(\nu)} + Y_1^\nu da_\nu \cdot F^{(\nu)} \\ &\equiv -F_1^{(\nu)} da_\nu + \underbrace{(-Y_1^\nu da_\nu)_D}_{= -D_1^{(\nu)} da_\nu} + Y_1^\nu da_\nu F^{(\nu)} \\ &\equiv 0 \pmod{O(x-a_\nu)} \end{aligned}$$

$\uparrow$ $Y_1^\nu = D_1^{(\nu)} + F_1^{(\nu)}$

- Prop : $Z^{(\nu)} \equiv [\Sigma^{(\nu)} F^{(\nu)-1}, F^{(\nu)} \frac{\partial \Lambda^{(\nu)}}{\partial x} F^{(\nu)-1}] + \frac{1}{x-a_\nu} d\Lambda_0^{(\nu)} \pmod{O(1)}$

(resp. $Z^{(\infty)} \equiv [\Sigma^{(\infty)} F^{(\infty)-1}, F^{(\infty)} \frac{\partial \Lambda^{(\infty)}}{\partial x} F^{(\infty)-1}] + \frac{1}{x} d\Lambda_0^{(\infty)} \pmod{O(\frac{1}{x^2})}$)

pf.) .. Note that by (△△),
$$F^{(\nu)} \frac{\partial \Lambda^{(\nu)}}{\partial x} F^{(\nu)-1} - A^{(\nu)} = -\frac{\partial F^{(\nu)}}{\partial x} F^{(\nu)-1} - F^{(\nu)} \left(\frac{\partial}{\partial x} \log D^{(\nu)} \right) F^{(\nu)-1} = O(1)$$

.. Also, by (■), $\frac{\partial}{\partial x} \Omega^{(\nu)} \equiv \frac{\partial}{\partial x} (F^{(\nu)} d'\Lambda^{(\nu)} \cdot F^{(\nu)-1}) \pmod{O(1)}$

So
$$\begin{aligned} Z^{(\nu)} &= dA^{(\nu)} - \frac{\partial}{\partial x} \Omega^{(\nu)} - [\Omega^{(\nu)}, A^{(\nu)}] \\ &\equiv [dF^{(\nu)} \cdot F^{(\nu)-1}, F^{(\nu)} \frac{\partial \Lambda^{(\nu)}}{\partial x} F^{(\nu)-1}] + F^{(\nu)} d\left(\frac{\partial \Lambda^{(\nu)}}{\partial x}\right) F^{(\nu)-1} \\ &\quad - [\frac{\partial F^{(\nu)}}{\partial x} \cdot F^{(\nu)-1}, F^{(\nu)} d'\Lambda^{(\nu)} F^{(\nu)-1}] - F^{(\nu)} \frac{\partial}{\partial x}(d'\Lambda^{(\nu)}) F^{(\nu)-1} - [\Omega^{(\nu)}, A^{(\nu)}] \\ &\equiv [\Sigma^{(\nu)} F^{(\nu)-1}, F^{(\nu)} \frac{\partial \Lambda^{(\nu)}}{\partial x} \cdot F^{(\nu)-1}] + \frac{1}{x-a_\nu} d\Lambda_0^{(\nu)} + R^{(\nu)} \pmod{O(1)} \end{aligned}$$

$$\rightsquigarrow R^{(\nu)} = \left[-F^{(\nu)}(\Omega^{(\nu)}F^{(\nu)})_D F^{(\nu)-1} + \Omega^{(\nu)} ,\ F^{(\nu)} \frac{\partial \Lambda^{(\nu)}}{\partial x} F^{(\nu)-1} \right]$$

$$- \left[\frac{\partial F^{(\nu)}}{\partial x} F^{(\nu)-1} ,\ F^{(\nu)} d'\Lambda^{(\nu)} \cdot F^{(\nu)-1} \right] - \left[\Omega^{(\nu)} , A^{(\nu)} \right]$$

$$= \left[\Omega^{(\nu)} ,\ \underbrace{F^{(\nu)} \frac{\partial \Lambda^{(\nu)}}{\partial x} F^{(\nu)-1} - A^{(\nu)}}_{\parallel} \right] - \left[\frac{\partial F^{(\nu)}}{\partial x} F^{(\nu)-1} ,\ F^{(\nu)} d'\Lambda^{(\nu)} \cdot F^{(\nu)-1} \right]$$

$$-\frac{\partial F^{(\nu)}}{\partial x} F^{(\nu)-1} - F^{(\nu)} \left(\frac{\partial}{\partial x} \log D^{(\nu)} \right) F^{(\nu)-1} = O(1)$$

$$\equiv \left[F^{(\nu)} d'\Lambda^{(\nu)} \cdot F^{(\nu)-1} ,\ -\frac{\partial F^{(\nu)}}{\partial x} F^{(\nu)-1} - F^{(\nu)} \left(\frac{\partial}{\partial x} \log D^{(\nu)} \right) F^{(\nu)-1} \right]$$

$$- \left[\frac{\partial F^{(\nu)}}{\partial x} F^{(\nu)-1} ,\ F^{(\nu)} d'\Lambda^{(\nu)} \cdot F^{(\nu)-1} \right]$$

$$\equiv -F^{(\nu)} \left[d'\Lambda^{(\nu)} ,\ \frac{\partial}{\partial x} \log D^{(\nu)} \right] F^{(\nu)-1} \equiv 0 \quad (\mathrm{mod}\ O(1))$$

- From the proposition and lemma, we see that the singularity of $Z^{(\nu)}$ at a_ν is at most $(x-a_\nu)^{-r_\nu}$. So we can write

$$\begin{cases} Z^{(\nu)}(x) = \dfrac{Z^{(\nu)}_{-r_\nu+1}}{(x-a_\nu)^{r_\nu}} + \cdots + \dfrac{Z^{(\nu)}_0}{x-a_\nu} + O(1) & \text{as } x \to a_\nu \\ Z^{(\infty)}(x) = Z^{(\infty)}_{-r_\infty+1} x^{r_\infty - 2} + \cdots + Z^{(\infty)}_{-1} + O\left(\frac{1}{x}\right) & \text{as } x \to \infty \end{cases} \sim (\bullet)$$

- Consider

$$\begin{cases} (Z^{(\nu)}_{-k})_{OD} & (\nu = 1, \ldots, m, \infty,\ k = 1, \ldots, r_\nu - 1) \\ Z^{(\nu)}_0 & (\nu = 1, \ldots, m) \\ H^{(\nu)} & \end{cases} \sim (\blacksquare\blacksquare)$$

.. Note that $(\blacksquare\blacksquare)$ consists of differential one forms $\underline{ds_i - \sum f_{ij}(s,t)\, dt_j}$, where

$$\begin{cases} s_i \leftrightarrow (A^{(\nu)}_{-k})_{OD} \quad (\nu = 1, \ldots, m, \infty;\ k = 1, \ldots, r_\nu - 1),\ A^{(\nu)}_0 \quad (\nu = 1, \ldots, m) \\ t_j \leftrightarrow a_1, \ldots, a_m,\ \Lambda^{(\nu)}_{-k} \quad (\nu = 1, \ldots, m, \infty,\ k = 1, \ldots, r_\nu) \end{cases}$$

.. Let $\mathcal{I}$ be the ideal of differential one forms generated by coefficients of $(\blacksquare\blacksquare)$ on the manifold $\mathcal{U} - \{\text{critical variety}\}$.

Prop: $(*)$ holds $\Longleftrightarrow$ $\mathcal{I} = 0$ $\rightsquigarrow$ this reduces $(*)$ to finitely many one forms.

pf) $(\Longleftarrow)$ Compare the coefficients of $(\bullet)$ and Prop.

Then we get
$$Z^{(\nu)} \equiv \frac{(Z^{(\nu)}_{-r_\nu+1})_D}{(x-a_\nu)^{r_\nu}} + \cdots + \frac{(Z^{(\nu)}_{-1})_D}{(x-a_\nu)^2}$$

$$\equiv \left[\left(\Gamma^{(\nu)}_1 (x-a_\nu) + \Gamma^{(\nu)}_2 (x-a_\nu)^2 + \cdots \right) F^{(\nu)-1} ,\ F^{(\nu)} \left(\frac{\Lambda^{(\nu)}_{-r_\nu}}{(x-a_\nu)^{r_\nu+1}} + \cdots + \frac{\Lambda^{(\nu)}_0}{x-a_\nu} \right) F^{(\nu)-1} \right]$$

$$+ \frac{d\Lambda^{(\nu)}_0}{x-a_\nu} + O(1) \quad (\mathrm{mod}\ \mathcal{I}).$$

- So comparing the coefficients of $(x-a_\nu)^{-r_\nu}$, we have

$$\underbrace{\left(Z^{(\nu)}_{-r_\nu+1}\right)_D}_{\text{diagonal}} \equiv \underbrace{\left[\Gamma_1^{(\nu)}, \Lambda^{(\nu)}_{-r_\nu}\right]}_{\text{off diagonal}} \pmod{\mathcal{J}} \rightsquigarrow \begin{cases} \left(Z^{(\nu)}_{-r_\nu+1}\right)_D \equiv 0 \\ \underbrace{\Gamma_1^{(\nu)} \equiv 0}_{(\text{Rmk})} \end{cases} \pmod{\mathcal{J}}$$

(Rmk: Here we use the generic assumption that $(\Lambda^{(\nu)}_{-r_\nu})_{\bar{i}\bar{i}} - (\Lambda^{(\nu)}_{-r_\nu})_{\bar{j}\bar{j}} \neq 0$.)

- Repeat the procedure, we have

$$\left(Z^{(\nu)}_{-r_\nu+2}\right)_D \equiv \left[\Gamma_2^{(\nu)}, \Lambda^{(\nu)}_{-r_\nu}\right] \pmod{\mathcal{J}} \rightsquigarrow \begin{cases} \left(Z^{(\nu)}_{-r_\nu+2}\right)_D \equiv 0 \\ \Gamma_2^{(\nu)} \equiv 0 \end{cases} \pmod{\mathcal{J}}$$

$$\cdots\cdots \longrightarrow \left(Z^{(\nu)}_{-r_\nu+1}\right)_D, \cdots \left(Z^{(\nu)}_{-1}\right)_D, \Gamma_1^{(\nu)}, \cdots, \Gamma^{(\nu)}_{r_\nu-1} \in \mathcal{J}$$

Also, we get $\underbrace{d\Lambda_0^{(\nu)}}_{\text{diagonal}} + \underbrace{\left[\Gamma^{(\nu)}_{r_\nu}, \Lambda^{(\nu)}_{-r_\nu}\right]}_{\text{off diagonal}} \equiv Z_0^{(\nu)} \equiv 0 \pmod{\mathcal{J}}$

Hence $\underline{d\Lambda_0^{(\nu)} \in \mathcal{J}}$.

- Similarly, at ∞, we have $\left(Z^{(\infty)}_{-r_\infty+1}\right)_D, \cdots, \left(Z^{(\infty)}_{-1}\right)_D \in \mathcal{J}$ and $d\Lambda_0^{(\infty)} \equiv \left(Z_0^{(\infty)}\right)_D \pmod{\mathcal{J}}$.

- Note that by definition, $Z^{(\infty)} = G^{(\nu)} Z^{(\nu)} G^{(\nu)-1} - [A^{(\infty)}, H^{(\nu)} G^{(\nu)-1}]$

$$\equiv G^{(\nu)} Z^{(\nu)} G^{(\nu)-1} \pmod{\mathcal{J}}$$

So all singular parts of $Z^{(\infty)}$ (resp. $Z^{(\nu)}$) at $a_1, \cdots, a_m, \infty$ lies in $\mathcal{J}$

$\rightsquigarrow Z^{(\infty)}, Z^{(\nu)} \in \mathcal{J}$

- Also, we see that $\underline{d\Lambda_0^{(\nu)}}$ $(\nu = 1, \cdots, m, \infty) \in \mathcal{J} \rightsquigarrow$ This shows the claim in <u>theorem 1</u>.

<u>Step 2</u>: <u>$d\mathcal{J} \subseteq \mathcal{J}$</u>

(Then by the Frobenius theorem, $\mathcal{J} = 0$ is completely integrable, also since the generator of $\mathcal{J}$ are of the form $ds_{\bar{i}} - \sum f_{\bar{i}j}(s,t)\, dt_j$, we know $\{t_j\}$ can be chosen to be independent variables.)

<u>Lemma</u>: (1) $\Phi^{(\nu)} := d\hat{Y}^{(\nu)} - \hat{Y}^{(\nu)} d'\Lambda^{(\nu)} - \Omega^{(\nu)} \hat{Y}^{(\nu)} \in \mathcal{J}$

$\downarrow$ similar to (Δ)

(2) $\begin{cases} d\Omega^{(\nu)} - \Omega^{(\nu)} \wedge \Omega^{(\nu)} \in \mathcal{J} \\ d\,\Theta^{(\nu)} - \Theta^{(\nu)} \wedge \Theta^{(\nu)} \in \mathcal{J} \end{cases}$

Assume lemma (2), by computation, we have $dZ^{(\nu)} = -\frac{\partial}{\partial x}\left(d\Omega^{(\nu)} - \Omega^{(\nu)2}\right) - \left[d\Omega^{(\nu)} - \Omega^{(\nu)2}, A^{(\nu)}\right] + [\Omega^{(\nu)}, Z^{(\nu)}]$ +

Also, $dH^{(\nu)} = -\left(d\Theta^{(\nu)} - \Theta^{(\nu)2}\right) G^{(\nu)} + \Theta^{(\nu)} H^{(\nu)}$

$\rightsquigarrow$ This proves $d\mathcal{J} \subseteq \mathcal{J}$

- Proof of lemma(1) ($\Xi^{(\nu)} \in \mathcal{J}$):

By computation, we have $\frac{\partial}{\partial x}\left(\Xi^{(\nu)} \Psi^{(\nu)-1}\right) = Z^{(\nu)} + [A^{(\nu)}, \Xi^{(\nu)}\Psi^{(\nu)-1}] - \Psi^{(\nu)}\frac{\partial}{\partial x}(d\Lambda^{(\nu)} - d'\Lambda^{(\nu)})\Psi^{(\nu)-1}$

So $\frac{\partial}{\partial x}\left(\Xi^{(\nu)}\Psi^{(\nu)-1}\right) - [A^{(\nu)}, \Xi^{(\nu)}\Psi^{(\nu)-1}] \equiv 0 \pmod{\mathcal{J}}$. $\quad$ ($\because d\Lambda_0^{(\nu)}, Z^{(\nu)} \in \mathcal{J}$)

Also $\Xi^{(\nu)}\Psi^{(\nu)-1} = O(x-a_\nu)$ (resp. $\Xi^{(\infty)}\Psi^{(\infty)-1} \in O(\frac{1}{x})$.)

- Since $\mathcal{J}$ has generators of the form $\underbrace{ds_i - \sum f_{ij}(s,t)\,dt_j}_{=:\theta_i}$, we may write

$$\Xi^{(\nu)}\Psi^{(\nu)-1} = \sum Z_i\theta_i + \sum Z_j' \, dx_j \quad \text{uniquely.}$$

Claim: $Z_j' = 0$. ($\rightsquigarrow \Xi^{(\nu)}\Psi^{(\nu)-1} \in \mathcal{J}$)

Note that we have $\frac{\partial}{\partial x}(Z_j') - [A^{(\nu)}, Z_j'] = 0$.

So if $Z_j' \neq 0$, then consider $\Psi^{(\nu)\prime}(x) := (I + Z_j'(x))\Psi^{(\nu)}(x)$.

$\rightsquigarrow$ We get $\frac{\partial}{\partial x}(\Psi^{(\nu)\prime}(x)) = A^{(\nu)}\Psi^{(\nu)\prime} - \Psi^{(\nu)\prime}\frac{d}{dx}\Lambda^{(\nu)}$ $\rightsquigarrow$ $\Psi^{(\nu)\prime}$ satisfies (Δ).

But the solution of (Δ) is <u>uniquely</u> solved $\xrightarrow{\times}$ Hence $Z_j' = 0$. #

- Proof of lemma(2) :

- Note that $d\Omega^{(\nu)} - \Omega^{(\nu)2} \equiv G^{(\nu)-1}\left(d\Omega^{(\infty)} - \Omega^{(\infty)2} - d\Theta^{(\nu)} + \Theta^{(\nu)2}\right)G^{(\nu)} \pmod{\mathcal{J}}$

So it suffices to show the lemma for $\Omega^{(\infty)}$, $\Theta^{(\nu)}$.

- $d\Omega^{(\infty)} \equiv d(\Psi^{(\infty)} d'\Lambda^{(\infty)} \cdot \Psi^{(\infty)-1}) + O(\frac{1}{x})$

$$= [\, d\Psi^{(\infty)} \cdot \Psi^{(\infty)-1}, \Psi^{(\infty)} d'\Lambda^{(\infty)} \Psi^{(\infty)-1}]_+ + O(\tfrac{1}{x})$$

$$\underset{\uparrow}{\equiv} [\, \Omega^{(\infty)}, \Psi^{(\infty)} d'\Lambda^{(\infty)} \Psi^{(\infty)-1}]_+ + O(\tfrac{1}{x})$$

Lemma(1) : $d\Psi^{(\infty)}\Psi^{(\infty)-1} \equiv -\Psi^{(\infty)} d'\Lambda^{(\infty)}\Psi^{(\infty)-1} + \Omega^{(\infty)} = O(\frac{1}{x}) \pmod{\mathcal{J}}$

$$\equiv \Omega^{(\infty)2} + O(\tfrac{1}{x}) \pmod{\mathcal{J}} \quad \text{as } x \to \infty$$

- Similar for a_ν, we have

$$d\Omega^{(\infty)} \underset{\uparrow}{\equiv} [\Theta^{(\nu)}, \Omega^{(\infty)}]_+ + [\Omega^{(\infty)} - \Theta^{(\nu)}, G^{(\nu)}\Psi^{(\nu)} d'\Lambda^{(\nu)} \Psi^{(\nu)-1} G^{(\nu)-1}]_+ + O(1) \pmod{\mathcal{J}}$$

Use lemma(1), $d\Lambda_0^{(\nu)} \in \mathcal{J}$

$$\equiv \Omega^{(\infty)2} + O(1) \pmod{\mathcal{J}} \quad \text{as } x \to a_\nu$$

$\Rightarrow d\Omega^{(\infty)} - \Omega^{(\infty)2} \equiv 0 \pmod{\mathcal{J}}$.

- As for $d\Theta^{(\nu)} - \Theta^{(\nu)2} \in \mathcal{J}$, the computation is straightforward (but complicated) so I omit it here.

#

5. Sketch of the proof of theorem 3

Recall:

- Theorem 2 states that the system
$$\begin{cases} dA^{(\mu)} = \dfrac{\partial \Omega^{(\mu)}}{\partial x} + [\Omega^{(\mu)}, A^{(\mu)}] & \sim (**) \\ dG^{(\mu)} = \Theta^{(\mu)} G^{(\mu)} \end{cases}$$
is a completely integrable system on the manifold $\langle a_1, \dots, a_m, (G^{(\nu)})_{ij}\ (i,j=1,\dots,n),\ (\Lambda^{(\nu)}_{-k})_{ii}\ (i=1,\dots,n,\ k=1,\dots,r_\nu),\ \underbrace{(A^{(\nu)}_{-k})_{ij}\ (i\neq j)}_{(*)}\ (k=1,\dots,r_\nu-1),\ (A^{(\nu)}_0)_{ij} \rangle_{\mathbb{C}} - \left(\{a_\nu = a_\mu \text{ for } \nu\neq\mu\} \cup \{(\Lambda^{(\nu)}_{-r_\nu})_{ii} = (\Lambda^{(\nu)}_{-r_\nu})_{jj} \text{ for } i\neq j\}\right)$

Moreover, the deformation parameter can be chosen as $\{a_1, \dots, a_m, (\Lambda^{(\nu)}_{-k})_{ii}\}$.

- Now theorem 3 is to show that the general solution of $(**)$ is meromorphic in $\{a_1, \dots, a_m, (\Lambda^{(\nu)}_{-k})_{ii}\}$
- General situation:

Let T be the universal covering of $\{(a_\mu, (\Lambda^{(\mu)}_{-k})_{jj})_{\substack{1\le\mu\le n\\ 1\le k\le r_\nu\\ 1\le j\le m}} \mid a_\mu \neq a_\nu\ (\mu\neq\nu),\ (\Lambda^{(\nu)}_{-r_\nu})_{ii} \neq (\Lambda^{(\nu)}_{-r_\nu})_{jj}\}$

Let N be the fiber bundle over T with fiber $((\Lambda^{(\nu)}_0)_{ii},\ \underbrace{G^{(\nu)}_{ij},\ (A^{(\nu)}_{-k})_{ij}\ (i\neq j),\ (A^{(\nu)}_0)_{ij}}_{\text{remaining singularity data}})$

Let M be the fiber bundle over T with fiber $((\Lambda^{(\nu)}_0)_{ii},\ C^{(\nu)}_{ij},\ x^{(\nu l)}_{ij})_{1\le\nu\le n}$

$C^{(\nu)}_{ij}$: connection matrices

$x^{(\nu l)}_{ij}$ is $((S^{(\nu)}_l)^{-1} - I)_{ij}$: Stokes matrices

remaining monodromy data

By solving the differential equation $\frac{dY}{dx} = A(x)Y$, we get a one-one holomorphic map $R: N \to M$.

Theorem 3':

R^{-1} is meromorphic, i.e. $Y(x)$ is meromorphic on M. $(R^{-1}: R(N) \to M)$.

(Then since a horizontal leaf of M represent a solution to isomonodromy deformation equation, theorem 3' implies theorem 3)

- The proof of theorem 3' consists of two parts:

(i) Solve the twins problem

(ii) Use the solution of twins problem to construct a Riemann Hilbert problem and solve it.

i) Solve the twins problem:

Given "twins of deformation parameters" $\begin{cases} (a_\mu, t^{(\mu)}_{j,\alpha})_{\substack{1\le j\le r_\mu\\ 1\le\alpha\le m}} \\ (b_\mu, s^{(\mu)}_{j,\alpha})_{\substack{1\le j\le r_\mu\\ 1\le\alpha\le m}} \end{cases}$ Assume $a_1, \dots, a_n, b_1, \dots, b_n$ are distinct.

a_ν part, $(\Lambda^{(\nu)}_{-k})_{jj}$ part

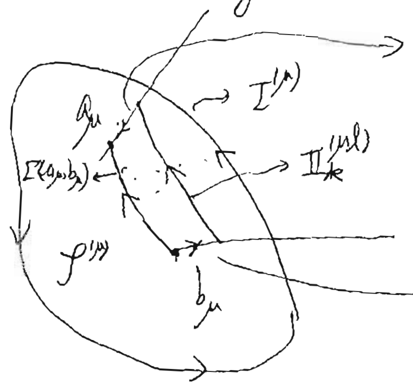
Also, assume $\begin{cases} t^{(\mu)}_{r_\mu,\alpha} \neq t^{(\mu)}_{r_\mu,\beta} \\ s^{(\mu)}_{r_\mu,\alpha} \neq s^{(\mu)}_{r_\mu,\beta} \end{cases}$ for $\alpha\neq\beta$ and $|t^{(\mu)}_{r_\mu,\alpha} - s^{(\mu)}_{r_\mu,\alpha}| \le$ small

· Set $\sigma^{(\mu l)}$ to be the permutation such that after the permutation $\sigma^{(\mu l)}(\alpha) > \sigma^{(\mu l)}(\beta)$ iff

$$Re^{(\mu)}_{\alpha\beta}(x) := Re\left(t^{(\mu)}_{r_\mu,\alpha}\frac{(x-a_\mu)^{-r_\mu}}{-r_\mu} - t^{(\mu)}_{r_\mu,\beta}\frac{(x-a_\mu)^{-r_\mu}}{-r_\mu}\right) < 0 \quad \text{for } x \text{ with } \arg(x-a_\mu) = \rho + \frac{\pi}{r_\mu}(l-1)$$

on the Stokes sector $\mathcal{S}^{(\mu l)}_{\rho\delta} := \{0<|x-a_\mu|<\delta,\ \rho+\pi\frac{l-1}{r_\mu}-\delta < \arg(x-a_\mu) < \rho+\pi\frac{l}{r_\mu}\}$.

· Consider the following contour :



$\gamma^{(\mu l)}_k := \{0<|x-a_\mu|<<1,\ \arg(x-a_\mu) = \rho+\frac{\pi}{r_\mu}(l-1)+k\varepsilon\}$

$\gamma'^{(\mu l)}_k := \{0<|x-b_\mu|<<1,\ \arg(x-b_\mu) = \rho-\frac{\pi}{r_\mu}l-k\varepsilon\}$

$(k=1,\dots,m-1)$

· Set $II^{(\mu l)} := II^{(\mu l)}_1$, then $I^{(\mu)}, II^{(\mu l)}$ divide $\mathbb{P}$ into $1+\sum_{\mu=1}^{n}\max(1, 2r_\mu)$ region.

· Set $\infty\in\mathcal{D}^{(0)}$ and $\mathcal{D}^{(\mu l)}$ contains
$$\begin{cases}\{0<|x-a_\mu|<<1 \mid \arg(x-a_\mu) = \rho+\frac{\pi l}{r_\mu}-\frac{\pi}{2r_\mu}\}\\ \{0<|x-b_\mu|<<1 \mid \arg(x-a_\mu) = \rho-\frac{\pi l}{r_\mu}-\frac{\pi}{2r_\mu}\}\end{cases}$$

* Choose $x_0\in\mathcal{D}^{(0)}$, the twins problem is to find an $m\times m$ matrix $Z(x)$ st.

(T1) $Z(x_0)=1$

(T2) $Z(x)$: holomorphic (single-valued) on $\mathcal{D}^{(0)}$

(T3) $Z(x)$ can be continued holomorphically to $\mathcal{D}^{(\mu 1)},\dots,\mathcal{D}^{(\mu, r_\mu+1)}$
Also in $\mathcal{D}^{(\mu l)}$, as $|x-a_\mu|\to 0$ or $|x-b_\mu|\to 0$ $Z(x)$ has an asymptotic expansion indep of l.

Set
$$\mathcal{E}^{(\mu)}_\alpha(x) := \frac{\exp\left(\sum_{j=1}^{r_\mu} t^{(\mu)}_{j,\alpha}\frac{(x-a_\mu)^{-j}}{-j} + t^{(\mu)}_{0,\alpha}\log(x-a_\mu)\right)}{\exp\left(\sum_{j=1}^{r_\mu} s^{(\mu)}_{j,\alpha}\frac{(x-b_\mu)^{-j}}{-j} + t^{(\mu)}_{0,\alpha}\log(x-a_\mu)\right)}$$

$$K^{(\mu\nu)}_{\alpha\beta}(y) := \begin{cases}(1-\delta_{\mu\nu})\left(C^{(\mu)}C^{(\nu)-1}\right)_{\alpha\beta} & \text{if } y\in I^{(\nu)}\\ \delta_{\mu\nu}\lambda^{(\nu l)}_{\alpha\beta} & \text{if } y\in II^{(\nu l)}_{\sigma^{(\nu l)}(\beta)}\\ 0 & \text{otherwise.}\end{cases}$$

$$\mathcal{K}^{(\mu\nu)}_{\alpha\beta}(x,y) := \frac{1}{2\pi i}\cdot\frac{1}{x-y}K^{(\mu\nu)}_{\alpha\beta}(y)\frac{\mathcal{E}^{(\mu)}_\alpha(y)}{\mathcal{E}^{(\nu)}_\beta(y)}$$

We claim that the solution $Z(x)$ of this problem is given by

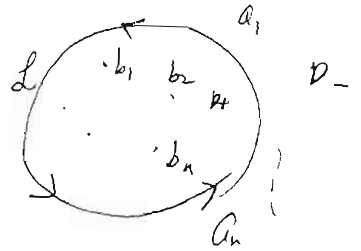
$$
\begin{aligned}
(Z(x))_{\alpha\beta} &= 2\pi i\,(x-x_0)\cdot \frac{(L_1(x,x_0))_{\alpha\beta}}{L_2} \\
&= 2\pi i\,(x-x_0)\,\frac{\displaystyle\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\sum_{\mu_1,\dots,\mu_l=1}^{n}\sum_{\alpha_1,\dots,\alpha_l=1}^{m}\int dx_1\cdots\int dx_l \times \det\begin{bmatrix} \delta_{\alpha\beta}\cdot\frac{1}{2\pi}\cdot\frac{-i}{x-x_0} & \left(((C^{(\mu_k)})^{-1})_{\alpha\alpha_k}\frac{1}{2\pi}\frac{i}{x_0-x_k}\Sigma^{(\mu_k)}_{\alpha_k}(x_k)\right)_{k=1,\dots,l} \\ \left(\frac{1}{2\pi i}\cdot\frac{1}{x_j-x}\cdot\frac{1}{\Sigma^{(\mu_j)}_{\alpha_j}(x_j)}C^{(\mu_j)}_{\alpha_j\beta}\right)_{j=1,\dots,l} & \left(K^{(\mu_j\mu_k)}_{\alpha_j\alpha_k}(x_j,x_k)\right)_{j,k=1,\dots,l}\end{bmatrix}}{\displaystyle\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\sum_{\mu_1,\dots,\mu_l=1}^{n}\sum_{\alpha_1,\dots,\alpha_l=1}^{m}\int dx_1\cdots\int dx_l \times \det\left(K^{(\mu_j\mu_k)}_{\alpha_j\alpha_k}(x_j,x_k)\right)_{j,k=1,\dots,l}}
\end{aligned}
$$

provided $\underline{L_2\neq 0}$. (The integral are all path integral over $\left(\bigcup_{\mu=1}^{n} I^{(\mu)}\right)\cup\left(\bigcup_{\substack{l=1,\dots,n\\ k=1,\dots,m+1}} II^{(\mu l)}_{k}\right)$)

(Remark: L_2 is convergent and is holomorphic with respect to the parameters by the Fredholm theory. Indeed, L_2 is the Fredholm determinant of the Fredholm integral equation

$$N(x)-\frac{1}{2\pi i}\int N(y)\,K(y)\,dy = I$$)

(ii) In the case $L_2\neq 0$, we consider the contour $\mathcal{L}$ below.



We want to solve $\begin{cases} R_-(x)\,Z(x) = R_+(x) & \text{if } x\in\mathcal{L} \\ R_-(x)-1 \to 0 & \text{as } |x|\to\infty \end{cases}$.

Then the solution $Y(x)$ to the original monodromy problem can be given by

$$Y(x)=\begin{cases} R_-(x)\,Z(x) & \text{if } x\in D_- \\ R_+(x) & \text{if } x\in D_+ \end{cases}$$

The solution to this R-H problem is given by the following standard procedure.

- $R_-(x)$ satisfies the Fredholm integral equation

$$R_-(x)-\frac{1}{2\pi i}\int dy\,R_-(y)\,\underbrace{H(y,x)}_{\frac{Z(y)Z(x)^{-1}}{y-x}} = 1 .$$

(This can be deduced by the Plemelj formula:

$$\frac{1}{2\pi i}\int_{\mathcal{L}}\frac{R_-(y)}{y-x}\,dy \longrightarrow \pm\frac{1}{2}R(x)+\frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_{\underbrace{\mathcal{L}-\mathcal{L}_\varepsilon}_{\mathcal{L}_\varepsilon\,=\,\mathcal{L}\cap\{|y-x|<\varepsilon\}}}\frac{R_-(y)}{y-x}\,dy \quad \text{as } x\to\mathcal{L} \text{ from } D^+ \text{ or } D^-)$$

- The matrix valued Fredholm integral equation has the solution of the following form: $R_-(x) = 1+\int_{\mathcal{L}}\frac{\Delta(y,x;1)}{\Delta(1)}\,dy$. , where

$$\Delta(\lambda) = \sum_{l=0}^{\infty} \frac{(-\lambda)^l}{l!} \sum_{\alpha_1 \cdots \alpha_l = 1}^{m} \int dx_1 \cdots \int dx_l \, H\begin{pmatrix} x_1 \cdots x_l \\ x_1 \cdots x_l \end{pmatrix}\begin{vmatrix} \alpha_1 \cdots \alpha_l \\ \alpha_1 \cdots \alpha_l \end{pmatrix}, \text{ where } H\begin{pmatrix} x_1 \cdots x_l \\ y_1 \cdots y_l \end{pmatrix}\begin{vmatrix} \alpha_1 \cdots \alpha_l \\ \beta_1 \cdots \beta_l \end{pmatrix} \quad (1)$$

the Fredholm determinant

$$= \det\left(H(x_j, y_k)_{\alpha_j \beta_k} \right)_{j,k=1,\dots,l}.$$

$$\Delta(y, x; \lambda)_{\alpha\beta} = \sum_{l=0}^{\infty} \frac{(-\lambda)^l}{l!} \sum_{\alpha_1 \cdots \alpha_l = 1}^{m} \int dx_1 \cdots \int dx_l \, H\begin{pmatrix} y & x_1 \cdots x_l \\ x & x_1 \cdots x_l \end{pmatrix}\begin{vmatrix} \alpha & \alpha_1 \cdots \alpha_l \\ \beta & \beta_1 \cdots \beta_l \end{pmatrix}$$

Now, if $\Delta(1) \neq 0$, $R_-(x)$ is indeed the solution to the R-H problem.

$\rightarrow$ $Y(x)$ is meromorphic wrt parameters provided $\Delta(1) \neq 0$, $I_2 \neq 0$.

(iii) Now if $I_2 = 0$, the idea here is to consider the twins problem with two more singularities $x_1, x_2 \in I^{(\mu_0)} \cup \left(\bigcup II_{k}^{(\mu_0)l} \right)$ for some μ_0.

Then the solution $Z'(x)$ to this problem plugged into (ii) solves the monodromy problem.

(iv) Verify $\Delta(1) \neq 0$ with respect to the parameters a_μ, b_μ, $t_{j\alpha}^{(\mu)}$, $s_{j\alpha}^{(\mu)}$, $\chi_{\alpha\beta}^{(\mu l)}$, $C_{\alpha\beta}^{(\mu)}$