Frob. Mfd
final presentation
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§ Semiuniversal unfolding

Let $f: (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ be st. 0 is an isolated singularity (we call f an isolated hypersurface singularity).

Def. An unfolding of f is $F: (\mathbb{C}^m \times \mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ st. $F|_{\mathbb{C}^m \times \{0\}} = f$.

$(M, 0) := (\mathbb{C}^n, 0)$ is its parameter space.

$(C, 0) := Z\underbrace{\left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_m}\right)}_{J_F} \subseteq (\mathbb{C}^m \times M, 0)$. It's the critical space of $(F, pr): (\mathbb{C}^m \times M, 0) \to (M, 0)$. Regard $(C, 0)$ as a cpx analy. space. $\mathcal{O}_{C,0} = \mathcal{O}_{\mathbb{C}^m \times M, 0} / J_F|_{(C,0)}$.

Note that $C \cap (\mathbb{C}^m \times \{0\}) = \{0\}$.

Def. (Kodaira-Spencer map) It's defined by

$$a_C: T_{M,0} \longrightarrow \mathcal{O}_{C,0}$$
$$X \longmapsto \widetilde{X}(F)|_{(C,0)},$$ where $\widetilde{X}$: lifting of X.

$\rightsquigarrow a_C|_0: T_0M \to \mathcal{O}_{\mathbb{C}^m, 0}/J_f$; it's called the reduced K-S map.

If $F_i: (\mathbb{C}^m \times M_i, 0) \to (\mathbb{C}, 0)$, $i = 1, 2$, are unfoldings of f, we say F_1 is induced from F_2 if

$$\begin{array}{ccc} (\mathbb{C}^m \times M_1, 0) & \xrightarrow{\exists \phi} & (\mathbb{C}^m \times M_2, 0) \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle pr_2} \\ (M_1, 0) & \xrightarrow[\exists \phi_{base}]{} & (M_2, 0) \end{array}$$
st. $\phi|_{\mathbb{C}^m \times \{0\}} = id$ and $F_1 = F_2 \circ \phi$.

Def. (i) F is a versal unfolding if any unfolding is induced from it.

(ii) F is semiuniversal if, moreover, the parameter space $(M, 0)$ is minimal.

Thm. (Thom, Mather) f: isolated hypersurf. sing. $\Rightarrow$ semiuni. unfolding exists!

Rmk. f: iso. hypersurf. sing. Then, F is versal $\Leftrightarrow$ $a_C|_0$ is surj ; F is semiuniversal $\Leftrightarrow$ $a_C|_0$ is iso.

Example. $f: (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ $F: (\mathbb{C}^3 \times \mathbb{C}^4, 0) \to (\mathbb{C}, 0)$ To. in . $\mathbb{C}[z]/$

Example. $f: (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ $F: (\mathbb{C}^3 \times \mathbb{C}^4, 0) \to (\mathbb{C}, 0)$. Then $\mathcal{O}_{C,0} \simeq \mathbb{C}[z]/(p'(z))$,

$(x, y, z) \mapsto x^2 + y^2 + z^5$, $(x, y, z, a_i) \mapsto x^2 + y^2 + \underbrace{z^5 + a_3 z^3 + a_2 z^2 + a_1 z + a_0}_{p(z)}$.

$a_C|_0: \frac{\partial}{\partial a_i} \mapsto z^i$, $\therefore$ iso.

The multiplication $\circ$ is defined by.

$p(z) \circ q(z) := p(z)\, q(z) \mod p'(z)$.

§ F-mfd assoc. to semiuni. unfoldings

Recall: An F-mfd is $(M, \circ, e)$, where

(i) M: conn. cpx mfd

(ii) $\circ: T_M \times T_M \to T_M$ comm. assoc. ; e: global unit field

(iii) $\mathrm{Lie}_{X\circ Y}(\circ) = X \circ \mathrm{Lie}_Y(\circ) + Y \circ \mathrm{Lie}_X(\circ)$ $\big(\Leftrightarrow [X\circ Y, Z\circ W] - [X\circ Y, Z]\circ W - [X \circ Y, W]\circ Z - X\circ[Y, Z\circ W] + X\circ[Y,Z]\circ W + X\circ[Y,W]\circ Z - Y\circ[X, Z\circ W] + Y\circ[X,Z]\circ W + Y\circ[X,W]\circ Z = 0 \quad \forall X,Y,Z,W\big)$

E is an Euler field if $\mathrm{Lie}_E(\circ) = \circ$.

Let $f: (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ be s.t. 0 is an isolated singularity (we call f an isolated hypersurface singularity). $F: (\mathbb{C}^m \times M, 0) \to (\mathbb{C}, 0)$ a semiuniv. unfolding.

Thm 1. (a) $(M, 0)$ can be equipped w/. a massive F-mfd str. (massive means $\circ$ is generically semisimple), (b) with Euler field.

Suppose $(M, \circ, e)$ is given.

Def. $\mathrm{Specan}(T_M) := \bigcup_{p\in M} \mathrm{Hom}_{\mathbb{C}\text{-alg}}(T_pM, \mathbb{C}) \subseteq T^*M$.

It's a cpx analy space: Consider $\varphi: \mathrm{Sym}_{\mathcal{O}_M} T_M \to T_M$ $\mathcal{O}_M$-alg homo. If t_i: loc. coor. on M, then

$$\mathrm{Sym}_{\mathcal{O}_M} T_M = \mathcal{O}(T^*M), \quad \cdot \longmapsto \circ$$

$\ker\varphi = \langle \partial_1 - 1, \partial_i\partial_j - \sum_k a_{ij}^k \partial_k \rangle_{\mathcal{O}_M}$.

Let $I := \langle \ker\varphi \rangle_{\mathcal{O}(T^*M)}$. $\Rightarrow \mathrm{Supp}\, I = \mathrm{Specan}(T_M)$

Rmk. $\pi: \mathrm{Specan}(T_M) \to M$ is a branched covering, branched above K = non s.s. locus.

$$L = \mathrm{Specan}(T_M)$$

Let y_i's be the coor. for fibers of T^*M, and $\alpha := \sum y_i dt_i$ in local coor.

Prop. Suppose $(M, \circ, e)$ is massive, and L is Lagrange, i.e. $\alpha|_{L_{reg}}$ is closed. Then $(M, \circ, e)$ is an F-mfd.

Pf. Consider the map $a: T_M \hookrightarrow (\pi_{T^*M})_* \mathcal{O}_{T^*M} \to \pi_* \mathcal{O}_L$ $\mathcal{O}_M$-alg iso.

$$X \longmapsto \alpha(X)|_L$$

$$X = \sum a_i \partial_i, \qquad \alpha\Big(\sum a_i \partial_{t_i} + \sum b_j \partial_{y_j}\Big)\Big|_L = \Big(\sum a_i y_i\Big)\Big|_L$$

Choose U nbd of $p \in M-K$. Then $L|_{\pi^{-1}(U)} = \coprod_{k=1}^n L_k$, $\pi: L_k \xrightarrow{\sim} U$. Let $\tilde{e}_i$ lifting of e_i on L_k. Compute

[illegible]

Rechoose coor. u_i's to be canonical coor, and $e_i = \frac{\partial}{\partial u_i}$

$$a([e_i, e_j]|_U)|_{L_k} = \alpha([\tilde{e}_i, \tilde{e}_j])|_{L_k} = 0. \quad \therefore [e_i, e_j] = 0. \quad M-K \text{ is F-mfd}. \therefore M \text{ is F-mfd}.$$

$$\alpha([\tilde{e}_i, \tilde{e}_j]) = \tilde{e}_i\alpha(\tilde{e}_j) - \tilde{e}_j\alpha(\tilde{e}_i) - d\alpha(\tilde{e}_i, \tilde{e}_j)$$

Rmk. If $(M, \circ, e)$ is massive, TFAE: (i) $(M, \circ, e)$ is F-mfd;

(ii) on $M-K$, $[e_i, e_j] = 0$;

(iii) $\alpha|_{L_{reg}}$ is closed.

$$C \subseteq (\mathbb{C}^m \times M, 0) \to (M, 0), \quad \pi_C: C \to (M,0)$$

Pf of thm 1(a). Consider the Kodaira-Spencer map $a_C: T_M \xrightarrow{\sim} (\pi_C)_* \mathcal{O}_C$, $X \mapsto (dF)(\tilde{X})|_C$. $\rightsquigarrow$ mult. on T_M. and, C is reduced $\Rightarrow$ $\circ$ is s.s. It remains to show $\alpha|_{L_{reg}}$ is closed. Define

$$q: C \to T^*M$$
$$z \mapsto \big(q(z): X \mapsto a_C(X)(z)\big) \in T^*_{\pi_C(z)}M.$$

Then $\mathrm{im}(q) = L$, and $(q^*\alpha)|_{C_{reg}} = dF|_{C_{reg}}$. $\therefore \alpha|_{L_{reg}}$ is closed.

Thm 1(b). $E := a_C^{-1}(F|_C)$ is an Euler field for the F-mfd $(M, \circ, e)$.

Pf. Take local coor. u_i's: cano. coor, and $\alpha = \sum y_i du_i$. Compute $a_C = q^* \circ a$

$d(a(\ldots)) = \sum y_i du_i = d(a(\text{Euler field}))|_L$

About last time: $\quad \text{Specan}(T_M)$

Prop. Let $(M,\circ,e)$ be massive, $L \overset{''}{} $ be Lagrange (i.e. $\alpha|_{L_{reg}}$ is closed). Then $(M,\circ,e)$ is an F-mfd.

Pf. Recall $a: T_M \to \pi_* \mathcal{O}_L$ $\mathcal{O}_M$-alg iso. Choose U: nbd of $p \in M-K$, $\therefore L|_{\pi^{-1}(U)} = \coprod_{k=1}^{n} L_k$ s.t. U
$X \mapsto \alpha(\tilde{X})|_L$

Choose coor. t_i's st. ∂_i's are idempotent at p. $\tilde{\partial_i}$: lifting of ∂_i. Compute

$$a([\partial_i,\partial_j]|_U)|_{L_k} = \alpha([\tilde{\partial_i},\tilde{\partial_j}])|_{L_k} = 0. \quad \therefore [\partial_i,\partial_j]|_U = 0. \quad \therefore M-K \text{ is F-mfd.}$$

$$\overset{\|}{\tilde{\partial_i}\underbrace{\alpha(\tilde{\partial_j})}_{\delta_{jk}} - \tilde{\partial_j}\underbrace{\alpha(\tilde{\partial_i})}_{\delta_{ik}} - d\alpha(\tilde{\partial_i},\tilde{\partial_j})} \qquad \therefore M \text{ is F-mfd.}$$

Thm 1.(a) $(M,0)$ has a massive F-mfd str.

$$\begin{tikzcd} C \subseteq (\mathbb{C}^m \times M, 0) \arrow[d] \arrow[dr, "\pi_C"] & \\ (M,0) & \end{tikzcd}$$

Pf. K-S map $a_C: T_M \xrightarrow{\sim} (\pi_C)_* \mathcal{O}_C$, $X \mapsto (dF)(\tilde{X})|_C \rightsquigarrow$ mult. on T_M.

And, C is sm. $\left(rk\left(\frac{\partial^2 F}{\partial x_i \partial x_j}, \frac{\partial^2 F}{\partial x_i \partial t_k}\right)\Big|_0 = m\right) \Rightarrow C|_0$ is reduced $\Rightarrow \circ$ is massive. Define

$q: C \to T^*M$
$z \mapsto (q(z): X \mapsto a_C(X)(z)) \in T^*_{\pi_C(z)} M.$

Then
- $q: C \xrightarrow{\sim} im(q) = L$
- $(q^*\alpha)|_{C_{reg}} = dF|_{C_{reg}} \Rightarrow \alpha|_{L_{reg}}$ is closed
- $a_C = q^* \circ a: T_M \to \pi_* \mathcal{O}_L \to (\pi_C)_* \mathcal{O}_C$

Thm 1.(b) $a_C^{-1}(F|_C)$ is an Euler field for the F-mfd $(M,\circ,e)$.

Pf. Take loc. coor. u_i's, and $\alpha = \sum y_i du_i$. Compute

$$d(a(a_C^{-1}(F|_C)))|_{L_{reg}} = d((q^*)^{-1}F|_C)|_{L_{reg}} = \alpha|_{L_{reg}} = \sum y_i du_i = d(a(\text{Euler field}))|_{L_{reg}}$$

$$\uparrow \quad d(a(\textstyle\sum u_i \partial_i))|_{L_{reg}} = d(y_i u_i)|_{L_{reg}} = \sum y_i du_i$$

Consider $\varphi: (\mathbb{C}^{n+1} \times M, 0) \to (\mathbb{C} \times M, 0)$ ($\dim = \mu$) Let $\mathcal{U}: T_t M \to T_t M$, $X \mapsto E \circ X$. Then
$(x,t) \mapsto (F(x,t), t)$.

$\varphi^{-1}(z,t)$ is singular $\Leftrightarrow (z,t) \in \varphi(C) =: \check{D} \subseteq \Delta \times M$.

Consider $H^n = \bigcup_{(z,t) \in \mathbb{C}\times M - \check{D}} H^n(\varphi^{-1}(z,t), \mathbb{C}) \to \mathbb{C}\times M - \check{D}$. ($\leftarrow$ bdle of rk μ) We may shrink φ to some $\mathcal{X} \to \Delta \times M$.

Rmk. **Thm.** (Milnor) Near singular fibers, each regular is homotopy equiv. to a bouquet of μ n-spheres.

Def. The Gauss–Manin connection on $\mathcal{H}$ is defined by

$$\nabla: \mathcal{H} \to \mathcal{H} \otimes \Omega_{\Delta\times M - \check{D}}, \quad \sigma_i \text{ basis of a local trivialization of } H^n.$$
$$\sigma = \sum \alpha_i \sigma_i \mapsto \sum \sigma_i \otimes d\alpha_i$$

Let $\Omega^{n+1}_{\mathcal{X}/\Delta\times M} := \Omega^{n+1}_{\mathcal{X}} / dF \wedge \Omega^n_{\mathcal{X}} + \sum_{i=1}^{n} dt_i \wedge \Omega^n_{\mathcal{X}}$ relative $(n+1)$-forms wrt. φ.

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\varphi"] \arrow[dr, "pr_M"'] & \Delta\times M \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

Let $\Omega_F := (pr_M)_* \Omega^{n+1}_{\mathcal{X}/\Delta\times M}$. $\rightsquigarrow$ $T_M \times \Omega_F \to \Omega_F$ $\rightsquigarrow \Omega_F$: free T_M-mod of rk 1.
$(X, [w]) \mapsto a(X)\cdot[w]$

Def. (Grothendieck) $\exists$ nondeg. pairing $\qquad \gamma: \{(x,t) \mid |\frac{\partial F}{\partial x_i}| = \delta\} \subseteq B^{n+1}_\varepsilon \times \{t\}$, $\delta \ll 1$

$$J_F: \Omega_F \times \Omega_F \to \mathcal{O}_M$$
$$([g_1 dx_0 \cdots dx_n], [g_2 dx_0 \cdots dx_n]) \mapsto \left(Res_{\mathcal{X}/M}\left[\frac{g_1 g_2 dx_0 \cdots dx_n}{\partial F/\partial x_0, \cdots, \partial F/\partial x_n}\right] : t \mapsto \frac{1}{(2\pi i)^{n+1}} \int_\gamma \frac{g_1 g_2 dx_0 \cdots dx_n}{\partial F/\partial x_0 \cdots \partial F/\partial x_n}\right)$$

Rmk. Under some choices (specifically: a monodromy inv. increasing filt. and a generator in its top degree graded piece), J_F induces a flat metric on T_M!

Lemma 2. Let $i: \Delta\times M - \check{D} \hookrightarrow \Delta\times M$. $\forall k\in\mathbb{Z}$, $\exists!$ extension $(\mathcal{H}^{(k)},\nabla)$ of $(\mathcal{H},\nabla)$, $\mathcal{H}^{(k)}\subseteq i_*\mathcal{H}$, st.

(1) $(\mathcal{H}^{(k)},\nabla)$ has a log pole along $\check{D}$

(2) the residue endomorphism along $\check{D}_{reg}$ is s.s, st. (i) for $\frac{n-1}{2}-k\neq 0$, egvls are $(\frac{n-1}{2}-k,0,\dots,0)$

(ii) for $\frac{n-1}{2}-k=0$, nil. w/. 2×2 Jordan block.

Rmk. $\mathcal{H}^{(0)} \simeq \varphi_*\Omega^{n+1}_{*/M}/dF\wedge d\varphi_*\Omega^{n-1}_{*/M}$.

$\mathcal{H}^{(-1)} \simeq \varphi_*\Omega^{n}_{*/M}/dF\wedge\varphi_*\Omega^{n-1}_{*/M} + d\varphi_*\Omega^{n-1}_{*/M}$. $\Rightarrow \varphi_*\Omega^{n+1}_{*/\Delta\times M} \simeq \mathcal{H}^{(0)}/\mathcal{H}^{(-1)}$

$\Rightarrow \Omega_F \simeq \pi_*\mathcal{H}^{(0)}/\pi_*\mathcal{H}^{(-1)}$ ($\pi:\Delta\times M\to M$ $\therefore \pi_*$ is exact)

Thm. (K. Saito) $\exists$ $\mathcal{O}_M$-bilin. pairing

∂_z^{-1} stands for integration along z direction

$$K_F: \pi_*\mathcal{H}^{(0)}\times\pi_*\mathcal{H}^{(0)} \to \mathcal{O}_M[[\partial_z^{-1}]]\cdot\partial_z^{-n-1}$$

$$(w_1,w_2)\longmapsto \sum_{k\geq 0} \underbrace{K_F^{(-k)}(w_1,w_2)}_{\in\,\mathcal{O}_M}\cdot\partial_z^{-n-1-k}, \text{ st.}$$

(1) $K_F^{(-k)}$ is $(-1)^k$-symm. pairing

$\nabla_{\partial_z}: \pi_*\mathcal{H}^{(k)} \xrightarrow{\sim} \pi_*\mathcal{H}^{(k+1)}$ $\forall k$

(2) For $w_1,w_2\in\pi_*\mathcal{H}^{(0)}$, $K_F(w_1,w_2)\cdot\partial_z^{-1} = K_F(\nabla_{\partial_z}^{-1}w_1,w_2) = K_F(w_1,-\nabla_{\partial_z}^{-1}w_2)$

(3) For $a,b\le 0$, $a+b<-k$, $K_F^{(-k)}(\pi_*\mathcal{H}^{(a)},\pi_*\mathcal{H}^{(b)})=0$

(4) For $w_1,w_2\in\pi_*\mathcal{H}^{(0)}$, $K_F(zw_1,w_2)-K_F(w_1,zw_2) = [z, K_F(w_1,w_2)]$, where $[z,\partial_z^{-k}] := k\partial_z^{-k-1}$

(5) For $w_1,w_2\in\pi_*\mathcal{H}^{(-1)}$, $X\in T_M$,

$$XK_F(w_1,w_2)=K_F(\nabla_X w_1,w_2)+K_F(w_1,\nabla_X w_2).$$

$N := \sum_{i\ge1}\frac{(-1)^{i-1}(h_u - id)^i}{i}$

§ Extension to $\mathbb{P}^1$.

$H \downarrow \mathbb{C}^\times$ holo v.b. of rk $= n$, w/. ∇: flat. A loop around 0 ⇝ monodromy h. Let $h = \underset{\text{s.s. part}}{h_s}\,\underset{\text{unipotent part}}{h_u} = h_u h_s$; $\underset{\text{nil. part}}{N} := \log h_u$

$\to \forall z\in\mathbb{C}^\times$, $H_z = \bigoplus_\lambda H_{z,\lambda}$, $H_{z,\lambda}$:= gen. egsp. of $h_s: H_z\to H_z$

Consider
$$\begin{array}{ccc} e^*H & \xrightarrow{pr} & H \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{e} & \mathbb{C}^\times \\ \zeta & \longmapsto & e^{2\pi i\zeta} \end{array}$$

Let $A\in\Gamma(\mathbb{C},e^*H)$ flat. ⇝ $pr\circ A: \mathbb{C}\to H$.

Def $H^\infty := \{pr\circ A \mid A\in\Gamma(\mathbb{C},e^*H) \text{ flat}\}$ the space of global flat multivalued sections.

$\to H^\infty = \bigoplus_\lambda H^\infty_\lambda$ egsp. decomp. of h.

Fix $A\in H^\infty_\lambda$, α st. $e^{-2\pi i\alpha}=\lambda$. Define

$$\mathbb{C} \to H$$
$$\zeta \longmapsto e(\alpha\zeta)\exp(-\zeta N)A(\zeta).$$

⇝ $es(A,\alpha): \mathbb{C}^\times \to H$

$z \longmapsto e(\alpha\zeta)\exp(-\zeta N)A(\zeta)$, where $e(\zeta)=z$.

The map $es(A,\alpha)$ is well-def: $e(\alpha(\zeta+1))\exp(-(\zeta+1)N)A(\zeta+1)$

$= \lambda^{-1}e(\alpha\zeta)\exp(-\zeta N)\exp(-N)\cdot hA(\zeta)$.

If $A\neq 0$, then $es(A,\alpha)$ is nowhere nonvanishing!

We have, $\forall\alpha$, $\Psi_\alpha: H^\infty_\lambda \xrightarrow{\sim} C^\alpha := \{es(A,\alpha)\}$ as v.s.

$A \longmapsto es(A,\alpha)$

Prop. (1) $z\cdot es(A,\alpha)=es(A,\alpha+1)$ $\quad \left(\overset{e(\zeta)}{\overset{\shortparallel}{z}}\cdot e(\alpha\zeta)\exp(-\zeta N)A(\zeta) = e((\alpha+1)\zeta)\exp(-\zeta N)A(\zeta)\right)$

(2) $\nabla_{\partial_z} es(A,\alpha) = \alpha\cdot es(A,\alpha-1) - \frac{N}{2\pi i} es(A,\alpha-1)$

$\Rightarrow$ (3) $(z\nabla_{\partial_z}-\alpha)es(A,\alpha) = -\frac{N}{2\pi i} es(A,\alpha)$ $\quad\Rightarrow C^\alpha$ is a gen. egsp. of $z\nabla_{\partial_z}$.

(4) $z: C^\alpha \to C^{\alpha+1}$ bij.

(5) $\nabla_{\partial_z}: C^\alpha\to C^{\alpha-1}$ bij $\Leftrightarrow \alpha\neq 0$.

Now, consider $(i_*\mathcal{H})_0$, $i:\mathbb{C}^\times\hookrightarrow\mathbb{C}$.

Def. $V^{>-\infty} := \bigoplus_{-1<\alpha\le 0}\mathbb{C}\{z\}[z^{-1}]C^\alpha \subseteq (i_*\mathcal{H})_0$ the space of all germs at 0 of sections of moderate growth.

identify C^α w/. its image in $(i_*\mathcal{H})_0$

Def. (Kashiwara-Magrange V-filtration) Introduce a decreasing, exaustive filtration on $V^{>-\infty}$ by

$$V^\alpha = \sum_{\beta\ge\alpha}\mathbb{C}\{z\}C^\beta = \bigoplus_{\alpha\le\beta<\alpha+1}\mathbb{C}\{z\}C^\beta, \quad (\text{index set}: \{\alpha \mid e^{-2\pi i\alpha} \text{ egvl of } h\})$$

$$V^{>\alpha} := \sum_{\beta>\alpha}\mathbb{C}\{z\}C^\beta = \bigoplus_{\alpha<\beta\le\alpha+1}\mathbb{C}\{z\}C^\beta,$$

$$Gr_V^\alpha := V^\alpha/V^{>\alpha} \simeq C^\alpha.$$

Note that each V^α is a free $\mathbb{C}\{z\}$-mod of rk n.

Def. (lattice) We call a rk n $\mathbb{C}\{z\}$-submod $\mathcal{L}_0$ of $V^{>-\infty}$ a lattice. Also def

$$Gr_V^\alpha\mathcal{L}_0 := (V^\alpha\cap\mathcal{L}_0+V^{>\alpha})/V^\alpha \subseteq Gr_V^\alpha\simeq C^\alpha.$$

$\rightarrow$ increasing, exhaustive filtration on H^∞:

$$F_pH^\infty_\lambda := \psi^{-1}_{\alpha+p}Gr_V^{\alpha+p}\mathcal{L}_0 = \psi_\alpha^{-1}z^{-p}Gr_V^{\alpha+p}\mathcal{L}_0 \text{ for } \lambda=e^{-2\pi i\alpha},\ -1<\alpha\le 0,$$

$$F_pH^\infty := \bigoplus_\lambda F_pH^\infty_\lambda.$$

increasing filt. on f.d. v.s. V

Def. (opposite filtration) We call $F_\cdot$ and $U_\cdot$ opposite if the following three equiv. condi. hold:

(i) $V=\bigoplus_p F_p\cap U_{-p}$

(ii) If $Gr_p^F Gr_q^U\neq 0$, then $p+q=0$

(iii) $\forall p$, $F_p=\bigoplus_{q\le p}F_q\cap U_{-q}$ and $U_p=\bigoplus_{q\le p}F_{-q}\cap U_q$.

Let $\mathcal{L}$ be $\mathcal{O}_{\mathbb{C}}$-free ext. of $\mathcal{H}$ w/. $\mathcal{L}_0\subseteq V^{>-\infty}$ (then, $\mathcal{L}_0$ is a $\mathbb{C}\{z\}$-lattice).

Thm3. (extension to $\mathbb{P}^1$) Given a monodromy inv. increasing filt. $U_\cdot$ on H^∞ st.

$F_\cdot H_1^\infty$ and $U_\cdot H_1^\infty$ are opposite,

$F_\cdot H_\lambda^\infty$ and $U_\cdot H_\lambda^\infty$ are opposite for $\lambda\neq 1$,

there exists $\mathcal{I}$: $\mathcal{O}_{\mathbb{P}^1}$-free ext. of $\mathcal{L}$, w/. log pole at ∞.

§ Statement of the Thm.

Thm F. Suppose we have $(U_\cdot, \gamma_1)$, where $U_\cdot$ is a monodromy inv. opposite filt. for $F^\cdot = F^{alg}_{n-\cdot}$, and γ_1 is a generator for $Gr^U_{[n-\alpha_1]} H^\infty_{e^{2\pi i\alpha_1}}$. Then this data induces a germ of a Frob. mfd on $(M,0)$.

Rmk. (1) The opposite filt here has a different meaning from p.5.

(2) $\alpha_1, \dots, \alpha_n \in (-1,n) \cap \mathbb{Q}$ are "spectral numbers" of the "Brieskorn lattice H_0''." We have

$$\dim Gr^U_{[n-\alpha_1]} H^\infty_{e^{-2\pi i\alpha_1}} = 1.$$

(3) $F^{alg}_\cdot$ is the filt. induced by the Brieskorn lattice H_0''.

Sketch of the proof (very sketchy at this moment):

The sheaf $\mathcal{H}^{(0)}$ on $\Delta \times M$ extends to locally free $\mathcal{O}_{\mathbb{P}^1 \times M}$-mod $\overline{\mathcal{H}^{(0)}}$. $\rightarrow$

$$e_1 : pr_* \overline{\mathcal{H}^{(0)}} \xrightarrow{\sim} \left(\overline{\mathcal{H}^{(0)}} / \tfrac{1}{z}\overline{\mathcal{H}^{(0)}}\right)\Big|_{\{\infty\}\times M}.$$

On the other hand, $\exists$ pairing

$$K_f : V^{>-1} \times V^{>-1} \to \mathbb{C}\{\{\partial_z^{-1}\}\}\partial_z^{-1},$$

where $\mathbb{C}\{\{\partial_z^{-1}\}\} = \left\{ \sum_{i=0}^\infty a_i \partial_z^{-i} \,\middle|\, \sum_{i=0}^\infty a_i \frac{1}{i!} z^i \in \mathbb{C}\{z\} \right\}$, and V comes from the cohomology bundle $H^n|_{\Delta^* \times \{0\}}$.

If we define

$$G^\alpha := \nabla_{\partial_z}^{[-\alpha]} \psi_{\alpha+[-\alpha]} \left(F^{[n-\alpha]} \cap U_{[n-\alpha]} H^\infty_{e^{-2\pi i\alpha}}\right) \subseteq C^\alpha,$$

then $K_f(G^\alpha, G^{n-1-\beta}) = \mathbb{C}\cdot\partial_z^{-n-1}$. So $\exists$ basis $s_1, \dots, s_n$ of $\bigoplus_\alpha G^\alpha$ st.

$$s_i \in G^{\alpha_i},$$
$$K_f(s_i, s_{n+1-j}) = \delta_{ij}\partial_z^{-n-1},$$
$$\left[\psi^{-1}_{\alpha_1+[-\alpha_1]}\left(\nabla_{\partial_z}^{-[-\alpha_1]} s_1\right)\right] = \gamma_1 \text{ in } Gr^U_{[n-\alpha_1]} H^\infty_{e^{2\pi i\alpha_1}}.$$

Moreover, $\exists$ iso

$$e_2 : \bigoplus_\alpha \mathcal{O}_M G^\alpha \xrightarrow{\sim} \left(\overline{\mathcal{H}^{(0)}} / \tfrac{1}{z}\overline{\mathcal{H}^{(0)}}\right)\Big|_{\{\infty\}\times M}.$$

Define $v_i := e_1^{-1} \circ e_2(s_i) \in pr_* \overline{\mathcal{H}^{(0)}}$. Then $\exists$ iso.

$$v : T_{M,0} \xrightarrow{\sim} \bigoplus_{i=1}^n \mathcal{O}_{M,0} v_i$$
$$X \longmapsto -\nabla_X \nabla_{\partial_z}^{-1} v_1.$$

Let $\delta_i := v^{-1}(v_i)$. Define the metric g via the iso.

$$T_{M,0} \to \Omega_{F,0}$$
$$X \mapsto a(X)\cdot \gamma^{(0)}(v_1)$$

and the Grothendieck pairing on $\Omega_{F,0}$. Then δ_i's are the flat v.f.

Thm F. Suppose we have $(\mathcal{U}_\cdot, Y_1)$, where $\mathcal{U}_\cdot$ is a monodromy inv. opposite filt. for $F := F^{alg}_{h-\cdot}$, and Y_1 is a generator for $Gr^{\mathcal{U}}_{[h-\alpha_1]} H^{\infty}_{e^{2\pi i \alpha_1}}$. Then, this data induces a germ of Frob. mfd on $(M,0)$.

§ Period map

Recall: We have $0 \to \mathcal{H}_0^{(-1)} \to \mathcal{H}_0^{(0)} \xrightarrow{r^{(0)}} \Omega_{F,0} \to 0$. This section shows $\exists$ splitting $\mathcal{H}_0^{(0)} = v(T_{M,0}) \oplus \mathcal{H}_0^{(-1)}$ of the above sequence, where $v: T_{M,0} \to \mathcal{H}_0^{(0)}$ is the "period map".

We introduce another description of $\mathcal{H}_0^{(-1)}, \mathcal{H}_0^{(0)}$.

Recall: $\varphi: \underset{(\mathbb{C}^{n+1} \times M, 0)}{\mathcal{X}} \to \Delta \times M$, $(x,t) \mapsto (F(x,t), t)$, $H^n := \bigcup_{(z,t) \in \Delta \times M - \check{D}} H^n(\varphi^{-1}(z,t), \mathbb{C}) \to \Delta \times M - \check{D}$.

$i: \Delta \times M - \check{D} \hookrightarrow \Delta \times M$

Let $\eta \in \Omega^n_{\mathcal{X},0}$. For sm. fiber $\varphi^{-1}(x,t)$, $\eta|_{\varphi^{-1}(x,t)}$ is a closed n-form. $\xrightarrow{\tau_1} \Omega^n_{\mathcal{X},0} \to (i_* \mathcal{H})_0$.

Let $\omega \in \Omega^{n+1}_{\mathcal{X},0}$. It induces the "Gelfand-Leray" form: for sm. fiber $\varphi^{-1}(x,t)$ it gives "$\frac{\omega}{dF}$" $|_{\varphi^{-1}(z,t)}$, which also equals the Poincaré residue of $\frac{\omega}{F-z}\Big|_{B^{n+1}_\varepsilon \times \{t\}}$. $\rightsquigarrow \tau_2: \Omega^{n+1}_{\mathcal{X},0} \to (i_*\mathcal{H})_0$.

$$\int_{T(\cdot)} \frac{\omega}{F-z}\Big|_{B^{n+1}_\varepsilon \times \{t\}} : H_n(\varphi^{-1}(z,t), \mathbb{C}) \to \mathbb{C}$$, where $T(\cdot)$ is a tubular nbd of $\cdot$. By Poincare duality

$\exists!$ form in $H^n(\varphi^{-1}(z,t), \mathbb{C})$.

Thm 4. (1) $\mathcal{H}_0^{(0)} = \operatorname{im} \tau_1$ and $\mathcal{H}^{(-1)} = \operatorname{im} \tau_2$.

(2) $\mathcal{H}_0^{(-1)} \subseteq \mathcal{H}_0^{(0)}$ is represented by $[\eta] \mapsto [dF \wedge \eta]$, $\eta \in \Omega^n_{\mathcal{X},0}$.

(3) $\nabla_X [dF \wedge \eta] = [dF \wedge Lie_X \eta] - [X(F) d\eta]$ for $X \in T_{M,0}$;

$\nabla_{\partial_z} [dF \wedge \eta] = [d\eta]$.

(4) $\nabla_{\partial_z}: \mathcal{H}_0^{(-1)} \xrightarrow{\sim} \mathcal{H}_0^{(0)}$.

Cor 5. Let $v_1 = [u(x,t) dx_0 \cdots dx_n] \in \mathcal{H}_0^{(0)}$, w/. $u(0) \neq 0$. Def

$$v: T_{M,0} \to \mathcal{H}_0^{(0)}$$
$$X \mapsto -\nabla_X \nabla_{\partial_z}^{-1} v_1$$ period map.

Then, $\mathcal{H}_0^{(0)} = v(T_{M,0}) \oplus \mathcal{H}_0^{(-1)}$. In particular, v is injective.

Pf. Recall: $T_{M,0} \times \Omega_{F,0} \to \Omega_{F,0}$, $(X, [\omega]) \mapsto a(X) \cdot [\omega]$ makes $\Omega_{F,0}$ a $T_{M,0}$-mod of rk 1. By assumption, $\Omega_{F,0} = \langle v_1 \rangle$. Def

$$\Omega_{F,0} \to \mathcal{H}_0^{(0)}, \quad [a(X) \cdot v_1] \mapsto v(X)$$. $\left(0 \to \mathcal{H}_0^{(-1)} \to \mathcal{H}_0^{(0)} \xrightarrow{r^{(0)}} \Omega_{F,0} \to 0 \text{ exact} \right)$

Check: $r^{(0)} v(X) = r^{(0)} \left(-\nabla_X [dF \wedge \nabla_{\partial_z}^{-1} v_1] \right)$
$= r^{(0)} \left(-[dF \wedge Lie_X \nabla_{\partial_z}^{-1} v_1] + [X(F) d \nabla_{\partial_z}^{-1} v_1] \right) = [X(F) d \nabla_{\partial_z}^{-1} v_1] = [X(F) v_1]$.

Let $\begin{matrix} H \\ \downarrow \\ \mathbb{C}^\times \end{matrix}$ v.b. of rk $=\mu$, $\mathcal{L}$: $\mathcal{O}_{\mathbb{C}}$-free ext. of $\mathcal{H}$ w/. $\mathcal{L}_0 \subseteq V^{>-\infty}$. Recall that, this induces a filt on H^∞: w/. ∇: flat

$$F_p H_\lambda^\infty := \psi_{\alpha+p}^{-1} Gr_V^{\alpha+p} \mathcal{L}_0 = \psi_\alpha^{-1} z^{-p} Gr_V^{\alpha+p} \mathcal{L}_0 \text{ for } \lambda = e^{-2\pi i \alpha}, -1<\alpha\le 0,$$

$$F_p H^\infty := \bigoplus_\lambda F_p H_\lambda^\infty.$$

<u>Thm 3.</u> Given a monodromy inv. increasing filt $U_\cdot$ on H^∞ st.

$F_\cdot H_1^\infty$ and $U_\cdot H_1^\infty$ are opposite,

$F_\cdot H_\lambda^\infty$ and $U_{\cdot+1} H_1^\infty$ are opposite for $\lambda \ne 1$,

there exists $\bar{\mathcal{L}}$: $\mathcal{O}_{\mathbb{P}^1}$-free ext. w/. log pole at ∞.

<u>pf</u> We have $H^\infty = \left(\bigoplus_p H_1^\infty \cap F_p \cap U_{-p}\right) \oplus \left(\bigoplus_{\lambda\ne 1} \bigoplus_p H_\lambda^\infty \cap F_p \cap U_{1-p}\right)$. Define

$$G^{\alpha+p} := z^p \psi_\alpha \left(H^\infty_{e^{-2\pi i\alpha}} \cap F_p \cap U_{(0 \text{ or } 1)-p}\right) \subseteq C^{\alpha+p} \qquad (\psi_\alpha : H_\lambda^\infty \xrightarrow{\sim} C^\alpha)$$

for $-1<\alpha\le 0$, $p\in\mathbb{Z}$.

Then $\cdot$ $C^\alpha = \bigoplus_{p\in\mathbb{Z}} z^{-p} G^{\alpha+p} \supseteq Gr_V^\alpha \mathcal{L}_0 = \bigoplus_{p\le 0} z^{-p} G^{\alpha+p} = G^\alpha \oplus z\, Gr_V^{\alpha-1} \mathcal{L}_0$,

$\cdot$ $N G^\alpha = (z\nabla_{\partial_z} - \alpha) G^\alpha \subseteq \bigoplus_{p\ge 0} z^{-p} G^{\alpha+p}$.

Now, choose v_i: basis of $\mathcal{L}_0$ st. principal parts of w_i is $s_i \in \bigoplus_\alpha G^\alpha$ and s_i's form basis of $\bigoplus G^\alpha$.
Eliminate the elementary parts in $\bigoplus_{\beta>\alpha_i} \bigoplus_{p\le 0} z^{-p} G^{\beta+p} \to v_i$'s st.

$$v_i \in \mathcal{L}_0 \cap \left(s_i + \bigoplus_{\beta>\alpha_i} \bigoplus_{p>0} z^{-p} G^{\beta+p}\right)$$

Then v_i's form a basis of $\mathcal{L}_0$ ($\because$ change of coor. matrix is upper triangular). Define $\bar{\mathcal{L}} := \bigoplus_i \mathcal{O}_{\mathbb{P}^1} v_i$. The fact that $\bar{\mathcal{L}}$ has a log pole at ∞ follows from the second "$\cdot$" above, $z\nabla_{\partial_z} = -w\nabla_{\partial_w}$, $z^{-p} = w^p$ and that the higher elementary parts of v_i lies in $\bigoplus_{\beta>\alpha_i} \bigoplus_{p>0} z^{-p} G^{\beta+p}$. $\quad (w = \frac{1}{z})$

§ Polarized mixed hodge structures and opposite filtrations

Let $H_{\mathbb{Q}}$: f.d. v.s, $S: H_{\mathbb{Q}} \times H_{\mathbb{Q}} \to \mathbb{Q}$ nondeg. bil., $(-1)^m$-symm ($m\in\mathbb{N}$), and $N \in End(H_{\mathbb{Q}})$, $N^{m+1}=0$, and $S(Na,b) + S(a,Nb) = 0$ $\forall\, a,b\in H_{\mathbb{Q}}$.

<u>Lemma</u> (weight filtration)

(1) $\exists !\, W_\cdot : 0 = W_{-1} \subseteq W_0 \subseteq \cdots \subseteq W_{2m} = H_{\mathbb{Q}}$ st. $N(W_\ell) \subseteq W_{\ell-2}$ and $N^\ell : Gr^W_{m+\ell} \xrightarrow{\sim} Gr^W_{m-\ell}$

(2) If $\ell+\ell' < 2m$, then $S(W_\ell, W_{\ell'}) = 0$

(3) Define the primitive subspace $P_{m+\ell}$'s by $\begin{cases} P_{m+\ell} = \ker(N^{\ell+1} : Gr^W_{m+\ell} \to Gr^W_{m-\ell-2}) & \text{if } \ell \ge 0 \\ P_{m+\ell} = 0 & \text{if } \ell<0. \end{cases}$ Then $Gr^W_{m+\ell} = \bigoplus_{i\ge 0} N^i P_{m+\ell+2i}$.

(4) $\exists\, S_\ell : Gr^W_{m+\ell} \times Gr^W_{m+\ell} \to \mathbb{Q}$ nondeg. symm bil., defined by $S_\ell(a,b) := S(\tilde{a}, N^\ell \tilde{b})$, where $a, b \in W_{m+\ell}$.

<u>Def.</u> (polarized mixed Hodge structure, PMHS) We say the above data is PMHS if $\exists F_\cdot$ on $H_{\mathbb{C}}$ st.

(1) $Gr^W_k = F^p Gr^W_k \oplus \overline{F^{k+1-p} Gr^W_k}$, i.e. $F^\cdot Gr^W_k$ gives a pure Hodge str. of weight k on Gr^W_k.

(2) $N(F^p) \subseteq F^{p-1}$

(3) $S(F^p, F^{m+1-p}) = 0$

(4) We have $\cdot$ $S_\ell(F^p P_{m+\ell}, F^{m+\ell+1-p} P_{m+\ell}) = 0$

$\cdot$ $i^{2p-m-\ell} S_\ell(u, \bar{u}) > 0$ if $u \in F^p P_{m+\ell} \cap \overline{F^{m+\ell-p} P_{m+\ell}}$, $u \ne 0$, i.e. the pure Hodge str. $F^\cdot P_{m+\ell}$ of weight $m+\ell$ on $P_{m+\ell}$ is polarized by S_ℓ.

For our case, consider $H^n|_{\Delta^\times \times \{0\}}$ ($\Delta^\times$: punctured disk)

<u>Lemma</u>. $\forall z \in \mathbb{C}^\times$, $\exists S: H^n(f^{-1}(z), \mathbb{C})_{\neq 1} \times H^n(f^{-1}(z), \mathbb{C})_{\neq 1} \to \mathbb{C}$ nondeg, $(-1)^n$-symm.

$$H^n(f^{-1}(z), \mathbb{C})_1 \times H^n(f^{-1}(z), \mathbb{C})_1 \to \mathbb{C} \quad \text{nondeg, } (-1)^{n+1}\text{-symm.}$$

Here, $H^n(f^{-1}(z), \mathbb{C})_1 := \ker(h_s - 1 : H^n(f^{-1}(z), \mathbb{C}) \to H^n(f^{-1}(z), \mathbb{C}))$; $H^n(f^{-1}(z), \mathbb{C})_{\neq 1}$ is similarly defined.

Now, $\because$
$$\begin{array}{ccc} e^*H|_{\Delta\times\{0\}} & \longrightarrow & H^n|_{\Delta\times\{0\}} \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{e} & \mathbb{C}^\times \end{array}$$
$\therefore$ $e^*H^n|_{\Delta\times\{0\}}$ is trivial bundle. $\rightsquigarrow$ $S: H^\infty_{\mathbb{Q}} \times H^\infty_{\mathbb{Q}} \to \mathbb{Q}$.

Again, S is $(-1)^n$-symm on $H^\infty_{\neq 1}$,
$(-1)^{n+1}$-symm on H^∞_1.

The space H^∞ equips with a PMHS; the filtration $F^\bullet$ is given by the "Brieskorn lattice".

<u>Def</u>. The Brieskorn lattice H_0'' is defined to be the image of
$$\Omega^{n+1}_{\mathbb{C}^{n+1},0} \to (i_{0,*}B)_0, \quad \omega \mapsto \text{"}\frac{\omega}{df}\Big|_{f^{-1}(z)}\text{"} \quad (\text{as in P.7})$$
where $B :=$ sheaf of holo. sec. of $H^n|_{\Delta^\times\times\{0\}}$, $i_0 : \Delta^\times \hookrightarrow \Delta$.

Similarly, one defines H_0' to be the image of $\Omega^n_{\mathbb{C}^{n+1},0} \to (i_{0,*}B)_0$.

<u>Thm</u>. H_0'' is a free $\mathbb{C}\{z\}$-mod of rk μ; equivalently, $\mathbb{C}\{z\}[z^{-1}]H_0'' = V^{>-\infty}$.

<u>Def</u>. Let $F^pH^\infty_\lambda := \psi_\alpha^{-1}\nabla_{\partial_z}^{n-p}\mathrm{Gr}_V^{\alpha+n-p}H_0''$, $F^pH^\infty = F^pH^\infty_\lambda$. $\to$ $0 = F^{n+1} \subseteq F^n \subseteq \cdots \subseteq F^0 = H^\infty$. This is called
$(\alpha \in (-1, 0], e^{-2\pi i\alpha} = \lambda)$ / Steenbrink's Hodge filtration.

<u>Thm</u>. Together with S and $-N$, this gives PMHS of weight n on $H^\infty_{\neq 1}$,
PMHS " $n+1$ on H^∞_1.

<u>Def</u>. (opposite filtration) An opposite filtration for a PMHS is $U_\bullet$ on $H_{\mathbb{C}}$ st.

(i) $H_{\mathbb{C}} = \oplus_p F^p \cap U_p$, ($\Leftrightarrow$ $F^p = \oplus_{q \geq p} F^p \cap U_q$ and $U_p = \oplus_{q \leq p} F^q \cap U_p$ $\Leftrightarrow$ $\mathrm{Gr}_F^p \mathrm{Gr}^U_q = 0$ for $p \neq q$)

(ii) $N(U_p) \subseteq U_{p-1}$,

(iii) $S(U_p, U_{m-1-p}) = 0$.

<u>Rmk</u>. (1) Opposite filtration exists! For example, define (Deligne)
$$I^{p,q} := (F^p \cap W_{p+q}) \cap \Big(\bar{F}^q \cap W_{p+q} + \sum_{j>0} \bar{F}^{q-j} \cap W_{p+q-j-1}\Big).$$
Then $U_p := \bigoplus_{i,q : i \leq p} I^{i,q}$ forms an oppo. filt.

(2) The space of all oppo. filt is iso. to $\mathbb{C}^{N_{opp}}$ for some $N_{opp} \in \mathbb{N}$.

(3) By condition (iii) in def and condition (iii) in def of PMHS, we know $S(F^p \cap U_p, F^q \cap U_q) = 0$ for $p+q \neq m$.
$\therefore$ $S: F^p \cap U_p \times F^{m-p} \cap U_{m-p} \to \mathbb{C}$ nondeg.

The pairing in P.9 also yields a pairing on $V^{>-1}$, denoted by K_f:

$$K_f : V^{>-1} \times V^{>-1} \to \mathbb{C}\{\{\partial_z^{-1}\}\}\partial_z^{-1}$$

(recall that $V^{>-1} = \oplus_{-1<\alpha\le 0} \mathbb{C}\{z\} C^\alpha$). Here, $\mathbb{C}\{\{\partial_z^{-1}\}\} := \{\sum_{i=0}^\infty a_i \partial_z^{-i} \mid \sum_{i=0}^\infty a_i \frac{1}{i!} z^i \in \mathbb{C}\{z\}\}$. K_f is defined as follows. Let $\alpha, \beta \in (-1, 0]$, $a \in C^\alpha$, $b \in C^\beta$, $g_1, g_2 \in \mathbb{C}\{\{\partial_z^{-1}\}\}$.

- For $\alpha+\beta \notin \mathbb{Z}$, $K_f(a,b) = 0$.
- For $\alpha+\beta = -1$, $K_f(a,b) = \frac{1}{(2\pi i)^n} S(\psi_\alpha^{-1}(a), \psi_\beta^{-1}(b)) \cdot \partial_z^{-1}$.
- For $\alpha = \beta = 0$, $K_f(a,b) := \frac{-1}{(2\pi i)^{n+1}} S(\psi_\alpha^{-1}(a), \psi_\beta^{-1}(b)) \partial_z^{-2}$.
- $K_f(g_1 a, g_2 b) := g_1(\partial_z^{-1}) g_2(-\partial_z^{-1}) K_f(a,b)$.
- $K_f(g_1 a, g_2 b) := \sum_{k\ge n} K_f^{(-k)}(g_1 a, g_2 b) \cdot \partial_z^{-n-1-k}$

Here, we have used the identity $\bigoplus_{-1<\alpha\le 0} \mathbb{C}\{z\} C^\alpha = \bigoplus_{-1<\alpha\le 0} \mathbb{C}\{\{\partial_z^{-1}\}\} \mathbb{C}^\alpha$.

We need the following theorem in the proof of thm F.

Thm. (1) $H_0'' \subseteq V^{>-1}$; $K_f^{(-k)}(H_0'', H_0'') = 0$ for $-n \le k \le -1$, i.e. $K_f(H_0'', H_0'') = \mathbb{C}\{\{\partial_z^{-1}\}\} \cdot \partial_z^{-n-1}$

(2) For $k \ge 0$, $K_f^{(k)}|_{H_0''}$ coincides with the restriction of $K_F^{(-k)}$ on $\mathcal{H}_0^{(0)}$ to H_0'' (recall $K_F^{(k)}$ is Saito's higher residue pairing; see P.4).

(3) $\nabla_{\partial_z}^{-1} H_0'' \subseteq H_0''$; in fact, $\nabla_{\partial_z}^{-1} H_0'' = H_0'$.

§ The proof

γ_1 is a generator of $Gr^{u}_{[n-\alpha_1]} H^\infty_{e^{-2\pi i \alpha_1}}$, whose dim. is 1. Here, $\alpha_1 := \min\{\alpha \mid Gr_V^\alpha H_0'' / z Gr_V^{\alpha-1} H_0'' \ne 0\}$

Proof of thm F. Statement: $(U., \gamma_1) \rightsquigarrow$ germ of Frob. mfd $(M, 0)$.

Extend $\mathcal{H}^{(0)}$ to $\overline{\mathcal{H}^{(0)}}$: $\mathcal{O}_{\mathbb{P}^1\times M}$-locally free mod. Let $pr : \mathbb{P}^1 \times M \to M$. $\rightsquigarrow$

$$\rho_1 : pr_* \overline{\mathcal{H}^{(0)}} \to \left(\overline{\mathcal{H}^{(0)}} \Big/ \tfrac{1}{z}\overline{\mathcal{H}^{(0)}}\right)\Big|_{\{\infty\}\times M}.$$

$G^\alpha := \nabla_{\partial_z}^{[-\alpha]} \psi_{\alpha+[-\alpha]}(F^{[n-\alpha]} \cap U_{[n-\alpha]} H^\infty_{e^{-2\pi i\alpha}}) \subseteq C^\alpha$.

On the other hand, by construction of $\overline{\mathcal{H}^{(0)}}$, $\overline{\mathcal{H}^{(0)}}|_{(\mathbb{P}^1-\Delta)\times M} = \oplus_\alpha \mathcal{O}_{(\mathbb{P}^1-\Delta)\times M} G^\alpha$. $\rightarrow$

$$\rho_2 : \bigoplus_\alpha \mathcal{O}_M \cdot G^\alpha \xrightarrow{\sim} \left(\overline{\mathcal{H}^{(0)}} \Big/ \tfrac{1}{z}\overline{\mathcal{H}^{(0)}}\right)\Big|_{\{\infty\}\times M}$$

Now, choose basis $s_1, \ldots, s_\mu$ of $\oplus_\alpha G^\alpha$ st.
- $s_i \in G^{\alpha_i}$
- $K_f(s_i, s_{\mu+1-j}) = \delta_{ij} \partial_z^{-n-1}$
- $[\psi^{-1}_{\alpha_1+[-\alpha_1]}(\nabla_{\partial_z}^{-[-\alpha_1]} s_1)] = \gamma_1$

Here, we used rmk (3) in P.9, and a fact,

$$\psi^{-1}_{\alpha_1+[-\alpha_1]} \circ \nabla_{\partial_z}^{-[-\alpha_1]} : \underbrace{G^{\alpha_1}}_{\dim 1} \xrightarrow{\sim} \underbrace{Gr^u_{[n-\alpha_1]} H^\infty_{e^{-2\pi i\alpha_1}}}_{\dim 1} \ni \gamma_1$$

Let $v_i := \rho_1^{-1} \circ \rho_2(s_i) \in pr_*\overline{\mathcal{H}^{(0)}}$. Then, $\nabla_X \nabla_{\partial_z}^{-1} v_i \in pr_* \overline{\mathcal{H}^{(0)}} = \bigoplus_{i=1}^{\mu} \mathcal{O}_M \cdot v_i$: Firstly, we have

$$\nabla_X \nabla_{\partial_z}^{-1}(v_i|_{\mathbb{C}\times M}) \in (pr|_{\mathbb{C}\times M})_*\left(\overline{\mathcal{H}^{(0)}}|_{\mathbb{C}\times M}\right)$$

by thm 4. For the germs at (∞, t), $t \in M$, by the construction again we have

$$v_i \in s_i + \tfrac{1}{z}\overline{\mathcal{H}^{(0)}}_{(\infty,t)}$$

$\Rightarrow \nabla_{\partial_z}^{-1} \in \overline{\mathcal{H}^{(0)}}_{(\infty,t)} + \nabla_{\partial_z}^{-1} s_i \Rightarrow \nabla_X \nabla_{\partial_z}^{-1} v_i \in \overline{\mathcal{H}^{(0)}}_{(\infty,t)}$ by the description of ∇_{∂_z} in thm 4.

Fact: $v_1 \in \mathcal{H}_0^{(0)}$ is represented by $[u(x,t)\, dx_0 \cdots dx_n]$ w/. $u(0) \neq 0$. $\therefore$ By Cor 5,

$$\begin{aligned} v: T_{M,0} &\to \mathcal{H}_0^{(0)} \\ X &\mapsto -\nabla_X \nabla_{\partial_z}^{-1} v_1 \end{aligned} \quad \text{is inj.}$$

$\therefore v: T_{M,0} \xrightarrow{\sim} \bigoplus_{i=1}^{\mu} \mathcal{O}_M \cdot v_i$. Let $\delta_i := v^{-1}(v_i)$. Define the metric g by

$$\begin{aligned} T_M \times T_M &\to \mathbb{C} \\ (X, Y) &\mapsto J_F\left(a(X) \cdot r^{(0)}(v_1), a(Y) \cdot r^{(0)}(v_1)\right). \end{aligned}$$

$\because K_F(v_i, v_{\mu+1-j}) = \delta_{ij} \partial_z^{-n-1} \Rightarrow K_F^{(0)}(v_i, v_{\mu+1-j}) = \delta_{ij}$

can prove $[\delta_i, \delta_j] = 0$ by some arguments.

$$\Rightarrow J_F(\underline{r^{(0)}(v_i)}, r^{(0)}(v_{\mu+1-j})) = \delta_{ij}$$

by the proof of cor 5

$$r^{(0)} v(\delta_i) = r^{(0)}(\nabla_{\delta_i} \nabla_{\partial_z}^{-1} v_1) = a(\delta_i) \cdot r^{(0)} v_1$$

$$\therefore g(\delta_i, \delta_{\mu+1-j}) = J_F(r^{(0)}(v_i), r^{(0)}(v_{\mu+1-j})) = \delta_{ij}.$$

Here, the fact $K_F(v_i, v_{\mu+1-j}) = \delta_{ij} \partial_z^{-n-1}$ needs some argument. $\forall k \geq 0$, one views $K_F^{(-k)}(v_i, v_{\mu+1-j}) \in \mathcal{O}_{M,0}$ as a power series. Want:

$$\left(X_1 \cdots X_m K_F^{(-k)}(v_i, v_j)\right)(0) = 0, \quad \forall m \geq 1, \ \forall X_i \in T_{M,0}.$$

Indeed,

$$\begin{aligned} (X_1 \cdots X_m K_F^{(-k)}(v_i, v_j))(0) &= K_F^{(-k)}(\nabla_{X_1} \cdots \nabla_{X_m} v_i, v_j)(0) + \cdots \\ &\overset{\text{P.4, thm, (5)}}{=} K_F^{(-k-m)}(\nabla_{X_1} \cdots \nabla_{X_m} \nabla_{\partial_z}^{-m} v_i, v_j)(0) + \cdots \\ &\overset{\text{P.4, thm, (2)}}{\in} K_F^{(-k-m)}\left(\bigoplus_{\ell=0}^{m-1} \nabla_{\partial_z}^{-\ell} \bigoplus_a \mathcal{O}_M \cdot v_a, v_j\right)(0) + \cdots \\ &\overset{\nabla_X \nabla_{\partial_z}^{-1} v_i \in \bigoplus \mathcal{O}_M \cdot v_j}{=} K_F^{(-k-m)}\left(\bigoplus_{\ell=0}^{m-1} \nabla_{\partial_z}^{-\ell} \bigoplus_a \mathbb{C}\, \underbrace{v_a^0}_{= v_a|_{\Delta \times \{0\}}}, v_j^0\right) + \cdots = 0 \end{aligned}$$

fact: v_i^0's $\in H_0''$. Then use P.10, thm, (2) and (3).