Integrable Deformations and Frobenius Manifolds

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2018.6

0 Basic Notations and Definitions

Let M be a connected complex manifold, $\mathcal{O}_M$ be the sheaf of holomorphic functions on M, and Z be a smooth hypersurface in M.
• For a holomorphic vector bundle E of rank d over M, it corresponds to a locally free sheaf of $\mathcal{O}_M$-modules of rank d, which will be denoted by $\mathcal{E}$, and is also called a "bundle." We define Θ_M to be the one corresponding to the tangent bundle TM.
• $\mathcal{O}_M(*Z)$ is the smallest sheaf containing all $\mathcal{O}_M(kZ)(k \in \mathbb{Z})$ as subsheaves, Ω_M^k is the sheaf of holomorphic k-forms on M, and $\Omega_M^k(*Z) := \Omega_M^k \otimes_{\mathcal{O}_M} \mathcal{O}_M(*Z)$
• A meromorphic bundle $\mathcal{M}$ on M with poles along Z is a locally free sheaf of $\mathcal{O}_M(*Z)$-modules of finite rank.
• For a holomorphic vector bundle E on X, it corresponds to a meromorphic bundle $\mathcal{E}(*Z) := \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(*Z)$.
• A meromorphic connection ∇ on a meromorphic bundle $\mathcal{M}$ with poles along Z is a $\mathbb{C}$-linear morphism $\mathcal{M} \to \Omega_M^1 \otimes_{\mathcal{O}_M} \mathcal{M}$ with its entries in $\Omega_M^1(*Z)$ with respect to a local frame.

1 Frobenius Structures Induces by Infinitesimal Period Mappings

Infinitesimal period mappings provide a way to construct Frobenius structures from a family of bundles on $\mathbb{P}^1$ with flat meromorphic connections. In this section, M is always a connected complex manifold.

1.1 Higgs Fields and the Induced Product Structures

Definition 1 (Higgs fields)**.** Let E be a holomorphic vector bundle on M. A Higgs field on E is an $\mathcal{O}_M$-linear morphism

$$\Phi\colon\ \mathcal{E} \to \Omega_M^1 \otimes_{\mathcal{O}_M} \mathcal{E}$$

with the integrability condition $\Phi \wedge \Phi = 0$.

For a holomorphic vector field ξ on an open subset U of M, we will write $\Phi_\xi\colon\ \mathcal{E}|_U \to \mathcal{E}|_U$ to denote the restricted morphism contracted with ξ.

Now assume $\Phi\colon \Theta_M \to \Omega_M^1 \otimes \Theta_M$ is a Higgs field on the tangent bundle TM. We can view it as a morphism $\Theta_M \otimes \Theta_M \to \Theta_M$ and see if it is symmetric.
The Higgs field can define a product structure on TM by $(\xi, \eta) \mapsto \xi \cdot \eta := -\Phi_\xi(\eta)$.

Proposition 1. Φ is symmetric $\Leftrightarrow$ the product $\cdot$ is associative and commutative.

Proof: On a local chart we can write $\Phi = \sum \Phi^i \otimes dz_i$ where $\Phi^i := \Phi_{\partial_i}$. Then the integrability condition is equivalent to that for all i and j, $\Phi^i \circ \Phi^j = \Phi^j \circ \Phi^i$. Then if Φ is symmetric,

$$\partial_i \cdot (\partial_j \cdot \partial_k) = \partial_i \cdot (\partial_k \cdot \partial_j) = \Phi^i(\Phi^k(\partial_j) = \Phi^k(\Phi^i(\partial_j) = \partial_k \cdot (\partial_i \cdot \partial_j) = (\partial_i \cdot \partial_j) \cdot \partial_k.$$

The commutativity is clear. □

1.2 Residue Endomorphisms

Let F be a bundle on M of rank $d := \dim M$. Then it induces a bundle $E := \pi^* F$ on $\mathbb{P}^1 \times M$ by the canonical projection $\pi\colon \mathbb{P}^1 \times M \to M$. Assume there is a flat meromorphic connection $\widetilde{\nabla}$ on E with a pole of order 1 along $\{0\} \times M$ and a logarithmic pole along $\{\infty\} \times M$. We will write $E_0 := i_0^* E$ where $i_0\colon M \simeq \{0\} \times M \hookrightarrow \mathbb{P}^1 \times M$ and likewise for E_∞. Of course $E_0 \simeq E_\infty \simeq F$.

• The restricted connection ∇ and the induced residue endomorphism R_∞ on E_∞:
In a local chart $U \times M$ with U a neighborhood of $\infty \in \mathbb{P}^1$, the connection matrix of $\widetilde{\nabla}$ with respect to a local frame has the form (in the chart ∞ is at $z^1 = 0$)

$$\Omega = \Omega^1 \frac{dz_1}{z_1} + \sum_{i \geq 2} \Omega^i dz_i,$$

where each Ω^j has holomorphic entries. Then we define a holomorphic connection ∇ on $\{0\} \times M$, whose local matrix representation is

$$\sum_{i \geq 2} \Omega^i(0, z_2, \cdots, z_{d+1}) dz_i.$$

Let $R_\infty\colon i_\infty^* \mathcal{E} \to i_\infty^* \mathcal{E}$ be the endomorphism on E_∞ whose local matrix representation is $\Omega^1(0, z_2, \cdots, z_d)$.
Fact 1. Regarded as a section of the bundle $\mathrm{Hom}(E_\infty, E_\infty)$ equipped with the natural flat connection induced by ∇, R_∞ is a horizontal section.

• The induced Higgs field Φ and the residue endomorphism R_0 on E_0:
In a local chart $U' \times M$ with U' a neighborhood of $0 \in \mathbb{P}^1$, the connection matrix of $\widetilde{\nabla}$ with respect to a local frame has the form

$$\Omega' = \frac{1}{z_1}(\Omega'^1 \frac{dz_1}{z_1} + \sum_{i \geq 2} \Omega'^i dz_i),$$

where each Ω'^j has holomorphic entries. Then we define an endomorphism-valued 1-form $\Phi\colon \mathcal{E}_0 \to \Omega^1_M \otimes \mathcal{E}_0$ whose local matrix representation is

$$\sum_{i \geq 2} \Omega'^i(0, z_2, \cdots, z_{d+1}) dz_i.$$

Let $R_0\colon i_0^* \mathcal{E} \to i_0^* \mathcal{E}$ be the endomorphism on E_0 whose local matrix representation is $\Omega'^1(0, z_2, \cdots, z_d)$.

1.3 Infinitesimal Period Mappings and the Induced Product Structure

Following the setting in the section 1.2, we regard all the objects ∇, Φ, R_0 and R_∞ on E_0. Besides, we further assume E_0 has a metric g and they satisfy

$$\nabla g = 0,\ \Phi^* = \Phi,\ R_0^* = R_0 \text{ and } R_\infty + R_\infty^* = -w \cdot \mathrm{id}_{E_0} \tag{*}$$

for some w where $(\cdot)^*$ is the adjoint with respect to the metric, and

$$\nabla^2 = 0,\ \nabla R_\infty = 0,\ \Phi \wedge \Phi = 0,\ [R_0, \Phi] = 0,\ \nabla \Phi = 0 \text{ and } \nabla R_0 + \Phi = [\Phi, R_\infty]. \tag{**}$$

For a ∇-horizontal section ω of E_0, we define the associated infinitesimal period mapping

$$\begin{aligned} \phi_\omega\colon TM &\to E_0 \\ \xi &\mapsto -\Phi_\xi(\omega). \end{aligned}$$

Such an ω is called **primitive** if ϕ_ω is an isomorphism, and **homogeneous** if ω is an eigenvector of R_∞.

Theorem 1. If E_0 admits a primitive and homogeneous section ω, then ϕ_ω equips M a Frobenius structure.

Proof: Since ϕ_ω is an isomorphism, we can carry on TM the structures on E_0 through it.

• The torsion-free flat connection ${}^{\omega}\nabla$ on TM: For $\xi \in TM$, we define

$$ {}^{\omega}\nabla\xi := \phi_{\omega}^{-1}\nabla(\phi_{\omega}(\xi)). $$

Indeed, fix a local ∇-horizontal frame of E_0 with a local coordinate $z^1, \cdots, z^d$, we can write $\Phi = \sum \Phi^i \otimes dz_i$. Then

$$ {}^{\omega}\nabla_{\partial_i}\partial_j = \phi_{\omega}^{-1}(\nabla_{\partial_i}(\phi_{\omega}(\partial_j))) = \phi_{\omega}^{-1}(\nabla_{\partial_i}(-\Phi^j(\omega))), $$

which is symmetric in i and j since $\nabla\Phi = 0$ and $\nabla\omega = 0$. Thus ${}^{\omega}\nabla$ is torsion-free.

Remark 1. Note that the torsion-freeness of ${}^{\omega}\nabla$ is equivalent to the ∇-horizontality of ϕ_{ω} since

$$ \nabla\phi_{\omega}(\xi,\eta) = \nabla_{\xi}\phi_{\omega}(\eta) + \nabla_{\eta}\phi_{\omega}(\xi) - \phi_{\omega}([\xi,\eta]). $$

• The commutative associative product structure with the ${}^{\omega}\nabla$-flat unit: For ξ and $\eta \in TM$, we define

$$ \xi \cdot \eta := \phi_{\omega}^{-1}(-\Phi_{\xi}(\phi_{\omega}(\eta))). $$

In a local coordinate,

$$ \partial_i \cdot \partial_j = \phi_{\omega}^{-1}(-\Phi_{\partial_i}(\phi_{\omega}(\partial_j)) = \phi_{\omega}^{-1}(\Phi^i(\Phi^j(\omega))) $$

and the conclusion follows with $\Phi \wedge \Phi = 0$.

Let $e := \phi_{\omega}^{-1}(\omega)$. Then

$$ {}^{\omega}\nabla e = \phi_{\omega}^{-1}(\nabla\omega) = 0 $$

and for any ξ,

$$ \xi \cdot e = \phi_{\omega}^{-1}(-\Phi_{\xi}(\phi_{\omega}(e))) = \phi_{\omega}^{-1}(-\Phi_{\xi}(\omega)) = \xi. $$

• The flat metric ${}^{\omega}g$:

For ξ and η on TM, we define

$$ {}^{\omega}g(\xi,\eta) := g(\phi_{\omega}(\xi), \phi_{\omega}(\eta)). $$

Then ${}^{\omega}\nabla^{\omega}g = 0$ since $\nabla g = 0$ and $\nabla\phi_{\omega} = 0$. Moreover by the torsion-freeness, ${}^{\omega}\nabla$ is the Levi-Civita connection of ${}^{\omega}g$.

• The euler vector field E: Let $E := \phi_{\omega}^{-1}(R_0(\omega))$ and say $R_{\infty}\omega = -q\omega$ by the homogeneity.

(1) ${}^{\omega}\nabla({}^{\omega}\nabla E) = 0$:

Locally

$$ \begin{aligned} {}^{\omega}\nabla_{\partial_i}(E) &= \phi_{\omega}^{-1}(\nabla_{\partial_i}(R_0(\omega))) \\ &= \phi_{\omega}^{-1}(\partial_i(R_0)(\omega)) \\ &= \phi_{\omega}^{-1}(([\Phi^i, R_{\infty}] - \Phi^i)(\omega)) \\ &= \phi_{\omega}^{-1}((-1-q)\Phi^i(\omega) - R_{\infty}(\Phi^i(\omega))) \\ &= (1+q)\,\partial_i + {}^{\omega}R_{\infty}(\partial_i) \end{aligned} $$

by $\nabla R_0 + \Phi = [\Phi, R_{\infty}]$, where ${}^{\omega}R_{\infty} := \phi_{\omega}^{-1} \circ R_{\infty} \circ \phi_{\omega}$. Hence ${}^{\omega}\nabla(E) = (1+q)\mathrm{id}_{TM} + {}^{\omega}R_{\infty}$.

By $\nabla R_{\infty} = 0$, $\nabla\phi_{\omega} = 0$ and the torsion-freeness of ${}^{\omega}\nabla$, we have

$$ \begin{aligned} {}^{\omega}\nabla^{\omega}R_{\infty}(\xi,\eta) &= \phi_{\omega}^{-1}(\nabla(R_{\infty} \circ \phi_{\omega})(\xi,\eta)) \\ &= \phi_{\omega}^{-1}(\nabla_{\xi}(R_{\infty}(\phi_{\omega}(\eta))) - \nabla_{\eta}(R_{\infty}(\phi_{\omega}(\xi))) - R_{\infty}(\phi_{\omega}([\xi,\eta]))) \\ &= \phi_{\omega}^{-1}(R_{\infty}(\nabla_{\xi}(\phi_{\omega}(\eta)) - \nabla_{\eta}(\phi_{\omega}(\xi))) - R_{\infty}(\phi_{\omega}([\xi,\eta]))) \\ &= \phi_{\omega}^{-1}(R_{\infty}(\phi_{\omega}([\xi,\eta])) - R_{\infty}(\phi_{\omega}([\xi,\eta]))) = 0. \end{aligned} $$

Thus ${}^{\omega}\nabla^{\omega}R_{\infty} = 0$, so ${}^{\omega}\nabla({}^{\omega}\nabla E) = 0$.

We have the new relations from the old ones (*):

$$ {}^{\omega}\nabla^{\omega}g = 0,\ \Phi^* = \Phi,\ ({}^{\omega}R_0)^* = {}^{\omega}R_0 \text{ and } {}^{\omega}R_{\infty} + ({}^{\omega}R_{\infty})^* = -w \cdot \mathrm{id}_{TM} $$

where the adjoint is respect to the metric ${}^{\omega}g$. Note these imply the symmetry of ${}^{\omega}\nabla c$ where $c(\xi_1, \xi_2, \xi_3) := {}^{\omega}g(\xi_1 \cdot \xi_2, \xi_3)$.

(2) $L_E({}^\omega g) = D\cdot {}^\omega g$ for some D, where L means the Lie derivative:
From (1), we have

$${}^\omega\nabla E = (1+q)\mathrm{id}_{TM} + {}^\omega R_\infty.$$

Then taking the adjoint gives

$$({}^\omega\nabla E)^* = (1+q)\mathrm{id}_{TM} + ({}^\omega R_\infty)^* = (1+q-w)\mathrm{id}_{TM} - {}^\omega R_\infty.$$

Then for ξ and η in TM,

$${}^\omega g({}^\omega\nabla_\xi E,\eta) + {}^\omega g(\xi, {}^\omega\nabla_\eta E) = {}^\omega g(({}^\omega\nabla E)(\xi),\eta) + {}^\omega g(({}^\omega\nabla E)^*(\xi),\eta) = (2+2q-w)^\omega g(\xi,\eta).$$

Let $D := 2+2q-w$ and we have

$$\begin{aligned}
L_E({}^\omega g)(\xi,\eta) &= E({}^\omega g(\xi,\eta)) - {}^\omega g(L_E\xi,\eta) - {}^\omega g(\xi, L_E\eta)\\
&= E({}^\omega g(\xi,\eta)) - {}^\omega g({}^\omega\nabla_E\xi - {}^\omega\nabla_\xi E,\eta) - {}^\omega g(\xi, {}^\omega\nabla_E\eta - {}^\omega\nabla_\eta E)\\
&= E({}^\omega g(\xi,\eta)) - {}^\omega g({}^\omega\nabla_E\xi,\eta) - {}^\omega g(\xi, {}^\omega\nabla_E\eta) + {}^\omega g({}^\omega\nabla_\xi E,\eta) + {}^\omega g(\xi, {}^\omega\nabla_\eta E)\\
&= {}^\omega\nabla_E({}^\omega g)(\xi,\eta) + (2+2q-w)^\omega g(\xi,\eta)\\
&= D\cdot {}^\omega g(\xi,\eta)
\end{aligned}$$

(3) $L_E(\cdot) = \cdot$ where $\cdot$ means the product structure:
First we claim that ${}^\omega R_0 := \phi_\omega^{-1}\circ R_0 \circ \phi_\omega$ is exactly the endomorphism $\xi \mapsto \xi\cdot E$. Indeed, by $[R_0,\Phi] = 0$,

$$\xi\cdot E = \phi_\omega^{-1}(-\Phi_\xi(\phi_\omega(\phi_\omega^{-1}(R_0(\omega)))) = \phi_\omega^{-1}(-R_0(\Phi_\xi(\omega))) = \phi_\omega^{-1}(R_0(\phi_\omega(\xi))) = {}^\omega R_0(\xi).$$

Since $\nabla\phi_\omega = 0$, the old relation $\nabla R_0 + \Phi = [\Phi, R_\infty]$ gives, after composing with ϕ_ω^{-1} and ϕ_ω,

$${}^\omega\nabla_\xi(\eta\cdot E) - ({}^\omega\nabla_\xi)\cdot E - \xi\cdot\eta = \xi\cdot({}^\omega\nabla_\eta E - (1+q)\eta) - ({}^\omega\nabla_{\xi\cdot\eta}E - (1+q)\xi\cdot\eta) = \xi\cdot {}^\omega\nabla_\eta E - {}^\omega\nabla_{\xi\cdot\eta}E.$$

By $\nabla\Phi = 0$, the above result simplifies to

$$L_E(\xi\cdot\eta) - (L_E\xi)\cdot\eta - \xi\cdot(L_E\eta) = \xi\cdot\eta.$$

Thus the theorem follows. □

2 Universal Semisimple Frobenius Structures

We aim at establishing the following theorem.
Theorem 2 ([Dub96]). There is a one-to-one correspondence

$$\{\text{semisimple simply connected Frobenius manifolds}\} \leftrightarrow \{(B_0^o, B_\infty, \omega^o, U) \text{ satisfying the } (\star) \text{ conditions}\}$$

with the $(\star)$ conditions that B_0^o is regularly semisimple, that $B_\infty + B_\infty^* = wI_d$ for some $w\in\mathbb{Z}$, that w^o is an eigenvector of B_∞, whose components don't vanish on the eigenbases of B_0^o, and that U is a simply connected open set of $\widetilde{X}_d \setminus \Theta_{\omega^o}$.

In the theorem, $X_d := \{(x^1,\cdots,x^d)\in\mathbb{C}^d\,|\,x^i \neq x^j \text{ for all } i<j\}$ and $\widetilde{X}_d$ is its universal cover. Fix $x^o = (x_1^o,\cdots,x_d^o)\in X_d$ and a lifted point $\widetilde{x}^o\in\widetilde{X}_d$, i.e., $\pi(\widetilde{x}^o) = x^o$ where $\pi\colon \widetilde{X}_d \to X_d$ is the covering map.
Proof: Suppose we are given $B_\infty + B_\infty^* = wI_d$ for some $w\in\mathbb{Z}$, $B_0^o = \mathrm{diag}(x_1^o,\cdots,x_d^o)$, thus regularly semisimple, and an eigenvector ω^o of B_∞, all components of which are non-zero.
Theorem 3 ([Mal83]). Given such B_0^o and B_∞, there exist a unique holomorphic bundle E on $\mathbb{P}^1\times\widetilde{X}_d$ and a flat meromorphic connection ∇ with a pole of order 1 along $\{0\}\times\widetilde{X}_d$ and a logarithmic pole along $\{\infty\}\times\widetilde{X}_d$, such that
(1) the restriction (E^o,∇^o) of (E,∇) at $\widetilde{x}^o$ has a global frame with respect to which the matrix representation of ∇^o is

$$\left(\frac{B_0^o}{z} + B_\infty\right)\frac{dz}{z};$$

(2) for any $\widetilde{x}\in\widetilde{X}_d$, the eigenvalues of the residue endomorphism R_0 at $\widetilde{x}$ are the components of $\pi(\widetilde{x})$.

Theorem 4 ([Sab08])**.** Let X be a connected complex analytic manifold and F a holomorphic vector bundle on $\mathbb{P}^1 \times X$ such that for any $x \in X$, the restriction $F|_{\mathbb{P}^1 \times \{x\}}$ has degree 0.
(1)(The nontriviality divisor) The set

$$\Theta := \{x \in M : F|_{\mathbb{P}^1 \times \{x\}} \text{ is non-trivial}\}$$

is $\emptyset$, X or a hypersurface of X.
(2)(The canonical identification between the restriction to 0 and ∞) We have

$$i_0^* F|_{\mathbb{P}^1 \times (X \setminus \Theta)} \simeq i_\infty^* F|_{\mathbb{P}^1 \times (X \setminus \Theta)}.$$

Theorem 5. ([Sab17]) Let X be a simply connected complex manifold and (F, ∇) a bundle on $D \times X$ with a pole of order 1 along $\{0\} \times X$. Suppose R_0 is the residue endomorphism and $(\hat{F}, \hat{\nabla})$ is its associated formal bundle.
(1)(the unique decomposition) If R_0 is regularly semisimple, $(\hat{F}, \hat{\nabla})$ has a unique decomposition to line bundles

$$(\hat{F}, \hat{\nabla}) \simeq \bigoplus_j (\hat{F}_j, \hat{\nabla}).$$

(2)(equivalence) For line bundles, the formalism $(F, \nabla) \mapsto (\hat{F}, \hat{\nabla})$ is an equivalence of categories.

By the theorem 3, we can, following the section 1.3, obtain ∇ and R_∞ on E_∞, Φ and R_0 on E_0. Via the theorem 4, we get a bundle E on $\widetilde{X}_d \setminus \Theta$ with objects ∇, Φ, R_∞ and R_0.
Since $\widetilde{X}_d$ is simply connected and ∇ is a flat connection on E_∞, it's trivial and thus we can find a ∇-horizontal ω on $\widetilde{X}_d$ such that $\omega(\widetilde{x}^o) = \omega^o$. Later we will let ω be its restriction to $\widetilde{X}_d \setminus \Theta$.
By the theorem 3 again, the residue R_0 is regular semisimple everywhere, so E_0, on $\{0\} \times \widetilde{X}_d$, can be decomposed to a direct sum of eigenbundles of rank one, each of which can be equipped with a flat connection by the theorem 5, and hence admits a global frame. We collect these d section, forming a global frame $\mathbf{e} = \{e_1, \cdots, e_d\}$ of E_0.
Restrict the frame on $\widetilde{X}_d \setminus \Theta$, also denoted by $\mathbf{e}$, and let ω^i be the components of ω with respect to $\mathbf{e}$. We set

$$\Theta_{\omega^o} := \Theta \cup \Big(\bigcup_{i=1}^{d} \{\text{the zero locus of } \omega^i\} \Big).$$

By our definition of Θ_{ω^o}, the sections

$$u^i := \omega^i e_i$$

form a basis of $E|_{\widetilde{X}_d \setminus \Theta_{\omega^o}}$. Then the infinitesimal period mapping associated to ω gives

$$\begin{aligned} \phi_\omega \colon T(\widetilde{X}_d \setminus \Theta_{\omega^o}) &\to E|_{\widetilde{X}_d \setminus \Theta_{\omega^o}} \\ \partial_i &\mapsto -\Phi_{\partial_i}(\omega) = u_i \end{aligned}$$

where the fact that $-\Phi_{\partial_i}(\omega) = u_i$ comes from the matrix representation of Φ with respect to $\mathbf{e}$, which will be explained in the proceeding sections.
Therefore, ϕ_ω is an isomorphism, and by the construction in the section 1.3, $\widetilde{X}_d \setminus \Theta_{\omega^o}$ admits a Frobenius structure. Note that we have the unit

$$e = \phi_\omega^{-1}(\omega) = \phi_\omega^{-1}(\sum u_i) = \sum \partial_i$$

and the Euler vector field

$$E = \phi_\omega^{-1}(R_0(\omega)) = \phi_\omega^{-1}(\sum x_i u_i) = \sum x_i \partial_i$$

for the matrix representation of R_0 with respect to $\mathbf{e}$ is $\mathrm{diag}(x_1, \cdots, x_d)$, which will also be explained.
Besides, we have

$$\partial_i \cdot \partial_j = \phi_\omega^{-1}(-\Phi_{\partial_i}(\phi_\omega(\partial_j))) = \phi_\omega^{-1}(-\Phi_{\partial_i}(u_j)) = \phi_\omega^{-1}(\delta_{ij} u_i) = \delta_{ij} \partial_i .$$

This proves one direction of the theorem.
Remark 2. We didn't check that the objects satisfy the condition (*) and (**), which would be clear after we show the solvability of the Birkhoff's problem in a family.

For the other way around, let M be a semisimple simply connected Frobenius manifold. Then we can define Φ by $\Phi_\xi(\eta) := -\xi \cdot \eta$, $R_0 := -\Phi(E)$, and $R_\infty := \nabla E$. The semisimplicity means that at each point R_0 is regularly semisimple, so its eigenvalues define d functions $(x_1, \cdots, x_d)\colon M \to X_d$.

Theorem 6 ([Mal83])**.** Let X be a simply connected complex manifold with a fixed base point $x^o \in X$, $\lambda_1, \cdots, \lambda_d$ d holomorphic functions $X \to \mathbb{C}$ such that $\lambda_i(x) \neq \lambda_j(x)$ for all $i \neq j$ and $x \in X$, (E^o, ∇^o) a bundle on D with a connection having a pole of order 1 at the origin, and the residue R_0^o whose eigenvalues are $\lambda_1(x^o), \cdots, \lambda_d(x^o)$. Then there exists a unique bundle (E, ∇) on $D \times X$ with a connection having a pole of order 1 along $\{0\} \times X$ such that

(1) for any $x \in X$, $R_0(x)$ has eigenvalues $\lambda_1(x), \cdots, \lambda_d(x)$, and

(2) $(E, \nabla)|_{D \times \{x^o\}} \simeq (E^o, \nabla^o)$.

By the theorem 6, we can as above construct a basis $\mathbf{e} = \{e_1 \cdots, e_d\}$, with respect to which the matrix of Φ is exactly $-dR_0$ (also will be clarified later), i.e.,

$$\Phi(e_i) = -dx_i \otimes e_i,$$

therefore, for all i and j,

$$e_i \cdot e_j = -\Phi_{e_j} e_i = L_{e_j}(x_i) \cdot e_i.$$

By the commutativity of the product, $L_{e_j}(x_i) = 0$ for $i \neq j$. Besides, $\lambda_i := L_{e_i}(x_i)$ is non-vanishing:
Write the unit vector field e as $e = \sum a_i e_i$. Then

$$e_i = e_i \cdot e = e_i \cdot \sum a_i e_i = a_i \lambda_i.$$

Thus $a_i \lambda_i \equiv 1$ so λ_i is non-vanishing.
Now we get a holomorphic map

$$(x_1, \cdots, x_d)\colon M \to X_d.$$

Since M is simply connected, it can be lifted to

$$f := (x_1, \cdots, x_d)\colon M \to \widetilde{X}_d,$$

which is a submersion, thus an open map, so it is proper. Since it's a proper local homeomorphism between locally compact Hausdorff spaces, it's a covering map, with the number of sheets

$$[\pi_1(f(M)) : \pi_1(M)] = 1$$

because M is simply connected and the image of a simply connected domain under a biholomorphic map is simply connected. Thus, f is isomorphic to an open subset of $\widetilde{X}_d$, and the conclusion of the theorem 2 follows. □

Remark 3. Note that by the canonicality of the product structure defined above, this $(\cdot, e, E)$ is independent of the choice of ω^o with the non-zero condition.

3 Birkhoff's Problem on $\mathbb{P}_1$

In this section, X is a simply-connected complex analytic manifold of dimension n.
We write $\mathbb{P}^1 = U_0 \cup U_\infty$ with $U_0 := \mathbb{P}^1 \setminus \{\infty\}$ and $U_\infty := \mathbb{P}^1 \setminus \{0\}$ and let τ and τ' be the coordinate on them respectively.
Let $D = B_r(0)$ be an open disc in $U_0 \simeq \mathbb{C}$ for some $r > 0$, and $(\widetilde{E}, \widetilde{\nabla})$ be a holomorphic bundle of rank d on $U := D \times X$, with a flat meromorphic connection $\widetilde{\nabla}$ having a pole of order 1 along $\{0\} \times X$, in the sense that its restriction to $(U_\infty \cap D) \times X$ is a flat connection on the holomorphic bundle $\widetilde{E}|_{(U_\infty \cap D) \times X}$.
Fix a point $x^o \in X$ and let $((\widetilde{E}^o, \widetilde{\nabla}^o)$ be the restriction of $(\widetilde{E}, \widetilde{\nabla})$ to $U^o := D \times \{x^o\}$.

Definition 2 (Birkhoff's problem)**.** Let $A(\tau)$ be a $d \times d$ matrix function on D with $\tau^{r+1} A(\tau)$ having holomorphic entries, one of which doesn't vanish at $\tau = 0$, for some $r \geq 0$. We say that Birkhoff's problem can be solved for $A(\tau)$ if there exists some $P(\tau) \in \mathrm{GL}_d(\mathcal{O}_U)$ such that the matrix $B(\tau) := P^{-1}AP + P^{-1}P'$ can be written as

$$B(\tau) = \frac{B_{-(r+1)}}{\tau^{-(r+1)}} + \cdots + \frac{B_{-1}}{\tau}$$

for some $B_{-1}, \cdots, B_{-(r+1)} \in M_d(\mathbb{C})$.

The following theorem says that there's a canonical solution to Birkhoff's problem for almost every values in $\mathbb{P}^1$ if we are given a solution for a particular value.

Theorem 7. Assume Birkhoff's problem can be solved for $(\widetilde{E}^o, \widetilde{\nabla}^o)$ at x^o, i.e., we can take a global frame ϵ^o of $\widetilde{E}^o$ with respect to which the connection matrix of $\widetilde{\nabla}^o$ can be written as

$$\Omega^o = (\frac{B_0^o}{\tau} + B_\infty)\frac{d\tau}{\tau}$$

for some B_0^o and $B_\infty \in M_d(\mathbb{C})$. Then there exist a hypersurface Θ of X not containing x^o and a unique basis ϵ of $\widetilde{\mathcal{E}}(*(D \times \Theta))$ which coincides with ϵ^o at x^o and in which the connection matrix of $\widetilde{\nabla}$ takes the form

$$\Omega = (\frac{B_0(x)}{\tau} + B_\infty)\frac{d\tau}{\tau} + \frac{C(x)}{\tau}$$

where $B_0(x)$ and $C(x)$ are a meromorphic matrix function and 1-form on X and holomorphic on $X \setminus \Theta$ with $B_0(x^o) = B_0^o$.

Proof:
First we prove the existence.
• Constructions of a bundle on $\mathbb{P}^1 \times X$ by sheaves gluing:
Let $D' \subseteq U_\infty$ be an open disc centered at ∞ such that $A := D \cap D' \neq \emptyset$. Recall τ' is the coordinate on D', i.e., $\tau' = 1/\tau$ on A. On $A \times X$, since $(\widetilde{E}, \widetilde{\nabla})|_{A\times X}$ is a flat bundle, it is determined by its monodromy. On the other hand, for X is simply-connected, $\pi_1(A \times X) = \pi_1(A)$. Thus $(\widetilde{E}, \widetilde{\nabla})|_{A\times X}$ is isomorphic to $p^*(\widetilde{E}^o, \widetilde{\nabla}^o)|_A$ where p is the projection $A \times X \to A$.
By the change of variable, the restriction of the trivial bundle $\mathcal{O}^d_{D'\times X}$ with the connection matrix

$$-(\tau' B_0^o + B_\infty)\frac{d\tau'}{\tau'}$$

to $A \times X$ is isomorphic to $(\widetilde{E}, \widetilde{\nabla})|_{A\times X}$ since they are both isomorphic to $p^*(\widetilde{E}^o, \widetilde{\nabla}^o)|_A$. Thus we can glue them up to get a bundle (E, ∇) on $\mathbb{P}^1 \times X$ with a flat meromorphic connection having poles on $\{0, \infty\} \times X$.

• The nontriviality divisor Θ of X :
Since $E|_{\mathbb{P}^1\times\{x^o\}} = E^o$ is trivial, its degree is 0 so we can apply the theorem 4 of the nontriviality divisor to X and get $\Theta \subsetneq X$ because $x^o \notin \Theta$.

• Extensions of a basis:
Since ∇ has a logarithmic pole along $\{\infty\} \times X$, there's an induced holomorphic flat connection ∇_∞ on $i^*_\infty \mathcal{E}$.
Fact 2 (the canonical identification between the restrictions to 0 and ∞)**.** Let π be the projection $\mathbb{P}^1 \times X \to X$. Then for $\mathcal{E}$ above, there exist canonical isomorphisms

$$\mathcal{E}(*\pi^{-1}\Theta) \simeq \pi^* i_0^* \mathcal{E}(*\pi^{-1}\Theta) \simeq \pi^* i_\infty^* \mathcal{E}(*\pi^{-1}\Theta)$$

where $i_0 \colon \{0\} \times X \hookrightarrow \mathbb{P}^1 \times X$ and $i_\infty \colon \{\infty\} \times X \hookrightarrow \mathbb{P}^1 \times X$.

By the triviality of $i^*_\infty \mathcal{E}$ and the above isomorphisms, we can extend ϵ^o to a basis ϵ of the bundle $\mathcal{E}(*\pi^{-1}\Theta)$.

• On U the connection matrix of ∇ takes the desired form.
Since the connection matrix Ω has order 1 at $\{0\} \times X$ in the basis ϵ, it can be written as

$$(\frac{B_0(x)}{\tau} + B_\infty(\tau, x))\frac{d\tau}{\tau} + C_0(\tau, x) + \frac{C(x)}{\tau}$$

where B_∞ and C_0 are holomorphic function and 1-form.
Note after a change of variables, at infinity the connection matrix is of the form

$$(-B_0(x)\tau' - B_\infty(\frac{1}{\tau'}, x))\frac{d\tau'}{\tau'} + C_0(\frac{1}{\tau'}, x) + C(x)\tau'.$$

Since the connection has a logarithmic pole at $\{\infty\} \times X$, we have B_∞ and C_0 are independent of τ', i.e., independent of τ.
Restricted to infinity, the basis ϵ is ∇_∞-horizontal, where the connection matrix is by definition $C_0(x)$, so $C_0(x) = 0$.
By the fact 1, because the endomorphism R_∞ is horizontal, we have

$$\nabla(R_\infty \epsilon^i) = R_\infty(\nabla \epsilon^i)$$

for $i = 1, \cdots, d$ if the basis ϵ consists of $\{\epsilon^1, \cdots, \epsilon^d\}$. Note the matrix representation of R_∞ with respect to the basis is $(-B_0(x)\tau' - B_\infty(x))|_{\tau'=0} = -B_\infty(x)$. Thus if we write $B_i :=$ the i-th column of $-B_\infty(x)$,

$$\nabla(R_\infty \epsilon^i) = \nabla(\epsilon \cdot B_i) = (\nabla \epsilon) \cdot B_i + \epsilon \cdot dB_i$$

and

$$R_\infty(\nabla \epsilon^i) = (\nabla \epsilon) \cdot B_i.$$

where $\epsilon := (\epsilon_1 \cdots \epsilon_d)$ and $\nabla \epsilon := (\nabla \epsilon_1 \cdots \nabla \epsilon_d)$. Thus $dB_i(x) = 0$ for all i, so B_∞ is constant in x. Hence we get the desired form

$$(\frac{B_0(x)}{\tau} + B_\infty)\frac{d\tau}{\tau} + \frac{C(x)}{\tau}.$$

For the uniqueness, suppose (ϵ', Θ') and (ϵ', Θ') satisfy the theorem. Let $\Theta := \Theta' \cup \Theta''$ and $X^o := X \setminus \Theta$. Then we get an isomorphism

$$\mathcal{O}^d_{D\times X}(*(D \times \Theta)) \xrightarrow{P} \mathcal{O}^d_{D\times X}(*(D \times \Theta))$$

via the base change P. Since the induced homomorphism $\pi_1(D \times X^o) \to \pi_1(\mathbb{P}^1 \times X^o)$ is an isomorphism, by the proof of existence, P can be extended holomorphicly to the isomorphism between $\mathcal{O}^d_{\mathbb{P}^1\times X}(*(\mathbb{P}^1 \times \Theta))$.
Note if the matrix of the two connections are Ω and Ω', we have $\Omega' = P^{-1}\Omega P + P^{-1}dP$, i.e.,

$$dP = P\Omega' - \Omega P.$$

Thus if we write P as column vectors, we can write the system as

$$dP = ((A_1 + \tau' A_2(x))\frac{d\tau'}{\tau'} + D(x)\tau') \cdot P$$

near the infinity. If write $P(x) = \sum_{l \geq l_0} \tau'^l P_l(x)$, then we have $dP_l = DP_{l-1}$.
Suppose $l_0 < 0$, then $dP_{l_0} = DP_{l_0-1} = 0$, so $P_{l_0} \equiv P_{l_0}(x^o) = 0$ since at x^o P is identity. Hence the entries of P are holomorphic. Hence $P(\tau', x) = P_0(x)$. Besides, $dP_0 = DP_{-1} = 0$, so $P(\tau', x) = P_0(x) \equiv P_0(x^o) = \mathrm{id}$. □

Remark 4. With respect to the basis ϵ, the matrix representation connection takes the form above, that of R_0 is $B_0(x)$, that of R_∞ is $-B_\infty$, and that of Φ is $C(x)$.
Besides, the integrability condition exactly tells

$$dC = 0,\ C \wedge C = 0,\ [B_0, C] = 0, \text{ and } dB_0 + C = [B_\infty, C],$$

which gives the condition (**) in the section 1.3.

We use the idea of this proof to prove the theorem 3 that was applied.

Theorem 3. Given $x^o = (x^o_1, \cdots, x^o_d) = \pi(\widetilde{x}^o) \in X_d$ and $B^o_0 := \mathrm{diag}(x^o_1, \cdots, x^o_d), B_\infty \in M_d(\mathbb{C})$, there exist a unique holomorphic bundle E on $\mathbb{P}^1 \times \widetilde{X}_d$ and a flat meromorphic connection ∇ with a pole of order 1 along $\{0\} \times \widetilde{X}_d$ and a logarithmic pole along $\{\infty\} \times \widetilde{X}_d$, such that
(1) the restriction (E^o, ∇^o) of (E, ∇) at $\widetilde{x}^o$ has a global frame with respect to which the matrix representation of ∇^o is

$$(\frac{B^o_0}{\tau} + B_\infty)\frac{d\tau}{\tau};$$

(2) for any $\widetilde{x} \in \widetilde{X}_d$, the eigenvalues of the residue endomorphism R_0 at $\widetilde{x}$ are the components of $\pi(\widetilde{x})$.

Proof:
We will again apply the theorem 6:

Theorem 6. Let X be a simply connected complex manifold with a fixed base point $x^o \in X$, $\lambda_1, \cdots, \lambda_d$ d holomorphic functions $X \to \mathbb{C}$ such that $\lambda_i(x) \neq \lambda_j(x)$ for all $i \neq j$ and $x \in X$, (E^o, ∇^o) a bundle on D with a connection having a pole of order 1 at the origin, and the residue R_0^o whose eigenvalues are $\lambda_1(x^o), \cdots, \lambda_d(x^o)$. Then there exists a unique bundle (E, ∇) on $D \times X$ with a connection having a pole of order 1 along $\{0\} \times X$ such that
(1) for any $x \in X$, $R_0(x)$ has eigenvalues $\lambda_1(x), \cdots, \lambda_d(x)$, and
(2) $(E, \nabla)|_{D \times \{x^o\}} \simeq (E^o, \nabla^o)$.

To begin with, let D be an open disk centered at the origin in $\mathbb{P}^1$, and let (E^o, ∇^o) be the trivial bundle $\mathcal{O}_D^d$ with the connection matrix of ∇^o being

$$(\frac{B_0^o}{\tau} + B_\infty)\frac{d\tau}{\tau}.$$

Define $\lambda_i := p_i \circ \pi$ where p_i is the projection $(x_1, \cdots, x_d) \in X_d \mapsto x_i \in \mathbb{C}$. Then by the theorem 6 we get a bundle $(\widetilde{E}, \widetilde{\nabla})$ on $D \times \widetilde{X}_d$. Then the argument of the previous proof works. □

4 Universal Integrable Deformations for Birkhoff's Problems

Let (E^o, ∇^o) be a bundle on $\mathbb{P}^1$ with a pole of order one along 0 and a logarithmic pole along ∞.

• Let X be a complex manifold. An integrable deformation of (E^o, ∇^o) parametrized by (X, x^o) is a bundle (E, ∇) with a flat meromorphic connection on $\mathbb{P}^1 \times X$ with a pole of order one along $\{0\} \times X$ and a logarithmic pole along $\{\infty\} \times X$ such that $(E, \nabla)|_{\mathbb{P}^1 \times \{x^o\}} = (E^o, \nabla^o)$.

• An integrable deformation (E, ∇) of (E^o, ∇^o) is called complete at x^o if for any other integrable deformation (E', ∇', x') of (E^o, ∇^o) parametrized by (X', x'), there exist neighborhoods V and V' of x^o and x' and an analytic map $f\colon (V', x') \to (V, x^o)$ such that $(E', \nabla')|_{\mathbb{P}^1 \times V'} = (\mathrm{id}_{\mathbb{P}^1} \times f)^*(E, \nabla)|_{\mathbb{P}^1 \times V}$. Moreover, such a deformation is called universal at x^o if such an f is unique.

4.1 Local Universal Deformations

Theorem 8. Let $B_0^o, B_\infty \in M_d(\mathbb{C})$ and (E^o, ∇^o) be the trivial bundle of rank d on $\mathbb{P}^1$ with the connection matrix

$$\Omega^o = (\frac{B_0^o}{\tau} + B_\infty)\frac{d\tau}{\tau}$$

in the canonical basis. If the matrix B_0^o is regular, i.e., its all eigenvalues have one Jordan block, then there exists a germ of universal deformation of (E^o, ∇^o).

Proof: Inspired by the ideas of the theorem 7, we consider the system, near x^o in a manifold X,

$$dC = 0,\ [B_0, C] = 0, \text{ and } dB_0 + C = [B_\infty, C] \qquad (***)$$

with $B_0(x^o) = B_0^o$. Locally $dC(x) = 0$ can be solved by $C(x) = d\Gamma(x)$ with $\Gamma(x^o) = 0$. Hence the system is equivalent to

$$[B_0, d\Gamma] = 0 \text{ and } d(B_0 + \Gamma) = [B_\infty, d\Gamma].$$

Note the second condition is exactly $B_0 = B_0^o - \Gamma + [B_\infty, \Gamma]$ since $\Gamma(x^o) = 0$, the system reduces to

$$[B_0^o - \Gamma + [B_\infty, \Gamma], d\Gamma] = 0.$$

• The system above is integrable on $M_d(\mathbb{C})$:
Consider vectors

$$\xi_1 = \sum_{i,j} \xi_1^{ij} \frac{\partial}{\partial \gamma_{ij}} \text{ and } \xi_2 = \sum_{i,j} \xi_2^{ij} \frac{\partial}{\partial \gamma_{ij}}.$$

Then to show the system $[B_0, d\Gamma] = 0$ is integrable, it suffices to verify that for any ξ_1 and ξ_2 annihilated by $[B_0, d\Gamma]_{mn}$ for all m and n, $(d[B_0, d\Gamma])_{mn}(\xi_1, \xi_2) = 0$ for all m and n also. Thus first we calculate (remember $B_0 = B_0^o - \Gamma + [B_\infty, \Gamma]$)

$$d[B_0, d\Gamma] = dB_0 \wedge d\Gamma + d\Gamma \wedge dB_0 = -2d\Gamma \wedge d\Gamma + 2[B_\infty, d\Gamma \wedge d\Gamma].$$

Let $\Xi_k := (\xi_k^{ij})_{i,j}$ for $k = 1$ and 2. Then ξ_1 and ξ_2 are annihilated by all the entries of $[B_0, d\Gamma]$ if and only if

$$[B_0, \Xi_1] = [B_0, \Xi_2] = 0,$$

and likewise (ξ_1, ξ_2) is annihilated by all the entries of $d[B_0, d\Gamma]$ if and only if

$$-2[\Xi_1, \Xi_2] + 2[B_\infty, [\Xi_1, \Xi_2]] = 0.$$

So it amounts to proving

$$[B_0, \Xi_1] = [B_0, \Xi_2] = 0 \Rightarrow -2[\Xi_1, \Xi_2] + 2[B_\infty, [\Xi_1, \Xi_2]] = 0.$$

Since B_0^o is regular, for x near x^o we have $B_0(x) = B_0^o - \Gamma + [B_\infty, \Gamma]$ is also regular for Γ near 0. For these Γ, that $[B_0, \Xi_1] = [B_0, \Xi_2] = 0$ means Ξ_1 and Ξ_2 are both polynomials in B_0. So Ξ_1 and Ξ_2 also commute and the claim follows.

• The integrable deformation:
Let Y be the integral submanifold of the system going through 0. Let (F, ∇) be the trivial bundle of rank d on Y with the connection matrix

$$(\frac{B_0}{\tau} + B_\infty)\frac{d\tau}{\tau} + \frac{d\Gamma}{\tau}$$

where $B_0 := B_0^o - \Gamma + [B_\infty, \Gamma]$. By the construction it's an integrable deformation of (E^o, ∇^o).

• The deformation is universal:
Let $(E, \widetilde{\nabla})$ be an integrable deformation of (E^o, ∇^o) parametrized by (X, x^o). Since the problem is local, we may assume X is a simply connected open subset in $\mathbb{C}^n$ and E is trivial, by the rigidity of trivial bundle on P^1. Then by the same argument as that in the theorem 7, there exists a unique basis of E in which the matrix of $\widetilde{\nabla}$ takes the form

$$(\frac{B_0(x)}{\tau} + B_\infty)\frac{d\tau}{\tau} + \frac{C(x)}{\tau}$$

with $B_0(x^o) = B_0$ and $dC(x) = 0$ (the latter again by the integrability condition). Hence there's a holomorphic map $\Gamma\colon X \to M_d(\mathbb{C})$ such that $d\Gamma = C$ and $\Gamma(x^o) = 0$. Since (B_0, C) also satisfies the system (***), the image of Γ is contained in Y, and by the construction $(E, \widetilde{\nabla}) = \Gamma^*(F, \nabla)$. Hence the deformation is complete. Besides, by the uniqueness of the basis, such a map $\Gamma\colon X \to V$ is unique. Hence its a universal deformation. □

4.2 Global Universal deformations

Recall that we have proven
Theorem 3. Given $x^o = (x_1^o, \cdots, x_d^o) = \pi(\tilde{x}^o) \in X_d$ and $B_0^o := \mathrm{diag}(x_1^o, \cdots, x_d^o), B_\infty \in M_d(\mathbb{C})$, there exist a unique holomorphic bundle E on $\mathbb{P}^1 \times \widetilde{X}_d$ and a flat meromorphic connection ∇ with a pole of order 1 along $\{0\} \times \widetilde{X}_d$ and a logarithmic pole along $\{\infty\} \times \widetilde{X}_d$, such that
(1) the restriction (E^o, ∇^o) of (E, ∇) at $\tilde{x}^o$ has a global frame with respect to which the matrix representation of ∇^o is

$$(\frac{B_0^o}{\tau} + B_\infty)\frac{d\tau}{\tau};$$

(2) for any $\tilde{x} \in \widetilde{X}_d$, the eigenvalues of the residue endomorphism R_0 at $\tilde{x}$ are the components of $\pi(\tilde{x})$.

Now we want to use this to get a global universal deformation under a more restricted condition than the regularity.

Theorem 9 ([Mal83]). Let $B_0^o, B_\infty \in M_d(\mathbb{C})$. If $B_0^o = \mathrm{diag}(x_1^o, \cdots, x_d^o)$ has distinct eigenvalues, then there exist a hyperplane Θ of $\widetilde{X}_d$ and a unique aolution (B_0, C), meromorphic on $\widetilde{X}_d$ with poles along Θ, of the system (***) such that $B_0(x^o) = B_0^o$ and for $\tilde{x} \in \widetilde{X}_d \setminus \Theta$, $B_0(\tilde{x})$ has eigenvalues $x_1, \cdots, x_d$ where $\pi(\tilde{x}) = (x_1, \cdots, x_d)$.

Proof: First we get a bundle (E, ∇) satisfying the conditions in the theorem 3. Then by the theorem 7 of the Birkhoff's problem, we can get a hyperplane Θ and a solution (B_0, C) of (***) meromorphic along Θ. By the condition (2) of the theorem 3, the existence follows.

For the uniqueness, assume $(B_0(\widetilde{x}), C(\widetilde{x}))$ is a solution of (***) satisfying the conditions of the theorem. Let V be a small open neighborhood of $\widetilde{x}^o$, on which there is a solution $\Gamma(\widetilde{x})$ such that $d\Gamma = C$ and $\Gamma(\widetilde{x}^o) = 0$. If let $(Y, 0)$ be the local universal deformation of $(E^o, \nabla^o) := (E, \nabla)|_{\mathbb{P}^1 \times \{\widetilde{x}^o\}}$, whose existence is by the previous theorem 8 since B_0^o now is in particular regular, then Γ define a map

$$f\colon (V, \widetilde{x}^o) \to (M_d(\mathbb{C}), 0)$$

has image in Y.
On the other hand, the eigenvalues of $B_0 := B_0^o - \Gamma + [B_\infty, \Gamma]$ defines a holomorphic map $(Y, 0) \to (X_d, x^o)$, and by shrinking Y to a simply connected neighborhood of 0, we may assume it can be lifted to

$$g\colon (Y, 0) \to (\widetilde{X}_d, \widetilde{x}^o).$$

By the construction, since for each $\widetilde{x} \in \widetilde{X}_d$, the eigenvalues of B_0 are exactly the component of $\pi(\widetilde{x})$, we have

$$g \circ f = \mathrm{id}_V.$$

Then we get the relation of their tangent maps

$$dg_0 \circ df_{\widetilde{x}^o} = \mathrm{id}_{T_0 V}.$$

Since both $T_0 V$ and $T_{\widetilde{x}^o} \widetilde{X}_d$ have dimensions d, g and f are inverse to each other in some open neighborhoods of 0 and $\widetilde{x}^o$. Thus f is uniquely determined by g, i.e., Γ, and hence C and B_0 are also uniquely determined by g. $\square$

Now we show that for the bundle (E, ∇) given by the theorem 3 and any $\widetilde{x} \in \widetilde{X}_d \setminus \Theta$, (E, ∇) induces a universal deformation of its restriction $(E, \nabla)|_{\mathbb{P}^1 \times \{\widetilde{x}\}}$.
In fact, for any $\widetilde{x} \in \widetilde{X}_d \setminus \Theta$, since the bundle constructed from $(E, \nabla)|_{\mathbb{P}^1 \times \{\widetilde{x}^o\}}$ and that from $(E, \nabla)|_{\mathbb{P}^1 \times \{\widetilde{x}\}}$ are isomorphic due to the uniqueness, it suffices to prove the universality at $\widetilde{x}^o$.
Let (E', ∇', x') be an integrable deformation of (E^o, ∇^o). Then the residue R_0' of the former is regularly semisimple in a small neighborhood V' of x' since it's an open condition. Then its eigenvalues define uniquely a map

$$f\colon (V', x') \to (\widetilde{X}_d, \widetilde{x}^o).$$

Then we have $f^*(E, \nabla)$ and $(E', \nabla')|_{\mathbb{P}^1 \times V'}$ are isomorphic by the unique from the initial condition (E^o, ∇^o).

5 The Proofs of the Two Facts

Let X be a simply connected complex manifold.
Fact 1. Let $(E, \widetilde{\nabla})$ be a bundle on $D \times X$ with $\widetilde{\nabla}$ flat and having a logarithmic pole along $\{\infty\} \times X$, and (E_∞, ∇) be its restriction to $\{\infty\} \times X$. Then regarded as a section of the bundle $\mathrm{Hom}(E_\infty, E_\infty)$ equipped with the natural flat connection induced by ∇, R_∞ is a horizontal section.

Proof: Let

$$\Omega = \frac{1}{z_1} \Omega^1(z) dz_1 + \sum_{i \geq 2} \Omega^i(z) dz_i$$

be the connection matrix of $\widetilde{\nabla}$ with respect to a local frame, where ∞ is at $z_1 = 0$. Then the matrix of ∇R_∞ is

$$\sum_{i \geq 2} \left(\frac{\partial \Omega^1(z)}{\partial z_i} + [\Omega^i(z), \Omega^1(z)]\right)|_{z_1 = 0}.$$

On the other hand, we have

$$[\Omega^1(z), \Omega^i(z)] = [z_1(\frac{1}{z_1}\Omega^1(z)), \Omega^i(z)] = \frac{\partial \Omega^1(z)}{\partial z_i} - z^1 \frac{\partial \Omega^i}{\partial z^1}.$$

for $i \geq 2$. Thus for $z_1 = 0$, we have

$$[\Omega^1(z), \Omega^i(z)]|_{z_1=0} = \frac{\partial \Omega^1(z)}{\partial z_i}|_{z_1=0}$$

so $\nabla R_\infty = 0$. □

Fact 2. Let E be a bundle on $\mathbb{P}^1 \times X$ and π be the projection $\mathbb{P}^1 \times X \to X$. If for some $x^o \in X$, $E|_{\mathbb{P}^1 \times \{x^o\}}$ is trivial, then there exist canonical isomorphisms

$$\mathcal{E}(*\pi^{-1}\Theta) \simeq \pi^* i_0^* \mathcal{E}(*\pi^{-1}\Theta) \simeq \pi^* i_\infty^* \mathcal{E}(*\pi^{-1}\Theta)$$

where $i_0 \colon \{0\} \times X \hookrightarrow \mathbb{P}^1 \times X$, $i_\infty \colon \{\infty\} \times X \hookrightarrow \mathbb{P}^1 \times X$, and Θ is the nontriviality divisor.

Proof: First we apply a general fact.

Fact. If $\mathcal{F}$ is an $\mathcal{O}_X$-coherent sheaf, locally free of rank d on $X \setminus \Theta$, then $\mathcal{O}_X(*\Theta) \otimes_{\mathcal{O}_X} \mathcal{F}$ is locally free of rank d as an $\mathcal{O}_X(*\Theta)$-module.

Applying the fact to the sheaf $\pi_* \mathcal{E}$, for $x^o \in \Theta$, let $e_1, \cdots, e_d$ be a basis of $\pi_* \mathcal{E}(*\Theta)_{x^o} = (\mathcal{O}_X(*\Theta) \otimes_{\mathcal{O}_X} \pi_* \mathcal{E})_{x^o}$. Then for a sufficiently small neighborhood V of x^o, the basis defines an isomorphism

$$\mathcal{O}^d_{\mathbb{P}^1 \times V}(*\pi^{-1}(\Theta)) \xrightarrow{\sim} \mathcal{E}(*\pi^{-1}(\Theta))|_{\mathbb{P}^1 \times V}.$$

Since the original problem can be check locally, this isomorphism makes the conclusion clear. □

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