Final Report for "Topics on Frobenius Manifolds" (Spring, 2018)

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Topic: Gromov-Witten Invariants and Quantum Cohomology

Ref:

Main reference:

- Fulton, Pandharipande - Note on Stable Maps and Quantum Cohomology, 1997

Others reference:

Manin - Frobenius Manifolds, Quantum Cohomology and Moduli Spaces, AMS.

Kock, Vainsencher - An Invitation to Quantum Cohomology, Birkhäuser.

Quantum Cohomology at the Mittag-Leffler Institute (Ed. by Paolo Aluffi)

Plan:

I. Moduli Spaces of Stable Maps

II. Gromov-Witten Invariants

III. Quantum Cohomology

$\rightsquigarrow$ The construction of moduli space of stable maps will leave to the next two talks

I. Moduli Spaces of Stable Maps:

1. Moduli of Stable Curves:

a) M_g : moduli space of projective non-sing. curves C of genus g over $\mathbb{C}$ mod. auto.

$\rightsquigarrow$ quasi-projective algebraic variety with $\dim M_g = 3g-3$ ($g \geq 2$)

Crucial fact: When $g=0$, $C \simeq \mathbb{P}^1$ $\mathrm{Aut}(C) = PGL(2;\mathbb{C})$ i.e. Möbius transf.

When $g=1$, $C \simeq \mathbb{C}/\Lambda$ Then $\mathrm{Aut}(C)$ contains the translation $z \longmapsto z+a$

Only when $g \geq 2$, $|\mathrm{Aut}(C)| < \infty$

$\rightsquigarrow$ M_g exists only when $g \geq 2$ and it is an alge. var. with orbifold sing.

b) $M_{g,n}$: moduli space of proj. non-sing. irred. curves C of genus g with n distinct marked pt. $p_1, \ldots, p_n \in C$

$\rightsquigarrow$ quasi-projective var. with $\dim M_{g,n} = 3g-3+n$

For $g=0$, $M_{g,n}$ exists if $n \geq 3$. $g=1$, $M_{g,n}$ exists if $n \geq 1$.

e.g. $M_{0,3} = \{(\mathbb{P}^1, 0, 1, \infty)\}$ $M_{0,4} = \mathbb{P}^1 \setminus \{0,1,\infty\}$

c) Compactification of $M_{g,n}$:

Def: A quasi-stable genus g n-pointed curve is a proj., connected nodal curve with arithmetic genus g with n distinct non-sing. marked pt. $p_1, \ldots, p_n$ on C. If $|\mathrm{Aut}(C, \{p_i\})| < \infty$, we say C is stable

(nodal: singularity are loc. cpx-analytically (formally) isom. to $xy=0$)

$\rightsquigarrow$ Consider $\varphi \in \mathrm{Aut}(C, \{p_i\})$ Taking normalization $\tilde{C} \xrightarrow{\pi} C$, φ lifts uniquely to $\tilde{\varphi} \in \mathrm{Aut}(\tilde{C}, \{q_i\})$, where $q_i := \pi^{-1}(p_i)$ ($\because$ q_i are non-sing.) and $\tilde{\varphi}$ fixes irred. comp.

$\Rightarrow$ $\tilde{\varphi}$ preserving special pts = preimage of singular pts and marked pt.

Thus, the stability condition can be reformulate into:

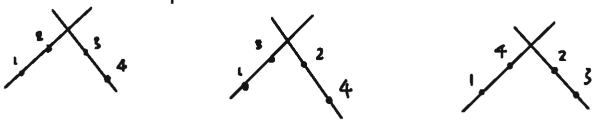
$(C, p_1, \ldots, p_n)$ is stable iff for each genus 0 component of $\tilde{C}$ has at least 3 special pts
genus 1 component of $\tilde{C}$ has at least 1 special pts

S: alge. scheme over $\mathbb{C}$ A family of (quasi-)stable over S is a flat, proj. morphism $\pi: \mathcal{C} \longrightarrow S$ with $\{p_i : S \longrightarrow \mathcal{C} \mid i=1,\ldots,n$, distinct section$\}$ s.t. $\forall s \in S$, $(\mathcal{C}_s, \{p_i(s)\})$ (geo. fiber) are (quasi-) stable curves

Thm: (Deligne-Mumford-Knudsen) $\exists$ coarse moduli space $\overline{M}_{g,n}$ of stable n-pointed genus g curve. $\overline{M}_{g,n}$: proj. variety of dim $3g-3+n$, and $M_{g,n} \underset{\text{open}}{\subseteq} \overline{M}_{g,n}$

Note that in the case of $g=0$, a stable n-pointed curve ($n \geq 3$) has no non-trivial automorphism $\longrightarrow$ $\overline{M}_{0,n}$ is a fine moduli space and is a non-sing. variety

In the genus 0 case, a n-pointed stable curve is just a tree of $\mathbb{P}^1$ satisfying stability condition. (Any two irred. components are either disjoint or intersects transversally at one pt) e.g. $\overline{M}_{0,3} = M_{0,3} = \{pt.\}$ $\overline{M}_{0,4} = \mathbb{P}^1$ The three added pts are represented by:

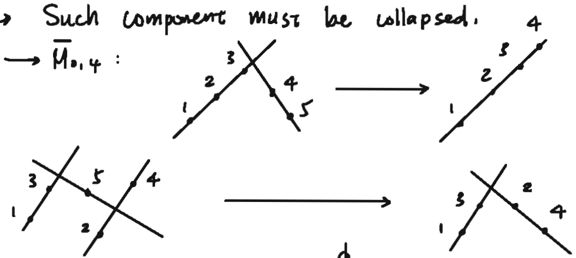


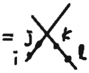
• Boundary of $\overline{M}_{0,n}$: $A \amalg B = \{1,\dots,n\}$: A partition of n with $|A|, |B| \geq 2$
$\rightsquigarrow$ A divisor $D(A|B)$. A generic pt. of $D(A|B)$ is represented by:

We use a finite set A to label the marked pt. For $B \subseteq A$ ($|B| \geq 3$),
we have a "forget points" morphism : $\overline{M}_{0,A} \longrightarrow \overline{M}_{0,B}$

On $M_{0,A}$, this is obvious. However, on the boundary $\overline{M}_{0,A} \setminus M_{0,A}$, the existence of this morphism not trivial since removing some pts may make a component unstable $\rightsquigarrow$ Such component must be collapsed.

e.g. $\overline{M}_{0,5} \longrightarrow \overline{M}_{0,4}$:



$\forall \{i,j,k,l\} \subseteq \{1,\dots,n\}$, we have : $\overline{M}_{0,n} \xrightarrow{\phi} \overline{M}_{0,\{i,j,k,l\}}$ Then $P(i,j|k,l) =$  $\in \overline{M}_{0,\{i,j,k,l\}}$

$\phi^{-1}(P(i,j|k,l))$ is a divisor on $\overline{M}_{0,n}$.

Fact (1) $\phi^{-1}(P(i,j|k,l)) = \sum D(A|B)$ over all partition $A \amalg B = \{1,\dots,n\}$ with $i,j \in A$, $k,l \in B$

Since $\overline{M}_{0,\{i,j,k,l\}} \simeq \mathbb{P}^1$ and on $\mathbb{P}^1$, any two points are linearly equiv. (Hartshorne, II.6.10.1)

$\Rightarrow$ The three boundary point in $\overline{M}_{0,\{i,j,k,l\}}$ are linearly equiv.

$\Rightarrow$ Their preimage are linearly equiv.

Thus, we have:

$$\sum_{\substack{i,j \in A \\ k,l \in B}} D(A|B) = \sum_{\substack{i,k \in A \\ j,l \in B}} D(A|B) = \sum_{\substack{i,l \in A \\ j,k \in B}} D(A|B) \quad \text{in } A^1(\overline{M}_{0,n}) \quad — (*)$$

Rmk: Keel proves that $D(A|B)$ generates the Chow ring together with (*) and $D(A|B) \cdot D(A'|B') = 0$ if there are no inclusion among A, B, A', B'

$\rightsquigarrow$ Gives a complete set of relations.

2. Stable Maps:

<u>Def</u>: X: non-sing. proj. var. $\beta \in H_2(X;\mathbb{Z})$ S: alge. scheme over $\mathbb{C}$

(1) A family of maps over S from n-pointed genus g curves to X consists of

(a) $\pi: C \longrightarrow S$ with sections $\{p_1, \dots, p_n\}$ is a family of quasi-stable curve over S

(b) A morphism $\mu: C \longrightarrow X$.

(2) Two families of maps over S: $(C \xrightarrow{\pi} S, \{p_i\}, \mu)$, $(C' \xrightarrow{\pi'} S, \{p_i'\}, \mu')$ are isom. if $\exists\ \tau: C \xrightarrow{\sim} C'$ s.t.

(3) $(C, \{p_i\}, \mu)$: maps from a n-pointed, quasi-stable curve to X

Automorphism of the maps is $\tau: C \xrightarrow{\sim} C$ s.t. $\mu \circ \tau = \mu$, $p_i = \tau(p_i)$

We say $(C, \{p_i\}, \mu)$ represents β if $\mu_*([C]) = \beta \in H_2(X, \mathbb{Z})$

We say $E \subset C$ is contracted by μ if E is mapped to a pt by μ.

<u>Def</u>: $(C, \{p_i\}, \mu)$ is stable if it satisfies the following equiv. condition

(i) $|Aut(C, \{p_i\}, \mu)| < \infty$

(ii) If $E \subset C$: irred. component contracted by μ, then $(E, \{\text{special pts on } E\})$ is a stable curve (i.e. If $E \simeq \mathbb{P}^1 / p_a(E) = 1$, E has at least $3/1$ special pts)

(i) $\Rightarrow$ (ii) If $\exists$ an unstable component E contracted by μ Then $|Aut(E, \{\text{special pt on } E\})| = \infty$

Then extends these automorphism to whole C by setting identity on other component

$\because$ E is sent to a pt. $\therefore$ These automorphism commutes w/ μ $\Rightarrow$ $\times$

(ii) $\Rightarrow$ (i) Suppose if we have an unstable irred. component E, then E must not be contracted by μ (def.) $\phi \in Aut(C, \{p_i\}, \mu)$ $E' = \phi(E)$ $\mu|_{E'} \circ \phi|_E = \mu|_E$

$\because$ X is proj. $\therefore$ May assume $X = \mathbb{P}^N$ $\mu: E \longrightarrow \mathbb{P}^N$ non-const.

$\because$ $\dim E = 1$, E irred. $\Rightarrow$ $\mu(E)$ has at most dim 1, irred. If $\dim \mu(E) = 0$, E is contracted by μ

If $\dim \mu(E) = 1$, then $\because$ $\forall p \in \mu(E)$, $\mu^{-1}(p)$ is proper closed subset of $E \Rightarrow \mu^{-1}(p)$ is finite ($\because$ E is proper)

Also, $\mu: E \longrightarrow \mathbb{P}^N$ is projective $\Rightarrow$ μ is finite Therefore, μ is branched cover on $\mu(E)$ $\Rightarrow$ Such $\phi|_E$ must be finite.

(4) A family of maps $(\pi: C \longrightarrow S, \{p_i\}, \mu)$ is stable if $\forall s \in S$, the geometric fiber $(C_s, \{p_i(s)\}) \longrightarrow X$ is stable.

We define a moduli functor $\overline{\mathcal{M}}_{g,n}(X, \beta)$: ($\mathbb{C}$-alge. scheme) $\longrightarrow$ (Set) by

$\overline{\mathcal{M}}_{g,n}(X, \beta)(S) = \{$ Isom. classes of familes of stable map of n-pointed, genus g curves over S to X reprecenting the class $\beta \}$

Thm 1 : $\exists$ projective, coarse moduli space $\overline{M}_{g,n}(X,\beta)$

i.e. $\overline{M}_{g,n}(X,\beta)$ is a scheme with natural transf. $\phi : \overline{\mathcal{M}}_{g,n}(X,\beta) \longrightarrow h_{\overline{M}_{g,n}(X,\beta)}$

s.t. (1) $\phi(\operatorname{Spec}\mathbb{C}) : \overline{\mathcal{M}}_{g,n}(X,\beta)(\operatorname{Spec}\mathbb{C}) \xrightarrow{\sim} \operatorname{Hom}(\operatorname{Spec}\mathbb{C}, \overline{M}_{g,n}(X,\beta))$

(2) Z: scheme $\psi : \overline{\mathcal{M}}_{g,n}(X,\beta) \to h_Z$, then $\exists!\ \gamma : \overline{M}_{g,n}(X,\beta) \longrightarrow Z$ s.t.

$\psi = \overline{\gamma} \circ \phi$ (where $\overline{\gamma} : h_{\overline{M}_{g,n}(X,\beta)} \longrightarrow h_Z$ defined by $f \mapsto \gamma \circ f$)

$\overline{M}^*_{g,n}(X,\beta) \subset \overline{M}_{g,n}(X,\beta)$: automorphism-free stable maps

$M^*_{g,n}(X,\beta) := \overline{M}^*_{g,n}(X,\beta) \cap M_{g,n}(X,\beta)$

Now, we focus on the case of moduli spaces of genus 0, n-pointed stable curves :

Def : A non-singular proj. variety X is convex if $\forall\ \mu : \mathbb{P}^1 \longrightarrow X$,

$H^1(\mathbb{P}^1, \mu^* T_X) = 0$, where T_X : tangent sheaf of X.

Thm 2 : X : proj. non-sing. convex variety

(i) $\overline{M}_{0,n}(X,\beta)$ is a normal projective variety (not necessarily irreducible)

of pure dim. $= \dim X + \int_\beta c_1(T_X) + n - 3$

(ii) $\overline{M}_{0,n}(X,\beta)$ is locally quot. of a non-sing. var. by finite group

(iii) $\overline{M}^*_{0,n}(X,\beta)$ is a non-sing., fine moduli space (for automorphism-free stable maps)

Def : The boundary of $\overline{M}_{0,n}(X,\beta)$ consists of $\{(C, \{p_i\}, \mu)\}$ with C reducible.

Thm 3 : X : non-sing., proj., convex variety The boundary of $\overline{M}_{0,n}(X,\beta)$ is a normal crossing divisor up to a finite gp quot.

$\searrow$ D : effective Cartier divisor D is simple normal crossing at p if X is reg. at p and $\exists$ nbd U and local coord. $x_1, \ldots, x_n$ (i.e. $\{\overline{x}_1, \ldots, \overline{x}_n\}$ forms a $k(p)$-basis of m_p / m_p^2)

s.t. $U \cap \operatorname{supp}(D) \subset (x_1 \cdots x_n = 0)$.

π : étale (smooth of rel. dim $= 0$) and $\pi(p') = p$

D is normal crossing at p if $\exists$ étale nbd $\pi : (X', p') \longrightarrow (X, p)$ s.t.

$\pi^{-1} D$ is snc at p.

Convention : $X = \mathbb{P}^r$, we write $\overline{M}_{g,n}(\mathbb{P}^r, d)$ instead of $\overline{M}_{g,n}(\mathbb{P}^r, d[\text{line}])$

Example : • When $\beta = 0$, then $\mu : C \longrightarrow X$ is a const. map $\Rightarrow \overline{M}_{0,n}(X, 0) \simeq \overline{M}_{0,n} \times X$

In particular, when $X = \{pt.\} \Rightarrow \beta = 0$. $\overline{M}_{0,n}(\{pt.\}, 0) \simeq \overline{M}_{0,n}$

• When X contains no rational curves, $\overline{M}_{0,n}(X,\beta) = \phi$ unless $\beta = 0$.

• $\overline{M}_{0,0}(\mathbb{P}^r, 1) = G(1, r)$

• $\overline{M}_{0,1}(\mathbb{P}^r, 1) =$ tautological line bundle over $G(1, r)$.

- Natural morphism :

(1) evaluation map : $\xi = (C \xrightarrow{\pi} S, \{p_i\}, \mu) \in \overline{M}_{g,n}(X,\beta)(S)$, define $\theta_i : \overline{M}_{g,n}(X,\beta) \to h_X$ by :

$$\begin{tikzcd} C \arrow[r, "\mu"] \arrow[d, "\pi"'] & X \\ S \arrow[u, bend left, "p_1,\dots,p_n"] & \end{tikzcd} \longmapsto S \xrightarrow{p_i} C \xrightarrow{\mu} X$$

By thm 1 $\Rightarrow \exists!\ ev_i : \overline{M}_{g,n}(X,\beta) \longrightarrow X$ s.t. $\theta_i = \overline{ev_i} \circ \phi$

Put $S = \operatorname{Spec} \mathbb{C} \Rightarrow ev_i : \overline{M}_{g,n}(X,\beta) \longrightarrow X$

$$[(C, p_1, \dots, p_n, \mu)] \mapsto \mu(p_i)$$

(2) forgetful map : $\because \overline{M}_{g,n}$ is a coarse moduli space of n-pointed genus g stable curve (for $2g-2+n>0$)

Given $\begin{tikzcd} C \arrow[r, "\mu"] \arrow[d, "\pi"'] & X \\ S \arrow[u, bend left, "p_1,\dots,p_n"] & \end{tikzcd}$ in $\overline{M}_{g,n}(X,\beta)(S) \Rightarrow \exists$ morphism $S \longrightarrow \overline{M}_{g,n}$

By universal property of $\overline{M}_{g,n}(X,\beta)$

$\Rightarrow \eta : \overline{M}_{g,n}(X,\beta) \longrightarrow \overline{M}_{g,n}$

<u>Rmk</u>: For X : homogeneous , Kim - Pandharipande proves that $\overline{M}_{0,n}(X,\beta)$ is in fact irreducible.

II. Gromov-Witten Invariants:

X: homogeneous variety i.e. $X = G/P$, where G: linear algebraic gp P: parabolic subgp of G

<u>Lemma 1</u>: X is a non-sing. proj. convex variety

<u>pf</u>: $\because G \curvearrowright X$ transitively $\therefore T_X$ is generated by global section.

Then $\mu^*(T_X)$ is g.b.g.s. for any morphism $\mu: \mathbb{P}^1 \longrightarrow X$

i.e. $\exists \{s_1, \ldots, s_n\} \in \Gamma(\mathbb{P}^1, \mu^* T_X)$ s.t. $\bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}\langle s_i \rangle \longrightarrow \mu^* T_X \longrightarrow 0$

Let $R := \ker\left(\bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}\langle s_i\rangle \longrightarrow \mu^* T_X\right)$

$\rightsquigarrow \quad 0 \longrightarrow R \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mu^* T_X \longrightarrow 0$ induces long exact seq. in cohomology

$\Rightarrow 0 \to H^0(\mathbb{P}^1, R) \to H^0(\mathbb{P}^1, \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}) \to H^0(\mathbb{P}^1, \mu^* T_X) \to H^1(\mathbb{P}^1, R)$

$\to H^1(\mathbb{P}^1, \bigoplus_i \mathcal{O}_{\mathbb{P}^1}) \longrightarrow H^1(\mathbb{P}^1, \mu^* T_X) \longrightarrow 0$

Note that $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0 \Rightarrow H^1(\mathbb{P}^1, \mu^* T_X) = 0$

Or, we can use Grothendieck splitting thm $\Rightarrow \mu^* T_X = \bigoplus \mathcal{O}_{\mathbb{P}^1}(d_i)$ Then $\mu^* T_X$ is g.b.g.s. $\Rightarrow \forall d_i \geq 0 \Rightarrow H^1(\mathbb{P}^1, \mu^* \mathbb{P}^1) = 0$ $\square$

Properties for homogeneous variety X:

(1) X: non-sing. projective and convex (lemma 1)

(2) The Chow theory and topological homology theory are isomorphic

(i.e. $A_\cdot(X) \simeq H_{2\cdot}(X; \mathbb{Z})$, $A^\cdot(X) \simeq H^{2\cdot}(X; \mathbb{Z})$)

(3) $\beta \in H_2(X; \mathbb{Z})$ effective class if β is represented by some genus 0 pointed stable map

Then the effective classes of X is a cone:

$$\left\{ \sum_{i=1}^{n} a_i \beta_i \;\middle|\; a_i \in \mathbb{Z}_{\geq 0},\ \beta_i \text{ is of the form } \iota_*[\mathbb{P}^1], \text{ for some } \iota: \mathbb{P}^1 \hookrightarrow X \right\}$$

<u>Notation</u>: $c \in H^*(X; \mathbb{Z})$, $\beta \in H_k(X; \mathbb{Z})$ $\int_\beta c :=$ deg. k component c_k evaluate at β

When $\beta = [V]$, $V \subseteq X$: pure-dim'l closed subvar., we denote $\int_V c$

Recall from I, we have evaluation maps: $ev_i: \overline{M}_{g,n}(X, \beta) \longrightarrow X \quad (i = 1, \ldots, n)$

Given $\gamma_1, \ldots, \gamma_n \in H^*(X, \mathbb{Z})$, $ev_1^*(\gamma_1) \cup \cdots \cup ev_n^*(\gamma_n) \in H^*(\overline{M}_{g,n}(X, \beta), \mathbb{Z})$

<u>Def</u>: n-pointed, genus 0 Gromov-Witten invariant is defined by:

$$I_\beta(\gamma_1, \ldots, \gamma_n) = \int_{\overline{M}_{0,n}(X,\beta)} ev_1^*(\gamma_1) \cup \cdots \cup ev_n^*(\gamma_n) \in \mathbb{Z}$$

<u>Rmk</u>: (1) If γ_i are homogeneous, then $I_\beta(\gamma_1, \ldots, \gamma_n) \neq 0$ only if $\sum \deg(\gamma_i) = \dim \overline{M}_{0,n}(X, \beta)$

(2) Given permutation $\sigma \in S_n$, $\overline{M}_{g,n}(X, \beta)(S) \longrightarrow \overline{M}_{g,n}(X, \beta)(S)$ induces via ϕ

$$\begin{array}{ccc} C \xrightarrow{\mu} X & & C \xrightarrow{\mu} X \\ \pi \downarrow \uparrow p_1, \ldots, p_n & \longmapsto & \pi \downarrow \uparrow p_{\sigma(1)}, \ldots, p_{\sigma(n)} \\ S & & S \end{array}$$

an automorphism $\overline{M}_{0,n}(X,\beta) \circlearrowleft \Sigma(\sigma) \Rightarrow ev_{\sigma(i)} = ev_i \circ \Sigma(\sigma)$

Therefore, $I_\beta(\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(n)}) = \int_{\overline{M}_{0,n}(X,\beta)} ev_1^*(\gamma_{\sigma(1)}) \cup \cdots \cup ev_n^*(\gamma_{\sigma(n)})$

$= \int_{\overline{M}_{0,n}(X,\beta)} (\Sigma(\sigma)^{-1})^* \left(ev_{\sigma(1)}^*(\gamma_{\sigma(1)}) \cup \cdots \cup ev_{\sigma(n)}^*(\gamma_{\sigma(n)})\right) = \int_{\overline{M}_{0,n}(X,\beta)} ev_1^*(\gamma_1) \cup \cdots \cup ev_n^*(\gamma_n)$

$\Rightarrow I_\beta(\gamma_1, \ldots, \gamma_n)$ is indep. of the permutation of γ_i's.

$\rightsquigarrow$ We may write GW inv. as $I_\beta(\gamma_1 \cdots \gamma_n)$ rather than $I_\beta(\gamma_1, \ldots, \gamma_n)$.

(3) 0-pointed invariant:

When $n=0$, we have 0-pointed GW inv. only when $\overline{M}_{0,0}(X,\beta)$ is zero-dim'l

By Thm 2 $\Rightarrow \dim X + \int_\beta c_1(T_X) = 3$

If $\dim X = 0$, then $\int_\beta c_1(T_X) = 0 \Rightarrow$ Impossible.

Now, if $\dim X > 0$, if $\beta = 0$, then $\overline{M}_{0,0}(X,0) = \phi$ ($\because$ Every map is const.
$\Rightarrow$ Every irred. component is contracted and there is no marked pt. to stablize them)

So, assume $\beta \neq 0$, we need following lemma:

<u>Lemma 2</u>: $\mu : \mathbb{P}^1 \longrightarrow X$ non-const morphism to a non-sing. proj., convex var.
, then $\int_{\mu_*[\mathbb{P}^1]} c_1(T_X) \geq 2$.

pf: Consider $d\mu : T_{\mathbb{P}^1} \longrightarrow \mu^* T_X \quad \because T_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(2) = \{$ homogeneous poly. of deg 2$\}$

$\rightsquigarrow$ We can pick a generic $s \in H^0(\mathbb{P}^1, T_{\mathbb{P}^1})$ s.t. $s(p_1) = s(p_2) = 0 \quad p_1, p_2 \in \mathbb{P}^1$ distinct

$\because \mu$ is non-const. $\therefore d\mu \neq 0 \Rightarrow d\mu(s) \in H^0(\mathbb{P}^1, \mu^* T_X) \neq 0$
and $d\mu(s)$ vanishes at least at p_1 and p_2.

Now, $\mu^* T_X \simeq \bigoplus \mathcal{O}_{\mathbb{P}^1}(\alpha_i) \quad \alpha_i \in \mathbb{Z}$ If $\alpha_j < 0$ for some j, then if $\alpha_j \leq -2$
$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\alpha_j)) \neq 0 \Rightarrow H^1(\mathbb{P}^1, \mu^* T_X) \neq 0 \Rightarrow \times$ If $\alpha_j = -1$, then consider
$\tilde{\mu} : \mathbb{P}^1 \xrightarrow[2:1]{} \mathbb{P}^1 \xrightarrow[\mu]{} X$, then $H^1(\mathbb{P}^1, \tilde{\mu}^* T_X) \neq 0 \Rightarrow \times$. Thus, $\alpha_j \geq 0 \ \forall j$

Then existence of $d\mu(s) \in H^0(\mathbb{P}^1, \mu^* T_X) \neq 0 \Rightarrow \exists j$ s.t. $\alpha_j \geq 2$

$\Rightarrow \int_{\mu_*[\mathbb{P}^1]} c_1(T_X) = \int_{\mathbb{P}^1} \mu^*(c_1(T_X)) = \int_{\mathbb{P}^1} c_1(\mu^* T_X) \geq \int_{\mathbb{P}^1} c_1(\mathcal{O}_{\mathbb{P}^1}(2)) = 2$ $\square$

By lemma 2 $\Rightarrow$ Only possibility is $\dim X = 1$, $\int_\beta c_1(T_X) = 2$

Hence, for homogeneous variety, 0-pointed invariants occur only $X \simeq \mathbb{P}^1$
$\Rightarrow I_1 = 1$. The unique 0-pointed invariants

<u>Lemma 3</u>: If $n\geq 1$, then $M^*_{0,n}(X,\beta) \subset \overline{M}_{0,n}(X,\beta)$ is a dense open set

pf: If $\beta=0$, then $\overline{M}_{0,n}(X,0)\neq\phi$ only if $n\geq 3$.

For $n\geq 3$, $\overline{M}_{0,n}(X,0)\simeq \overline{M}_{0,n}\times X$ Then $\overline{M}^*_{0,n}(X,0)=\overline{M}^*_{0,n}\times X=\overline{M}_{0,n}\times X$ $=\overline{M}_{0,n}(X,0)$. $\Rightarrow$ $M^*_{0,n}(X,0) \underset{\text{dense}}{\subset} \overline{M}_{0,n}(X,0)$.

If $\beta\neq 0$, by thm 3 $\Rightarrow$ $M_{0,n}(X,\beta) \underset{\text{open}}{\overset{\text{dense}}{\subset}} \overline{M}_{0,n}(X,\beta)$

$[(\mathbb{P}^1,\{p_i\},\mu)]\in M_{0,n}(X,\beta)$ $\because$ $\beta\neq 0$ $\therefore$ μ is non-const.

Let $A\subset PGL(2;\mathbb{C})$ s.t. $\forall g\in A$, $g\circ\mu=\mu$. Then $|A|<\infty$

$\Rightarrow$ $\exists\, U \underset{\text{open}}{\subseteq} \mathbb{P}^1$ s.t. $U=\{x\in\mathbb{P}^1 \mid \mathrm{Stab}_A x=\{id\}\}$

Take $p_1',\ldots\ldots,p_n'\in U$ $\Rightarrow$ $(\mathbb{P}^1,\{p_i'\},\mu)$ is automorphism-free

$\Rightarrow$ For generic marked pts $\{p_i'\}\subseteq\mathbb{P}^1$, $(\mathbb{P}^1,\{p_i'\},\mu)$ is auto-free

$\Rightarrow$ $M^*_{0,n}(X,\beta) \underset{\text{dense}}{\overset{\text{open}}{\subset}} M_{0,n}(X,\beta)$ $\square$

Now, say $\Gamma_1,\ldots\ldots,\Gamma_n$: pure dimensional subvar. of X $[\gamma_i]$: corresponding class in $H^*(X;\mathbb{Z})$ via Poincaré duality. Then $\deg(\gamma_i)=\mathrm{codim}(\Gamma_i)$

Assume $\sum_{i=1}^n \mathrm{codim}(\Gamma_i)=\dim(X)+\int_\beta c_1(T_X)+n-3$

<u>Lemma 4</u>: For $n\geq 0$, $g_1,\ldots\ldots,g_n$: generic element of G, the scheme theoretic intersection $ev_1^{-1}(g_1\Gamma_1)\cap\ldots\ldots\cap ev_n^{-1}(g_n\Gamma_n)$ is a finite set of reduced pts supported in $M_{0,n}(X,\beta)$, and we have:

$$I_\beta(\gamma_1,\ldots\ldots,\gamma_n)=\#\left(ev_1^{-1}(g_1\Gamma_1)\cap\ldots\ldots\cap ev_n^{-1}(g_n\Gamma_n)\right) \quad (*)$$

pf: If $n=0$, then $I_1=1$ on $\mathbb{P}^1$, and $\overline{M}_{0,0}(\mathbb{P}^1,1)=M^*_{0,0}(\mathbb{P}^1,1)=\{pt.\}$

If $n\geq 1$, $G^n:=G\times\cdots\cdots\times G$ $X^n:=X\times\cdots\cdots\times X$ Then $G^n \curvearrowright X^n$ transitive

Then we have $ev:=(ev_1,\ldots\ldots,ev_n):\overline{M}_{0,n}(X,\beta)\longrightarrow X^n$

<u>Step 1</u>: apply Kleiman's thm to:

$$\begin{array}{ccc} & & \overline{M}_{0,n}(X,\beta) \\ & & \big\downarrow ev \\ \Gamma:=\Gamma_1\times\cdots\cdots\times\Gamma_n & \hookrightarrow & X^n \end{array}$$

Then note that if $\dim(\Gamma^\sigma\times_{X^n}\overline{M}^{0,n}(X,\beta))=\dim\Gamma+\overline{M}^{0,n}(X,\beta)-\dim(X^n)$

$=\dim\overline{M}^{0,n}(X,\beta)-\sum_{i=1}^n \mathrm{codim}(\Gamma_i)=0$

$\Rightarrow$ $\exists\, V_1 \underset{\text{open}}{\subseteq} G^n$ s.t. $\forall\sigma\in V_1$, $\Gamma^\sigma\times_{X^n}\overline{M}^{0,n}(X,\beta)$ is either ϕ or dim 0.

<u>Step 2</u>: consider $M^*_{0,n}(X,\beta)^c:=\overline{M}_{0,n}(X,\beta)\setminus M^*_{0,n}(X,\beta)$, whose codim ≥ 1

Apply Kleiman's thm to:

$$\begin{array}{ccc} & & M^*_{0,n}(X,\beta)^c \\ & & \big\downarrow \\ \Gamma & \hookrightarrow & X^n \end{array}$$

Then for $V_2 \underset{\text{open}}{\subset} G^n$, $\sigma \in V_2$, if $\Gamma^\sigma \times_{X^n} M^*_{0,n}(X,\beta)^c \neq \phi$, then $\dim(\Gamma^\sigma \times_{X^n} M^*_{0,n}(X,\beta)^c)$
$= \dim\Gamma + \dim M^*_{0,n}(X,\beta)^c \setminus \dim(X^n) = \dim(M^*_{0,n}(X,\beta)^c) \setminus \sum_{i=1}^n \operatorname{codim}(\Gamma_i) < 0$
$\Rightarrow \Gamma^\sigma \times_{X^n} M^*_{0,n}(X,\beta)^c = \phi$ for $\sigma \in V_2$

Thus, for $(g_1, \dots, g_n) \in V_1 \cap V_2 \underset{\text{open}}{\overset{\text{dense}}{\subset}} G^n$, $\bigcap_{i=1}^n ev_i^{-1}(g_i\Gamma_i)$ is a 0-dim'l and supported in $M^*_{0,n}(X,\beta)$

<u>Step 3</u>: Let $\Gamma_i' := \operatorname{sing}(\Gamma_i)$: sing. locus of Γ_i, $\Gamma' := (\Gamma_i')_{i=1}^n$

Next, we apply Kleiman's thm again to
$$\begin{array}{c} M^*_{0,n}(X,\beta) \\ \downarrow ev \\ \Gamma' \hookrightarrow X^n \end{array}$$
, and counting dim.

$\Rightarrow \exists V_3 \underset{\text{open}}{\subset} G^n$ s.t. $\bigcap_{i=1}^n ev_i^{-1}(g_i\Gamma_i') = \phi$ or $\dim < 0$ for $(g_i)_{i=1}^n \in V_3$.

Thus, we may assume Γ_i are non-sing.

<u>Step 4</u>: Finally, we apply Kleiman's thm to
$$\begin{array}{c} M^*_{0,n}(X,\beta) \\ \downarrow \\ \Gamma \hookrightarrow X^n \end{array}$$
$\Rightarrow \bigcap_{i=1}^n ev_i^{-1}(g_i\Gamma_i)$ is regular

$\Rightarrow$ The intersection is a finite set of reduced pts supp. in $M^*_{0,n}(X,\beta)$

<u>Step 5</u>: Observe that we have the fibered product:

$$\begin{tikzcd} \bigcap_{i=1}^n ev_i^{-1}(g_i\Gamma_i) \arrow[r] \arrow[d] & \overline{M}_{0,n}(X,\beta) \times \prod_{i=1}^n g_i\Gamma_i \arrow[d] \\ \overline{M}_{0,n}(X,\beta) \arrow[r, "\iota = (id, ev)"] & \overline{M}_{0,n}(X,\beta) \times X^n \end{tikzcd}$$

$\Rightarrow [\bigcap_{i=1}^n ev_i^{-1}(g_i\Gamma_i)] = \iota^*[\overline{M}_{0,n}(X,\beta) \times \prod_{i=1}^n g_i\Gamma_i] = (id, ev)^*[\overline{M}_{0,n}(X,\beta) \times \prod_{i=1}^n g_i\Gamma_i]$

$= [\overline{M}_{0,n}(X,\beta)] \cap [\prod_{i=1}^n ev_i^*(g_i\Gamma_i)]$ in $A_0(\overline{M}_{0,n}(X,\beta))$

$= I_\beta(\gamma_1, \dots, \gamma_n)$ □

• Three Basic Properties:

(I) $\beta = 0$, $\overline{M}_{0,n}(X,\beta) \simeq \overline{M}_{0,n} \times X$ and $ev_i = p : \overline{M}_{0,n} \times X \longrightarrow X$ for $i = 1, \dots, n$

$ev_1^*(\gamma_1) \cup \dots \cup ev_n^*(\gamma_n) = p^*(\gamma_1 \cup \dots \cup \gamma_n)$

$I_\beta(\gamma_1, \dots, \gamma_n) = \int_{\overline{M}_{0,n} \times X} p^*(\gamma_1 \cup \dots \cup \gamma_n) = \int_{p_*[\overline{M}_{0,n} \times X]} \gamma_1 \cup \dots \cup \gamma_n$

For $n = 0, 1, 2$, $\overline{M}_{0,n} = \phi$. If $n > 3$, then $\dim \overline{M}_{0,n} > 0 \Rightarrow p_*([\overline{M}_{0,n} \times X] = 0$

Only case: $n = 3$, $\overline{M}_{0,n} = \{pt\}$

$\Rightarrow I_0(\gamma_1, \gamma_2, \gamma_3) = \int_X \gamma_1 \cup \gamma_2 \cup \gamma_3$

(II). $\gamma_1 = 1 \in H^0(X)$

If $\beta \neq 0$, $ev_1^*(\gamma_1) \cup \cdots \cup ev_n^*(\gamma_n) = ev_2^*(\gamma_2) \cup \cdots \cup ev_n^*(\gamma_n)$

$\Rightarrow \exists\, \omega \in H^{\cdot}(\overline{M}_{0,n-1}(X,\beta))$ s.t. $ev_2^*(\gamma_2) \cup \cdots \cup ev_n^*(\gamma_n) = \phi_1^*\omega$,

where $\phi_1 : \overline{M}_{0,n}(X,\beta) \longrightarrow \overline{M}_{0,n-1}(X,\beta)$ forget one point

$$[(C, p_1, \ldots, p_n, \mu)] \longmapsto [(C, p_2, \ldots, p_{n-1}, \mu)]$$

$$I_\beta(1, \gamma_2, \ldots, \gamma_n) = \int_{\overline{M}_{0,n}(X,\beta)} \phi_1^*\omega = \int_{(\phi_1)_*([\overline{M}_{0,n}(X,\beta)])} \omega$$

$\because \dim \overline{M}_{0,n}(X,\beta) = 3g-3+n > \dim \overline{M}_{0,n-1} = 3g-3+(n-1)$

$\therefore (\phi_1)_*([\overline{M}_{0,n}(X,\beta)]) = 0$

$\Rightarrow I_\beta(1, \gamma_2, \ldots, \gamma_n) = 0$

If $\beta = 0$, we go back to case (I), where $n=3$

$$\Rightarrow I_0(1, \gamma_2, \gamma_3) = \int_X \gamma_2 \cup \gamma_3$$

(III) $\gamma_1 \in H^2(X)$, $\beta \neq 0$, then we have:

$$I_\beta(\gamma_1, \ldots, \gamma_n) = \left(\int_\beta \gamma_1\right) \cdot I_\beta(\gamma_2, \ldots, \gamma_n)$$

pf: Consider $\psi : \overline{M}_{0,n}(X,\beta) \longrightarrow X \times \overline{M}_{0,n-1}(X,\beta)$ by $\psi = ev_1 \times \phi_1$, ϕ_1: forget first marked pt.

Since X is homogeneous, $A_*(X \times \overline{M}_{0,n-1}(X,\beta)) \cong A_*(X) \otimes A_*(\overline{M}_{0,n-1}(X,\beta))$

Write $\psi_*([\overline{M}_{0,n}(X,\beta)]) = \beta' \times [\overline{M}_{0,n-1}(X,\beta)] + \alpha$, $\beta' \in A_1(X)$

α: Some homology class supported over a proper closed subset of $\overline{M}_{0,n-1}(X,\beta)$

$$I_\beta(\gamma_1, \ldots, \gamma_n) = \int_{[\overline{M}_{0,n}(X,\beta)]} ev_1^*\gamma_1 \cup \cdots \cup ev_n^*(\gamma_n) = \int_{\psi_*([\overline{M}_{0,n}(X,\beta)])} \gamma_1 \times (ev_2^*(\gamma_2) \cup \cdots ev_n^*(\gamma_n))$$

$$= \int_{\beta' \times [\overline{M}_{0,n-1}(X,\beta)] + \alpha} \gamma_1 \times (ev_2^*(\gamma_2) \cup \cdots \cup ev_n^*(\gamma_n)) = \left(\int_{\beta'} \gamma_1\right) \cdot \int_{\overline{M}_{0,n-1}(X,\beta)} ev_2^*\gamma_2 \cup \cdots \cup ev_n^*\gamma_n + \int_\alpha \gamma_1 \times (\underbrace{\cancel{ev_2^*}}_{0}(\gamma_2) \cup \cdots \cup ev_n^*(\gamma_n))$$

$$= \left(\int_{\beta'} \gamma_1\right) \cdot I_\beta(\gamma_2, \ldots, \gamma_n)$$

<u>Claim</u>: $\beta' = \beta$.

Consider a generic pt. $\{s = [(\overset{\mathbb{P}^1}{\overset{\wr}{C}}, \{p_2, \ldots, p_n, \mu\})]\} \in \overline{M}_{0,n-1}(X,\beta)$

$$\begin{tikzcd}
C \arrow[r, "j"] \arrow[d, "\mu \times pt."] & \overline{M}_{0,n}(X,\beta) \arrow[d, "\psi"] \\
X \times \{s\} \arrow[r, "j"] \arrow[d] & X \times \overline{M}_{0,n-1}(X,\beta) \arrow[d, "p_2"] \\
\{s\} \arrow[r, "j"] & \overline{M}_{0,n-1}(X,\beta)
\end{tikzcd}$$

$j^*\psi_*[\overline{M}_{0,n}(X,\beta)] = (\mu, pt)_* j^*[\overline{M}_{0,n}(X,\beta)] = (\mu, pt)_*[C]$

$= \beta \times \{s\}$

$j^*\psi_*[\overline{M}_{0,n}(X,\beta)] = j^*(\beta' \times [\overline{M}_{0,n-1}(X,\beta)] + \alpha)$

$= \beta' \times \{s\}$

$\Rightarrow \beta = \beta'$

III, Quantum Cohomology:

X: homogenoous variety Then the Schubert classes form a natural basis for $H^*(X;\mathbb{Z})$

<u>Notation</u>: Fix a basis $\{\underset{\underset{H^0(X)}{\cap}}{T_0=1}, \underbrace{T_1, \dots, T_p}_{H^2(X)}, \underbrace{T_{p+1}, \dots, T_m}_{\text{additive basis of other cohomology}}\}$ of $H^*(X;\mathbb{Z})$

For $1 \le i, j \le m$, we define $g_{ij} := \int_X T_i \cup T_j$ intersection form on X w.r.t. the natural basis. $\rightsquigarrow$ This plays the role of flat metric in Frobenius mfd

Define $(g^{ij}) = (g_{ij})^{-1}$ Via $A^*(X \times X) \simeq A^*X \otimes A^*X$ $\quad \Delta \subset X \times X \xrightarrow{p_2} X$, $\quad X \times X \xrightarrow{p_1} X$

$$[\Delta] = \sum g^{ij} T_i \otimes T_j$$

$$T_i \cup T_j = \sum_{e,f} I_0(T_i \cdot T_j \cdot T_e) g^{ef} T_f$$

<u>Lemma 1</u>: $\gamma \in H^*(X;\mathbb{Z})$, $\gamma^n := \gamma \cup \dots \cup \gamma_n$ Given $n \in \mathbb{N}$, there are only finitely many effective $\beta \in H_2(X;\mathbb{Z})$ s.t. $I_\beta(\gamma^n) \ne 0$

pf: $\because$ X is homigeneus var. $\therefore$ $\beta \in H^2(X;\mathbb{Z})$, $\exists a_1, \dots, a_p \ge 0$ s.t. $\beta = \sum_{i=1}^p a_i \beta_i$, where β_i : effective classes

By lemma II-2 $\Rightarrow \int_{\beta_i} c_1(T_X) \ge 2$ Thus, given $N \in \mathbb{N}$, $\exists$ finitely many effective class β s.t. $\int_\beta c_1(T_X) \le N$.

Now, for $I_\beta(\gamma^n) \ne 0$, then $\dim \overline{M}_{0,n}(X,\beta) \le n \cdot \dim X$

(Otherwise, $\dim \overline{M}_{0,n}(X,\beta) > n \cdot \dim X \Rightarrow \deg($ top homogeneous component of $\gamma^n) < \dim \overline{M}_{0,n}(X,\beta))$

$$\Rightarrow \dim \overline{M}_{0,n}(X,\beta) = \dim X + \int_\beta c_1(T_X) + n - 3 \le n \dim X$$

$$\Rightarrow \int_\beta c_1(T_X) \le (n-1)\dim X + 3 - n \quad \square$$

Now, define "potential" by: $\Phi(\gamma) = \sum_{n \ge 3} \sum_\beta \frac{1}{n!} I_\beta(\gamma^n)$

Now, write $\gamma = \sum_{i=0}^m \gamma_i T_i$ By lemma 15 $\Rightarrow \Phi(\gamma) = \Phi(y_0, \dots, y_m)$ becomes a formal power series in $\mathbb{Q}[[y]] := \mathbb{Q}[[y_0, \dots, y_m]]$:

$$\Phi(y_0, \dots, y_m) = \sum_{n_0 + \dots + n_m \ge 3} \sum_\beta I_\beta(T_0^{n_0}, T_1^{n_1}, \dots, T_m^{n_m}) \frac{y_0^{n_0}}{n_0!} \cdots \frac{y_m^{n_m}}{n_m!}$$

Define $\Phi_{ijk} = \dfrac{\partial^3 \Phi}{\partial y_i \partial y_j \partial y_k}$, for $0 \le i, j, k \le m$

<u>Check</u>: $\Phi_{ijk} = \sum_{n\geq 0}\sum_{\beta} \frac{1}{n!} I_\beta(\gamma^n \cdot T_i \cdot T_j \cdot T_k)$ — $(*)$

Note: $$\frac{\partial \Phi}{\partial y^0} = \sum_{n_0+\dots+n_m\geq 3} \sum_\beta I_\beta(T_0^{n_0} \cdots T_m^{n_m}) \frac{y_0^{n_0-1}}{(n_0-1)!}\frac{y_1^{n_1}}{n_1!}\cdots\frac{y_m^{n_m}}{n_m!}$$

$$= \sum_{n_0+\dots+n_m\geq 2} \sum_\beta I_\beta(T_0^{n_0+1} \cdots T_m^{n_m}) \frac{y_0^{n_0}}{n_0!}\frac{y_1^{n_1}}{n_1!}\cdots\frac{y_m^{n_m}}{n_m!} = \sum_{n\geq 2}\sum_\beta \frac{1}{n!} I_\beta(\gamma^n \cdot T_0)$$

shifted n_0 by 1

Similar for other indices and higher derivative.

Define a "quantum product" by: $T_i * T_j = \sum_{e,f} \Phi_{ije} g^{ef} T_f$

Then extends $*$ $\mathbb{Q}[[y]]$-linearly to $H^*(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}[[y]] \rightsquigarrow \mathbb{Q}[[y]]$-algebra

Commutativity : Obvious. $\because \Phi_{ijk} = \Phi_{jik} \Rightarrow T_j * T_i = \sum_{e,f} \Phi_{jie} g^{ef} T_f = \sum_{e,f} \Phi_{ije} g^{ef} T_f$

T_0 is identity: $T_0 * T_j = \sum_{e,f} \Phi_{0je} g^{ef} T_f$

By $(*)$, $\Phi_{0je} = \sum_{n\geq 0}\sum_\beta \frac{1}{n!} I_\beta(\gamma^n \cdot 1 \cdot T_j \cdot T_e)$

By property (II) of GW inv. $\Rightarrow I_\beta(\gamma^n \cdot 1 \cdot T_j \cdot T_e) \neq 0$ only if $\beta = 0$, $n = 0$

$\rightarrow \Phi_{0je} = I_0(1\cdot T_j \cdot T_e) = \int_X T_j \cup T_e = g_{je}$

$\Rightarrow T_0 * T_j = \sum_{e,f} g_{je} g^{ef} T_f = \sum_{e,f} \delta_j^f T_f = T_j$

<u>Thm4</u>: The quantum product is associative

$(T_i * T_j) * T_k = \sum_{e,f} \Phi_{ije} g^{ef} T_f * T_k = \sum_{e,f}\sum_{c,d} \Phi_{ije} g^{ef} \Phi_{fkc} g^{cd} T_d$

$T_i * (T_j * T_k) = \sum_{e,f} \Phi_{jke} g^{ef} T_i * T_f = \sum_{e,f}\sum_{c,d} \Phi_{jke} g^{ef} \Phi_{ifc} g^{cd} T_d$

$\Rightarrow \sum_{e,f} \Phi_{ije} g^{ef} \Phi_{fkl} = \sum_{e,f} \Phi_{jke} g^{ef} \Phi_{ifl} = \sum_{e,f} \Phi_{jke} g^{ef} \Phi_{fil}$ $\forall l \rightsquigarrow$ WDVV eqn

$F(i,j|k,l) := \sum_{e,f} \Phi_{ije} g^{ef} \Phi_{fkl}$ Then WDVV $\Leftrightarrow F(i,j|k,l) = F(j,k|i,l)$

By $(*) \Rightarrow$

$$F(i,j|k,l) = \sum_{e,f}\sum_{n_1,n_2\geq 0}\sum_{\beta_1,\beta_2} \frac{1}{n_1!n_2!} I_{\beta_1}(\gamma^{n_1}\cdot T_i \cdot T_j \cdot T_e) g^{ef} I_{\beta_2}(\gamma^{n_2} T_k \cdot T_l \cdot T_f)$$

<u>Fact</u>: The boundary of $\overline{M}_{0,n}(X,\beta)$ consists of divisor $D(A,B;\beta_1,\beta_2)$

where i) $A \cup B = \{1,\dots,n\}$: A partition of $\{1,\dots,n\}$ iii) If $\beta_1 = 0$, then $|A| \geq 2$

ii) $\beta_1 + \beta_2 = \beta$, β_i : effective classes $(\beta_2 = 0)$ $(|B| \geq 2)$

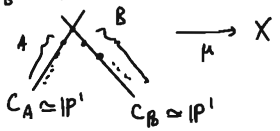
$D(A,B;\beta_1,\beta_2)$ = locus of maps $\mu : C_A \cup C_B \longrightarrow X$ s.t.

(a) $C = C_A \cup C_B$, C_A, C_B : qsi-stable curves of genous 0 meeting at a pt.

(b) The markings of A/B lie on C_A / C_B

(c) $\mu_A = \mu|_{C_A}$ represents β_1 $\quad \mu_B = \mu|_{C_B}$ represents β_2

Generic pt. of $D(A,B;\beta_1,\beta_2)$:



Then $D(A,B;\beta_1,\beta_2) \simeq \overline{M}_{0,A\cup\{\cdot\}}(X,\beta_1) \times_X \overline{M}_{0,B\cup\{\cdot\}}(X,\beta_2)$

$\overline{M} := M_{0,n}(X,\beta)$ $\quad \overline{M}_A := \overline{M}_{0,A\cup\{\cdot\}}(X,\beta_1)$, $\overline{M}_B := \overline{M}_{0,B\cup\{\cdot\}}(X,\beta_2)$ $\quad D := D(A,B;\beta_1,\beta_2)$

$\overline{M}_A \times \overline{M}_B := \overline{M}_{0,A\cup\{\cdot\}}(X,\beta_1) \times_{\mathrm{Spec}\,\mathbb{C}} \overline{M}_{0,B\cup\{\cdot\}}(X,\beta_2)$

$$\begin{tikzcd}
\overline{M} \arrow[d, "ev := ev_1 \times \cdots \times ev_n"'] & D \arrow[l, "\alpha"', "\text{emb.}"] \arrow[r, "\iota", "\text{inclusion}"'] \arrow[d, "\eta"] & \overline{M}_A \times \overline{M}_B \arrow[d, "ev' := ev_A \times ev_B \times ev_\cdot \times ev_\cdot"] \\
X^n & X^{n+1} \arrow[l, "p"', "\substack{\text{proj.}\\ \text{(forget last}\\ \text{factor)}}"] \arrow[r, "\delta", "{(x_1,\dots,x_{n+1}) \mapsto (x_1,\dots,x_{n+1},x_{n+1})}"'] & X^{n+2}
\end{tikzcd}$$

<u>Lemma 2</u>: $\forall\, \gamma_1, \dots, \gamma_n \in H^\cdot(X)$ $\quad \iota_* \circ \alpha^*(ev_1^*\gamma_1 \cup \dots \cup ev_n^*\gamma_n)$

$$= \sum_{e,f} g^{ef} \Big(\prod_{a\in A} ev_a^*(\gamma_a)\, ev_\cdot^*(T_e)\Big) \times \Big(\prod_{b\in B} ev_b^*(\gamma_b)\, ev_\cdot^*(T_f)\Big)$$

pf: $\iota_* \circ \alpha^*(ev_1^*\gamma_1 \cup \dots \cup ev_n^*(\gamma_n)) = \iota_* \circ \alpha^* \circ ev^*(\gamma_1 \times \cdots \times \gamma_n)$

$= \iota_* \circ (ev \circ \alpha)^*(\gamma_1 \times \cdots \times \gamma_n) = \iota_* \circ \eta^* \circ p^*(\gamma_1 \times \cdots \times \gamma_n)$

$= \iota_* \eta^*(\gamma_1 \times \cdots \times \gamma_n \times [X]) = ev'^* \circ \delta_*(\gamma_1 \times \cdots \times \gamma_n \times [X])$

$= ev'^*(\gamma_1 \times \cdots \times \gamma_n \times [\Delta]) = \sum_{e,f} g^{ef} ev'^*(\gamma_1 \times \cdots \times \gamma_n \times T_e \times T_f)$

$= \sum_{e,f} g^{ef} \Big(\prod_{a\in A} ev_a^*(\gamma_a) \cdot ev_\cdot^*(T_e)\Big) \times \Big(\prod_{b\in B} ev_b^*(\gamma_b) \cdot ev_\cdot^*(T_f)\Big)$ $\square$

Now, fix $\beta \in H_2(X)$ $\gamma_1, \dots, \gamma_n \in H^*(X)$ $q, r, s, t \in \{1, \dots, n\}$: distinct

$$G(q,r|s,t) := \sum_{\substack{\beta_1+\beta_2=\beta \\ A \amalg B = \{1,\dots,n\} \\ q,r\in A\ s,t\in B}} \sum_{e,f} I_{\beta_1}\left(\prod_{a\in A} \gamma_a T_e\right) g^{ef} I_{\beta_2}\left(\prod_{b\in B} \gamma_b T_f\right)$$

By lemma 2 $\Rightarrow$ $$G(q,r|s,t) = \sum_{\substack{A \amalg B=\{1,\dots,n\} \\ \beta_1+\beta_2=\beta \\ q,r\in A,\ s,t\in B}} \int_{D(A,B|\beta_1,\beta_2)} ev_1^*(\gamma_1) \cup \dots \cup ev_n^*(\gamma_n)$$

<u>Lemma 3</u>: For $i, j, k, l \in \{1, \dots, n\}$, distinct

$$D(i,j|k,l) = \sum_{\substack{A\amalg B=\{1,\dots,n\} \\ \beta_1+\beta_2=\beta \\ i,j\in A,\ k,l\in B}} D(A,B;\beta_1,\beta_2)$$ Then $D(i,j|k,l) \sim D(i,l|j,k)$ as divisors.

pf: $\overline{M}_{0,n}(X,\beta) \xrightarrow{\eta} \overline{M}_{0,n} \xrightarrow{\phi} \overline{M}_{0,\{i,j,k,l\}} \simeq \mathbb{P}^1$

Note that $D(i,j|k,l) \subset \overline{M}_{0,n}(X,\beta)$ is the preimage of $P(i,j|k,l) \subset \overline{M}_{0,\{i,j,k,l\}}$ via $\phi\circ\eta$

(<u>Fact</u>: Convexity $\Rightarrow$ All components in the preimage appear w/ multiplicity 1)

$\because$ $P(i,j|k,l) \sim P(i,l|j,k)$ $\therefore$ $D(i,j|k,l) \sim D(i,l|j,k)$ $\square$

Hence, by lemma 3 $\Rightarrow$ $$G(q,r|s,t) = \int_{D(q,r|s,t)} ev_1^*(\gamma_1)\cup\dots\cup ev_n^*(\gamma_n)$$

$$= \int_{D(r,s|q,t)} ev_1^*(\gamma_1)\cup\dots\cup ev_n^*(\gamma_n) = G(r,s|q,t) \rightsquigarrow G(q,r|s,t) = G(r,s|q,t) \text{ --- } (**)$$

Now, apply $(**)$ to: $\gamma_i = \gamma$ for $i = 1, \dots, n-4$ $\gamma_{n-3} = T_i$ $\gamma_{n-2} = T_j$ $\gamma_{n-1} = T_k$ $\gamma_n = T_l$

$q = n-3$, $r = n-2$, $s = n-1$, $t = n$

$$G(q,r|s,t) = \sum_{e,f} \sum_{\substack{n_1+n_2=n \\ \beta_1+\beta_2=\beta}} \binom{n-4}{n_1-2} I_{\beta_1}(\gamma^{n_1-2} T_i T_j T_e) g^{ef} I_{\beta_2}(\gamma^{n_2-2} T_k T_l T_f)$$

↘ # of partition of A and B which A has n_1 element B has n_2 element

$$= \sum_{e,f} \sum_{\substack{n_1+n_2=n \\ \beta_1+\beta_2=\beta}} \frac{(n-4)!}{(n_1-2)!(n_2-2)!} I_{\beta_1}(\gamma^{n_1-2} T_i T_j T_e) g^{ef} I_{\beta_2}(\gamma^{n_2-2} T_k T_l T_f)$$

$$= n! \sum_{e,f} \sum_{\substack{n_1+n_2=n-4 \\ \beta_1+\beta_2=\beta}} \frac{1}{n_1!\, n_2!} I_{\beta_1}(\gamma^{n_1-2} T_i T_j T_e) g^{ef} I_{\beta_2}(\gamma^{n_2-2} T_k T_l T_f)$$

Now, $G(q,r|s,t) = G(r,s|q,t) \Rightarrow$

$$\sum_{e,f} \sum_{\substack{n_1+n_2=n-4 \\ \beta_1+\beta_2=\beta}} \frac{1}{n_1!\,n_2!} I_{\beta_1}(\gamma^{n_1-2}\, T_i T_j T_e)\, g^{ef}\, I_{\beta_2}(\gamma^{n_2-2}\, T_k T_\ell T_f)$$

$$= \sum_{e,f} \sum_{\substack{n_1+n_2=n-4 \\ \beta_1+\beta_2=\beta}} \frac{1}{n_1!\,n_2!} I_{\beta_1}(\gamma^{n_1-2}\, T_j T_k T_e)\, g^{ef}\, I_{\beta_2}(\gamma^{n_2-2}\, T_i T_\ell T_f)$$

$\rightsquigarrow$ sum over n $\Rightarrow$ $F(i,j|k,\ell) = F(j,k|i,\ell)$ $\square$

Recall: Last time, we assume the existence of moduli spaces of stable maps, and we define GW inv. and quantum cohomology.

Now, we turn to prove the existence of $\overline{M}_{g,n}(X,\beta)$ and its properties.

$\overline{\mathcal{M}}_{g,n}(X,\beta) : \{\text{Scheme}/\mathbb{C}\} \longrightarrow \{\text{Set}\}$

$S \longmapsto \{\text{isom. class of } n\text{-pointed, genus } g \text{ stable maps over } S\}$

Thm 1: $\exists$ proj. coarse moduli space $\overline{M}_{g,n}(X,\beta)$ for the moduli functor $\overline{\mathcal{M}}_{g,n}(X,\beta)$

For $g=0$;

Thm 2: X: proj. non-sing. convex variety

(i) $\overline{M}_{0,n}(X,\beta)$ is a normal projective variety (not necessarily irreducible) of pure dim. $= \dim X + \int_\beta c_1(T_X) + n - 3$

(ii) $\overline{M}_{0,n}(X,\beta)$ is locally quot. of a non-sing. var. by finite group

(iii) $\overline{M}^*_{0,n}(X,\beta)$ is a non-sing., fine moduli space (for automorphism-free stable maps)

For boundary $\overline{M}_{0,n}(X,\beta) \setminus M_{0,n}(X,\beta)$:

Thm 3: $\overline{M}_{0,n}(X,\beta) \setminus M_{0,n}(X,\beta)$ is a normal crossing divisor up to a finite gp quot.

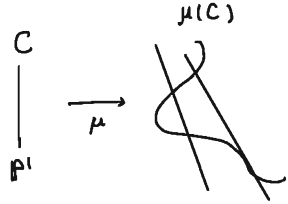
Goal of 2nd talk: Construction of the case $X = \mathbb{P}^r$, $\beta = d[\text{line}]$ i.e. $\overline{M}_{0,n}(\mathbb{P}^r, d)$

Idea: To understand the image of C under μ in $\mathbb{P}^r$, we take a basis of linear form (hyperplane = linear form) (Coord. on $\mathbb{P}^r$) to cut them, and record the intersection

Assume the intersection does NOT lie in special pts

$\Rightarrow$ We obtain more non-sing. marked pt.

Note that μ has degree d $\Rightarrow$ generically, the image should intersects the hyperplane at d many points.



$\rightarrow$ This becomes a curve with $n + d(r+1)$-marked pt.

First, Construct such moduli space (Rigidification)

Second: We change different basis, and glue $\overline{M}_{0,n}(\mathbb{P}^n, d)$ from the moduli spaces above.

• Rigidification:

For $r=0$, $M_{g,n}(\mathbb{P}^0, 0) \simeq \bar{M}_{g,n}$ for $d=0$, $\bar{M}_{g,n}(\mathbb{P}^r, 0) \simeq \bar{M}_{g,n} \times \mathbb{P}^r$

For $(g,n,r,d) = (0,0,1,1) \Rightarrow \bar{M}_{0,0}(\mathbb{P}^1, 1) = \mathrm{Spec}\,\mathbb{C}$

Assume $r>0$, $d>0$, $(g,n,r,d) \neq (0,0,1,1)$:

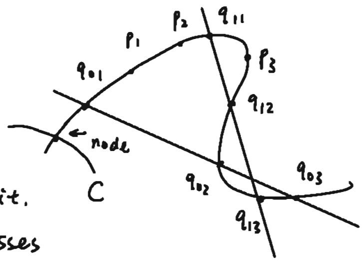
Let $\mathbb{P}^r = \mathbb{P}(V)$ $V^* = H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$ $\bar{t} = (t_0, \dots, t_r)$: basis of V^*

Def: A $\bar{t}$-rigid stable family of degree d maps from n-pointed genus g curves to $\mathbb{P}^r$ consists of: $(\mathcal{C} \xrightarrow[\pi]{} S, \{p_i\}_{1\le i\le n}, \{q_{i,j}\}_{0\le i\le r,\ 1\le j\le d}, \mu)$

(i) $(\mathcal{C} \xrightarrow[\pi]{} S, \{p_i\}_{i=1}^n, \mu)$: stable maps to $\mathbb{P}^r$

(ii) $(\mathcal{C} \xrightarrow[\pi]{} S, \{p_i\}_{i=1}^n, \{q_{i,j}\}_{0\le i\le r,\ 1\le j\le q})$: stable $n+d(r+1)$-pointed curves over S

(iii) $\mu^*(t_i) = q_{i1} + \cdots + q_{id}$ as effective Cariter divisor.



(iii) $\Rightarrow$ The family intersects $\{t_i = 0\}$ transversally.

(ii) $\Rightarrow$ hyperplane intersects at non-sing, unmarked pt.

Rmk: If $(g,n,r,d) = (0,0,1,1) \Rightarrow n + d(r+1) = 2$

$\rightsquigarrow$ No stable 2-pointed genus 0 curve $\rightsquigarrow$ We avoid it.

Define: $\bar{\mathcal{M}}_{g,n}(\mathbb{P}^r, d, \bar{t})$: moduli functor for isom. classes of $\bar{t}$-rigid, n-pointed genus g degree d stable maps over S

Prop1: $\exists$ quasi-proj. coarse moduli space $\bar{M}_{g,n}(\mathbb{P}^r, d, \bar{t})$

For $g=0$, $\bar{M}_{0,n}(\mathbb{P}^r, d, \bar{t})$ is fine moduli space and non-sing.

Idea: Datum of $\bar{t}$-rigid stable family $\rightsquigarrow S \longrightarrow \bar{M}_{0,m}$ $m = n + r(d+1)$

Step1: Image of S lies in a universal loc. closed subscheme $B \subset \bar{M}_{0,m}$

Step2: Note that when $S = \mathrm{Spec}\,\mathbb{C}$, the image in B records: domain curve C, marked pt. $\{p_i\}$, and pull-back divisor $(q_{i1} + \cdots + q_{id})_{0\le i\le r}$ of hyperplanes $\bar{t} = \{t_0, \dots, t_r\}$ via μ.

This does NOT enough to reconstruct $\mu \rightsquigarrow$ This only determine μ up to $(\mathbb{C}^*)^r \curvearrowright \mathbb{P}^r$ diagonally

$\rightarrow$ Recorded in the total space of $\mathbb{C}^*$-bundle over B

So, this is the required moduli space.

pf: $(g=0)$ $m := n + d(r+1)$ $\overline{M}_{0,m}$: moduli space of genus 0, m-pointed stable

Recall:

Thm: (Deligne-Mumford-Knudsen) $\exists$ coarse moduli space $\overline{M}_{g,n}$ of stable n-pointed genus g curve. $\overline{M}_{g,n}$: proj. variety of dim $3g-3+n$, and $M_{g,n} \underset{\text{open}}{\subseteq} \overline{M}_{g,n}$

For the case of $g=0$, a stable n-pointed curve $(n \geq 3)$ has no non-trivial automorphism $\rightsquigarrow$ $\overline{M}_{0,n}$ is a fine moduli space and is a non-sing. variety

$\pi : \overline{U}_{0,m} \longrightarrow \overline{M}_{0,m}$: universal family ($\because$ $\overline{M}_{0,m}$ is fine)

with sections $\{p_i : \overline{M}_{0,m} \longrightarrow \overline{U}_{0,m}\}_{i=1}^{n}$ $\{q_{ij} : \overline{M}_{0,m} \longrightarrow \overline{U}_{0,m}\}_{\substack{0\le i\le r \\ 1\le j\le d}}$

$\because$ $\overline{U}_{0,m}$ is non-sing $\therefore$ $q_{i,1} + \cdots + q_{i,d}$ as a divisor $\rightsquigarrow$

$\mathcal{H}_i := \mathcal{O}_{\overline{U}_{0,m}}(q_{i,1} + \cdots + q_{i,d})$, $0 \le i \le r$

$s_i \in H^0(\overline{U}_{0,m}, \mathcal{H}_i)$ representing the divisor.

$\forall\ \gamma : X \longrightarrow \overline{M}_{0,m}$, consider

$$\begin{tikzcd} X \times_{\overline{M}_{0,m}} \overline{M}_{0,m} \arrow[r, "\bar{\gamma}"] \arrow[d, "\pi_X"] & \overline{U}_{0,m} \arrow[d, "\pi"] \\ X \arrow[r, "\gamma"] & \overline{M}_{0,m} \end{tikzcd}$$

Def: $\gamma : X \longrightarrow \overline{M}_{0,m}$ <u>$\mathcal{H}$-balanced</u> if

(1) $\forall\ i = 1, \dots, r$, $(\pi_X)_*(\bar{\gamma}^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1}))$ is loc. free

(2) $\forall\ i = 1, \dots, r$, $\pi_X^*(\pi_X)_*\bar{\gamma}^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1}) \xrightarrow{\sim} \bar{\gamma}^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$

Claim1: $\exists B \subset \overline{M}_{0,m}$: universal locally closed subscheme s.t.

(a) $\iota : B \hookrightarrow \overline{M}_{0,m}$ is $\mathcal{H}$-balanced

(b) $\forall\ \mathcal{H}$-balanced morphism $\gamma : X \longrightarrow \overline{M}_{0,m}$ factors thru B uniquely

Assume such B exists first.

$$\begin{tikzcd} \overline{U}_B \arrow[r, "\bar{\iota}"] \arrow[d, "\pi_B"] & \overline{U}_{0,m} \arrow[d, "\pi"] \\ B \arrow[r, hook, "\iota"] & \overline{M}_{0,m} \end{tikzcd}$$

Let $\mathcal{G}_i := (\pi_B)_*\bar{\iota}^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$: line bundle on B

$\rightsquigarrow$ $\tau_i : Y_i \longrightarrow B$ $\mathbb{C}^*$-bundle associated to $\mathcal{G}_i$. Now, $\tau_i^*\mathcal{G}_i$ has a tautological tautological section $\rightsquigarrow$ $\tau_i^*\mathcal{G}_i$ is trivial.

Consider $Y := Y_1 \times_B Y_2 \times_B Y_3 \times \cdots \times_B Y_r$ with $p_i : Y \longrightarrow Y_i$ $\tau : Y \longrightarrow B$

$$\begin{tikzcd} \overline{U}_Y \arrow[r, "\bar{\tau}"] \arrow[d, "\pi_Y"] & \overline{U}_B \arrow[r, "\bar{\iota}"] \arrow[d, "\pi_B"] & \overline{U}_{0,m} \arrow[d, "\pi"] \\ Y \arrow[r, "\tau"] & B \arrow[r, hook, "\iota"] & \overline{M}_{0,m} \end{tikzcd}$$

Claim2: On $\bar{U}_Y$, $\exists$ canonical isom. $\bar{\tau}^*\iota^*\mathcal{H}_i \cong \bar{\tau}^*\iota^*\mathcal{H}_0 =: \mathcal{L}$ for $i=1,\dots,r$

pf: $\bar{\tau}\bar{\iota}^*(\mathcal{H}_i\otimes\mathcal{H}_0^{-1}) \underset{\uparrow\ \iota \text{ is } \mathcal{H}\text{-balanced}}{\cong} \bar{\tau}^*\pi_B^*(\pi_B)_*\bar{\iota}^*(\mathcal{H}_i\otimes\mathcal{H}_0) \simeq \pi_Y^*\tau^*(\pi_B)_*\bar{\iota}^*(\mathcal{H}_i\otimes\mathcal{H}_0)$

$\simeq \pi_Y^*\tau^*\mathcal{G}_i$

$\simeq \pi_Y^*(\tau_i\circ p_i)^*\mathcal{G}_i \simeq \pi_Y^* p_i^*\tau_i^*\mathcal{G}_i$ $\because$ $\tau_i^*\mathcal{G}_i$ is canonically trivial

$\Rightarrow$ $\bar{\tau}^*\tau^*\mathcal{H}_i\otimes\mathcal{H}_0^{-1}$ is canonically trivial. $\square$

Now, $\bar{\tau}^*\bar{\iota}^*(S_i)$ is a section of $\mathcal{L}$, for $i=0,\dots,r$

$\because$ $\{q_{ij}\}$ are distinct $\therefore$ $S_0,\dots,S_r$ have no common zeros

Define $\mu: \bar{U}_Y \longrightarrow \mathbb{P}^r$ as following:

Consider $V^* \longrightarrow H^0(\mathcal{L}) \quad \rightsquigarrow \quad V^*\otimes\mathcal{O}_{\bar{U}_Y} \longrightarrow \mathcal{L}$

$t_i \longmapsto \bar{\tau}^*\bar{\iota}^*(S_i) \qquad t_i\otimes f_i \longmapsto f_i\,\bar{\tau}^*\bar{\iota}^*(S_i)$

$\rightsquigarrow$ $\mu: \bar{U}_Y \longrightarrow \mathbb{P}^r$

Claim3: $(\bar{U}_Y \xrightarrow{\pi_Y} Y, \{p_i\}_{i=1}^{n}, \{q_{ij}\}_{\substack{0\le i\le r\\ 1\le j\le d}}, \mu)$ is a universal family of $\bar{t}$-rigid stable maps $\rightsquigarrow$ $\bar{U}_Y = \bar{M}_{0,n}(\mathbb{P}^r, d, \bar{t})$

pf: First, notice that for $y\in Y$, $C=(\bar{U}_Y)_y$ is a $m=n+d(r+1)$-pointed, genus 0 stable curve with marked pts $\{p_i(y)\}$, $\{q_{ij}(y)\}$

We need to prove the stability of the map $(\pi_Y: \bar{U}_Y \longrightarrow Y, \{p_i\}, \mu)$

Let $E\subset C$: irreducible component. If E is contracted by μ

i.e. $\dim\mu(E)=0$ Then none of $\{q_{ij}\}$ lies on E

Since C is a m-pointed stable $\Rightarrow$ Each component must have three special pts and no $\{q_{ij}\}$ $\Rightarrow$ $(C, \{p_i\}, \mu)$ is stable.

$\Rightarrow$ $(\bar{U}_Y \xrightarrow{\pi_Y} Y, \{p_i\}, \{q_{ij}\})$ is a $\bar{t}$-rigid stable family.

Next, we need to show: $\bar{U}_Y \xrightarrow{\pi_Y} Y$ is universal

Pick any $(\pi: \mathcal{C} \longrightarrow S, \{p_i\}, \{q_{ij}\}, \nu)$: $\bar{t}$-rigid stable family

$\because$ $(\pi: \mathcal{C}\longrightarrow S, \{p_i\}, \{q_{ij}\})$ m-pointed genus 0 stable curves, and $\bar{M}_{0,m}$ represents the moduli functor $\bar{\mathcal{M}}_{0,m}$ $\Rightarrow$ $\exists$ $\lambda: S \longrightarrow \bar{M}_{0,m}$ s.t. $(S\times_{\bar{M}_{0,m}}\bar{U}_{0,m} \longrightarrow S, \{p_i\}, \{q_{ij}\})$ $\simeq (\mathcal{C} \xrightarrow[\pi]{} S, \{p_i\}, \{q_{ij}\})$

$$\begin{array}{ccc} \mathcal{C}\cong S\times_{\bar{M}_{0,m}}\bar{U}_{0,m} & \xrightarrow{\bar{\lambda}} & \bar{U}_{0,m} \\ \downarrow{\scriptstyle \pi_S} & & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow{\lambda} & \bar{M}_{0,m} \end{array}$$

Claim4: λ is $\mathcal{H}_i$-balanced

First, recall that: $\mathcal{H}_i := \mathcal{O}_{\bar{U}_{0,m}}(q_{i1}+\dots+q_{id})$

$\rightsquigarrow$ $(\bar{\lambda}^*\mathcal{H}_i, \bar{\lambda}(S_i))$: line bundle and its section on $\mathcal{C}$

and $Z(\bar{\lambda}(S_i)) = q_{i1}+\dots+q_{id}$: effective Cartier divisor on $\mathcal{C}$

On the other hand, ν is induced by: $\psi: V^* \longrightarrow H^0(\mathcal{C}, \nu^*(\mathcal{O}_{\mathbb{P}(V)}(1)))$
$t_i \longmapsto z_i := \psi(t_i)$

$\because \begin{matrix}\mathcal{C}\\ \downarrow\\ S\end{matrix}$ $\bar{t}$-rigid $\Rightarrow (\nu^*(\mathcal{O}_{\mathbb{P}(V)}(1), z_i)$ gives the Cartier divisor $q_{i1} + \cdots + q_{id}$

$$\begin{tikzcd} \mathbb{P}(V) & \mathcal{C} \arrow[l, "\nu"'] \arrow[r, "\bar{\lambda}"] \arrow[d, "\pi_S"] & \bar{U}_{0,m} \arrow[d, "\pi"] \\ & S \arrow[r, "\lambda"] & \bar{M}_{0,m} \end{tikzcd}$$

$\rightarrow \exists!$ isom. $\bar{\lambda}^* \mathcal{H}_i \simeq \nu^* \mathcal{O}_{\mathbb{P}(V)}(1) \quad \forall i = 0, \dots, r$

So, $\bar{\lambda}^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1}) \simeq \nu^* \mathcal{O}_{\mathbb{P}(V)} \simeq \mathcal{O}_{\mathcal{C}}$ — (1)

$\Rightarrow \pi_{S*} \mathcal{O}_{\mathcal{C}} \simeq \mathcal{O}_S$ (This is proved in below)

So, λ is $\mathcal{H}_i$-balanced.

By universal property of $B \Rightarrow S \xrightarrow{\lambda} B \hookrightarrow \bar{M}_{0,m}$, and $\exists$ canonical

$\mathcal{O}_S \simeq \pi_{S*}(\bar{\lambda}^* \mathcal{H}_i \otimes \mathcal{H}_0^{-1}) \simeq \lambda^* \mathcal{G}_i$ — (2)

$\Rightarrow$ This gives a nowhere vanishing canonical section of $\lambda^* \mathcal{G}_i$ over S

$\rightsquigarrow S \longrightarrow \Upsilon_i \rightsquigarrow S \longrightarrow \Upsilon = \Upsilon_1 \times_B \Upsilon_2 \times_B \cdots \times_B \Upsilon_r$

$$\begin{tikzcd} \bar{U}_B \arrow[r, "\bar{\iota}"] \arrow[d, "\pi_B"'] & \bar{U}_{0,m} \arrow[d, "\pi"] \\ B \arrow[r, "\iota"'] & \bar{M}_{0,m} \end{tikzcd}$$

$\rightsquigarrow S \times_\Upsilon \bar{U}_\Upsilon = S \times_\Upsilon (\Upsilon \times_{\bar{M}_{0,m}} \bar{U}_{0,m}) \simeq S \times_{\bar{M}_{0,m}} \bar{U}_{0,m} \simeq \mathcal{C}$

$\rightarrow$ This gives the universality of $\bar{U}_\Upsilon \longrightarrow \Upsilon$ $\square$

Generalities: $\pi: \mathcal{C} \longrightarrow S$ flat family of quasi-stable curves

For $s \in S$, we have natural map $R^i \pi_* \mathcal{O}_{\mathcal{C}} \otimes k(s) \longrightarrow H^i(\mathcal{C}_s, (\mathcal{O}_{\mathcal{C}})_s)$

For $i = 0$, $\pi_* \mathcal{O}_{\mathcal{C}} \otimes k(s) \longrightarrow H^0(\mathcal{C}_s, (\mathcal{O}_{\mathcal{C}})_s)$

By Cohomology and base change, to prove $\varphi^0(s)$ is isomorphism, it suffices to check that it is surjective.

Note that C_s is a quasi-stable curve $\Rightarrow C_s$ is connected

$\Rightarrow H^0(\mathcal{C}_s, (\mathcal{O}_{\mathcal{C}})_s) \simeq \mathbb{C}$, and $\pi_* \mathcal{O}_{\mathcal{C}} \otimes k(s) \longrightarrow H^0(\mathcal{C}_s, (\mathcal{O}_{\mathcal{C}})_s)$ non-zero

$\Rightarrow \pi_* \mathcal{O}_{\mathcal{C}} \otimes k(s) \longrightarrow H^0(\mathcal{C}_s, (\mathcal{O}_{\mathcal{C}})_s)$ onto. $\Rightarrow \pi_* \mathcal{O}_{\mathcal{C}} \otimes k(s) \simeq \Gamma(\mathcal{C}_s, (\mathcal{O}_{\mathcal{C}})_s)$

$\Rightarrow \pi_* \mathcal{O}_{\mathcal{C}}$ is locally free

$\rightsquigarrow \mathcal{O}_S \simeq \pi_*(\mathcal{O}_{\mathcal{C}})$ ($\mathcal{O}_S \hookrightarrow \pi_* \mathcal{O}_{\mathcal{C}}$ since π is onto; $\pi_* \mathcal{O}_{\mathcal{C}} \otimes k(s) \simeq H^0(\mathcal{C}_s, (\mathcal{O}_{\mathcal{C}})_s)$ + NAK $\Rightarrow$ onto)

N: line bundle on $S \Rightarrow N \simeq N \otimes \mathcal{O}_S \simeq N \otimes \pi_* \mathcal{O}_{\mathcal{C}} \simeq \pi_*(\mathcal{O}_{\mathcal{C}} \otimes \pi^* N) \simeq \pi_* \pi^* N$

pf of claim 1:

If $\mathcal{L}, \mathcal{M}$: line bundle on $\mathcal{C}$, $\exists$ line bundle N on S s.t. $\mathcal{L} \otimes \mathcal{M}^{-1} \simeq \pi^* N$

iff (a) $\pi_*(\mathcal{L} \otimes \mathcal{M}^{-1})$ is locally free

(b) $\pi^* \pi_*(\mathcal{L} \otimes \mathcal{M}^{-1}) \xrightarrow{\sim} \mathcal{L} \otimes \mathcal{M}^{-1}$ is an isom.

($\Rightarrow$) $\exists$ line bundle N on S s.t. $\mathcal{L} \otimes \mathcal{M}^{-1} \simeq \pi^* N \Rightarrow \pi_*(\mathcal{L} \otimes \mathcal{M}^{-1}) \simeq \pi_* \pi^* N \simeq N$

$\Rightarrow$ $\pi_*(\mathcal{L} \otimes \mathcal{M}^{-1})$ is loc. free.

$\pi^* \pi_*(\mathcal{L} \otimes \mathcal{M}^{-1}) \simeq \pi^* \pi_* \pi^* N \simeq \pi^* N \simeq \mathcal{L} \otimes \mathcal{M}^{-1}$

($\Leftarrow$) Set $N = \pi_*(\mathcal{L} \otimes \mathcal{M}^{-1})$ (a) $\Rightarrow$ N is a line bundle on S

(b) $\Rightarrow \pi^* N = \pi^* \pi_*(\mathcal{L} \otimes \mathcal{M}^{-1}) \simeq \mathcal{L} \otimes \mathcal{M}^{-1}$.

<u>Def</u>: $\mathcal{L}_s$: line bundle on the geometric fiber $\mathcal{C}_s$ of π $\mathcal{C}_s = \bigcup_i Y_i$

multidegree of $\mathcal{L}_s := (\deg(\mathcal{L}_s|_{Y_i}))_i$

Claim 1 is established via the following lemma:

<u>Lemma 1</u>: $\mathcal{L}, \mathcal{M}$: line bundle on $\mathcal{C}$ s.t. $\operatorname{multideg}(\mathcal{M}_s) = \operatorname{multideg}.(\mathcal{L}_s)$ on each geom. fiber $\mathcal{C}_s$, then $\exists !$ closed subscheme $T \hookrightarrow S$ s.t.

(I) $\exists N$: l.b. on T s.t. $\mathcal{L}_T \otimes \mathcal{M}_T^{-1} \simeq \pi^* N$

(II) If $R \to S$ and N : l.b. on R s.t. $\mathcal{L}_R \otimes \mathcal{M}_R^{-1} \simeq \pi^* N$, then

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & S \\ & T \arrow[u] \end{tikzcd}$$

proof of lemma is postponed to the third talk.

Now, for claim 1, we need to first restrict to an open subscheme of $\overline{M}_{0,m}$ s.t. $\mu^* t_i$ intersects C transversally

Then apply lemma 1 $\Rightarrow$ We obtain our required B.

- Construction of $\overline{M}_{0,n}(\mathbb{P}^r, d)$:

Given $\mu: C \longrightarrow \mathbb{P}^r$ pointed stable map, it may not rigid for a given basis $\bar{t}$ of $V^* = H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$

However, by Bertini's thm (that hyperplane intersection avoid nodes and marked pts, also intersects transversally), μ is rigid in some basis $\bar{t}'$

<u>Idea</u>: Construct $\overline{M}_{0,n}(\mathbb{P}^r, d)$ by glueing quot. of $\overline{M}_{0,n}(\mathbb{P}^r, d, \bar{t})$ for different $\bar{t}$

$\overline{M}(\bar{t}) := \overline{M}_{0,n}(\mathbb{P}^r, d, \bar{t})$, and $(\mathcal{U} \longrightarrow \overline{M}(\bar{t}), \{p_i\}, \{q_{i,j}\}, \mu)$: universal family of $\bar{t}$-rigid genus 0 stable maps

$G = G_{d,r} := \underbrace{S_d \times \cdots \times S_d}_{r+1\text{-copies}}$ S_d: symmetric of d letters

Then given $\sigma \in G$, $(\pi: \mathcal{U} \longrightarrow \overline{M}(\bar{t}), \{p_i\}, \{q_{i,\sigma(j)}\}, \mu)$ is also a $\bar{t}$-rigid family over $M(\bar{t})$ By universal property $\Rightarrow \sigma: \overline{M}(\bar{t}) \longrightarrow \overline{M}(\bar{t})$

Thus, $G \curvearrowright \overline{M}(\bar{t})$.

$\because$ $\overline{M}(\bar{t})$: quasi-proj. G: finite $\Rightarrow$ $\overline{M}(\bar{t})/G$: quasi-proj.

Now, for $\bar{t} = \{t_0, t_1, \ldots, t_r\}$. $\bar{t}' = \{t_0', t_1', \ldots, t_r'\}$: two bases of V^*

$i=0,\ldots,r$ $D_i := Z(\mu^* t_i')$: associated divisor of $\mu^* t_i' = q_{i1}' + \cdots + q_{id}'$

Then we already have the universal family:

$$\begin{tikzcd} D_i \subset \mathcal{U} \arrow[r, "\mu"] \arrow[d, "\pi"] & \mathbb{P}^r \\ \overline{M}(\bar{t}) & \end{tikzcd}$$

Then consider $\overline{M}(\bar{t})_{tr} \subset \overline{M}(\bar{t})$: maximal open subscheme s.t. $\Omega_{D_i/\overline{M}(\bar{t})} = 0$ $\forall i$, i.e. $\pi: D_i \longrightarrow \overline{M}(\bar{t})_{tr}$ is smooth of rel. dim 0 = étale

$\overline{M}(\bar{t}, \bar{t}') \subset \overline{M}(\bar{t})_{tr}$: maximal open subscheme s.t. $\{p_i\}$ and D_i are disjoint

$\rightsquigarrow$ $\overline{M}(\bar{t}, \bar{t}')$ is G-inv., we can consider $\overline{M}(\bar{t}, \bar{t}')/G$

$$\mathcal{E} := \prod_{i=0}^{r} (\underbrace{D_i \times_{\overline{M}(\bar{t},\bar{t}')} \times \cdots \times_{\overline{M}(\bar{t},\bar{t}')} D_i}_{d\text{-copies}} \setminus \Delta)$$

$\rightsquigarrow$ $\mathcal{E}$ carries a $G \times G$-action and $\mathcal{E}/G \simeq \overline{M}(\bar{t}, \bar{t}')$

<u>Prop 4</u>: $\exists$ canonical isom. $\overline{M}(\bar{t}, \bar{t}')/G \simeq \overline{M}(\bar{t}', \bar{t})/G$.

pf: We have natural morphism $\mathcal{E} \xrightarrow{\varphi} \overline{M}(\bar{t}, \bar{t}')$, and the fiber of φ over $(C, \{p_i\}, \{q_{i,j}\}, \mu)$ is $\{q_{ij}'\}_{\substack{0 \le i \le r \\ 1 \le j \le d}}$ $\rightsquigarrow$ $\mathcal{E} \xrightarrow{\psi} \overline{M}(\bar{t}')$ ψ is G-equiv. for G-action on $\mathcal{E}$ (permuting the product and D_i) and $\{q_{ij}'\}$-permutation G-action on $\overline{M}(\bar{t}')$ $\rightsquigarrow$

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\psi"] \arrow[d, "\psi"'] & \overline{M}(\bar{t}') \\ \overline{M}(\bar{t}', \bar{t}) \arrow[ur] & \end{tikzcd}$$

by construction

$\Rightarrow \bar{M}(\bar{t},\bar{t}') \simeq \mathcal{E}/G \xrightarrow{\tilde{\Psi}} \bar{M}(\bar{t}',\bar{t})/G$ Note that $\tilde{\Psi}$ is G-equiv. (resp. permutation of $\{q_{ij}\}$) $\Rightarrow \bar{M}(\bar{t},\bar{t}')/G \simeq \mathcal{E}/G\times G \xrightarrow{\bar{\Psi}} \bar{M}(\bar{t}',\bar{t})/G$.

The inverse is done by exchanging $\bar{t}$ and $\bar{t}'$ in above construction. $\square$

<u>Glueing data</u>:

$\{\bar{M}(\bar{t})/G\}_{\bar{t}}$: basis of V^*

$\bar{\Psi}_{\bar{t}\bar{t}'}: \bar{M}(\bar{t},\bar{t}')/G \xrightarrow{\sim} \bar{M}(\bar{t}',\bar{t})/G$ (cocycle condition)

$\overset{\text{glue}}{\rightsquigarrow} \bar{M}_{0,n}(\mathbb{P}^r, d)$

· Addendum for Rigidification:

Recall: Last time, for $\bar{t}$: basis of $V^* = H^0(\mathbb{P}^r, \mathcal{O}(1))$, in the construction of moduli space of $\bar{t}$-rigid stable maps $\overline{M}_{0,n}(\mathbb{P}^r, d, \bar{t})$, we need the concept of $\mathcal{H}$-balanced map: $m := n + d(r+1)$ $\pi: \overline{U}_{0,m} \longrightarrow \overline{M}_{0,m}$ with sections $\{p_i\}_{i=1}^{n}$, $\{q_{ij}\}_{0\le i\le r,\, 1\le j\le d}$ $\mathcal{H}_i := \mathcal{O}_{\overline{U}_{0,m}}(q_{i1}+\cdots+q_{id})$ $i = 0, \dots, r$

$\gamma: X \longrightarrow \overline{M}_{0,m}$ $\mathcal{H}$-balanced if

$$\begin{tikzcd} X \times_{\overline{M}_{0,m}} \overline{U}_{0,m} \arrow[r, "\bar{\gamma}"] \arrow[d, "\pi_X"] & \overline{U}_{0,m} \arrow[d, "\pi"] \\ X \arrow[r, "\gamma"] & \overline{M}_{0,m} \end{tikzcd}$$

(a) $\forall\, i = 1, \dots, r$ $(\pi_X)_*(\bar{\gamma}^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1}))$ loc. free

(b) $\forall\, i = 1, \dots, r$, $(\pi_X)^*(\pi_X)_*(\bar{\gamma}^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})) \underset{\substack{\uparrow \\ \text{canonical}}}{\simeq} \bar{\gamma}^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$

We state the following claim: Claim1: $\exists\, B \subset \overline{M}_{0,m}$: locally closed subscheme s.t. (a) $\iota: B \hookrightarrow \overline{M}_{0,m}$ is $\mathcal{H}$-balanced

(b) $\forall$ $\mathcal{H}$-balanced morphism $\gamma: X \longrightarrow \overline{M}_{0,m}$,

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[dr] & \overline{M}_{0,m} \\ & B \arrow[u, "\iota"'] \end{tikzcd}$$

Now, for $\pi: \mathcal{C} \longrightarrow S$: family of prestable curve over S

(i.e. π is flat, proj. and $\mathcal{C}_s$: connected, cpx nodal curve with $P_a(C) = g$)

$\rightsquigarrow$ $\mathcal{O}_S \cong \pi_* \mathcal{O}_{\mathcal{C}}$ canonically.

$\forall\, N$: l.b. on S, $N \simeq N \otimes \mathcal{O}_S \simeq N \otimes \pi_* \mathcal{O}_{\mathcal{C}} \simeq \pi_*(\mathcal{O}_{\mathcal{C}} \otimes \pi^* N) \simeq \pi_* \pi^* N$

Suppose $\mathcal{L}, \mathcal{M}$: l.b. on $\mathcal{C}$, then $\exists\, N$: l.b. on S s.t. $\mathcal{L} \otimes \mathcal{M}^{-1} \simeq \pi^* N$

iff (1) $\pi_*(\mathcal{L} \otimes \mathcal{M}^{-1})$ is loc. free

(2) $\pi^* \pi_*(\mathcal{L} \otimes \mathcal{M}^{-1}) \simeq \mathcal{L} \otimes \mathcal{M}^{-1}$

Def: $\mathcal{L}_s$: l.b. on the geom. fiber $\mathcal{C}_s$ of π, $\operatorname{multideg}(\mathcal{L}_s) = (\deg(\mathcal{L}_s|_E))_{E \underset{\text{comp}}{\overset{\text{irred.}}{\subset}} C_s}$

The claim is established by the following prop:

Prop: $\mathcal{L}, \mathcal{M}$: l.b. on $\mathcal{C}$ s.t. $\operatorname{multideg}.(\mathcal{L}_s) = \operatorname{multideg}.(\mathcal{M}_s)$, then $\exists!\, T \underset{\text{closed}}{\subset} S$ s.t. (I) $\exists$ l.b. N on T s.t. $\mathcal{L}_T \otimes \mathcal{M}_T^{-1} \simeq \pi^* N$

(II) $(R \xrightarrow{f} S, K)$: $\begin{tikzcd} R \times_S \mathcal{C} \arrow[r, "\bar{f}"] \arrow[d, "\pi_R"] & \mathcal{C} \arrow[d, "\pi"] \\ R \arrow[r, "f"] & S \end{tikzcd}$, : l.b. on R s.t. $\underbrace{\bar{f}^* \mathcal{L} \otimes \bar{f}^* \mathcal{M}^{-1}}_{\bar{f}(\mathcal{L} \otimes \mathcal{M}^{-1})} \simeq \pi_R^* K$,

, then

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & S \\ & T \arrow[u] \end{tikzcd}$$

pf: It suffices to prove for $\mathcal{L}$: l.b. on $\mathcal{C}$ with $\operatorname{multideg}(\mathcal{L}_s) = 0$, $\forall\, s \in S$, then $\exists!\, T \underset{\text{closed}}{\subset} S$ s.t. (I) $\exists$ l.b. N on T s.t. $\mathcal{L}_T \simeq \pi^* N$

(II) $R \xrightarrow{f} S$ any morphism s.t. $\exists$ l.b. K on R and $\pi_R^* K \simeq \bar{f}^* \mathcal{L}$, then

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[dr] & S \\ & T \arrow[u] \end{tikzcd}$$

First, the uniqueness of T follows from the universal property (II)

Now, suppose $\exists$ open covering $\{V_i\}$ of S s.t. the prop holds for each V_i.

Then we obtain a closed subscheme $T_i \subset V_i$.

On $V_i \cap V_j$, $T_i \cap (V_i \cap V_j)$ and $T_j \cap (V_i \cap V_j)$ satisfy the prop. for $V_i \cap V_j$

By uniqueness $\Rightarrow T_i \cap (V_i \cap V_j) = T_j \cap (V_i \cap V_j)$

Thus, $\exists T \underset{\text{closed}}{\subset} S$ s.t. $T \cap V_i = T_i$.

Now, assume S is affine, $S = \operatorname{Spec} A$ A: f.g. $\mathbb{C}$-alge.

$\pi : \mathcal{C} \longrightarrow S = \operatorname{Spec} A$ proj. flat

It suffices to prove that $\forall s \in S$, closed pt. $\exists$ nbd of s s.t. the prop holds

Recall in the proof of cohomology and base change, we have a finite cpx $L^\bullet$ of f.g. A-mod. s.t. L° : A-flat, L^k free for $k \geq 1$ gives $\pi_* \mathcal{L}$ universally

$\because$ $\therefore$ is A-flat $\Rightarrow L^\circ$ is loc. free $\Rightarrow$ Restriction to some nbd of s, we may assume $L^\bullet$ are finite free A-mod.

$$0 \to L^\circ \xrightarrow{\phi} L^1 \to L^2 \to \cdots \quad \rightsquigarrow \quad L^1 \xrightarrow{{}^t\phi} L^\circ \longrightarrow M \longrightarrow 0, \text{ where } M = \operatorname{coker}({}^t\phi)$$

$$\Rightarrow L^1 \otimes_A B \xrightarrow[{}^t\phi_B]{} L^\circ \otimes_A B \to M \otimes_A B \longrightarrow 0, \quad \forall B : A\text{-alge.}$$

$$\rightsquigarrow 0 \to \operatorname{Hom}_B(M \otimes_A B, B) \longrightarrow L^\circ \otimes_A B \xrightarrow{\phi_B} L^1 \otimes_A B$$

Thus,
$$\begin{array}{ccc} \operatorname{Spec} B \times_S \mathcal{C} & \xrightarrow{\bar{f}} & \mathcal{C} \\ \pi_B \downarrow & & \downarrow \pi \\ \operatorname{Spec} B & \xrightarrow{f} & \operatorname{Spec} A \end{array} \qquad (\pi_B)_*(\bar{f}^*\mathcal{L})(\operatorname{Spec} B) = H^\circ(\operatorname{Spec} B \times_S \mathcal{C}, \bar{f}^*\mathcal{L})$$

$$= H^\circ(\operatorname{Spec} B \times_S \mathcal{C}, \mathcal{L} \otimes B) \simeq h^\circ(L^\bullet \otimes B) = \ker(\phi_B)$$

$= \operatorname{Hom}_B(M \otimes_A B, B) \simeq \operatorname{Hom}_A(M, B)$

Now, let $F = \{ s \in S \mid \mathcal{L}_s \simeq \mathcal{O}_{\mathcal{C}_s} \}$. Then $\mathcal{L}_s$ is trivial iff $\dim H^\circ(\mathcal{C}_s, \mathcal{L}_s) \geq 1$ and $\dim H^\circ(\mathcal{C}_s, \mathcal{L}_s^{-1}) \geq 1$, ($\because$ $\mathcal{C}_s$ cpx proj. $\Rightarrow$ complete)

By semiconti. thm, F is closed. For $Y \setminus F$, take $T = \phi$.

$\rightarrow$ It remains to prove: $\forall s \in F$, closed pt. $\exists$ nbd of s s.t. prop holds

If $s \in F$, closed pt. then since multdeg$(\mathcal{L}_s) = 0$, $\mathcal{L}_s \simeq \mathcal{O}_{\mathcal{C}_s} \Leftrightarrow$ $\dim_{k(s)} H^\circ(\mathcal{C}_s, \mathcal{L}_s) = 1$,

Thus, $1 = \dim_{k(s)} H^\circ(\mathcal{C}_s, \mathcal{L}_s) = \dim_{k(s)}(M \otimes_A k(s)) = \dim_{k(s)} M_s / m_s M_s$

By Nakayama lemma $\Rightarrow \exists r \in M_s$ s.t. $M_s = A_s \langle r \rangle$

$\Rightarrow \exists$ nbd U of s s.t. $\widetilde{M}(U) = \mathcal{O}_U \langle r \rangle$

Restriction to this nbd U, we may assume $M = A/I$, where $I \trianglelefteq A$.

$T' = \mathrm{Spec}(A/I) \underset{\text{closed}}{\subset} S$

$$\begin{tikzcd} \mathcal{C}\times_S T' \arrow[r, "\bar{\iota}"] \arrow[d, "\pi_{T'}"] & \mathcal{C} \arrow[d] \\ \mathrm{Spec}(A/I) = T' \arrow[r, hook, "\iota"] & S \end{tikzcd}$$

$(\pi_{T'})_*(\bar{\iota}^*\mathcal{L})(T') \cong \mathrm{Hom}_A(A/I, A/I) \simeq A/I \Rightarrow (\pi_{T'})_*(\bar{\iota}^*\mathcal{L}) \simeq \mathcal{O}_{T'}$

$\Rightarrow (\pi_{T'})_*(\bar{\iota}^*\mathcal{L})$ free.

Now, consider the natural homo. $(\pi_{T'})^*(\pi_{T'})_*(\bar{\iota}^*\mathcal{L}) \xrightarrow{\lambda} \bar{\iota}^*\mathcal{L}$ on $\mathcal{C}_{T'}$

$\because$ both sides are loc. free of rk 1, then λ_z is isom. for $z \in \mathcal{C}_{T'}$

iff λ_z is onto iff $[(\pi_{T'})^*(\pi_{T'})_*(\bar{\iota}^*\mathcal{L})_z] \otimes_{\mathcal{O}_z} k(z) \twoheadrightarrow (\bar{\iota}^*\mathcal{L})_z \otimes_{\mathcal{O}_z} k(z)$ is onto

For $s \in T'$, closed pt,

$$\begin{tikzcd} \mathcal{C}_s \arrow[r] \arrow[d, "\pi_s"] & \mathcal{C}_{T'} \arrow[r, "\bar{\iota}"] \arrow[d, "\pi_{T'}"] & \mathcal{C} \arrow[d, "\pi"] \\ \{s\} \arrow[r, hook] & T' \arrow[r, hook, "\iota"] & S \end{tikzcd}$$

$\because H^0(T', (\pi_{T'})_*\bar{\iota}^*\mathcal{L}) = \mathrm{Hom}_A(A/I, A/I) \twoheadrightarrow \mathrm{Hom}_A(A/I, A/m_s) = H^0(\mathcal{C}_s, \mathcal{L}_s)$

and $\mathcal{L}_s$ is trivial $\Rightarrow \lambda_z$ is isom. at $z \in \mathcal{C}_s$

On the other hand, $Z = \{z \in \mathcal{C}_T \mid \lambda_z \text{ is not iso.}\} = \mathrm{supp}(\ker\lambda) \cup \mathrm{supp}(\mathrm{coker}\lambda)$

$\underset{\text{closed}}{\subset} \mathcal{C}_{T'}$ and $Z \cap \mathcal{C}_s = \phi \quad \forall s \in T'$.

$\because \mathcal{C} \xrightarrow[\pi]{} S$ proj. $\Rightarrow$ proper $\quad \therefore \pi(Z)$ is a closed subset of S,

not containing s. $\Rightarrow \exists$ affine nbd V of s s.t. $V \cap \pi(Z) = \phi$

Restriction to V, may assume $M \cong A/I$, and define $T = \mathrm{Spec}(A/I)$

Then (I) is satisfied for T.

For (II), for any $R \xrightarrow{f} S$ s.t.

$$\begin{tikzcd} R \arrow[rr, "f"] \arrow[dr] & & S \\ & T \arrow[ur] & \end{tikzcd}$$

is local condition on R. May assume $R = \mathrm{Spec}\,B$

, for some A-alge. B. Also, we may also assume K is trivial on Z.

Then

$$\begin{tikzcd} \mathcal{C}_R \arrow[r, "\bar{f}"] \arrow[d, "\pi_R"] & \mathcal{C} \arrow[d, "\pi"] \\ R \arrow[r, "f"] & S \end{tikzcd}$$

$\bar{f}^*\mathcal{L} \simeq \pi_R^*\mathcal{O}_Z \cong \mathcal{O}_{\mathcal{C}_R}$ and $(\pi_R)_*\bar{f}^*\mathcal{L} = (\pi_R)_*\mathcal{O}_{\mathcal{C}_R} \simeq \mathcal{O}_R$

Hence, $B \simeq \mathrm{Hom}_A(A/I, B) \Rightarrow I \cdot B = 0$ Thus,

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \\ & A/I \arrow[ur] & \end{tikzcd}$$

$\rightsquigarrow$

$$\begin{tikzcd} \mathrm{Spec}\,B \arrow[rr] \arrow[dr] & & \mathrm{Spec}\,A \\ & \mathrm{Spec}(A/I) \arrow[ur] & \end{tikzcd}$$

$\blacksquare$

• Addendum for Glueing:

$\overline{M}(\bar{t}) := \overline{M}_{0,n}(\mathbb{P}^r, d, \bar{t})$, and $(\mathcal{U} \longrightarrow \overline{M}(\bar{t}), \{p_i\}, \{q_{ij}\}, \mu)$: universal family of $\bar{t}$-rigid genus 0 stable map

$G = G_{d,r} := \underbrace{S_d \times \cdots \times S_d}_{r+1\text{-copies}}$ S_d: symmetric of d letters

Then given $\sigma \in G$, $(\pi: \mathcal{U} \longrightarrow \overline{M}(\bar{t}), \{p_i\}, \{q_{i,\sigma(j)}\}, \mu)$ is also a $\bar{t}$-rigid family over $M(\bar{t})$ By universal property $\Rightarrow \sigma: \overline{M}(\bar{t}) \longrightarrow \overline{M}(\bar{t})$

Thus, $G \curvearrowright \overline{M}(\bar{t})$.

$\because$ $\overline{M}(\bar{t})$: quasi-proj. G: finite $\Rightarrow \overline{M}(\bar{t})/G$: quasi-proj.

Now, for $\bar{t} = \{t_0, t_1, \ldots, t_r\}$, $\bar{t}' = \{t_0', t_1', \ldots, t_r'\}$: two bases of V^*

$D_i := Z(\mu^* t_i')$: associated divisor of $\mu^* t_i' = q_{i1}' + \cdots + q_{id}'$, for $i = 0, \ldots, r$

Then we already have the universal family: $D_i \subset \mathcal{U} \xrightarrow{\mu} \mathbb{P}^r$, $\pi\downarrow$, $\overline{M}(\bar{t})$

Then consider $\overline{M}(\bar{t})_{tr} \subset \overline{M}(\bar{t})$: maximal open subscheme s.t. $\Omega_{D_i/\overline{M}(\bar{t})} = 0$ $\forall i$. i.e. $\pi: D_i \longrightarrow \overline{M}(\bar{t})_{tr}$ is smooth of rel. dim 0 = étale

$\overline{M}(\bar{t}, \bar{t}') \subset \overline{M}(\bar{t})_{tr}$: maximal open subscheme s.t. $\{p_i\}$ and D_i are disjoint

Then $\sigma \in G$, σ acts on $\overline{M}(\bar{t})$ by permuting the pts of intersection of hyperplanes $\Rightarrow \overline{M}(\bar{t}, \bar{t}')$ is G-inv.

Thus, we may speak of $\overline{M}(\bar{t}, \bar{t}')/G$ to be the quoti.

Let $\mathcal{E} := \prod_{i=0}^{r} (\underbrace{D_i \times_{\overline{M}(\bar{t},\bar{t}')} \cdots \times_{\overline{M}(\bar{t},\bar{t}')} D_i}_{d \text{ copies}} \setminus \Delta)$

Then by construction, $\mathcal{E} \longrightarrow \overline{M}(\bar{t}, \bar{t}')$ is étale, proj. Also, the fiber of $\mathcal{E}$ over $(C, \{p_i\}, \{q_{ij}\}, \mu)$ is the set $\{q_{ij}'\}$ of pts mapped by μ to $\{t_i' = 0\}$

$\Rightarrow \mathcal{E} \longrightarrow \overline{M}(\bar{t}, \bar{t}')$: quasi-finite. Therefore, $\mathcal{E} \longrightarrow \overline{M}(\bar{t}, \bar{t}')$ is a finite, étale.

Also, $\mathcal{E}/G \simeq \overline{M}(\bar{t}, \bar{t}')$ $\Rightarrow$ Regard $\mathcal{E}$ as étale Galois cover of $\overline{M}(\bar{t}, \bar{t}')$ with Galois gp G, and $\mathcal{E}$: a $\bar{t}'$-rigid stable family

$\rightsquigarrow \mathcal{E} \longrightarrow \overline{M}(\bar{t}')$

Prop: $\exists$ canonical isom. $\overline{M}(\bar{t}, \bar{t}')/G \simeq \overline{M}(\bar{t}', \bar{t})/G$

pf: Note that $\mathcal{E} \longrightarrow \overline{M}(\bar{t}')$ is G-equiv. for Galois G-action on $\mathcal{E}$ and $\{q_{ij}'\}$ permutation G-action on $\overline{M}(\bar{t}')$

Also, by construction,
$$\begin{tikzcd} \mathcal{E} \arrow[rr] \arrow[dr] & & \bar{M}(\bar{t}') \\ & \bar{M}(\bar{t}',\bar{t}) \arrow[ur] & \end{tikzcd} \Rightarrow \bar{M}(\bar{t},\bar{t}') \simeq \mathcal{E}/\text{Galois} \xrightarrow{\Psi} \bar{M}(\bar{t}',\bar{t})/G$$

$\because$ Ψ is G-inv. for $\{q_{ij}\}$-permutation $\Rightarrow$ $\bar{M}(\bar{t},\bar{t}')/G \longrightarrow \bar{M}(\bar{t}',\bar{t})/G$

Exchange the role of $\bar{t}$ and $\bar{t}'$ $\Rightarrow$ We obtain the inverse map $\square$

$\rightsquigarrow$ $\mathcal{E}$ = moduli space of $\bar{t},\bar{t}'$-rigid stable map

Patching: $\{\bar{M}(\bar{t})/G\}_{\bar{t}:\text{basis of } V^*}$

$$\Psi_{\bar{t}\bar{t}'}: \bar{M}(\bar{t},\bar{t}')/G \xrightarrow{\sim} \bar{M}(\bar{t}',\bar{t})/G$$

$$\xrightarrow[\text{glueing}]{} \bar{M}_{0,n}(\mathbb{P}^r, d)$$

$\square$

<u>Rmk1</u>: $\bar{M}_{0,n}(\mathbb{P}^r, d)$ is (1) of finite type over $\mathbb{C}$
(2) separated and proper
(3) projective.

<u>Rmk2</u>: When $g=0$, $\bar{M}(\bar{t})$: non-sing. quasi-proj. var.
$\rightarrow$ $\bar{M}(\mathbb{P}^r, d)$: locally a quot. of non-sing. var. by finite gp

<u>Lemma</u>: $\xi \in \bar{M}(\bar{t})$ s.t. $G_{d,r}$-action is not free, then ξ corresponds to a stable map with non-trivial automorphism.

Note that:
$$G_{d,r} \curvearrowright \bar{M}(\bar{t}) \quad \because \bar{M}(\bar{t}) \text{ is fine} \Rightarrow G_{d,r} \curvearrowright \mathcal{U} \longrightarrow \bar{M}(\bar{t})$$
by isom. on the stable maps over $\bar{M}(\bar{t})$.

If $\exists\, \gamma \neq 1$ s.t. $\gamma\cdot\xi = \xi$. Say $\xi = [C, \{p_i\}, \{q_{ij}\}, \mu] \in \bar{M}(\bar{t})$

Then γ induces $\tilde{\gamma}: C \longrightarrow C$ Then $\tilde{\gamma}$ is non-trivial on the marked pts $\{q_{ij}\}$. $\Rightarrow$ C has non-trivial auto. $\square$

• Construction of $\overline{M}_{0,n}(X,\beta)$:

Recall: For X: proj. var, we have:

Thm 1: $\exists$ proj. coarse moduli space $\overline{M}_{g,n}(X,\beta)$

$\because X$ proj. fix an imbedding $\iota: X \hookrightarrow \mathbb{P}^r$ Let $\iota_*(\beta) = d\,[\text{line}]$

Lemma: $\exists$ closed subscheme $\overline{M}_{g,n}(X,\beta,\bar{t}) \subset \overline{M}_{g,n}(\mathbb{P}^r, d, \bar{t})$ s.t. for $(\pi: \mathcal{C} \to S, \{p_i\}, \{q_{ij}\}, \mu)$: $\bar{t}$-rigid stable family of genus g, n-pointed degree d maps to $\mathbb{P}^r$, then $S \longrightarrow \overline{M}_{g,n}(\mathbb{P}^r, d, \bar{t})$ iff $\mathcal{C} \xrightarrow{\mu} \mathbb{P}^r$

$$\begin{tikzcd} S \arrow[rr] \arrow[dr] & & \overline{M}_{g,n}(\mathbb{P}^r, d, \bar{t}) \\ & \overline{M}_{g,n}(X,\beta,\bar{t}) \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[rr, "\mu"] \arrow[dr, "\mu"'] & & \mathbb{P}^r \\ & X \arrow[ur, "\iota"'] & \end{tikzcd}$$

and $\forall s \in S$, $\mu_*[\mathcal{C}_s] = \beta \in H_2(X;\mathbb{Z})$

pf: For $g=0$, $(\pi_M : \mathcal{U} \to \overline{M}_{0,n}(\mathbb{P}^r, d, \bar{t}), \{p_i\}, \{q_{ij}\}, \mu)$: universal family over $\overline{M}_{0,n}(\mathbb{P}^r, d, \bar{t})$

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\mu"] \arrow[d, "\pi_M"'] & \mathbb{P}^r \\ \overline{M}_{0,n}(\mathbb{P}^r, d, \bar{t}) & \end{tikzcd}$$

For $k>0$, $\mu^*\mathcal{O}_{\mathbb{P}^r}(k)$ is an loc. free sheaf on $\mathcal{U}$

Claim: On genus 0 curve, any v.b. g.b.g.s has no higher cohomology.

pf: For $C = \mathbb{P}^1$, E: v.b. on $\mathbb{P}^1 \Rightarrow E \simeq \bigoplus_i \mathcal{O}(m_i)$ $m_i \in \mathbb{Z}$

$\because E$ is g.b.g.s. $\therefore \forall m_i \geq 0 \Rightarrow H^i(\mathbb{P}^1, E) = 0$ for $i>0$.

Now, for C = tree of $\mathbb{P}^1$'s. We prove by induction on irred. comp. of $\mathbb{P}^1$:

$C = C_1 \cup C_2$ $j_i : C_i \hookrightarrow C$, we have: $0 \to \mathcal{O}_C \to (j_1)_*\mathcal{O}_{C_1} \oplus (j_2)_*\mathcal{O}_{C_2} \to j_*\mathcal{O}_p \to 0$

$C_i \simeq \mathbb{P}^1$ $C_1 \cap C_2 = \{p\}$ $j: \{p\} \hookrightarrow C$

$\to 0 \to E \to (j_1)_*\mathcal{O}_{C_1} \otimes E \oplus (j_2)_*\mathcal{O}_{C_2} \otimes E \to j_*\mathcal{O}_p \otimes E \to 0$

For $i>1$, $H^{i-1}(C, j_*\mathcal{O}_p \otimes E) = H^{i-1}(C, j_*(\mathcal{O}_p \otimes j^*E)) = H^{i-1}(\{p\}, \mathcal{O}_p \otimes j^*E)$

$= 0$ (By Grothendieck vanishing)

$H^i(C, (j_1)_*\mathcal{O}_{C_1} \otimes E) \oplus (j_2)_*\mathcal{O}_{C_2} \otimes E) = H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1} \otimes j_k^*E) = 0$ for $k=1,2$

$\Rightarrow H^i(C, E) = 0$ for $i>1$

Now, $\Gamma(C, (j_1)_*\mathcal{O}_C \otimes E) \oplus \Gamma(C, (j_2)_*\mathcal{O}_{C_2} \otimes E) \twoheadrightarrow \Gamma(C, j_*\mathcal{O}_p \otimes E)$

$\Rightarrow H^1(C, E) = 0$ $\square$

Also, $(\pi_M)_*(\mu^*\mathcal{O}_{\mathbb{P}^r}(k))_p \twoheadrightarrow H^0(\mathcal{U}, \mu^*\mathcal{O}_{\mathbb{P}^r}(k) \otimes k(p))$

By cohomology and base change, $(\pi_M)_*(\mu^*\mathcal{O}_{\mathbb{P}^r}(k))$ is a v.b. with fiber $H^0(\mathcal{C}_s, \mu^*\mathcal{O}_{\mathbb{P}^r}(k))$

Now, let $\mathcal{I}_X$: ideal sheaf of $X \overset{\iota}{\hookrightarrow} \mathbb{P}^r$ $\mathcal{I}_X = \ker(\iota^\# : \mathcal{O}_{\mathbb{P}^r} \longrightarrow \iota_* \mathcal{O}_X)$

$\rightsquigarrow 0 \to \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}^r} \longrightarrow \iota_* \mathcal{O}_X$ By Serre's thm, take $l >> 0$ s.t.

$\mathcal{I}_X(l)$ is g.b.g.s. $\Rightarrow$ $0 \to \mathcal{I}_X(l) \longrightarrow \mathcal{O}_{\mathbb{P}^r}(l)$ Then global section of $\mathcal{I}_X(l)$

gives global section of $(\pi_M)_* \mu^* \mathcal{O}_{\mathbb{P}^r}(l)$, say $s_1, \dots, s_r$

$$Z = Z(s_1) \cap \dots \cap Z(s_r) \quad \text{(Scheme-theoretic intersection)}$$

$\to$ $Z \subset \overline{M}_{0,n}(\mathbb{P}^r, d, \bar{\tau})$ and $\mu^{-1}|_{\pi_M^{-1}(Z)} : \pi_M^{-1}(Z) \longrightarrow \mathbb{P}^r$

$$\begin{tikzcd} \pi_M^{-1}(Z) \arrow[r] \arrow[dr] & \mathbb{P}^r \\ & X \arrow[u, hook, "\iota"'] \end{tikzcd}$$

$\because$ Z is of finite type over $\mathbb{C}$

$\therefore$ Z is quasi-cpt. $\Rightarrow$ $Z = Z_1 \cup Z_2 \cup \dots \cup Z_r$ Z_i : connected components

Then since for each i, pts in Z_i represent the same homlogy class

Z_β = union of the components of Z consisting of maps representing

$\beta \in H_2(X;\mathbb{Z})$. Take $\overline{M}_{0,n}(X, \beta, \bar{\tau}) = Z_\beta$ $\square$

Clearly, $\overline{M}_{0,n}(X, \beta, \bar{\tau})$ is G-inv. $\Rightarrow$ We can form $\overline{M}_{0,n}(X, \beta, \bar{\tau})/G$

Then patching as $X = \mathbb{P}^r$ case $\rightsquigarrow$ We obtain a closed

subscheme $\overline{M}_{0,n}(X, \beta) \subset \overline{M}_{0,n}(\mathbb{P}^r, d)$

Now, for different choices of $\iota, \iota' : X \hookrightarrow \mathbb{P}^r$

Universal property of $\overline{M}_{0,n}(X, \beta)$ $\Rightarrow$ They coincide.

<u>Rmk</u>: Projectivity of $\overline{M}_{0,n}(X, \beta)$ follows from proj. of $\overline{M}_{0,n}(\mathbb{P}^r, d)$

$\square$