

Final Report for "Topics on Frobenius Manifolds" (Spring, 2018)

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Topic: Gromov-Witten Invariants and Quantum Cohomology

Ref:

Main reference:

• Fulton, Pandharipande - Note on Stable Maps and Quantum Cohomology, 1997

Others reference:

Marin - Frobenius Manifolds, Quantum Cohomology and Moduli Spaces, AMS.

Kock, Vainsencher - An Invitation to Quantum Cohomology, Birkhäuser.

Quantum Cohomology at the Mittag-Leffler Institute (Ed. by Paolo Aluffi)

Plan:

I. Moduli Spaces of Stable Maps

II. Gromov-Witten Invariants

III. Quantum Cohomology

→ The construction of moduli space of stable maps will leave to the next two talks

I. Moduli Spaces of Stable Maps :

1. Moduli of Stable Curves :

a) M_g : moduli space of projective non-sing. curves C of genus g over \mathbb{C} mod. auto.

\rightarrow quasi-projective algebraic variety with $\dim M_g = 3g - 3$ ($g \geq 2$)

Crucial fact: When $g=0$, $C \cong \mathbb{P}^1$ $\text{Aut}(C) = \text{PGL}(2; \mathbb{C})$ i.e. Möbius transf.

When $g=1$, $C \cong \mathbb{C}/\Lambda$ Then $\text{Aut}(C)$ contains the translation $z \mapsto z+a$

Only when $g \geq 2$, $|\text{Aut}(C)| < \infty$

$\rightarrow M_g$ exists only when $g \geq 2$ and it is an alge. var. with orbifold sing.

b) $M_{g,n}$: moduli space of proj. non-sing. ^{irred.} curves C of genus g with n distinct marked pt. $p_1, \dots, p_n \in C$

\rightarrow quasi-projective var. with $\dim M_{g,n} = 3g - 3 + n$

For $g=0$, $M_{0,n}$ exists if $n \geq 3$. $g=1$, $M_{1,n}$ exists if $n \geq 1$.

e.g. $M_{0,3} = \{(\mathbb{P}^1, 0, 1, \infty)\}$ $M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ (formally) singularity are loc. cpx-analytically

isom. to $x^2y=0$ with arithmetic genus g

c) Compactification of $M_{g,n}$:

Def: A ^{quasi-stable} ^{genus g} n -pointed curve is a proj., connected nodal curve with n distinct non-sing. marked pt. p_1, \dots, p_n on C . If $|\text{Aut}(C, \{p_i\})| < \infty$, we say C is stable

\rightarrow Consider $\varphi \in \text{Aut}(C, \{p_i\})$ Taking normalization $\tilde{C} \xrightarrow{\pi} C$, φ lifts uniquely to $\tilde{\varphi} \in \text{Aut}(\tilde{C}, \{q_i\})$, where $q_i = \pi^{-1}(p_i)$ ($\because q_i$ are non-sing.) and $\tilde{\varphi}$ fixes irred. comp.

$\Rightarrow \tilde{\varphi}$ preserving special pts = preimage of singular pts and marked pt.

Thus, the stability condition can be reformulate into :

(C, p_1, \dots, p_n) is stable iff for each genus 0 component of \tilde{C} has at least 3 special pts
genus 1 component of \tilde{C} has at least 1 special pts

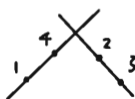
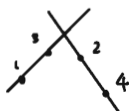
S : alge. scheme over \mathbb{C} A family of (quasi-)stable over S is a morphism $\pi: C \rightarrow S$ with $\{p_i: S \rightarrow C \mid i=1, \dots, n\}$ distinct section s.t. $\forall s \in S$, $(C_s, \{p_i|_s\})$ are ^{flat, proj.} ^{(quasi-) stable curves}

Thm: (Deligne-Mumford-Knudsen) \exists coarse moduli space $\bar{M}_{g,n}$ of stable n -pointed genus g curve. $\bar{M}_{g,n}$: proj. variety of dim $3g - 3 + n$, and $M_{g,n} \subset_{\text{open}} \bar{M}_{g,n}$

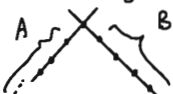
Note that in the case of $g=0$, a stable n -pointed curve ($n \geq 3$) has no non-trivial automorphism $\rightarrow \bar{M}_{0,n}$ is a fine moduli space and is a non-sing. variety

In the genus 0 case, a n -pointed stable curve is just a tree of \mathbb{P}^1 satisfying stability condition. (Any two irred. components are either disjoint or intersects transversally at one pt)

e.g. $\bar{M}_{0,3} = M_{0,3} = \{pt.\}$ $\bar{M}_{0,4} = \mathbb{P}^1$ The three added pts are represented by:



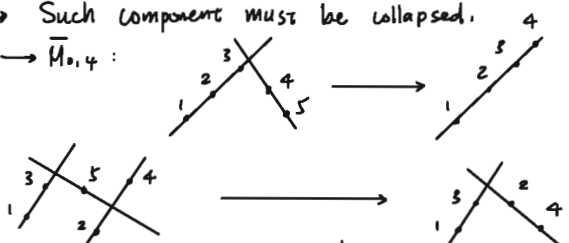
• Boundary of $\bar{M}_{0,n}$: $A \cup B = \{1, \dots, n\}$: A partition of n with $|A|, |B| \geq 2$
 \rightarrow A divisor $D(A|B)$. A general pt. of $D(A|B)$ is represented by:



We use a finite set A to label the marked pt. For $B \in A$ ($|B| \geq 3$),
 We have a "forget points" morphism: $\bar{M}_{0,A} \rightarrow \bar{M}_{0,B}$

On $M_{0,A}$, this is obvious. However, on the boundary $\bar{M}_{0,A} \setminus M_{0,A}$, the existence of this morphism is not trivial since removing some pts may make a component unstable \rightarrow Such component must be collapsed.

e.g. $\bar{M}_{0,5} \rightarrow \bar{M}_{0,4}$:



$\forall \{i, j, k, \ell\} \subseteq \{1, \dots, n\}$, we have: $\bar{M}_{0,n} \xrightarrow{\phi} \bar{M}_{0, \{i, j, k, \ell\}}$ Then $P(i, j | k, \ell) = \sum_{i, j \in A} \sum_{k, \ell \in B} D(A|B) \in \bar{M}_{0, \{i, j, k, \ell\}}$

$\phi^*(P(i, j | k, \ell))$ is a divisor on $\bar{M}_{0,n}$.
Fact (1) $\phi^*(P(i, j | k, \ell)) = \sum D(A|B)$ over all partition $A \cup B = \{1, \dots, n\}$ with $k, \ell \in B$

Since $\bar{M}_{0, \{i, j, k, \ell\}} = \mathbb{P}^1$ and on \mathbb{P}^1 , any two points are linearly equiv. (Hartshorne, II.6.10.1)
 \Rightarrow The three boundary point in $\bar{M}_{0, \{i, j, k, \ell\}}$ are linearly equiv.
 \Rightarrow Their preimage are linearly equiv.

Thus, we have:

$$\sum_{\substack{i, j \in A \\ k, \ell \in B}} D(A|B) = \sum_{\substack{i, k \in A \\ j, \ell \in B}} D(A|B) = \sum_{\substack{i, \ell \in A \\ j, k \in B}} D(A|B) \text{ in } A^1(\bar{M}_{0,n}) - \langle \star \rangle$$

Rmk: Keel proves that $D(A|B)$ generates the Chow ring together with $\langle \star \rangle$ and $D(A|B) \cdot D(A'|B') = 0$ if there are no inclusion among A, B, A', B'
 \rightarrow Gives a complete set of relations.

2. Stable Maps:

Def: X : non-sing. proj. var. $\beta \in H_2(X; \mathbb{Z})$ S : alge. scheme over \mathbb{C}

(1) A family of maps over S from n -pointed genus g curves to X consists of

(a) $\pi: C \rightarrow S$ with sections $\{p_1, \dots, p_n\}$ is a family of quasi-stable curve over S

(b) A morphism $\mu: C \rightarrow X$.

(2) Two families of maps over S : $(C \xrightarrow{\pi} S, \{p_i\}, \mu)$, $(C' \xrightarrow{\pi'} S, \{p'_i\}, \mu')$ are isom. if $\exists \tau: C \xrightarrow{\sim} C'$ s.t.

(3) $(C, \{p_i\}, \mu)$: maps from a n -pointed, quasi-stable curve to X

Automorphism of the maps is $\tau: C \xrightarrow{\sim} C$ s.t. $\mu \circ \tau = \mu$, $p_i = \tau(p_i)$

We say $(C, \{p_i\}, \mu)$ represents β if $\mu_*([C]) = \beta \in H_2(X; \mathbb{Z})$

We say $E \subset C$ is contracted by μ if E is mapped to a pt by μ .

Def: $(C, \{p_i\}, \mu)$ is stable if it satisfies the following equiv. condition

(i) $|\text{Aut}(C, \{p_i\}, \mu)| < \infty$

(ii) If $E \subset C$: irred. component contracted by μ , then $(E, \{\text{special pts on } E\})$ is a stable curve (i.e. If $E \cong \mathbb{P}^1/p_a(E)=1$, E has at least 3/1 special pts)

(iii) \Rightarrow (ii) If \exists an unstable component E contracted by μ Then $|\text{Aut}(E, \{\text{special pt on } E\})| = \infty$

Then extends these automorphism to whole C by setting identity on other component

$\therefore E$ is sent to a pt. \therefore These automorphism commutes w/ $\mu \Rightarrow \times$

(iii) \Rightarrow (ii) Suppose if we have an unstable irred. component E , then E must not be

contracted by μ (def.) $\phi \in \text{Aut}(C, \{p_i\}, \mu)$ $E' = \phi(E)$ $\mu|_{E'} \circ \phi|_E = \mu|_E$

$\therefore X$ is proj. \therefore May assume $X = \mathbb{P}^N$ $\mu: E \rightarrow \mathbb{P}^N$ non-const.

$\because \dim E = 1$, E irred. $\Rightarrow \mu(E)$ has at most $\dim 1$, irred. If $\dim \mu(E) = 0$, E is contracted by μ

If $\dim \mu(E) = 1$, then $\forall p \in \mu(E)$, $\tilde{\mu}^{-1}(p)$ is proper closed subset of $E \Rightarrow \tilde{\mu}^{-1}(p)$ is finite ($\because E$ is proper)

Also, $\mu: E \rightarrow \mathbb{P}^N$ is projective $\Rightarrow \mu$ is finite. Therefore, μ is

branched cover on $\mu(E) \Rightarrow$ Such $\phi|_E$ must be finite.

(4) A family of maps $(\pi: C \rightarrow S, \{p_i\}, \mu)$ is stable if $\forall s \in S$, the geometric fiber $(C_s, \{p_i(s)\}) \rightarrow X$ is stable.

We define a moduli functor $\bar{M}_{g,n}(X, \beta): (\mathbb{C}\text{-alge. scheme}) \rightarrow (\text{Set})$ by

$\bar{M}_{g,n}(X, \beta)(S) = \{ \text{Isom. classes of families of stable map of } n\text{-pointed, genus } g \text{ curves over } S \text{ to } X \text{ representing the class } \beta \}$

Thm 1: \exists projective, coarse moduli space $\overline{M}_{g,n}(X, \beta)$

i.e. $\overline{M}_{g,n}(X, \beta)$ is a scheme with natural transf. $\phi: \overline{M}_{g,n}(X, \beta) \rightarrow h_{\overline{M}_{g,n}(X, \beta)}$

s.t. (1) $\phi(\text{Spec } \mathbb{C}) = \overline{M}_{g,n}(X, \beta)(\text{Spec } \mathbb{C}) \xrightarrow{\cong} \text{Hom}(\text{Spec } \mathbb{C}, \overline{M}_{g,n}(X, \beta))$

(2) Z : scheme $\psi: \overline{M}_{g,n}(X, \beta) \rightarrow h_Z$, then $\exists!$ $\gamma: \overline{M}_{g,n}(X, \beta) \rightarrow Z$ s.t.

$\psi = \gamma \circ \phi$ (where $\bar{\gamma}: h_{\overline{M}_{g,n}(X, \beta)} \rightarrow h_Z$ defined by $f \mapsto \tau \circ f$)

$\overline{M}_{g,n}^*(X, \beta) \subset \overline{M}_{g,n}(X, \beta)$: automorphism-free stable maps

$\overline{M}_{g,n}^*(X, \beta) := \overline{M}_{g,n}^*(X, \beta) \cap \overline{M}_{g,n}(X, \beta)$

Now, we focus on the case of moduli spaces of genus 0, n -pointed stable curves:

Def: A non-singular proj. variety X is convex if $\forall \mu: \mathbb{P}^1 \rightarrow X$,

$H^1(\mathbb{P}^1, \mu^* T_X) = 0$, where T_X : tangent sheaf of X .

Thm 2: X : proj. non-sing. convex variety

(i) $\overline{M}_{0,n}(X, \beta)$ is a normal projective variety (not necessarily irreducible)

of pure dim. = $\dim X + \int_{\mathbb{P}^1} c_1(T_X) + n - 3$

(ii) $\overline{M}_{0,n}(X, \beta)$ is locally quot. of a non-sing. var. by finite group

(iii) $\overline{M}_{0,n}^*(X, \beta)$ is a non-sing., fine moduli space (for automorphism-free stable maps)

Def: The boundary of $\overline{M}_{0,n}(X, \beta)$ consists of $\{(C, \{p_i\}, \mu)\}$ with C reducible.

Thm 3: X : non-sing., proj., convex variety. The boundary of $\overline{M}_{0,n}(X, \beta)$ is a

normal crossing divisor up to a finite gp quot.

$\rightarrow D$: effective Cartier divisor D is simple normal crossing at p if X is reg. at p and \exists nbd U and local coord. x_1, \dots, x_n (i.e. $\{\bar{x}_1, \dots, \bar{x}_n\}$ forms a k_p -basis of $\mathfrak{m}_p/\mathfrak{m}_p^2$) s.t. $U \cap \text{supp}(D) \subset (x_1 \cdots x_n = 0)$.

D is normal crossing at p if \exists étale nbd $\pi: (X', p') \rightarrow (X, p)$ s.t.

$\pi^* D$ is snc at p .

Convention: $X = \mathbb{P}^r$, we write $\overline{M}_{g,n}(\mathbb{P}^r, d)$ instead of $\overline{M}_{g,n}(\mathbb{P}^r, d[\text{line}])$

Example: When $\beta = 0$, then $\mu: \mathbb{C} \rightarrow X$ is a const. map $\Rightarrow \overline{M}_{0,n}(X, 0) = \overline{M}_{0,n} \times X$

In particular, when $X = \{\text{pt.}\} \Rightarrow \beta = 0$. $\overline{M}_{0,n}(\{\text{pt.}\}, 0) \simeq \overline{M}_{0,n}$

• When X contains no rational curves, $\overline{M}_{0,n}(X, \beta) = \emptyset$ unless $\beta = 0$.

• $\overline{M}_{0,0}(\mathbb{P}^r, 1) = \mathbb{G}(1, r)$

• $\overline{M}_{0,1}(\mathbb{P}^r, 1) = \text{tautological line bundle over } \mathbb{G}(1, r)$.

• Natural morphism:

(1) evaluation map: $\mathcal{J} = (\downarrow_S \pi, \{P_i\}, \mu) \in \bar{M}_{g,n}(X, \beta)(S)$, define $\theta_i: \bar{M}_{g,n}(X, \beta) \rightarrow h_X$ by:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\mu} & X \\ \uparrow \pi & & \downarrow \pi \\ S & \xrightarrow{\mu} & X \end{array} \quad \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} \begin{array}{ccc} S & \xrightarrow{P_i} & \mathbb{C} \xrightarrow{\mu} X \end{array}$$

By thm 1 $\Rightarrow \exists!$ $ev_i: \bar{M}_{g,n}(X, \beta) \rightarrow X$ s.t. $\theta_i = \bar{ev}_i \circ \phi$

Put $S = \text{Spec } \mathbb{C} \Rightarrow ev_i: \bar{M}_{g,n}(X, \beta) \rightarrow X$

$$[(\mathbb{C}, P_1, \dots, P_n, \mu)] \mapsto \mu(P_i)$$

(2) forgetful map: $\because \bar{M}_{g,n}$ is a coarse moduli space of n -pointed genus g stable curve (for $2g-2+n > 0$)

Given $\mathbb{C} \xrightarrow{\mu} X$ in $\bar{M}_{g,n}(X, \beta)(S) \Rightarrow \exists$ morphism $S \rightarrow \bar{M}_{g,n}$

$\uparrow \pi$
By universal property of $\bar{M}_{g,n}(X, \beta)$

$$\Rightarrow \eta: \bar{M}_{g,n}(X, \beta) \rightarrow \bar{M}_{g,n}$$

Rmk: For X : homogeneous, Kim-Pandharipande proves that $\bar{M}_{0,n}(X, \beta)$ is in fact irreducible.

II. Gromov-Witten Invariants:

X : homogeneous variety i.e. $X = G/P$, where G : linear algebraic gp P : parabolic subgp of G

Lemma 1: X is a non-sing. proj, convex variety

pf: $\because G \rightarrow X$ transitively $\therefore T_x$ is generated by global section.

Then $\mu^*(T_X)$ is g.b.g.s. for any morphism $\mu: \mathbb{P}^1 \rightarrow X$

i.e. $\exists \{s_1, \dots, s_n\} \in \Gamma(\mathbb{P}^1, \mu^*T_X)$ s.t. $\bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1} \langle s_i \rangle \rightarrow \mu^*T_X \rightarrow 0$

Let $\mathcal{R} := \ker \left(\bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1} \langle s_i \rangle \rightarrow \mu^*T_X \right)$

$\rightarrow 0 \rightarrow \mathcal{R} \rightarrow \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1} \rightarrow \mu^*T_X \rightarrow 0$ induces long exact seq. in cohomology

$$\Rightarrow 0 \rightarrow H^*(\mathbb{P}^1, \mathcal{R}) \rightarrow H^*(\mathbb{P}^1, \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}) \rightarrow H^*(\mathbb{P}^1, \mu^*T_X) \rightarrow H^1(\mathbb{P}^1, \mathcal{R})$$

$$\rightarrow H^1(\mathbb{P}^1, \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}) \rightarrow H^1(\mathbb{P}^1, \mu^*T_X) \rightarrow 0$$

Note that $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0 \Rightarrow H^1(\mathbb{P}^1, \mu^*T_X) = 0$

Or, we can use Grothendieck splitting thm $\Rightarrow \mu^*T_X = \bigoplus \mathcal{O}_{\mathbb{P}^1}(d_i)$ Then μ^*T_X is g.b.g.s. $\Rightarrow \forall d_i \geq 0 \Rightarrow H^1(\mathbb{P}^1, \mu^*T_X) = 0$

Properties for homogeneous variety X :

(1) X : non-sing. projective and convex (lemma 1)

(2) The Chow theory and topological homology theory are isomorphic

(i.e. $A_*(X) \cong H_{2*}(X; \mathbb{Z})$, $A^*(X) \cong H^{2*}(X; \mathbb{Z})$)

(3) $\beta \in H_2(X; \mathbb{Z})$ effective class if β is represented by some genus 0 pointed stable map

Then the effective classes of X is a cone:

$$\left\{ \sum_{i=1}^n a_i \beta_i \mid a_i \in \mathbb{Z}_{\geq 0}, \beta_i \text{ is of the form } L_{\beta}[\mathbb{P}^1], \text{ for some } \iota: \mathbb{P}^1 \hookrightarrow X \right\}$$

Notation: $c \in H^*(X; \mathbb{Z})$, $\beta \in H_2(X; \mathbb{Z})$ $\int_{\beta} c := \text{deg. } k \text{ component } c_k \text{ evaluate at } \beta$

When $\beta = [V]$, $V \subseteq X$: pure-dim'l closed subvar., we denote $\int_V c$

Recall from I, we have evaluation maps: $ev_i: \bar{M}_{g,n}(X, \beta) \rightarrow X$ ($i=1, \dots, n$)

Given $r_1, \dots, r_n \in H^*(X, \mathbb{Z})$, $ev_1^*(r_1) \cup \dots \cup ev_n^*(r_n) \in H^*(\bar{M}_{g,n}(X, \beta), \mathbb{Z})$

Def: n -pointed, genus 0 Gromov-Witten invariant is defined by:

$$I_{\beta}(r_1, \dots, r_n) = \int_{\bar{M}_{0,n}(X, \beta)} ev_1^*(r_1) \cup \dots \cup ev_n^*(r_n) \in \mathbb{Z}$$

Rmk: (1) If r_i are homogeneous, then $I_{\beta}(r_1, \dots, r_n) \neq 0$ only if $\sum \deg(r_i) = \dim \bar{M}_{0,n}(X, \beta)$

(2) Given permutation $\sigma \in S_n$, $\bar{M}_{g,n}(X, \beta)(S) \xrightarrow{\phi} \bar{M}_{g,n}(X, \beta)(S)$ induce via ϕ

$$\begin{array}{ccc} C \xrightarrow{\mu} X & \xrightarrow{\quad} & C \xrightarrow{\mu} X \\ \pi \downarrow \int_{p_1, \dots, p_n} & \xrightarrow{\quad} & \pi \downarrow \int_{p_{\sigma(1)}, \dots, p_{\sigma(n)}} \end{array}$$

An automorphism $\bar{M}_{0,n}(X, \beta) \xrightarrow{\Sigma(\sigma)} \Sigma(\sigma) \Rightarrow ev_{\sigma(i)} = ev_i \circ \Sigma(\sigma)$

Therefore, $I_\beta(\gamma_{\sigma(1)}, \dots, \gamma_{\sigma(n)}) = \int_{\bar{M}_{0,n}(X, \beta)} ev_1^*(\gamma_{\sigma(1)}) \cup \dots \cup ev_n^*(\gamma_{\sigma(n)})$

$$= \int_{\bar{M}_{0,n}(X, \beta)} (\Sigma(\sigma)^{-1})^* (ev_{\sigma(1)}^*(\gamma_{\sigma(1)}) \cup \dots \cup ev_{\sigma(n)}^*(\gamma_{\sigma(n)})) = \int_{\bar{M}_{0,n}(X, \beta)} ev_1^*(\gamma_1) \cup \dots \cup ev_n^*(\gamma_n)$$

$\Rightarrow I_\beta(\gamma_1, \dots, \gamma_n)$ is indep. of the permutation of γ_i 's.

\sim We may write GW mv. as $I_\beta(\gamma_1, \dots, \gamma_n)$ rather than $I_\beta(\gamma_{\sigma(1)}, \dots, \gamma_{\sigma(n)})$.

(3) 0-pointed invariant:

When $n=0$, we have 0-pointed $\in W$ inv. only when $\bar{M}_{0,0}(X, \beta)$ is zero-dim'l

By Thm 2 $\Rightarrow \dim X + \int_\beta c_1(T_X) = 3$

If $\dim X = 0$, then $\int_\beta c_1(T_X) = 0 \Rightarrow$ Impossible.

Now, if $\dim X > 0$, if $\beta = 0$, then $\bar{M}_{0,0}(X, 0) = \phi$ (Every map is const.)

\Rightarrow Every irred. component is contracted and there is no marked pt. to stabilize them)

So, assume $\beta \neq 0$, we need following lemma:

Lemma 2: $\mu: \mathbb{P}^1 \rightarrow X$ non-const morphism to a non-sing, proj., convex var.

, then $\int_{\mu^*} c_1(T_X) \geq 2$.

pf: Consider $d\mu: T_{\mathbb{P}^1} \rightarrow \mu^* T_X \quad \therefore T_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(2) = \{ \text{homogeneous poly. of deg } 2 \}$

\rightarrow We can pick a generic $s \in H^0(\mathbb{P}^1, T_{\mathbb{P}^1})$ s.t. $s(p_1) = s(p_2) = 0$ $p_1, p_2 \in \mathbb{P}^1$ distinct

$\therefore \mu$ is non-const. $\therefore d\mu \neq 0 \Rightarrow d\mu(s) \in H^0(\mathbb{P}^1, \mu^* T_X) \neq 0$

and $d\mu(s)$ vanishes at least at p_1 and p_2 .

Now, $\mu^* T_X \simeq \bigoplus \mathcal{O}_{\mathbb{P}^1}(\alpha_j)$ $\alpha_j \in \mathbb{Z}$ If $\alpha_j < 0$ for some j , then if $\alpha_j \leq -2$

$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\alpha_j)) \neq 0 \Rightarrow H^1(\mathbb{P}^1, \mu^* T_X) \neq 0 \Rightarrow$ ~~✗~~ If $\alpha_j = -1$, then consider

$\tilde{\mu}: \mathbb{P}^1 \xrightarrow{\substack{\rightarrow \\ \substack{\cong \\ \mu}} \mathbb{P}^1} X$, then $H^1(\mathbb{P}^1, \tilde{\mu}^* T_X) \neq 0 \Rightarrow$ ~~✗~~. Thus, $\alpha_j \geq 0 \forall j$

Then existence of $d\mu(s) \in H^0(\mathbb{P}^1, \mu^* T_X) \neq 0 \Rightarrow \exists j$ s.t. $\alpha_j \geq 2$

$$\Rightarrow \int_{\mu^*} c_1(T_X) = \int_{\mathbb{P}^1} \mu^*(c_1(T_X)) = \int_{\mathbb{P}^1} c_1(\mu^* T_X) \geq \int_{\mathbb{P}^1} c_1(\mathcal{O}_{\mathbb{P}^1}(2)) = 2 \quad \square$$

By lemma 2 \Rightarrow Only possibility is $\dim X = 1$. $\int_\beta c_1(T_X) = 2$

Hence, for homogeneous variety, 0-pointed invariants occur only $X \simeq \mathbb{P}^1$

$\Rightarrow I_1 = 1$. The unique 0-pointed invariants

Lemma 3: If $n \geq 1$, then $M_{0,n}^*(X, \beta) \subset \bar{M}_{0,n}(X, \beta)$ is a dense open set

pf: If $\beta = 0$, then $\bar{M}_{0,n}(X, 0) \neq \emptyset$ only if $n \geq 3$.

For $n \geq 3$, $\bar{M}_{0,n}(X, 0) \cong \bar{M}_{0,n} \times X$ Then $M_{0,n}^*(X, 0) = \bar{M}_{0,n}^* \times X = \bar{M}_{0,n} \times X = \bar{M}_{0,n}(X, 0) \Rightarrow M_{0,n}^*(X, 0) \subset_{\text{dense}} \bar{M}_{0,n}(X, 0)$.

If $\beta \neq 0$, by thm 3 $\Rightarrow M_{0,n}(X, \beta) \subset_{\text{open}} \bar{M}_{0,n}(X, \beta)$

$[(\mathbb{P}^1, \{p_i\}, \mu)] \in M_{0,n}(X, \beta) \because \beta \neq 0 \therefore \mu$ is non-const.

Let $A \subset \text{PGL}(2; \mathbb{C})$ s.t. $\forall g \in A, g \circ \mu = \mu$. Then $|A| < \infty$

$\Rightarrow \exists U \subset_{\text{open}} \mathbb{P}^1$ s.t. $U = \{x \in \mathbb{P}^1 \mid \text{Stab}_A x = \{\text{id}\}\}$

Take $p'_1, \dots, p'_n \in U \Rightarrow (\mathbb{P}^1, \{p'_i\}, \mu)$ is automorphism-free

\Rightarrow For generic marked pts $\{p'_i\} \in \mathbb{P}^1, (\mathbb{P}^1, \{p'_i\}, \mu)$ is auto-free

$\Rightarrow M_{0,n}^*(X, \beta) \subset_{\text{dense}} M_{0,n}(X, \beta)$ a

Now, say $\Gamma_1, \dots, \Gamma_n$: pure dimensional subvar. of X [Γ_i]: corresponding class in $H^1(X; \mathbb{Z})$ via Poincaré duality. Then $\deg(\Gamma_i) = \text{codim}(\Gamma_i)$

Assume $\sum_{i=1}^n \text{codim}(\Gamma_i) = \dim(X) + \int c_1(T_X) + n - 3$

Lemma 4: For $n \geq 0, g_1, \dots, g_n$: generic element of G , the scheme theoretic intersection $ev_1^{-1}(g_1 \Gamma_1) \cap \dots \cap ev_n^{-1}(g_n \Gamma_n)$ is a finite set of reduced pts supported in $M_{0,n}(X, \beta)$, and we have:

$$I_\beta(\gamma_1, \dots, \gamma_n) = \# \{ev_1^{-1}(g_1 \Gamma_1) \cap \dots \cap ev_n^{-1}(g_n \Gamma_n)\} \quad (*)$$

pf: If $n=0$, then $I_1 = 1$ on \mathbb{P}^1 , and $\bar{M}_{0,0}(\mathbb{P}^1, 1) = M_{0,0}^*(\mathbb{P}^1, 1) = \{\text{pt.}\}$

If $n \geq 1, G^n := G \times \dots \times G, X^n := X \times \dots \times X$ Then $G^n \curvearrowright X^n$ transitive

Then we have $ev := (ev_1, \dots, ev_n): \bar{M}_{0,n}(X, \beta) \rightarrow X^n, \bar{M}_{0,n}(X, \beta)$

Step 1: apply Kleiman's thm to:

$$\Gamma := \Gamma_1 \times \dots \times \Gamma_n \hookrightarrow X^n \xrightarrow{ev}$$

Then note that if $\dim(\Gamma^\epsilon \times_{X^n} \bar{M}_{0,n}^{a,n}(X, \beta)) = \dim \Gamma + \bar{M}_{0,n}^{a,n}(X, \beta) - \dim(X^n)$
 $= \dim \bar{M}_{0,n}^{a,n}(X, \beta) - \sum_{i=1}^n \text{codim}(\Gamma_i) = 0$

$\Rightarrow \exists V_{\text{open}} \subset G^n$ s.t. $\forall \sigma \in V_1, \Gamma^\epsilon \times_{X^n} \bar{M}_{0,n}^{a,n}(X, \beta)$ is either \emptyset or dim 0.

Step 2: consider $M_{0,n}^*(X, \beta)^c := \bar{M}_{0,n}(X, \beta) \setminus M_{0,n}^*(X, \beta)$, whose $\text{codim} \geq 1$

Apply Kleiman's thm to:

$$\Gamma^c \hookrightarrow \bar{M}_{0,n}^*(X, \beta)^c$$

Then for $V_2 \subset_{\text{open}} G^n$, $\sigma \in V_2$, if $\Gamma^\sigma \times_{X^n} M_{0,n}^*(X, \beta)^c \neq \emptyset$, then $\dim(\Gamma^\sigma \times_{X^n} M_{0,n}^*(X, \beta)^c) = \dim \Gamma + \dim M_{0,n}^*(X, \beta)^c - \dim(X^n) = \dim(M_{0,n}^*(X, \beta)^c) - \sum_{i=1}^n \text{codim}(\Gamma_i) < 0$

$\Rightarrow \Gamma^\sigma \times_{X^n} M_{0,n}^*(X, \beta)^c = \emptyset$ for $\sigma \in V_2$

Thus, for $(g_1, \dots, g_n) \in V_1 \cap V_2 \stackrel{\text{dense}}{\subset} G^n$, $\bigcap_{i=1}^n \text{ev}_i^{-1}(g_i; \Gamma_i)$ is a 0-dim'l and supported in $M_{0,n}^*(X, \beta)$

Step 3: Let $\Gamma'_i := \text{sing}(\Gamma_i)$: sing. locus of Γ_i , $\Gamma' := (\Gamma'_i)_{i=1}^n$, $M_{0,n}^*(X, \beta)$

Next, we apply Kleiman's thm again to $\Gamma' \hookrightarrow X^n$, and counting dim.

$\Rightarrow \exists V_3 \subset_{\text{open}} G^n$ s.t. $\bigcap_{i=1}^n \text{ev}_i^{-1}(g_i; \Gamma'_i) = \emptyset$ or $\dim < 0$ for $(g_i)_{i=1}^n \in V_3$.

Thus, we may assume Γ_i are non-sing.

Step 4: Finally, we apply Kleiman's thm to $\Gamma \hookrightarrow X^n \Rightarrow \bigcap_{i=1}^n \text{ev}_i^{-1}(g_i; \Gamma_i)$ is regular

\Rightarrow The intersection is a finite set of reduced pts supp. in $M_{0,n}^*(X, \beta)$

Step 5: Observe that we have the fibred product:

$$\begin{array}{ccc} \bigcap_{i=1}^n \text{ev}_i^{-1}(g_i; \Gamma_i) & \longrightarrow & \bar{M}_{0,n}(X, \beta) \times_{i=1}^n g_i; \Gamma_i \\ \downarrow & & \downarrow \\ \bar{M}_{0,n}(X, \beta) & \xrightarrow{L = (\text{id}, \text{ev})} & \bar{M}_{0,n}(X, \beta) \times X^n \\ \Rightarrow [\bigcap_{i=1}^n \text{ev}_i^{-1}(g_i; \Gamma_i)] & = & L^* [\bar{M}_{0,n}(X, \beta) \times_{i=1}^n g_i; \Gamma_i] = (\text{id}, \text{ev})^* [\bar{M}_{0,n}(X, \beta) \times_{i=1}^n g_i; \Gamma_i] \\ & = & [\bar{M}_{0,n}(X, \beta)] \cap [\prod_{i=1}^n \text{ev}_i^* (g_i; \Gamma_i)] \text{ in } A_0(\bar{M}_{0,n}(X, \beta)) \end{array}$$

$$= I_\beta(r_1, \dots, r_n) \quad a$$

• Three Basic Properties:

(I) $\beta=0$, $\bar{M}_{0,n}(X, \beta) \simeq \bar{M}_{0,n} \times X$ and $\text{ev}_i = p: \bar{M}_{0,n} \times X \rightarrow X$ for $i=1, \dots, n$
 $\text{ev}_i^*(r_1) \cup \dots \cup \text{ev}_n^*(r_n) = p^*(r_1 \cup \dots \cup r_n)$

$$I_\beta(r_1, \dots, r_n) = \int_{\bar{M}_{0,n} \times X} p^*(r_1 \cup \dots \cup r_n) = \int_{p^*[\bar{M}_{0,n} \times X]} r_1 \cup \dots \cup r_n$$

for $n=0, 1, 2$, $\bar{M}_{0,n} = \emptyset$ If $n \geq 3$, then $\dim \bar{M}_{0,n} > 0 \Rightarrow p_*[\bar{M}_{0,n} \times X] = 0$

Only case: $n=3$, $\bar{M}_{0,3} = \{pt\}$

$$\Rightarrow I_0(r_1, r_2, r_3) = \int_X r_1 \cup r_2 \cup r_3$$

(II) $\gamma_i = 1 \in H^0(X)$

If $\beta \neq 0$, $ev_1^*(\gamma_1) \cup \dots \cup ev_n^*(\gamma_n) = ev_2^*(\gamma_2) \cup \dots \cup ev_n^*(\gamma_n)$

$\Rightarrow \exists \omega \in H^1(\bar{M}_{0,n-1}(X, \beta))$ s.t. $ev_2^*(\gamma_2) \cup \dots \cup ev_n^*(\gamma_n) = \phi_1^* \omega$,

where $\phi_1: \bar{M}_{0,n}(X, \beta) \rightarrow \bar{M}_{0,n-1}(X, \beta)$ forget one point

$$[(C, p_1, \dots, p_n, \mu)] \mapsto [(C, p_2, \dots, p_{n-1}, \mu)]$$

$$I_\beta(1, \gamma_2, \dots, \gamma_n) = \int_{\bar{M}_{0,n}(X, \beta)} \phi_1^* \omega = \int_{(\phi_1)_*([\bar{M}_{0,n}(X, \beta)])} \omega$$

$$\therefore \dim \bar{M}_{0,n}(X, \beta) = 3g - 3 + n > \dim \bar{M}_{0,n-1} = 3g - 3 + (n-1)$$

$$\therefore (\phi_1)_*([\bar{M}_{0,n}(X, \beta)]) = 0$$

$$\Rightarrow I_\beta(1, \gamma_2, \dots, \gamma_n) = 0$$

If $\beta = 0$, we go back to case (I), where $n=3$

$$\Rightarrow I_0(1, \gamma_2, \gamma_3) = \int_X \gamma_2 \cup \gamma_3$$

(III) $\gamma_i \in H^*(X)$, $\beta \neq 0$, then we have:

$$I_\beta(\gamma_1, \dots, \gamma_n) = (\int_\beta \gamma_1) \cdot I_\beta(\gamma_2, \dots, \gamma_n)$$

Def: Consider $\psi: \bar{M}_{0,n}(X, \beta) \rightarrow X \times \bar{M}_{0,n-1}(X, \beta)$ by $\psi = ev_1 \times \phi_1$, ϕ_1 : forget first marked pt

Since X is homogeneous, $A_*(X \times \bar{M}_{0,n-1}(X, \beta)) \cong A_*(X) \otimes A_*(\bar{M}_{0,n-1}(X, \beta))$

Write $\psi_*([\bar{M}_{0,n}(X, \beta)]) = \beta' \times [\bar{M}_{0,n-1}(X, \beta)] + \alpha$, $\beta' \in A_1(X)$

α : some homology class supported over a proper closed subset of $\bar{M}_{0,n-1}(X, \beta)$

$$I_\beta(\gamma_1, \dots, \gamma_n) = \int_{[\bar{M}_{0,n}(X, \beta)]} ev_1^* \gamma_1 \cup \dots \cup ev_n^* \gamma_n = \int_{\psi_*([\bar{M}_{0,n}(X, \beta)])} \gamma_1 \times (ev_2^*(\gamma_2) \cup \dots \cup ev_n^*(\gamma_n))$$

$$= \int_{\beta' \times [\bar{M}_{0,n-1}(X, \beta)] + \alpha} \gamma_1 \times (ev_2^*(\gamma_2) \cup \dots \cup ev_n^*(\gamma_n)) = \left(\int_{\beta'} \gamma_1 \right) \cdot \int_{\bar{M}_{0,n-1}(X, \beta)} ev_2^* \gamma_2 \cup \dots \cup ev_n^* \gamma_n + \int_\alpha \gamma_1 \times (ev_2^* \gamma_2 \cup \dots \cup ev_n^* \gamma_n)$$

$$= (\int_\beta \gamma_1) \cdot I_\beta(\gamma_2, \dots, \gamma_n)$$

Claim: $\beta' = \beta$.

Consider a general pt. $\{s\} = \{[(C, \{p_2, \dots, p_n, \mu\})] \in \bar{M}_{0,n-1}(X, \beta)$

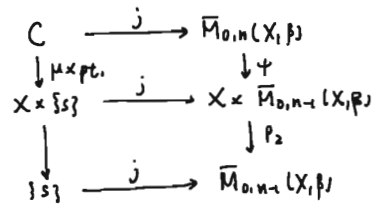
$$j^* \psi_*([\bar{M}_{0,n}(X, \beta)]) = (\mu, p_2, \dots, p_n) \# j^*[\bar{M}_{0,n-1}(X, \beta)] = (\mu, p_2, \dots, p_n) \# [C]$$

$$= \beta \times \{s\}$$

$$j^* \psi_*([\bar{M}_{0,n}(X, \beta)]) = j^*(\beta' \times [\bar{M}_{0,n-1}(X, \beta)] + \alpha)$$

$$= \beta' \times \{s\}$$

$$\Rightarrow \beta = \beta'$$



III, Quantum Cohomology:

X : homogeneous variety Then the Schubert classes form a natural basis for $H^*(X; \mathbb{Z})$

Notation: Fix a basis $\{\tau_0=1, \tau_1, \dots, \tau_p, \tau_{p+1}, \dots, \tau_m\}$ of $H^*(X; \mathbb{Z})$
 $H^*(X)$ $H^*(X)$ additive basis of other cohomology

For $1 \leq i, j \leq m$, we define $g_{ij} := \int_X \tau_i \cup \tau_j$ intersection form on X w.r.t. the natural basis. \rightarrow This plays the role of flat metric in Frobenius mfd

Define $(g^{ij}) = (g_{ij})^{-1}$ via $A^*(X \times X) \simeq A^*X \otimes A^*X \xrightarrow{\Delta} X \times X \xrightarrow{p_2} X$
 $[\Delta] = \sum g^{ij} \tau_i \otimes \tau_j$

$$\tau_i \cup \tau_j = \sum_{e, f} I_0(\tau_i, \tau_j, \tau_e) g^{ef} \tau_f$$

Lemma: $\gamma \in H^*(X; \mathbb{Z})$, $\gamma^n = \gamma \cup \dots \cup \gamma$ Given $n \in \mathbb{N}$, there are only finitely many effective $\beta \in H_2(X; \mathbb{Z})$ s.t. $I_\beta(\gamma^n) \neq 0$

Pf: $\because X$ is homogeneous var. $\therefore \beta \in H^2(X; \mathbb{Z})$, $\exists a_1, \dots, a_p \geq 0$ s.t. $\beta = \sum_{i=1}^p a_i \beta_i$

where β_i : effective classes

By lemma II-2 $\Rightarrow \int_{\beta_i} c_1(T_X) \geq 2$ Thus, given $N \in \mathbb{N}$, \exists finitely many effective class β s.t. $\int_{\beta} c_1(T_X) \leq N$.

Now, for $I_\beta(\gamma^n) \neq 0$, then $\dim \bar{M}_{0,n}(X, \beta) \leq n \cdot \dim X$

(Otherwise, $\dim \bar{M}_{0,n}(X, \beta) > n \cdot \dim X \Rightarrow \deg(\text{top homogeneous component of } \gamma^n) < \dim \bar{M}_{0,n}(X, \beta)$
 $\Rightarrow \dim \bar{M}_{0,n}(X, \beta) = \dim X + \int_{\beta} c_1(T_X) + n - 3 \leq n \dim X$

$$\Rightarrow \int_{\beta} c_1(T_X) \leq (n-1) \dim X + 3 - n$$

Now, define "potential" by: $\Phi(\gamma) = \sum_{n \geq 3} \sum_{\beta} \frac{1}{n!} I_\beta(\gamma^n)$

Now, write $\gamma = \sum_{i=0}^m \gamma_i \tau_i$ By lemma 15 $\Rightarrow \Phi(\gamma) = \Phi(\gamma_0, \dots, \gamma_m)$ becomes a formal power series in $\mathbb{Q}[[\gamma]] = \mathbb{Q}[[\gamma_0, \dots, \gamma_m]]$:

$$\Phi(\gamma_0, \dots, \gamma_m) = \sum_{n_0, \dots, n_m \geq 3} \sum_{\beta} I_\beta(\tau_0^{n_0}, \tau_1^{n_1}, \dots, \tau_m^{n_m}) \frac{\gamma_0^{n_0}}{n_0!} \dots \frac{\gamma_m^{n_m}}{n_m!}$$

Define $\Phi_{ijk} = \frac{\partial^3 \Phi}{\partial \gamma_i \partial \gamma_j \partial \gamma_k}$, for $0 \leq i, j, k \leq m$

Check: $\Phi_{ijk} = \sum_{n \geq 0} \sum_{\beta} \frac{1}{n!} I_{\beta}(r^n \cdot T_i \cdot T_j \cdot T_k) - (*)$

Note: $\frac{\partial \Phi}{\partial y^0} = \sum_{n_0 + \dots + n_m \geq 0} \sum_{\beta} I_{\beta}(T_0^{n_0} \dots T_m^{n_m}) \frac{y_0^{n_0-1}}{(n_0-1)!} \frac{y_1^{n_1}}{n_1!} \dots \frac{y_m^{n_m}}{n_m!}$

shifted n_i by 1 \rightarrow $= \sum_{n_0 + \dots + n_m \geq 2} \sum_{\beta} I_{\beta}(T_0^{n_0+1} \dots T_m^{n_m}) \frac{y_0^{n_0}}{n_0!} \frac{y_1^{n_1}}{n_1!} \dots \frac{y_m^{n_m}}{n_m!} = \sum_{n \geq 2} \sum_{\beta} \frac{1}{n!} I_{\beta}(r^n \cdot T_0)$

Similar for other indices and higher derivative.

Define a "quantum product" by: $T_i * T_j = \sum_{e,f} \Phi_{ije} g^{ef} T_f$

Then extends $*$ $\mathbb{Q}[[y]]$ -linearly to $H^*(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}[[y]] \rightarrow \mathbb{Q}[[y]]$ -algebra

Commutativity: Obvious. $\therefore \Phi_{ijk} = \Phi_{jik} \Rightarrow T_j * T_i = \sum_{e,f} \Phi_{jie} g^{ef} T_f = \sum_{e,f} \Phi_{ij} g^{ef} T_f$

T_0 is identity: $T_0 * T_j = \sum_{e,f} \Phi_{0je} g^{ef} T_f$

By (*), $\Phi_{0je} = \sum_{n \geq 0} \sum_{\beta} \frac{1}{n!} I_{\beta}(r^n \cdot 1 \cdot T_j \cdot T_e)$

By property (II) of GW mv. $\Rightarrow I_{\beta}(r^n \cdot 1 \cdot T_j \cdot T_e) \neq 0$ only if $\beta = 0, n = 0$

$\Rightarrow \Phi_{0je} = I_0(1 \cdot T_j \cdot T_e) = \int_X T_j \cup T_e = g_{je}$

$\Rightarrow T_0 * T_j = \sum_{e,f} g_{je} g^{ef} T_f = \sum_{e,f} \delta_j^f T_f = T_j$

Thm 4: The quantum product is associative

$(T_i * T_j) * T_k = \sum_{a,f} \Phi_{ija} g^{ef} T_f * T_k = \sum_{e,f} \sum_{c,d} \Phi_{ija} g^{ef} \Phi_{fkc} g^{cd} T_d$

$T_i * (T_j * T_k) = \sum_{a,f} \Phi_{jka} g^{ef} T_i * T_f = \sum_{e,f} \sum_{c,d} \Phi_{jka} g^{ef} \Phi_{ifc} g^{cd} T_d$

$\Rightarrow \sum_{e,f} \Phi_{ija} g^{ef} \Phi_{fkl} = \sum_{e,f} \Phi_{jka} g^{ef} \Phi_{ifl} = \sum_{e,f} \Phi_{jka} g^{ef} \Phi_{fil} \quad \forall l \rightsquigarrow \text{WDVV eqn}$

$F(i,j|k,l) := \sum_{e,f} \Phi_{ije} g^{ef} \Phi_{fkl}$ Then $\text{WDVV} \Leftrightarrow F(i,j|k,l) = F(j,i|l,e)$

By (*) \Rightarrow

$F(i,j|k,l) = \sum_{e,f} \sum_{n_1, n_2 \geq 0} \sum_{\beta_1, \beta_2} \frac{1}{n_1! n_2!} I_{\beta_1}(r^{n_1} \cdot T_i \cdot T_j \cdot T_e) g^{ef} I_{\beta_2}(r^{n_2} \cdot T_k \cdot T_l \cdot T_f)$

Fact: The boundary of $\overline{M}_{0,n}(X, \beta)$ consists of divisor $D(A, B; \beta_1, \beta_2)$

where i) $A \cup B = \{1, \dots, n\}$: A partition of $\{1, \dots, n\}$ iii) If $\beta_1 = 0$, then $|A| \geq 2$

ii) $\beta_1 + \beta_2 = \beta$, β_i : effective classes ($\beta_2 = 0$) ($|B| \geq 2$)

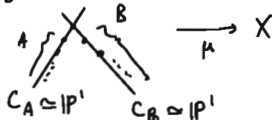
$D(A, B; \beta_1, \beta_2) = \text{locus of maps } \mu: C_A \cup C_B \rightarrow X \text{ s.t.}$

(a) $C = C_A \cup C_B$, C_A, C_B : quasi-stable curves of genus 0 meeting at a pt.

(b) The markings of A/B lie on C_A/C_B

(c) $\mu_A = \mu|_{C_A}$ represents β_1 $\mu_B = \mu|_{C_B}$ represents β_2

Generic pt. of $D(A, B; \beta_1, \beta_2)$:



Then $D(A, B; \beta_1, \beta_2) \simeq \bar{M}_0, \text{Aut}_{\mathbb{Z}}(X, \beta_1) \times_X \bar{M}_0, \text{Aut}_{\mathbb{Z}}(X, \beta_2)$

$\bar{M} := \bar{M}_{0,n}(X, \beta)$ $\bar{M}_A := \bar{M}_{0, \text{Aut}_{\mathbb{Z}}}(X, \beta_1)$, $\bar{M}_B := \bar{M}_{0, \text{Aut}_{\mathbb{Z}}}(X, \beta_2)$ $D := D(A, B; \beta_1, \beta_2)$

$\bar{M}_A \times \bar{M}_B := \bar{M}_{0, \text{Aut}_{\mathbb{Z}}}(X, \beta_1) \times_{\text{Spec } \mathbb{C}} \bar{M}_{0, \text{Aut}_{\mathbb{Z}}}(X, \beta_2)$

$$\begin{array}{ccccc}
 \bar{M} & \xleftarrow{\text{emb.}} & D & \xrightarrow{\text{inclusion}} & \bar{M}_A \times \bar{M}_B \\
 \downarrow \text{ev}_i = \text{ev}_1 \times \dots \times \text{ev}_n & & \downarrow \eta & & \downarrow \text{ev}' := \text{ev}_A \times \text{ev}_B \times \text{ev}_\bullet \times \text{ev}_\bullet \\
 X^n & \xleftarrow{p} & X^{n+1} & \xrightarrow{s} & X^{n+2} \\
 \text{proj.} & & \text{forget last} & & (X_1, \dots, X_{n+1}) \mapsto (X_1, \dots, X_{n+1}, X_{n+1}) \\
 \text{factor} & & & &
 \end{array}$$

Lemma 2: $\forall r_1, \dots, r_n \in H^*(X)$ $L_* \circ \alpha^* (ev_1^* r_1 \cup \dots \cup ev_n^* r_n)$

$$= \sum_{a, f} g^{a, f} \left(\prod_{a \in A} ev_a^*(\gamma_a) \cdot ev_\bullet^*(T_a) \right) \times \left(\prod_{b \in B} ev_b^*(\gamma_b) \cdot ev_\bullet^*(T_f) \right)$$

$$\text{pf: } L_* \circ \alpha^* (ev_1^* r_1 \cup \dots \cup ev_n^* r_n) = L_* \circ \alpha^* \circ ev^* (r_1 \times \dots \times r_n)$$

$$= L_* \circ (ev \circ \alpha)^* (r_1 \times \dots \times r_n) = L_* \circ \eta^* \circ p^* (r_1 \times \dots \times r_n)$$

$$= L_* \eta^* (r_1 \times \dots \times r_n \times [X]) = ev^* \circ \xi_* (r_1 \times \dots \times r_n \times [X])$$

$$= ev^* (r_1 \times \dots \times r_n \times [\Delta]) = \sum_{a, f} g^{a, f} ev^* (r_1 \times \dots \times r_n \times T_a \times T_f)$$

$$= \sum_{a, f} g^{a, f} \left(\prod_{a \in A} ev_a^*(\gamma_a) \cdot ev_\bullet^*(T_a) \right) \times \left(\prod_{b \in B} ev_b^*(\gamma_b) \cdot ev_\bullet^*(T_f) \right)$$

Now, fix $\beta \in H_2(X)$, $r_1, \dots, r_n \in H^*(X)$, $q, r, s, t \in \{1, \dots, n\}$: distinct

$$G(q, r | s, t) := \sum_{\substack{p_1 + p_2 = \beta \\ A \cup B = \{1, \dots, n\} \\ q, r \in A, s, t \in B}} \sum_{e, f} I_{\beta_1} \left(\prod_{a \in A} r_a T_a \right) g^{e, f} I_{\beta_2} \left(\prod_{b \in B} r_b T_b \right)$$

By lemma 2 $\Rightarrow G(q, r | s, t) = \sum_{\substack{A \cup B = \{1, \dots, n\} \\ \beta_1 + \beta_2 = \beta \\ q, r \in A, s, t \in B}} \int_{D(A, B | \beta_1, \beta_2)} e v_1^*(r_1) \cup \dots \cup e v_n^*(r_n)$

Lemma 3: For $i, j, k, l \in \{1, \dots, n\}$, distinct

$$D(i, j | k, l) = \sum_{\substack{A \cup B = \{1, \dots, n\} \\ \beta_1 + \beta_2 = \beta \\ i, j \in A, k, l \in B}} D(A, B | \beta_1, \beta_2) \text{ Then } D(i, j | k, l) \sim D(i, l | j, k) \text{ as divisors.}$$

pf: $\bar{M}_{0, n}(X, \beta) \xrightarrow{\eta} \bar{M}_{0, n} \xrightarrow{\phi} \bar{M}_{0, \{i, j, k, l\}} \cong \mathbb{P}^1$



Note that $D(i, j | k, l) \subset \bar{M}_{0, n}(X, \beta)$ is the preimage of $P(i, j | k, l) \subset \bar{M}_{0, \{i, j, k, l\}}$

via $\phi \circ \eta$

(Fact: Convexity \Rightarrow All components in the preimage appear w/ multiplicity 1)

$$\therefore P(i, j | k, l) \sim P(i, l | j, k) \therefore D(i, j | k, l) \sim D(i, l | j, k) \quad \square$$

Hence, by lemma 3 $\Rightarrow G(q, r | s, t) = \int_{D(q, r | s, t)} e v_1^*(r_1) \cup \dots \cup e v_n^*(r_n)$

$$= \int_{D(r, s | q, t)} e v_1^*(r_1) \cup \dots \cup e v_n^*(r_n) = G(r, s | q, t) \rightsquigarrow G(q, r | s, t) = G(r, s | q, t) \text{ (**)}$$

Now, apply (**) to: $r_i = r$ for $i=1, \dots, n-4$, $r_{n-3} = T_i$, $r_{n-2} = T_j$, $r_{n-1} = T_k$, $r_n = T_l$

$$q = n-3, r = n-2, s = n-1, t = n$$

$$G(q, r | s, t) = \sum_{e, f} \sum_{\substack{n_1 + n_2 = n \\ \beta_1 + \beta_2 = \beta}} \binom{n-4}{n_1-2} I_{\beta_1} (r^{n_1-2} T_i T_j T_k) g^{e, f} I_{\beta_2} (r^{n_2-2} T_k T_l T_f)$$

$$= \sum_{e, f} \sum_{\substack{n_1 + n_2 = n \\ \beta_1 + \beta_2 = \beta}} \frac{(n-4)!}{(n_1-2)!(n_2-2)!} I_{\beta_1} (r^{n_1-2} T_i T_j T_k) g^{e, f} I_{\beta_2} (r^{n_2-2} T_k T_l T_f)$$

of partition of A and B which A has n_1 element
B has n_2 element

$$= n! \sum_{e, f} \sum_{\substack{n_1 + n_2 = n-4 \\ \beta_1 + \beta_2 = \beta}} \frac{1}{n_1! n_2!} I_{\beta_1} (r^{n_1-2} T_i T_j T_k) g^{e, f} I_{\beta_2} (r^{n_2-2} T_k T_l T_f)$$

Now, $G(q, r | s, t) = G(r, s | q, t) \Rightarrow$

$$\sum_{e,f} \sum_{\substack{n_1+n_2=n-4 \\ \beta_1+\beta_2=\beta}} \frac{1}{n_1!n_2!} I_{\beta_1}(\gamma^{n_1-2} T_i T_j T_e) g^{ef} I_{\beta_2}(\gamma^{n_2-2} T_k T_l T_f)$$

$$= \sum_{e,f} \sum_{\substack{n_1+n_2=n-4 \\ \beta_1+\beta_2=\beta}} \frac{1}{n_1!n_2!} I_{\beta_1}(\gamma^{n_1-2} T_j T_k T_e) g^{ef} I_{\beta_2}(\gamma^{n_2-2} T_i T_l T_f)$$

$$\Rightarrow \text{Sum over } n \Rightarrow F(i, j | k, l) = F(j, k | i, l) \quad a$$