Complex Analysis II, Final Reports

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Week I

[1] June 9 黃哲宏 Big Picard Theorem

[2] June 11 李昱陞 Modular Forms and Moduli Problem

[3] June 11 林肱慶 (Confluent) Hypergeometric Functions

Week II

[4] June 16 黃庭瀚 Sum of Squares

[5] June 16 古晉丞 Fundamental Groups and Covering Spaces

[6] June 18 高尉庭 Topological Classification of Compact Surfaces

[7] June 18 李自然 Frobenius Method for ODE with Regular Singularities

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[9] June 19 江泓 Asymptotic of Airy Function

Week III

[10] June 23 唐爾晨 Mandelbrot Sets and Julia Sets

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Julia and Mandelbrot set, 唐爾晨

Notation For $f: U \to U$, $f^{(n)} := \overbrace{f \circ f \circ \cdots \circ f}^{n}$ denote its n-th iteration

Recall • A normal family $\mathcal{F}$ of meromorphic functions on region $\Omega \subseteq \mathbb{C}$:
for any $\{f_n\} \subseteq \mathcal{F}$, $\exists$ subsequence $\{f_{n_k}\}$ uniformly converges on cpt subsets.

• Montel's thm :
if $\mathcal{F}$ is a family of meromorphic func. omitting 3 values, then $\mathcal{F}$ is normal.

(proof) by a fractional linear transform, we can assume $\mathcal{F}$ omit $0, 1, \infty$.
use the fact that $\lambda: \mathbb{H} \to \mathbb{C}\setminus\{0,1\}$ is a universal cover,
and that $\mathbb{D} \xrightarrow[\text{conformal}]{\varphi} \mathbb{H}$ to lift to $\tilde{g}$ for each $g \in \mathcal{F}$

$$\begin{tikzcd} & \mathbb{D} \arrow[d, "\lambda\varphi"] \\ \Omega \arrow[ur, "\tilde{g}"] \arrow[r, "g"] & \mathbb{C}\setminus\{0,1\} \end{tikzcd}$$

Now $\forall$ seq. $\{f_n\}$ in $\mathcal{F}$, $\{\tilde{f}_n\}$ is unif. bdd. $\Rightarrow$ equicontinuity by Cauchy thm
So $\exists$ subseq. $\{\tilde{f}_{n_j}\} \to f$ uniformly on cpt set. $f: \Omega \to \overline{\mathbb{D}}$
① if in fact $f: \Omega \to \mathbb{D}$, then $\{f_{n_j}\} \to \lambda\circ\varphi\circ f$ unif. on cpt set.
② if not, by Hurwitz thm, $f \equiv \text{const}$, then $\{f_{n_j}\} \to 0$ or 1 or ∞ unif. ※.

Def'n For a meromorphic function $f: \mathbb{C}^* \to \mathbb{C}^*$ (i.e. a rational function)
• Fatou set $\mathcal{F}(f) := \{ z \in \mathbb{C}^* \mid \{f^{(n)}\}$ form a normal family on a nbd of $z \}$
• Julia set $\mathcal{J}(f) := \mathbb{C}^* \setminus \mathcal{F}(f)$

Facts,
1. Fatou set is open, Julia set is closed
2. both sets are totally f-invariant i.e. $f(\mathcal{J}) = \mathcal{J} = f^{-1}(\mathcal{J})$ ...
3. Julia set is nonempty

(proof)
if $\mathcal{J}(f) = \phi$ then $\{f^{(n)}\}$ is a normal family on $\mathbb{C}^*$ (cptness argument)
$\therefore \exists$ subseq. $\{f^{(n_j)}\} \to g$ uniformly on $\mathbb{C}^*$. So g is also rational function.
Since (# of zeros of $f^{(n_j)}$) $\to \infty$ but (# of zeros of g) $< \infty$. $\rightarrow\!\leftarrow$

4. either $\mathcal{J} = \mathbb{C}^*$ or $\mathcal{J}$ has no interior. (modify Thm 2)

Technique Change of coordinates : let $\varphi: U \xrightarrow{\sim} V$ be a conformal map.
For $f: U \to U$, we associate $g = \varphi f \varphi^{-1}: V \to V$
We say f, g are conjugation equivalent. ($f \underset{\varphi}{\simeq} g$)
Behavior of functions are similar under conjugation:

Facts: For $f \underset{\varphi}{\approx} g$, a is critical pt of $f \Leftrightarrow \varphi(a)$ is critical pt of g
a is fixed pt $\Leftrightarrow \varphi(a)$ is fixed pt
And a $\varphi(a)$ has the same multiplier,

Def'n: if $f(a)=a$ is a fixed pt, its multiplier $:= f'(a)$
- attracting fixed pt : $0<|f'(a)|<1$
- repelling " : $|f'(a)|>1$
- super-attracting " : $|f'(a)|=0$

Theorem 1: if a is an attracting/repelling fixed pt of f with multiplier λ
then $\exists$ conformal map φ from nbd of a to nbd of 0, s.t. $f \underset{\varphi}{\approx} \lambda z$.

(proof) First. Consider the case when a : attracting fixed pt.
W.L.O.G let $a=0$, define $\varphi_n(z) = \lambda^{-n} f^{(n)}(z) = z +$,
then $\varphi_n \circ f = \lambda^{-n} f^{(n+1)} = \lambda \varphi_{n+1}$, we claim $\varphi_n \to \varphi$ unif. on a nbd of 0
$|f(z)-\lambda z| \le C|z|^2$ in $|z|\le\delta$ for some C, δ
So $|f(z)| \le (|\lambda|+C\delta)|z|$ and replace δ smaller s.t. $|\lambda|+C\delta<1$
So $|f(z)| \le \delta$ if $|z|\le\delta$, and $|f^{(n)}(z)| \le (|\lambda|+C\delta)^n|z|$, if $|z|\le\delta$
pick δ smaller s.t. $(|\lambda|+C\delta)^2<|\lambda|$.
$\Rightarrow |\varphi_{n+1}(z)-\varphi_n(z)| = \left|\frac{f(f^{(n)}(z))-\lambda f^{(n)}(z)}{\lambda^{n+1}}\right| \le \frac{C|f^{(n)}(z)|^2}{|\lambda^{n+1}|} = \frac{C}{|\lambda|}\left(\frac{|\lambda|+C\delta}{|\lambda|}\right)^n|z|^2$

by Wierс M-test $\varphi_n \to \varphi$ unif. on $|z|\le\delta$ and φ is conformal
$\varphi f \varphi^{-1}(x) = \lambda x$ ⊠
If 0 is repelling, then $\frac{1}{f}$ has 0 as an attracting fixed pt.
So $\frac{1}{f} \underset{\varphi}{\approx} \frac{1}{\lambda} z$ and so $f \underset{\varphi}{\approx} \lambda z$

Proposition 1. if a is (super) attracting fixed pt.
Define $A(a) := \{z\in\mathbb{C} \mid f^{(n)}(z) \to a\}$, called the basin of attraction of a
is nonempty, open, $A(a) \subseteq \mathcal{F}(f)$ and $\partial A(a) = \mathcal{J}(f)$

(proof) $|f(z)-a| < C|z-a|$ for some $C<$ for $|z-a|<\delta$ hence $B(a,\delta) \subseteq A(a) \cap \mathcal{F}(f)$
if $x\in A(a)$ then $f^{(n)}(x) \in B(a,\delta)$ if $n\gg1$ so $A(a) = \bigcup_{1}^{\infty} f^{(n)^{-1}}(B(a,\delta))$ open, $\subseteq \mathcal{F}(f)$
For $\mathbb{C}^*\setminus\overline{A(a)}$, $\{f^{(n)}\}$ omit $A(a)$ on each component.
$\Rightarrow \mathbb{C}^*\setminus\overline{A(a)} \subseteq \mathcal{F}(f)$ so $\mathcal{J}(f) \subseteq \partial A(a)$.

$\forall x \in \partial A(a)$, and nbd U of x $\quad f^{(n)}(z) \to a$ on $U \cap A(a)$ but $f^{(n_j)}(x)$ doesn't

$\Rightarrow \{f^{(n)}\}$ not normal on U $\quad \therefore x \in \mathcal{J}(f)$

Proposition 2 If a : repelling fixed pt, then $a \in \mathcal{J}(f)$

(proof) ~~if not $\{f^{(n_j)}\} \to g$ unif. on cpt nbd.~~

~~So $f^{(n_j)\prime}(a) \to g'(a)$ but $f^{(n_j)\prime}(a) = (f'(a))^{n_j} \to \infty$,~~ $\times$

Theorem I'. (Boettcher) If a : Super-attracting fixed pt of f.

then $\exists \varphi$ conformal defined near a s.t. $f \underset{\varphi}{\approx} z^p$ $\quad (p = \mathrm{ord}_a(f-a))$

(proof)

by $z' = c_1^{\frac{1}{1-p}}(z-a)$ let $f = z^p + \cdots$

$\exists C, \delta > 0$ s.t. $|f(z)| \le C|z|^p$ in $|z| \le \delta \Rightarrow |f^{(n)}(z)| \le (C|z|)^{p^n}$

Set $\varphi_n(z) = (f^{(n)}(z))^{p^{-n}} = (z^{p^n} + \cdot)^{p^{-n}} = z(1 + \cdot\ )$

$\varphi_n \circ f = (f^{(n)} \circ f)^{p^{-n}} = \varphi_{n+1}^p$, remains to claim $\varphi_n \to \varphi$ unif on a nbd.

$\frac{\varphi_{n+1}}{\varphi_n} = \left(\frac{\varphi_1 \circ f^{(n)}}{f^{(n)}}\right)^{p^{-n}} = (1 + O(|f^{(n)}|))^{p^{-n}} = 1 + O(p^{-n})O((C|z|)^{p^n}) = 1 + O(p^{-n})$

if we let $|z| \le C^{-1}$

So $\prod_{n=1}^{\infty} \frac{\varphi_{n+1}}{\varphi_n}$ conv. unif. So $\varphi_n \to \varphi$ unif. $\divideontimes$

Extend φ (= Boettcher Coordinate) Functional equation for φ. $\quad \varphi(f(z)) = \varphi(z)^p$

$\Rightarrow \log|\varphi(f(z))| = p \log|\varphi(z)|$

$\therefore$ We can extend $\log|\varphi(z)|$ to $A(a)$ being harmonic.

Polynomial Case From now on, Consider only poly. f of deg $d \ge 2$.

∞ : super-attracting, and $f(z) = \infty \Leftrightarrow z = \infty$. (and $p = d$)

$\Rightarrow$ In $A(\infty)$, $\log|\varphi|$ has only log pole at ∞ and harmonic elsewhere

Observe as $z \to \partial A(\infty) = \mathcal{J}$, $\log|\varphi(z)| \to 0$

$\therefore \log|\varphi(z)| = G(z, \infty)$ is the green func. on $A(\infty)$!

Fact. 1. $\mathcal{J}(f)$ is bdd now, so is cpt.

2. $A(\infty)$ is Connected (i.e it has no bdd component)

(proof) $f(\mathcal{J}) = \mathcal{J}$ bdd. so $\forall$ bdd component V of $\mathbb{C}^* \setminus \mathcal{J}$

V is bd by $\mathcal{J}$, max. prin. $\Rightarrow f(V)$ bd by $f(\mathcal{J}) = \mathcal{J}$.

$\Rightarrow f^{(n)}(x)$, $x \in V$ never conv. to ∞, so $V \cap A(\infty) = \phi$.

Theorem 2. (fractal nature of Julia set)

① if U open s.t. $U \cap \mathcal{J} \neq \phi$, then $\exists\, m$ s.t. $\mathcal{J} \subseteq U \cup \cdots \cup f^{(m)}(U)$

② $\forall x \in \mathcal{J}$, $\bigcup_n f^{(n)^{-1}}(x)$ is dense in $\mathcal{J}$

(proof) $\{f^{(n)}\}$ cannot be normal on $U \xRightarrow{\text{Montel thm}} \{f^{(n)}\}$ omit at most 1 value in $\mathbb{C}$

Case 1: $\{f^{(n)}\}$ omit no value $\Rightarrow \mathcal{J} \subseteq \mathbb{C} = \bigcup_n f^{(n)}(U)$ and by cptness of $\mathcal{J}$.

Case 2: $\{f^{(n)}\}$ only omit $y \in \mathbb{C}$, then $f(z) = y \Rightarrow z = y$

So $f(z) = y + k^{p}(z-y)^{p}$, then $f^{(n)}(z) = y + k^{p^n}(z-y)^{p^n}$, $(p = \deg f \geq 2)$

So $\exists$ nbd of y s.t. $f^{(n)}(z) \to y$ unif $\Rightarrow y \in \mathcal{F}$, so $\mathcal{J} \subseteq \bigcup_n f^{(n)}(U)$.

For any open set V s.t. $V \cap \mathcal{J} \neq \phi$, by above, $x \in f^{(m)}(V)$ for some m.

hence $f^{(m)^{-1}}(x) \cap V \neq \phi$, this proves $\bigcup_n f^{(n)^{-1}}(x) \subseteq \mathcal{J}$ dense in $\mathcal{J}$

$\to$ Boundary scanning method & inverse iteration method for drawing a Julia set by computer.

Theorem 3. All iterations of critical points remain bounded $\Leftrightarrow \mathcal{J}$ connected.

(proof) ① if critical points $\notin A(\infty)$, recall Böttcher coord. φ. Green func. G.

φ originally defined for $|z| > R \gg 0$, we want to extend to all $A(\infty)$

On $A(\infty)$, a root function h of f is locally defined, and can be continued along all arcs.

We start from the curve $\{G(z) = R \cdot d\}$

and define $\varphi(z) = \varphi(f(z))^{1/d}$ along this curve, for $\{G(z) = R\}$.

then $\forall R > 0$ we can do this and at last φ is defined on $\{G(z) > 0\} = A(\infty)$

As $z \to \partial A(\infty)$, $|\varphi(z)| \to 1$, and no value taken twice.

$\Rightarrow \varphi : A(\infty) \xrightarrow{\sim} \mathbb{C}^* \setminus \mathbb{D}$ conformal. so $A(\infty)$ simply connected so $\mathcal{J} = \partial A(\infty)$ connected.

② By the same method we can extend φ until the curve $\{G(z) = G(c_0)\}$

c_0 being a critical pt.

By Roche's thm, $f(z) = f(c_0)$ has ≥ 2 roots at c_0, implies.

$f(z) = f(c_0) + \varepsilon$ has ≥ 2 roots near c_0 if $0 < \varepsilon \ll 1$.

$\Rightarrow \{G(z) = G(c_0)\} \cap (\text{nbd of } c_0)$ has ≥ 4 curves linked to c_0,

So $G^{-1}([0, G(c_0)))$ is divided into ≥ 2 disjoint open sets

$\Rightarrow \mathcal{J} \subseteq G^{-1}(0)$ is disconnected.

Per-Duet

Theorem 4. All iterations of critical points $\longrightarrow \infty$ $\Rightarrow$ $\mathcal{J}$ is totally disconnected.

(proof) take large disk $D \supseteq \mathcal{J}$ s.t. $f(\mathbb{C}^* \setminus D) \subset \mathbb{C}^* \setminus \overline{D}$,

Find N large s.t. $f^{(N)}$ maps all critical points to $\mathbb{C}^* \setminus \overline{D}$,

$\forall n \geq N$, $f^{(n)}$ has no critical values in $\overline{D}$, so an inverse g_n is defined: $\overline{D} \longrightarrow D$ once chosen a branch.

For any $x \in \mathcal{J}$, pick g_n s.t. $g_n(f^{(n)}(x)) = x$.

$\{g_n\}$ unif. bdd on $\overline{D}^+$ $\xRightarrow{\text{Montel thm}}$ subseq $\{g_{n_k}\} \longrightarrow g$ on $\overline{D}$.

but $\forall z \in D \cap A(\infty)$ $g_n(z) \longrightarrow \tilde{z} \in \partial A(\infty) = \mathcal{J}$.

So $g(D \cap A(\infty)) \subset \mathcal{J}$, but g is an open mapping, and $\mathrm{Int}(\mathcal{J}) = \phi$.

$\Rightarrow g \equiv \text{Const.} = x$ on $\overline{D}$ we conclude: $\begin{cases} g_n(D) \ni x \\ g_n(\partial D) \cap \mathcal{J} = \phi, \\ \mathrm{diam}(g_n(\overline{D})) \longrightarrow 0 \end{cases}$

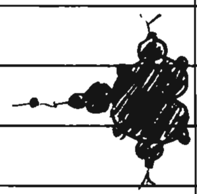
Thus, $\mathcal{J}$ is totally disconnected.

Note: Equivalence definitions for a cpt set K of $\mathbb{R}^n$ to be $\boxed{\text{totally disconnected}}$.

(1) K contains no continuum (2) $\forall x \in K, \forall \varepsilon > 0, \exists E \subseteq K$ s.t. $d(E, K \setminus E) > 0$ and $x \in E$ and $\mathrm{diam}(E) \longrightarrow 0$

Corollary For the simple case of deg $= 2$ polynomial,
$\exists$ only 1 critical point, so either $\mathcal{J}$ connected or totally disconnected.

Definition $\mathcal{M} := \{ c \in \mathbb{C} \mid (z^2+c)^{(n)}(0) \text{ is bounded w.r.t. } n \in \mathbb{N} \}$ is called the
$= \{ c \in \mathbb{C} \mid \mathcal{J}(z^2+c) \text{ connected} \}$ Mandelbrot Set



where we denote z^2+c as f_c, $\mathcal{J}(z^2+c)$ as $\mathcal{J}_c$

The study of Mandelbrot set is often a correspondence between parameter space c and dynamic space z.

Proposition 3 $c \in \mathcal{M} \Leftrightarrow |f_c^{(n)}(0)| \leq 2$, $\forall n \in \mathbb{N}$

Also $\mathcal{M}$ is cpt and $\mathbb{C} \setminus \mathcal{M}$ is connected.

(proof) ① if $\exists n$ s.t. $r = f_c^{(n)}(0)$ satisfy $|r| > 2$ (assume n smallest)

On $|z| = |r|$, $|z^2+c| \geq |r|^2 - |r| = (|r|-1)|z|$

then $\left|\frac{z}{z^2+c}\right| \leq \frac{1}{|r|-1}$ on $|z| = |r|$ and $\longrightarrow 0$ as $z \longrightarrow \infty$ $\xRightarrow{\text{Max. principle}}$ inequality holds $\forall |z| \geq |r|$

$\Rightarrow |f_c^{(n+k)}(0)| \geq (|r|-1)^k |r| \longrightarrow \infty$ as $k \longrightarrow \infty$ so $c \notin \mathcal{M}$

~~The other side~~ is from def'n. of $\mathcal{M}$

② So $M \subseteq \{|c| \leq 2\}$. Also $M = \bigcap_{n=1}^{\infty} \{c \in \mathbb{C} \mid |P_c^{(n)}(0)| \leq 2\}$ is closed $\Rightarrow M$ cpt

For all bdd region U s.t. $\partial U \subset M$,

$\forall n \in \mathbb{N}$, $|P_c^{(n)}(0)| \leq 2$ on $\partial U \Rightarrow$ by max. principle, on $c \in U$,

hence $\forall c \in U$, $|P_c^{(n)}(0)| \leq 2$ for all $n \Rightarrow U \subseteq M$

$\therefore$ $\mathbb{C} \setminus M$ has only unbounded components which is connected.

Proposition. $\{c \mid f_c$ has attracting fixed points$\}$ is a cardiod (心臓線) $C \subset M$

(proof) easy caculation gives $C = \{\frac{\lambda}{2} - \frac{\lambda^2}{4} \mid |\lambda| < 1\}$

for each $c \in C$ f_c has attracting fixed pt $\Rightarrow J_c$ not totally disconnected.

More facts Each collection of c s.t. f_c has "attracting n-cycle" also corresponds to a finite disjoint union of disks in M

Theorem 5. M is connected.

(proof) For each $c \in \mathbb{C} \setminus M$, we have Böttcher coordinate $\varphi_c(z)$

since 0 is the only critical pt. of f_c, $\varphi_c(z)$ can be extended to $\{z \mid G_c(z) > G_c(0)\}$, analytically.

in particular, $G_c(c) = 2G_c(0) > G_c(0)$. so $\varphi_c(c)$ is defined

where $\varphi_c(c) = c \prod_{n=1}^{\infty} \left(\frac{f_c^{(n)}(c)}{f_c^{(n-1)}(c)^2} \right)^{2^{-n}} = c \prod_{n=0}^{\infty} \left(1 + \frac{c}{f_c^{(n)}(c)^2} \right)^{2^{-n-1}}$ is analytic

denote $\Phi(c) = \varphi_c(c)$, it has simple pole at ∞.

$\log|\Phi(c)| = G_c(c) = 2G_c(0) \longrightarrow 0$ as $c \longrightarrow M$. ($G_c(z)$ jointly cont. in c, z)

$\therefore |\Phi(c)| \longrightarrow 1$ and by argument principle. Φ takes all values in $\mathbb{C} \setminus \overline{\mathbb{D}}$ once.

Hence $\Phi: \mathbb{C}^* \setminus M \xrightarrow{\sim} \mathbb{C}^* \setminus \overline{\mathbb{D}}$ hence $\mathbb{C}^* \setminus M$ simply connected $\Rightarrow M$ connected

Complex Analysis Ch XII (Gamelin)

Complex Dynamics (Carleson & Gamelin)

§ Asymptotics (2) Partition function

林東成
2015 6 23

Def (Stein, chapter 10 p.293)

If $n \in \mathbb{N}$, let $p(n)$ denote the numbers of ways n can be written as a sum of positive integers

n	0	1	2	3	4	5	6	7	$\cdots$
$p(n)$	1	1	2	3	5	7	11	15	$\cdots$

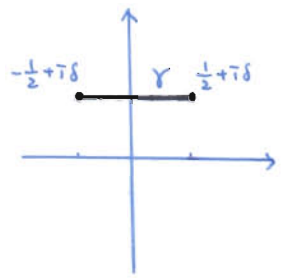
Theorem (Stein, chapter 10 p 293)

If $|x|<1$, then $\displaystyle\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$

Theorem (Hardy–Ramanujan formula, 1918)

(1) $p(n) \sim \dfrac{1}{4\sqrt{3}\,n} e^{K\sqrt{n}}$ as $n \to \infty$, where $K = \sqrt{\frac{2}{3}}\,\pi$

(2) More precisely, $p(n) = \dfrac{1}{2\pi\sqrt{2}} \dfrac{d}{dn}\left(\dfrac{e^{K(n-\frac{1}{24})^{1/2}}}{(n-\frac{1}{24})^{1/2}} \right) + O(e^{\frac{K}{2}\sqrt{n}})$



<pf>

Recall $\displaystyle\sum_{n=0}^{\infty} p(n)\omega^n = \prod_{k=1}^{\infty} \frac{1}{1-\omega^n}$

Write $\omega = e^{2\pi i \zeta}$ $\zeta \in \mathbb{H}$ Then $\displaystyle\sum_{n=0}^{\infty} p(n) e^{2\pi i n\zeta} =: f(\zeta) = \prod_{n=1}^{\infty} \frac{1}{1-e^{2\pi i n\zeta}}$

$\Rightarrow p(n) = \displaystyle\int_{\gamma} f(\zeta) e^{-2\pi i n \zeta}\, d\zeta$ (δ determined later)

γ ← unit length

Recall (Stein, chapter 10 p.292)

Dedekind eta function : $\eta(z) = e^{\frac{\pi i z}{12}} \displaystyle\prod_{n=1}^{\infty}(1-e^{2\pi i n z})$ for $\mathrm{Im}(z)>0$

Prop

If $\mathrm{Im}(z)>0$, then $\eta(-1/z) = \sqrt{z/i}\;\eta(z)$

Therefore, $f(\zeta) = e^{\frac{\pi i\zeta}{12}}\eta(\zeta)^{-1} \Rightarrow e^{\frac{\pi i(-1/\zeta)}{12}} f(-1/\zeta)^{-1} = \sqrt{\zeta/i}\; e^{\frac{\pi i \zeta}{12}} f(\zeta)^{-1} \Rightarrow f(\zeta) = \sqrt{\zeta/i}\; e^{\frac{\pi i(\zeta+\frac{1}{\zeta})}{12}} f(-1/\zeta)$ —— (*)

When $\zeta \to 0$ since $\mathrm{Im}(-1/\zeta) \to \infty$, $f(-1/\zeta) \to 1$

Take $f_1(\zeta) := \sqrt{\zeta/i}\; e^{\frac{\pi i(\zeta+\frac{1}{\zeta})}{12}}$ to approximate $f(\zeta)$ (they have same behavior near $\zeta=0$)

since $|f(x+iy)| \le f(iy)$ and $f(iy)\uparrow$ as $y \downarrow$

Now write $p(n) = p_1(n) + E(n)$

$$\begin{cases} p_1(n) = \displaystyle\int_{\gamma} \sqrt{\zeta/i}\; e^{\frac{\pi i(\zeta+\frac{1}{\zeta})}{12}} e^{-2\pi i n\zeta}\, d\zeta \\ E(n) = \displaystyle\int_{\gamma} \sqrt{\zeta/i}\; e^{\frac{\pi i(\zeta+\frac{1}{\zeta})}{12}} e^{-2\pi i n\zeta} \left(f(-1/\zeta)-1\right) d\zeta \end{cases}$$

estimate the error $E(n)$:

- If $\zeta \in \gamma$ $\left| \sqrt{\zeta/i}\; e^{\frac{\pi i(\zeta+\frac{1}{\zeta})}{12}} e^{-2\pi i n\zeta} \right| \le C e^{2\pi n\delta} e^{\frac{\pi}{12}\frac{\delta}{\delta^2+x^2}}$
- On the other hand, from $f(\zeta) = 1 + O(e^{-2\pi y})$, $y \ge 1$, we know that $|f(-1/\zeta)-1| \le Ce^{-2\pi \underbrace{\frac{\delta}{\delta^2+x^2}}_{\mathrm{Re}(i/\zeta)}}$ if $\frac{\delta}{\delta^2+x^2} \ge 1$

  As for $y \le 1$, we already know that $|f(\zeta)| \le f(iy) \le Ce^{\frac{\pi}{12y}}$ from (*)

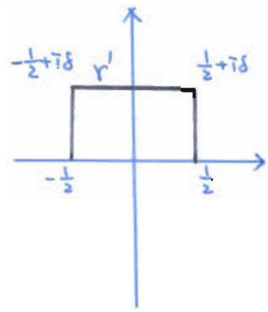
  Therefore, $f(-1/\zeta)-1| \le O(e^{\frac{\pi}{12}\frac{\delta^2+x^2}{\delta}}) = O(e^{\frac{\pi}{48\delta}})$ for $y = \frac{\delta}{\delta^2+x^2} \le 1$ ($\because |x|<\frac{1}{2}$)

From above, when $\frac{\delta^2}{\delta^2+x^2} \gg 1$ leads to contribution of $O(e^{2\pi n\delta})$

when $\frac{\delta^2}{\delta^2+x^2} \leq 1$, leads to contribution of $O(e^{2\pi n\delta} e^{\frac{\pi}{48\delta}})$

$\Rightarrow E(n) = O(e^{2\pi n\delta} e^{\frac{\pi}{48\delta}})$ By A.M $\geq$ G.M "=" holds for $2\pi n\delta = \frac{\pi}{48\delta} \Rightarrow \delta = \frac{1}{4\sqrt{6}} \frac{1}{\sqrt{n}}$

$\therefore$ When $\delta = \frac{1}{4\sqrt{6}}\frac{1}{\sqrt{n}}$, $\underline{E(n) = O(e^{\frac{K}{2}\sqrt{n}})}$



After the estimation of $E(n)$, we now want to change the contour γ into γ'

since for $z \in \pm\frac{1}{2} + it\delta$, $0 \leq t \leq 1$, $\sqrt{z/i}\, e^{\frac{\pi i}{12z}}$ is $O(1)$ (smaller than allowed error)

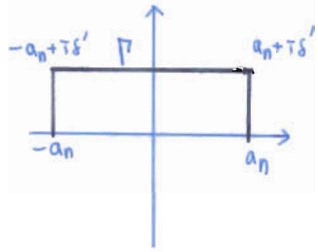
$\therefore$ Now we change $p_1(n)$ as $\int_{\gamma'} \sqrt{z/i}\, e^{\frac{\pi i (z+\frac{1}{z})}{12}} e^{-2\pi i n z}\, dz$ —— (☆)

We make a change of variables $z \mapsto \mu z$

$$p_1(n) = \int_{\Gamma} \sqrt{\mu z/i}\, e^{\frac{\pi i}{12}(\mu z + \frac{1}{\mu z})} e^{-2\pi i n \mu z}\, dz\, \mu = \int_{\Gamma} \sqrt{\mu z/i}\, \mu e^{(\frac{i\pi}{12}\mu - 2\pi i n\mu) z + \frac{i\pi}{12\mu}\frac{1}{z}}\, dz$$

Here we want to make $p_1(n)$ of the form $e^{Ai(\frac{1}{z}-z)}$: $\begin{cases} A = \frac{\pi}{12\mu} \\ A = 2\pi n\mu - \frac{\pi\mu}{12} \end{cases}$, $\begin{cases} A = \frac{\pi}{\sqrt{6}}(n-\frac{1}{24})^{1/2} \\ \mu = \frac{1}{2\sqrt{6}}(n-\frac{1}{24})^{-1/2} \end{cases}$

Where $\Gamma = \mu^{-1}\gamma'$; $a_n = \frac{1}{2}\mu^{-1} = \sqrt{6}(n-\frac{1}{24})^{1/2}$ $\delta' = \delta\mu^{-1} = \frac{1}{2\sqrt{n}}(n-\frac{1}{24})^{1/2}$



<u>Method</u> (steepest descent)

<u>Recall</u> (Stein, chapter 8 ex 2)

If $F(z)$ is holomorphic near z_0 $F(z_0) = F'(z_0) = 0 \neq F''(z_0)$

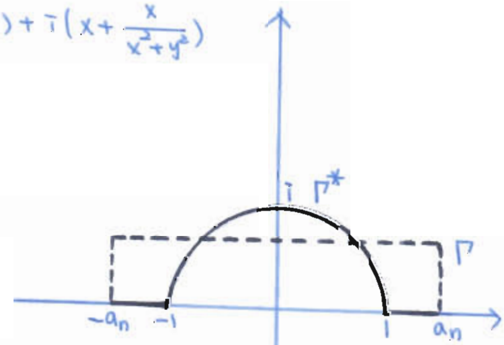
Then $\exists\, \Gamma_1\ \Gamma_2$ pass z_0 and orthogonal to each other near z_0

and $F|_{\Gamma_1}$: real with minimum at z_0, $F|_{\Gamma_2}$ real with maximum at z_0

let $F(z) = i(z - \frac{1}{z})$ Write $z = x+iy$, then $F(z) = (-y + \frac{y}{x^2+y^2}) + i(x + \frac{x}{x^2+y^2})$

- $F(z)$ has critical point at $z = i$ when $z \in \mathbb{H}$
- $\mathrm{Im}(F(z)) = 0$ if $x = 0$ or $x^2+y^2 = 1$

$F|_{x=0} = \frac{1}{y} - y$ (max at i) $F_{x^2+y^2=1} = 2xi$ min at i)



Therefore, $p_1(n) = \mu^{\frac{3}{2}} \int_{\Gamma^*} e^{-AF(z)} \sqrt{z/i}\, dz$

On real axis, $\int e^{-AF(z)}\sqrt{z/i}\, dz$ is bounded by $\sup_{|z| \leq a_n} z^{\frac{1}{2}}$, so the integral is of $O(1)$, which can be ignored)

As for the point on unit circle

let $z = e^{i\theta}$, then $dz = ie^{i\theta}d\theta$, $i(z-\frac{1}{z}) = -2\sin\theta$

$$p_1(n) = -\int_0^{\pi} \mu^{3/2} e^{2A\sin\theta} e^{i\frac{3\theta}{2}} \sqrt{i}\, d\theta = \mu^{\frac{3}{2}} \int_{-\pi/2}^{\pi/2} e^{2A\cos\theta} (\cos\tfrac{3\theta}{2} + i\sin\tfrac{3\theta}{2})\, d\theta \quad \text{—— } (**)$$

By Prop 2.1 ($\int_a^b e^{-s\Phi(x)}\psi(x)\, dx = e^{-s\Phi(x_0)} (\frac{A}{\sqrt{s}} + O(\frac{1}{s}))$ $A = \sqrt{2\pi}\frac{\psi(x_0)}{(\Phi''(x_0))^{1/2}}$ where $x_0 \in (a,b)$ s.t. $\Phi'(x_0) = 0$, $\Phi''(x_0) > 0$)

Here, $\Phi(\theta) := -\cos\theta$ $\theta_0 = 0 \Rightarrow \Phi(\theta_0) = -1$ $\Phi''(\theta_0) = 1$ Choose $\psi(x) = \cos(\frac{3\theta}{2}) + i\sin(\frac{3\theta}{2})$ then $\psi(\theta_0) = 1$

Therefore $p_1(n)$ contributes $\mu^{3/2}\frac{\sqrt{2\pi}}{(2s)^{1/2}}(1 + O(\frac{1}{\sqrt{s}}))$ where $s = \frac{\pi}{\sqrt{6}}(n-\frac{1}{24})^{1/2}$ $K = \pi\sqrt{\frac{2}{3}}$ $\mu = \frac{\sqrt{6}}{12}(n-\frac{1}{24})^{-1/2}$

$\therefore p(n) = \frac{1}{4n\sqrt{3}} e^{K\sqrt{n}} (1 + O(\frac{1}{\sqrt[4]{n}}))$ #

Now, back to (☆), define $P_1(n) = \int_{\gamma_n'} \sqrt{z/i}\, e^{\frac{\pi i (z + \frac{1}{z})}{12}} e^{-2\pi i n z}\, dz$ (variation of contour γ' to γ_n')

Write $P_1(n) = \frac{d}{dn} q(n) + e(n)$ where $q(n) = \frac{1}{2\pi} \int_{\gamma'} (z/i)^{1/2} e^{\frac{\pi i (z + \frac{1}{z})}{12}} e^{-2\pi i n z}\, dz$

After the same works (i.e. $P \mapsto P^*$, $z \mapsto \mu z$), we have $q(n) = \frac{\mu^{\frac{1}{2}}}{2\pi} \int_{P^*} e^{-AF(z)} (z/i)^{-1/2}\, dz$

where $F(z) = i(z - \frac{1}{z})$ $\quad A = \frac{\pi}{\sqrt{6}} (n - \frac{1}{24})^{1/2}$ $\quad \mu = \frac{1}{2\sqrt{6}} (n - \frac{1}{24})^{-1/2}$

let $z = e^{i\theta}$ then $P_1(n) = \frac{-\mu^{1/2}}{2\pi} \int_0^{\pi} e^{2A\sin\theta} e^{i\frac{\theta}{2}} i^{\frac{3}{2}}\, d\theta = \frac{\mu^{1/2}}{2\pi} \int_{-\pi/2}^{\pi/2} e^{2A\cos\theta} (\cos\frac{\theta}{2} + \cancelto{0}{i\sin\frac{\theta}{2}})\, d\theta$

$$= \frac{\mu^{1/2}}{2\pi} e^{2s} \int_{-\sqrt{2}/2}^{\sqrt{2}/2} e^{-4Ax^2}\, dx$$

(note that $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$
write $x = \sin\frac{\theta}{2}$)

$$= \frac{\mu^{1/2}}{2\pi} e^{2s} \left\{ \int_{-\infty}^{\infty} e^{-4Ax^2}\, dx + O\left(\int_{\sqrt{2}/2}^{\infty} e^{-4Ax^2}\, dx \right) \right\}$$

$$= \frac{\mu^{1/2}}{2\pi} e^{2s} \left\{ \frac{\sqrt{\pi}}{2\sqrt{A}} + O(e^{-2A}) \right\}$$

Therefore, $\frac{d}{dA}\left(\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} e^{-4Ax^2}\, dx \right) = \frac{d}{dA}\left(\frac{\sqrt{\pi}}{2\sqrt{A}} \right) + O(e^{-2s})$ and $e(n)$ is $O(1)$

$\therefore P(n) = \frac{d}{dn}\left(\mu^{1/2} \frac{e^{2s}}{\pi} \frac{\sqrt{\pi}}{2\sqrt{A}} \right) + O(e^{-2s}) = \frac{1}{2\pi\sqrt{2}} \frac{d}{dn}\left(\frac{e^{K(n - \frac{1}{24})^{1/2}}}{(n - \frac{1}{24})^{1/2}} \right) + O(e^{\frac{K}{2}\sqrt{n}})$ #

Remark

- Asymptotic
  guiding principle: (1) Deformation of contour (2) Laplace's method (3) Generating function
- Approximate an Integral
  Laplace's method $\int_a^b e^{Mf(x)}\, dx$ $M \gg 0$ $\longleftrightarrow$ Steepest descent $\int_\gamma f(z) e^{\lambda g(z)}$ $\lambda \gg 0$
-

n	61	62	63
$P(n)$	1121505	1300156	1505499
$\frac{1}{2\pi\sqrt{2}} \frac{d}{dn}\left(\frac{e^{K(n-\frac{1}{24})^{1/2}}}{(n-\frac{1}{24})^{1/2}} \right)$	1121538.992	1300121.359	1505535.606

- If we compute $P(n)$ by $\int_P \frac{f(z)}{z^{n+1}}$ instead of $\int_\gamma f(z) e^{-2\pi i n z}\, dz$,

  we can still get the same result

  (reference http // plms oxfordjournals org content s2-17/1 75.full.pdf)

Reference

Stein, complex analysis

Appendix (Asymptotic formulae in combinatory analysis by H Hardy and S Ramanujan)

- Euler identity $\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots} = 1 + \frac{x}{(1-x)^2} + \frac{x^4}{(1-x)^2(1-x^2)^2} + \cdots$

$\Rightarrow e^{A\sqrt{n}} < p(n) < e^{B\sqrt{n}}$ for some $A, B > 0$ for large n Hence, $A\sqrt{n} < \log p(n) < B\sqrt{n}$

Question $\overset{?}{\exists} C$ s.t. $\log p(n) \sim C\sqrt{n}$ (YES!)

Theorem ("Tauberian")

If $g(x) = \sum a_n x^n$ with positive coefficient and $\log g(x) \sim \frac{A}{1-x}$ when $x \to 1$

Then $\log s_n = \log(a_0 + a_1 + \cdots + a_n) \sim 2\sqrt{An}$ as $n \to \infty$

Since $c = \lim \frac{\log p(n)}{\sqrt{n}}$ and if we write $g(x) = (1-x)f(x) = \sum\{p(n) - p(n-1)\}x^n = \frac{1}{(1-x^2)(1-x^3)\cdots}$

Then $g(x)$ is of positive coefficient and $\log g(x) = \sum_{k=2}^{\infty} \frac{1}{1-x^k} \sim \frac{1}{1-x}\sum_{v=1}^{\infty}\frac{1}{v^2} = \frac{\pi^2/6}{1-x}$ as $x \to$

(by using $v x^{v-1}(1-x) < 1 - x^v < v(1-x) \Rightarrow \frac{1}{1-x}\sum\frac{x^{2v}}{v} < \log g(x) < \frac{1}{1-x}\sum\frac{x^{v+1}}{v^2}$)

Therefore, $\log p(n) = a_0 + a_1 + \cdots + a_n \sim c\sqrt{n}$ where $c = \frac{2\pi}{\sqrt{6}}$ $g(x) \sim \sqrt{\frac{1-x}{2\pi}}\, e^{\frac{\pi^2}{6(1-x)}}$

auxiliary function $F_a(x) := \frac{1}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\psi_a(n)x^n$ where $\psi_a(n) := \frac{d}{dn}\left(\frac{\cosh\left(k(n-\frac{1}{24})^{\frac{1}{2}}\right) - 1}{(n-\frac{1}{24})^{\frac{1}{2}}}\right)$ $a > 0$

(the "principle branch" of F is regular for all plane except for $x = 1$)

by transformed into an integral by means of a general function given by Lindelöf

and $F(x) - \chi(x)$ is regular for $x = 1$ where $\chi(x) = \frac{x^{\frac{1}{24}}}{\sqrt{2\pi}}\sqrt{\log(\tfrac{1}{x})}\left(e^{\frac{\pi^2}{6\log(\frac{1}{x})}} - 1\right)$

Compare $\chi(x)$ and $f(x)$ and apply Cauchy's theorem on $f - F$

we get $p(n) = \frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{e^{k(n-\frac{1}{24})^{\frac{1}{2}}}}{(n-\frac{1}{24})^{\frac{1}{2}}}\right) + O(e^{\frac{k}{2}\sqrt{n}})$

come from $z = 1$ from $z = -1$

Similarly, $p(n) = \frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{e^{k(n-\frac{1}{24})^{\frac{1}{2}}}}{(n-\frac{1}{24})^{\frac{1}{2}}}\right) + \frac{(-1)^n}{2\pi}\frac{d}{dn}\left(\frac{e^{\frac{1}{2}k(n-\frac{1}{24})^{\frac{1}{2}}}}{(n-\frac{1}{24})^{\frac{1}{2}}}\right) + \frac{\sqrt{3}}{\pi\sqrt{2}}\cos\left(\frac{2}{3}\pi n - \frac{1}{18}\pi\right)\frac{d}{dn}\left(\frac{e^{\frac{1}{3}k(n-\frac{1}{24})^{\frac{1}{2}}}}{(n-\frac{1}{24})^{\frac{1}{2}}}\right)$

$+ \frac{\sqrt{2}}{\pi}\cos\left(\frac{1}{2}\pi n - \frac{1}{8}\pi\right)\frac{d}{dn}\left(\frac{e^{\frac{1}{4}k(n-\frac{1}{24})^{\frac{1}{2}}}}{(n-\frac{1}{24})^{\frac{1}{2}}}\right) + \cdots$

from $z = e^{\frac{2}{3}\pi i}$

from $z = e^{\frac{1}{3}\pi i}$

Dirichlet Theorem with Density　廖律恩

- **Introduction**: Given $m, a \in \mathbb{N}$, with $(m,a)=1$, define $\pi_a(x) := \# \left\{ p \in \mathbb{N} \;\middle|\; \begin{array}{l} p \text{ is a prime} \le x \\ p \equiv a \pmod m \end{array} \right\}$

our goal is to prove the following theorem.

$$\pi_a(x) \sim \frac{x}{\phi(m) \log x}$$

- **Thought**:

Recall that we can prove $\#\{p \text{ is a prime}\} = \infty$ by showing that $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\cdots+\frac{1}{p}+\cdots \to \infty$

If we want to prove $\# \left\{ \begin{array}{l} p \text{ is a prime} \\ p \equiv 1 \pmod 4 \end{array} \right\} = \infty$

we may use

$$S_1 = \sum \frac{1}{p} = \frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{p}+\cdots \to \infty$$

$$S_2 = \frac{1}{2} + \sum_{p>2} \frac{(-1)^{\frac{p-1}{2}}}{p} = \frac{1}{2}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots \quad \text{bounded}$$

we want

then we get $\displaystyle\sum_{p \equiv 1 \pmod 4} \frac{1}{p}$ is also unbounded, so

$$\# \left\{ \begin{array}{l} p \text{ is a prime} \\ p \equiv 1 \pmod 4 \end{array} \right\} = \infty$$

· Characters of finite abelian groups: (Serre)

· Def: G is an abelian group, a character of G is a $\underset{\text{group}}{\widehat{\ }}$ homomorphism $\chi: G \longrightarrow \underset{\uparrow}{\mathbb{C}^*}$ multiplicative group of $\mathbb{C}$.

* in our case, we may take $G = \mathbb{Z}/m\mathbb{Z}^*$

· It is easy to see that all characters form an abelian group $\mathrm{Hom}(G, \mathbb{C}^*)$, denoted by $\hat{G}$.

Ques: How may characters?

· Lemma1: Let H be a supgroup of G, then every character of H extends to a character of G

proof: Induction on $[G:H]$

① $[G:H]=1 \Rightarrow H=G$, nothing to prove.

② $[G:H]>1$, choose $x \in G-H$, with n the smallest interger >1 s.t. $x^n \in H$ Now for any character χ of H, we may find $w \in \mathbb{C}^*$ s.t $w^n = \chi(x^n)$ and define character χ' of $H' = (H, x)$

$$\chi'(h') = \chi(h)\, w^a \quad \text{where } h' = \underset{H}{h}\, \underset{a}{x^a}$$

$\chi'(h')$ is well-defined and it is a character of H'. Since $[G:H'] < [G:H]$ we're done

2

- If we define the restriction $\rho : \hat{G} \to \hat{H}$, then we just learned that ρ is surjective. Moreover, since $\ker\rho$ are the characters act trivial on H, hence $\ker\rho \cong \widehat{G/H}$. then we have exact sequence

$$\{1\} \to \widehat{G/H} \to \hat{G} \to \hat{H} \to \{1\}$$

- Theorem 1: $\hat{G}$ have the same order as G

proof: Induction on n, the order of G

We know by above that $|\hat{G}| = |\widehat{G/H}| \cdot |\hat{H}|$, hence by induction we're done. $\triangle$

- Lemma 2: For any $x \in G$, $x \neq 1$, there exists a character χ of G s.t $\chi(x) \neq 1$

proof: Consider $H = (x)$, since H is cyclic with order n, we can see $\chi(x^t) := e^{\frac{2\pi i}{n} t}$ is a character of H by Lemma 1, χ can be extend to a character χ' of G, with $\chi'(x) = \chi(x) = e^{\frac{2\pi i}{n}} \neq 1$ $\triangle$

- For $G = \mathbb{Z}/m\mathbb{Z}^*$, we extend any character χ of $\mathbb{Z}/m\mathbb{Z}^*$ to define on $\mathbb{Z}/m\mathbb{Z}$

$$\bar{\chi}(n) = \begin{cases} \chi(n), & \text{if } (n,m) = 1 \\ 0, & \text{otherwise} \end{cases}$$

*Theorem 2. Let $n = \text{ord } G$, χ a character, then

(1) $$\sum_{x\in G}\chi(x) = \begin{cases} n, & \text{if } \chi = \chi_0 \leftarrow \text{identity}. \\ 0, & \text{if } \chi \neq \chi_0 \end{cases}$$

(2) $$\sum_{\chi\in\hat{G}}\chi(x) = \begin{cases} n, & \text{if } x = 1 \in G \\ 0, & \text{if } x \neq 1 \end{cases}$$

proof

(1) If $\chi = \chi_0$, $\sum_{x\in G}\chi(x) = \overbrace{1+1+\cdots+1}^{n} = n$

If $\chi \neq \chi_0$, then $\exists\, y \in G$ s.t. $\chi(y) \neq 1$, then

$$\chi(y)\cdot\sum_{x\in G}\chi(x) = \sum_{x\in G}\chi(xy) = \sum_{x'\in G}\chi(x)$$

$$\Rightarrow \sum_{x\in G}\chi(x) = 0$$

(2) By the same arguement of (1), if $x \neq 1$, then $\exists\, \psi \in \hat{G}$ s.t. $\psi(x) \neq 1$ (by Lemma 2)

$$\Rightarrow \psi(x)\sum_{\chi\in\hat{G}}\chi(x) = \sum_{\chi\in\hat{G}}\psi\chi(x) = \sum_{\chi\in\hat{G}}\chi(x)$$

$$\Rightarrow \sum_{\chi\in\hat{G}}\chi(x) = 0$$ ¥

- Prime Number Theorem (Stein)
- $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$, $\Lambda(n)=\begin{cases}\log p, & \text{if } n=p^m,\ p \text{ a prime}\\ 0, & \text{otherwise}\end{cases}$

  $\psi(x)=\sum_{1\le n\le x}\Lambda(x)$, $\psi_1(x)=\int_1^x \psi(u)\,du$

- Analytic continuation of ζ to $\mathrm{Re}(s)=\sigma>0$

$$\zeta(s)=\frac{1}{s-1}+H(s),\quad H(s)=\sum_{n=1}^{\infty}\delta_n(s)$$

$$\delta_n(s)=\int_n^{n+1}\left(\frac{1}{n^s}-\frac{1}{x^s}\right)$$

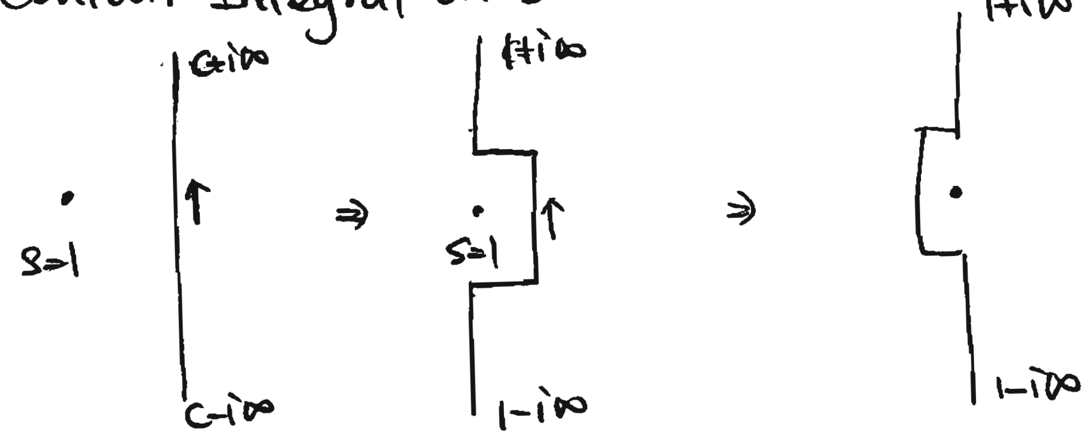
- $\sigma>1$ and t is real, we have

$$\log\left|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)\right|\ge 0$$

$$\Rightarrow\begin{cases}\zeta(1+it)\neq 0 \quad \forall t\in\mathbb{R} \ (\text{So } \zeta \text{ has no zeros for } \sigma\ge 1)\\ \text{Estimate of } \left|\frac{\zeta'(s)}{\zeta(s)}\right|\end{cases}$$

- $$\psi_1(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds$$

- Contour Integral on s



The L-function $L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$ on $Re(s)>0$

Observe that for $Re(s)>1$, $\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$ converges absolutely since $\left|\frac{\chi(n)}{n^s}\right| \le \frac{1}{n^{\sigma}}$ where $s=\sigma+it$

and so the Eular Product Formula implies

$$L(s,\chi)=\prod_{p\text{ prime}}\frac{1}{1-\frac{\chi(p)}{p^s}}$$

Thm 3 $\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$ Converges on $Re(s)>0$, more over, $L(s,\chi)$ is holomorphic on $Re(s)>0$. (if $\chi\neq\chi_0$)

proof:

① Let $A_{N,M}=\sum_{n=N}^{M}\chi(n)$, then $|A_{N,M}|\le\phi(m)$

$$\Rightarrow \sum_{n=N}^{M}\frac{\chi(n)}{n^s} = \sum_{n=N}^{M-1}A_{N,n}\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)+A_{N,M}\cdot\frac{1}{M^s}$$

as ~~we~~ ~~can~~ see $\left|\frac{1}{n^s}-\frac{1}{(n+1)^s}\right|\le\frac{|s|}{n^{s+1}}$

$$\text{so}\left|\sum_{n=N}^{M}\frac{\chi(n)}{n^s}\right|\le\phi(m)\sum_{n=N}^{M-1}\frac{|s|}{n^{s+1}}+\frac{\phi(m)}{M^s}\xrightarrow{\text{as } M,N\to\infty}0$$

$\Rightarrow \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$ converges on $Re(s)>0$

② Also we can see from above that $\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$ Converge uniformly on every compact subset of $Re(s)>0$

$\Rightarrow \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$ is holomorphic by Thm 5.2, Chap 2 in Stein.

- Thm 4: $L(s,\chi_0)$ extends to a meromorphic function for $\mathrm{Re}(s)>0$, it has a unique simple pole at $s=1$ with residue $\frac{\phi(m)}{m}$.

proof: Obviously $L(s,\chi_0) = \zeta(s)\prod_{p|m}(1-\frac{1}{p^s})$ for $\mathrm{Re}\, s>1$

Since $\zeta(s)$ extends to a meromorphic function on $\mathrm{Re}\, s>0$, so does $L(s,\chi_0)$ (and so only a unique simple pole at 1)

On the other hand,

$$\mathrm{res}(L(s,\chi_0),1) = \prod_{p|m}(1-\frac{1}{p})\,\mathrm{res}(\zeta(s),1) = \frac{\phi(m)}{m}$$

Now we know $L(s,\chi)$ is defined as a holo. ($\chi_0 \Rightarrow$ mero) function on $Re(s)>0$. We still need to define $\log L(s,\chi)$:

Since for $Re(s)>1$, we have $\left|\frac{\chi(p)}{p^s}\right|<1$, so take principle branch to define

$$\log \frac{1}{1-\frac{\chi(p)}{p^s}} = \sum \frac{\left(\frac{\chi(p)}{p^s}\right)^n}{n} \quad \left(\log\frac{1}{1-a} := \sum \frac{a^n}{n}\right)$$

and define

$$\log L(s,\chi) = \sum_p \log \frac{1}{1-\frac{\chi(p)}{p^s}}$$

notice that $\underset{\substack{\uparrow \\ \text{from Eular Product}}}{L(s,\chi)\neq 0}$ on $Re(s)>1 \Rightarrow \log$ is well-defined.

$$\log L(s,\chi) = \sum_p \log \frac{1}{1-\frac{\chi(p)}{p^s}}$$

$$= \sum_p \sum_n \frac{\chi(p)^n p^{-ns}}{n} = \sum_{n,p} \frac{\chi(p)^n}{n\, p^{ns}}$$

the series $\sum_{n,p} \frac{\chi(p)^n}{n\cdot p^{ns}}$ is obviously convergent,

so $\frac{L'(s,\chi)}{L(s,\chi)} = -\sum_{n\geq 1}^{\infty} \frac{\chi(n)\Lambda(n)}{n^s}$

(Remark: $\frac{\zeta'(s)}{\zeta(s)} = -\sum \frac{\Lambda(n)}{n^s}$)

• Lemma 3 $\log|L^3(\sigma,\chi_0)L^4(\sigma+ti,\chi)L(\sigma+2ti,\chi^2)| \geq 0$ $(\sigma>1)$

proof:

$$\log|L^3(\sigma,\chi_0)L^4(\sigma+ti,\chi)L(\sigma+2ti,\chi^2|$$
$$= 3\log|L(\sigma,\chi_0)| + 4\log|L(\sigma+ti,\chi)| + \log|L(\sigma+2ti,\chi^2)|$$
$$= 3\log L(\sigma,\chi_0) + 4\,\mathrm{Re}\log L(\sigma+ti,\chi) + \mathrm{Re}\log L(\sigma+2ti,\chi^2)$$
$$= \sum_{n,p}\left(\frac{3\chi_0(p^n)}{np^{n\sigma}} + \mathrm{Re}\,\frac{4\chi(p^n)}{n\,p^{n(\sigma+ti)}} + \mathrm{Re}\,\frac{\chi^2(p^n)}{n\,p^{n(\sigma+2ti)}}\right)$$
$$= \sum_{\substack{n,p\\ n\nmid m}} \frac{3+4\cos(\theta_n)+\cos 2\theta_n}{n\,p^{n\sigma}}$$
$$= \sum_{\substack{n,p\\ n\nmid m}} \frac{2(1+\cos\theta_n)^2}{n\,p^{n\sigma}} \geq 0$$

∯

where $\theta_n = \eta(p^n) - t\log(p^n)$

$\chi(p^m) = e^{i\eta(p^m)}$

• Thm 5 $L(s,\chi)$ does not vanish on the line $\sigma=1$

proof Prove by contradiction: Suppose $\exists t_0\in\mathbb{R}$ s.t. $L(1+it_0,\chi)=0$. Since $L(s,\chi)$ is holomorphic

$\Rightarrow$ $|L(1+it_0,\chi)|^4 \leq C(\sigma-1)^4$ as $\sigma\to 1$

and since $s=1$ is a pole for $L(s,\chi_0)$ (by Thm 4) so that $|L(\sigma,\chi_0)|^3 \leq c'(\sigma-1)^{-3}$ as $\sigma\to 1$

Finally, $|L(\sigma+2it_0)|$ remains bounded as $\sigma\to 1$

$\Rightarrow \left| L^3(\sigma,\chi_0)\, L^4(\sigma+it_0,\chi)\, L(\sigma+2it_0,\chi^2)\right| \to 0$ as $\sigma\to 1$

contradicted to Lemma 3 ☆

- <u>Estimate of $\dfrac{L'(s,\chi)}{L(s,\chi)}$</u>

- Thm 6 : Suppose $s=\sigma+it$ with $\sigma,t\in\mathbb{R}$ Then for each σ_0, $0\leq\sigma_0\leq 1$, and every $\varepsilon>0$, there exists a constant C_ε so that

(i) $|L(s,\chi)|\leq C_\varepsilon |t|^{1-\sigma_0+\varepsilon}$, if $\sigma_0\leq\sigma$ and $|t|\geq 1$

(ii) $|L'(s,\chi)|\leq C_\varepsilon |t|^{\varepsilon}$, if $1\leq\sigma$, and $|t|\geq 1$

Proof : The proof is basicly same with Prop 2.7 Chap 6. Stein

① For $\chi=\chi_0$, it is straight foward from Thm 4 and Prop 2.7 Chap 6 Stein

② If $\chi\neq\chi_0$, then we know that

$$L(s,\chi) = \sum_{n=1}^{\infty} A(n)\underbrace{\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)}_{= \delta_n(s)} \quad \text{where } A(n)=\sum_{i\leq n}\chi(i)$$

then $\begin{cases} |\delta_n(s)| \le \frac{|s|}{n^{\sigma+1}} & \text{by mean value thm} \\ |\delta_n(s)| \le \frac{2}{n^{\sigma}} \end{cases}$

$$\Rightarrow |\delta_n(s)| \le \left(\frac{|s|}{n^{\sigma+1}}\right)^{\delta} \left(\frac{2}{n^{\sigma_0}}\right)^{1-\delta} \le \frac{2|s|^{\delta}}{n^{\sigma_0+\delta}}$$

choose $\delta = 1-\sigma_0+\varepsilon$

$$\Rightarrow |L(s,\chi)| \le \phi(m)\, 2 \cdot |s|^{1-\sigma_0+\varepsilon} \sum \frac{1}{n^{1+\varepsilon}}$$

so (i) is proved

(since when $\sigma > 1+r_0$ with $r_0>0$, $L(s,\chi)$ is bdd.)

② $$L'(s,\chi) = \frac{1}{2\pi r} \int_0^{2\pi} L(s+re^{i\theta},\chi)\, e^{i\theta}\, d\theta$$

Choose $r = \varepsilon$ and apply (i) $\leftarrow \sigma_0 = 1-\varepsilon$ to get

$$|L'(s,\chi)| \le \frac{1}{2\pi\varepsilon} \cdot 2\pi\, C_\varepsilon |t|^{\varepsilon}$$

∅

· Thm 7. For every $\varepsilon>0$, we have $\left|\frac{1}{L(s,\chi)}\right| \le C_\varepsilon |t|^{\varepsilon}$ when $s=\sigma+it$, $\sigma \ge 1$ and $|t| \ge 1$

proof:

① From Lemma 3 we know $\forall \sigma \ge 1, |t| \ge 1$

$$1 \le |L^3(\sigma,\chi_0) L^4(\sigma+it,\chi) L(\sigma+2it,\chi^2)|$$

$$\Rightarrow |L(\sigma+it,\chi)| \ge C' \cdot (\sigma-1)^{\frac{3}{4}} \cdot |t|^{-\frac{\varepsilon}{4}} \longrightarrow (*)$$

② Consider two cases:

(i) If $\sigma - 1 \geq A|t|^{-5\varepsilon}$ (where A is a appropriate const we will choose later)

$$\Rightarrow \quad |L(\sigma+it,\chi)| \geq A'|t|^{-4\varepsilon}$$

(ii) If $\sigma - 1 \leq A|t|^{-5\varepsilon}$, then choose $\sigma' > \sigma$ with

$$\sigma' - 1 = A \cdot |t|^{-5\varepsilon}$$

$$\Rightarrow \quad |L(\sigma+it,\chi)| \geq |L(\sigma'+it,\chi)| - |L(\sigma'+it,\chi) - L(\sigma+it,\chi)|$$

but from mean-value thm, and Thm 7

$$|L(\sigma'+it,\chi) - L(\sigma+it,\chi)| \leq c'' \cdot |\sigma'-\sigma| \, |t|^{\varepsilon} \leq c'' \cdot |\sigma'-1| \cdot |t|^{\varepsilon}$$

also from (✡), take $\sigma = \sigma'$ get

$$|L(\sigma+it,\chi)| \geq c'(\sigma'-1)^{\frac{3}{4}}|t|^{-\frac{\varepsilon}{4}} - c''(\sigma'-1)|t|^{\varepsilon}$$

Choose $A = \left(c'/_{2c''}\right)^{4}$, and recall $\sigma'-1 = A|t|^{-5\varepsilon}$

$$\Rightarrow \quad c'(\sigma'-1)^{\frac{3}{4}}\,|t|^{-\frac{\varepsilon}{4}} = 2c''(\sigma'-1)\,|t|^{\varepsilon}$$

so $\quad |L(\sigma+it,\chi)| \geq A'' \cdot |t|^{-4\varepsilon}$

✡

Proof of Dirichlet Theorem with Density

$$\Lambda_a(n) := \begin{cases} \log p, & \text{if } n = p^k \text{ and } p \equiv a \pmod m \\ 0, & \text{otherwise} \end{cases}$$

$$\psi_a(x) := \sum_{n \le x} \Lambda_a(n)$$

$$\psi_{1a}(x) := \int_1^x \psi_a(u)\,du$$

- Thm 8: $\psi_a(x) \sim \frac{x}{\phi(m)} \Rightarrow \pi_a(x) \sim \frac{x}{\phi(m)\log x}$

Pf: It suffices to show

$1 \le \liminf \pi_a(x) \cdot \frac{\phi(m)\log x}{x}$ and $\limsup \pi_a(x) \frac{\phi(m)\log x}{x} \le$

① $$\psi_a(x) = \sum_{\substack{p \le x \\ p \equiv a \pmod m}} \left[\frac{\log x}{\log p}\right] \log p \le \sum_{\substack{p \le x \\ p \equiv a \pmod m}} \frac{\log x}{\log p} \log p$$

$$= \pi_a(x) \log x$$

$$\Rightarrow \frac{\phi(m)\psi_a(x)}{x} \le \pi_a(x) \frac{\phi(m)\log x}{x}$$

hence $\liminf \pi_a(x) \frac{\phi(m)\log x}{x} \ge 1$

② Fix $0 < \alpha < 1$, note that

$$\psi_a(x) \ge \sum_{\substack{p \le x \\ p \equiv a \pmod m}} \log p \ge \sum_{\substack{x^\alpha < p \le x \\ p \equiv a \pmod m}} \log p \ge (\pi_a(x) - \pi_a(x^\alpha)) \log x^\alpha$$

$$\Rightarrow \psi_a(x) + \alpha \pi_a(x^\alpha) \log x \ge \alpha \pi_a(x) \log x$$

$$\Rightarrow \alpha\, \pi_a(x) \frac{\phi(m) \log x}{x} \le \frac{\psi_a(x)\phi(m)}{x} + \alpha \cdot \frac{x^\alpha \log x \cdot \phi(m)}{x}$$

$$\Rightarrow \alpha \cdot \limsup_{x\to\infty} \pi_a(x) \frac{\phi(m)\log x}{x} \le 1$$

then let $\alpha \to 1$ we complete the proof ✠.

· Thm 9. $\psi_{1a}(x) \sim \dfrac{x^2}{2\phi(m)} \Rightarrow \psi_a(x) \sim \dfrac{x}{\phi(m)}$

Pf: Since $\psi_a(x)$ is increasing, hence if $1<\beta$

$$\psi_a(x) \le \frac{1}{(\beta-1)x} \int_x^{\beta x} \psi_a(u)\,du$$

$$\Rightarrow \frac{\phi(m)\psi_a(x)}{x} \le \frac{\phi(m)}{(\beta-1)x^2} \left[\psi_{1a}(\beta x) - \psi_{1a}(x)\right]$$

$$\Rightarrow \limsup_{x\to\infty} \frac{\phi(m)\psi_a(x)}{x}$$

$$\le \limsup_{x\to\infty} \frac{\phi(m)}{\beta-1}\left[\frac{\psi_{1a}(\beta x)}{\beta x}\beta^2 - \frac{\psi_{1a}(x)}{x^2}\right]$$

$$= \frac{1}{\beta-1}\cdot\left(\frac{1}{2}\beta^2 - \frac{1}{2}\right) = \frac{1}{2}(\beta+1)$$

let $\beta \to 1$ get $\limsup_{x\to\infty} \dfrac{\phi(m)\psi_a(x)}{x} \le 1$

Similarly $\liminf_{x\to\infty} \dfrac{\phi(m)\psi_a(x)}{x} \ge 1$ come from

$$\frac{1}{(1-\alpha)x}\int_{\alpha x}^{x} \psi_a(u)\,du \le \psi_a(x) \quad ✠$$

· Recall Lemma 2.4, chap 7, Stein:

If $c>0$, then

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{a^s}{s(s+1)}\, ds = \begin{cases} 0, & \text{if } 0<a\le 1 \\ 1-\frac{1}{a}, & \text{if } 1\le a \end{cases}$$

· Thm 10: $\psi_{1a}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \left(\sum_{\chi} \frac{-\chi(a^{-1}) L'(s,\chi)}{\phi(m) L(s,\chi)} \right) ds$ on $c>1$

pf.

① First note that

$$\sum_{\chi} \chi(a^{-1}) \frac{L'(s,\chi)}{L(s,\chi)} = \sum_{\chi} \chi(a^{-1}) \sum_{n=1}^{\infty} \frac{-\chi(n)\Lambda(n)}{n^s}$$

$$= \sum_{n=1}^{\infty} \frac{-\chi(a^{-1}n)\Lambda(n)}{n^s}$$

$$\overset{\text{Thm 2}}{=} -\phi(m) \sum_{n=1}^{\infty} \frac{\Lambda_a(n)}{n^s}$$

② $\psi_{1a}(x) = \int_0^x \psi_a(u)\, du$

$$= \sum_{n\le x} \int_0^x \Lambda_a(n) f_n(u)\, du \quad f_n(u) = \begin{cases} 1, & \text{if } n\le u \\ 0, & \text{otherwise} \end{cases}$$

$$= \sum_{n\le x} \Lambda_a(n) \int_n^x du$$

$$= \sum_{n\le x} \Lambda_a(n)(x-n)$$

③ $$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s+1}}{s(s+1)}\left(\sum_{\chi}\frac{-\chi(a^{-1})L'(s,\chi)}{\phi(m)L(s,\chi)}\right)ds$$

$$= \frac{x}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s}}{s(s+1)}\sum_{n\geq 1}^{\infty}\frac{\Lambda_a(n)}{n^s}$$

$$= x\sum_{n=1}^{\infty}\Lambda_a(n)\cdot\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\left(\frac{x}{n}\right)^s ds}{s(s+1)}$$

$$= x\sum_{n\leq x}\Lambda_a(n)\left(1-\frac{n}{x}\right)$$

$$= \sum_{n\leq x}\Lambda_a(n)(x-n) = \psi_{1a}(x)$$

Notice that $\displaystyle\sum_{n=1}^{\infty}\int_{c-i\infty}^{c+i\infty}\left|\frac{x^s}{s(s+1)}\cdot\frac{\Lambda_a(n)}{n^s}\right|ds$

$$\leq \sum_{n\geq 1}^{\infty}\frac{\Lambda_a(n)}{n^c}\int_{c-i\infty}^{c+i\infty}\frac{x^c}{s(s+1)}ds$$

$$\leq A\sum_{n=1}^{\infty}\frac{\Lambda_a(n)}{n^c} < \infty$$

· If we set $g_a(s) = \dfrac{x^{s+1}}{s(s+1)}\cdot\left(\sum_{\chi}\dfrac{-\chi(a^{-1})}{\phi(m)}\cdot\dfrac{L'(s,\chi)}{L(s,\chi)}\right)$

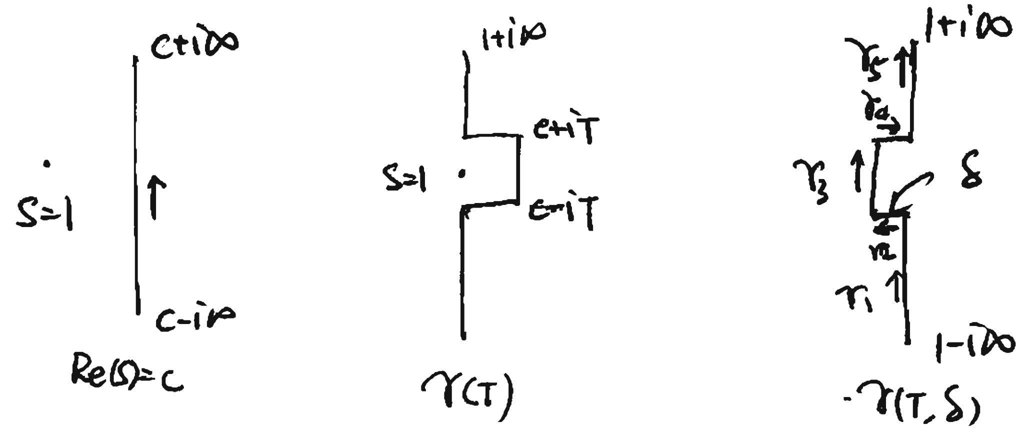
then $\displaystyle\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}g_a(s)\,ds = \psi_{1a}(x)$

since $L(s,\chi)$ is holo if $\chi\neq\chi_0$ — arguement principle

$L(s,\chi_0$ has a simple pole at $s=1$

we know $g_a(s)$ has a simple pole of orde $\dfrac{x^2}{2\phi(m)}$

· Thm 11 $\psi_{1a}(x) \sim \dfrac{x^2}{2\phi(m)}$



Pf

① By Cauchy Thm, and that $\left|\dfrac{L'(s,\chi)}{L(s,\chi)}\right| \le A|t|^{\eta}$ ← $s=\sigma+it$, $\sigma \ge 1$, $|t| \ge 1$

for any fixed $\eta > 0$ (by Thm 6, Thm 7)

we can see $\dfrac{1}{2\pi i}\displaystyle\int_{1+i\infty}^{c+i\infty} g_a(s)\,ds = 0$

so $\psi_{1a}(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} g_a(s)\,ds = \dfrac{1}{2\pi i}\int_{\gamma(T)} g_a(s)\,ds$

② By residue thm,

$$\frac{1}{2\pi i}\int_{\gamma(T)} g_a(s)\,ds = \frac{x^2}{2\phi(m)} + \frac{1}{2\pi i}\int_{\gamma(T,\delta)} g_a(s)\,ds$$

T, δ would be determined later

to have $\displaystyle\int_{\gamma(T,\delta)} g_a(s)\,ds \to 0$

③ For γ_1, γ_5: we may take $\left|\frac{L'(s,\chi)}{L(s,\chi)}\right| \le A \cdot |t|^{\frac{1}{2}}$

then we have $\left|\frac{1}{2\pi i}\int_{\gamma_1} g_a(s)\,ds\right| \le Cx^2\int_T^{\infty}\frac{|t|^{\frac{1}{2}}}{t^2}\,dt$

So we can choose T so large s.t.

$$\text{R.H.S.} \le \frac{\varepsilon}{2}x^2 \text{ for a fixed } \varepsilon > 0.$$
$$\forall x$$

γ_5 is similar.

④ For γ_3. Choose δ so small so that $L(s,\chi) \ne 0$ $\forall s \in \gamma_3$ $\forall \chi$

(because we have proved that $L(s,\chi) \ne 0$ on $1+it$)

note that $|x^{1+s}| = x^{2-\delta}$

hence $\left|\frac{1}{2\pi i}\int_{\gamma_3} g_a(s)\,ds\right| \le C_T x^{2-\delta}$

depend on T.

⑤ For γ_2, γ_4:

$$\left|\frac{1}{2\pi i}\int_{\gamma_2} g_a(s)\right| \le C_T'\int_{1-\delta}^{1} x^{1+\sigma}\,d\sigma \le C_T'\frac{x^2}{\log x}$$

⑥ From ③, ④, ⑤, we get

$$\left|\frac{\psi_{1a}(x)}{x^2} - \frac{1}{2\phi(m)}\right| \le 2\varepsilon + C_T x^{-\delta} + C_T'\frac{1}{\log x}$$

so $\psi_{1a}(x) \sim \dfrac{x^2}{2\phi(m)}$ ✠

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Dirichlet's Principle

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2015.06.25

Notation. Let A, B be subsets of a topological space. We say $A \subset\subset B$ if $\overline{A}$, the closure of A, is contained in $\mathring{B}$, the interior of B.

Let (Ω, z) be a coordinate patch of a Riemann surface S. Then for some $a \in \mathbb{C}$ and $r > 0$, if $B(a; r) \subset\subset z(\Omega)$, then we call $B_z(a; r) := z^{-1}(B(a; r))$ a z-disk.

1 Das Dirichletsche Integral

Notation (p.107)**.** Let S denote a connected (oriented) Riemann surface. Anything related to "K" denotes a z-disk for some z. In particular, we arbitrarily fix a point $p_0 \in S$, a coordinate map z_0 with $z_0(p_0) = 0$, and some appropriate $0 < R_0 < R_0'$. Then we call $K_0 := B_{z_0}(p_0; R_0)$ the **hole**, call $K_0' := B_{z_0}(p_0; R_0')$ the **lid**, call $K_0' \backslash \overline{K_0}$ the **lock-ring**, and call $S \backslash \overline{K_0}$ the **punched surface**.

Recall (p93, p72)**.** For $\eta = (\eta_1 \mathrm{d}x + \eta_2 \mathrm{d}y)$ and $\xi = (\xi_1 \mathrm{d}x + \xi_2 \mathrm{d}y)$ being two 1-forms, we define $[\eta, \xi] := \eta \wedge (^*\xi) = (\eta_1\xi_1 + \eta_2\xi_2)(\mathrm{d}x \wedge \mathrm{d}y)$, which is symmetric and bilinear on the two inputs.

Definition (p.97)**.** Let $A \subseteq S$ be a region, and $v, w \in \mathcal{C}^1(A)$. The Dirichlet integral is defined to be $D_A(v, w) := \int_A [\mathrm{d}v, \mathrm{d}w]$. If $v = w$, we denote the integral by $D_A(v) := D_A(v, v) \geq 0$. The set of **admissible functions** is defined to be $\mathfrak{M}(A) := \{v \in \mathcal{C}^1(\mathring{A}) \cap \mathcal{C}^0(\overline{A}) \,:\, D_A(v) < \infty\}$

Notation (p.114)**.** For $v \in \mathfrak{M}(K)$, define $\overline{v}$ to be the harmonic function on K that agrees with v on ∂K (which may be derived from Poisson's integration formula).

Lemma 1 (p.97)**.** $\forall v \in \mathfrak{M}(K)$, $D_K(v) - D_K(\overline{v}) = D_K(v - \overline{v}) \geq 0$.

(hint: $D_K(\overline{v}, v - \overline{v}) = 0$)

Theorem 2 (p.106)**.** *Let Φ be a harmonic function on the lid which is regular in the lock-ring, and satisfies $\frac{\partial \Phi}{\partial n} = 0$ along ∂K_0. There exists a harmonic function U such that U is regular in $S \backslash \overline{K_0}$ and that $U - \Phi$ is regular in K_0.*

Definition (p.108)**.** The set of **competing functions** is defined to be

$$\mathscr{F} := \{(v, v^*) \,:\, v \in \mathfrak{M}(S \backslash \overline{K_0}),\ v^* \in \mathfrak{M}(K_0'),\ v \equiv v^* + \Phi \text{ in } K_0' \backslash \overline{K_0}\}$$

Whenever there is no ambiguity, we tend to use v in place of (v, v^*). We define the **potential** to be $D(v) := D_{S\backslash\overline{K_0}}(v) + D_{K_0}(v^*)$.

Remark (p.108)**.** The potential can be also derived by the following process: Let a smoothing function λ be fixed, which is identically 1 in the hole, and vanishes outside the lid. We define the 2-forms $\Psi = (1-\lambda)[\mathrm{d}v, \mathrm{d}v] + \lambda[\mathrm{d}v^*, \mathrm{d}v^*]$ over S, and that $\Psi' = \lambda\left([\mathrm{d}v, \mathrm{d}v] - [\mathrm{d}v^*, \mathrm{d}v^*]\right)$ over $K_0' \backslash \overline{K_0}$. Then $D(v)$ can be given by the sum of $D_\lambda(v) := \int_S \Psi$ and $D_\lambda'(v) := \int_{K_0'\backslash\overline{K_0}} \Psi'$.

Fact 3 (pp.108–109)**.**

1. $\forall v \in \mathscr{F}$, $0 \leq D(v) < \infty$.
2. If U exists, then $(u, u^*) := (U|_{S\backslash\overline{K_0}}, U|_{K_0'} - \Phi) \in \mathscr{F}$.
3. If Φ can be extended on an open disk K that contains the closure of the lid, then there exists a cut-off function λ such that $\lambda|_{K_0'} \equiv 1$ and $\lambda|_{S\backslash K} \equiv 0$. Therefore the pair (v_0, v_0^*) which is defined by $v_0^* \equiv 0$, $v_0 \equiv \lambda\Phi$ on $K\backslash K_0$, and $v_0 \equiv 0$ on $S\backslash K$ is a competing function.

   In summary, we are free to assume $\mathscr{F} \neq \varnothing$

4. Let K be contained in the lid or the punched surface. Suppose that $v_1, v_2 \in \mathscr{F}$ coincide outside of K. That is, $v_1 \equiv v_2$ and $v_1^* \equiv v_2^*$ respectively on each of their domains except on K. Then

$$D(v_1) - D(v_2) = \begin{cases} D_K(v_1) - D_K(v_2) & \text{whenever } K \subseteq S\backslash\overline{K_0} \\ D_K(v_1^*) - D_K(v_2^*) & \text{whenever } K \subseteq K_0' \end{cases}$$

   (hint: for the second case, apply Green's theorem)

Observation 4 (p.110)**.** $\mathscr{F} = v_0 + \mathfrak{M}(S)$ in the following senses:

First, for all $v_1, v_2 \in \mathscr{F}$, $v_1 - v_2$ and $v_1^* - v_2^*$ agree on the lock-ring, so they define an admissible function on S. Conversely, for all $v \in \mathscr{F}$ and $w \in \mathfrak{M}(S)$, $(v + w, v^* + w)$ lies in $\mathscr{F}$. Therefore for a fixed member $v_0 \in \mathscr{F}$, there is a one-to-one correspondence $\mathscr{F} \leftrightarrow \mathfrak{M}(S)$, $v \mapsto v - v_0$

Second, define $T := K_0 + (S\backslash\overline{K_0})$ to be the direct sum of spaces, which may be identified with $S\backslash\partial K_0$ sometimes. We identify $v \in \mathscr{F}$ with the corresponding function in $\mathcal{C}^1(T)$, which is defined by

$$p \mapsto \begin{cases} v(p) & \text{if } p \in S\backslash\overline{K_0} \\ v^*(p) & \text{if } p \in K_0 \end{cases}$$

and satisfies $D_T(v) = D(v) < \infty$. Thus $v \in \mathfrak{M}(T)$.

Finally, notice that $({}^{\mathfrak{M}(T)}/_{\sim}, D_T(\cdot,\cdot))$ is a inner-product space over $\mathbb{R}$, where the equivalence relation $\sim$ presents "$v_1 \sim v_2 \Leftrightarrow v_1 - v_2 = const.$" In addition, $\mathfrak{M}(S)$, which is included in $\mathfrak{M}(T)$ by restriction, is a subspace. Therefore we can handle the problem as a problem of orthogonal projection: find $v_{/\!/} = w \in \mathfrak{M}(S)$ so that the norm of $v_\perp = u = v - w$ is minimized.

Proposition 5 (p.110, due to Beppo Levi)**.** *Define* $d := \inf\{D(v) : v \in \mathscr{F}\}$. *Then for all* $v_1, v_2 \in \mathscr{F}$*,*

$$\sqrt{D_S(v_1 - v_2)} \leq \sqrt{D(v_1) - d} + \sqrt{D(v_2) - d}$$

Proof. As mentioned, we identify $\mathscr{F}$ as a subset of $\mathfrak{M}(T)$.

Let $\lambda \in \mathbb{R}$. If $\lambda \neq -1$, then $\frac{\lambda v_1 + v_2}{\lambda + 1} \in \mathscr{F}$. Hence $D_T(\frac{\lambda v_1 + v_2}{\lambda+1}) = D(\frac{\lambda v_1 + v_2}{\lambda+1}) \geq d$, so $D_T(\lambda v_1 + v_2) \geq (\lambda+1)^2 d$. The last inequality remains valid when $\lambda = -1$.

In summary, the quadratic function on λ

$$\lambda^2(D_T(v_1) - d) + 2\lambda(D_T(v_1, v_2) - d) + (D_T(v_2) - d)$$

is always ≥ 0. Hence we have the discriminant

$$(D_T(v_1, v_2) - d)^2 - (D_T(v_1) - d)(D_T(v_2) - d) \leq 0$$

It follows that

$$\begin{aligned}
0 &\leq D_T(v_1 - v_2) \\
&= D_T(v_1) - 2D_T(v_1, v_2) + D_T(v_2) \\
&= (D_T(v_1) - d) + (D_T(v_2) - d) - 2(D_T(v_1, v_2) - d) \\
&\leq (D_T(v_1) - d) + (D_T(v_2) - d) + 2\sqrt{(D_T(v_1) - d)(D_T(v_2) - d)} \\
&= \left(\sqrt{D_T(v_1) - d} + \sqrt{D_T(v_2) - d}\right)^2
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \sqrt{D_T(v_1 - v_2)} \leq \sqrt{D_T(v_1) - d} + \sqrt{D_T(v_2) - d} \\
&\Rightarrow \sqrt{D_S(v_1 - v_2)} \leq \sqrt{D(v_1) - d} + \sqrt{D(v_2) - d}
\end{aligned}$$

□

Corollary (p.111)**.** If a minimizing function exists, it is unique up to an additive constant.

Notation (p.111)**.** $\lim\limits_{v}$ means the limitation taken as $D(v) \to d$ among those $v \in \mathscr{F}'$, where $\mathscr{F}' := \left\{ v \in \mathscr{F} : \int_{\partial K_0} v^* \, \mathrm{d}s = 0 \right\}$.

2 Fourierreihe

Let $K = B_z(0; R)$ be a fixed z-disk, and $z = x + iy = re^{i\theta}$. For all $v, w \in \mathfrak{M}(K)$, define $J_{z,K}(v, w) := \iint_{z(K)} v(z)w(z)\,\mathrm{d}x\,\mathrm{d}y$, and that $J_{z,K}(v) := J_{z,K}(v, v)$.

Let $u = \overline{v}$ be the harmonic function on K that agree with $v \in \mathfrak{M}(K)$ on ∂K. Then u is the real part of an analytic function $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Hence

$$\begin{aligned} u(z) = \mathrm{Re}(f(z)) &= \sum_{n=0}^{\infty} (\mathrm{Re}(c_n)\mathrm{Re}(z^n) - \mathrm{Im}(c_n)\mathrm{Im}(z^n)) \\ &= a_0 + \sum_{n=1}^{\infty} (a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)) \end{aligned}$$

where $a_n = \mathrm{Re}(c_n)$ and $b_n = -\mathrm{Im}(c_n)$. Notice that $\int_0^{2\pi} f(re^{i\theta})e^{-ni\theta}\,\mathrm{d}\theta = 2\pi r^n c_n$ for $n \geq 0$, and $= 0$ for $n < 0$. Hence for all $n > 0$,

$$\begin{aligned} a_n &= \frac{1}{2\pi r^n}\mathrm{Re}\left(\int_0^{2\pi} f(re^{i\theta})e^{-ni\theta}\,\mathrm{d}\theta\right) \\ &= \frac{1}{2\pi r^n}\mathrm{Re}\left(\int_0^{2\pi} f(re^{i\theta})(e^{-ni\theta} + e^{ni\theta})\,\mathrm{d}\theta\right) \\ &= \frac{1}{2\pi r^n}\int_0^{2\pi} \mathrm{Re}\left(f(re^{i\theta})(2\cos(n\theta))\right)\,\mathrm{d}\theta \\ &= \frac{1}{\pi r^n}\int_0^{2\pi} u(re^{i\theta})\cos(n\theta)\,\mathrm{d}\theta & \text{, and similarly,} \\ b_n &= \frac{1}{\pi r^n}\int_0^{2\pi} u(re^{i\theta})\sin(n\theta)\,\mathrm{d}\theta \end{aligned}$$

Note that $a_0 = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,\mathrm{d}\theta$

Define $P_n = \mathrm{Re}(z^n) = r^n\cos(n\theta), Q_n = \mathrm{Im}(z^n) = r^n\sin(n\theta) \in \mathfrak{M}(K)$. Observe that $\mathrm{d}P_n = {}^*\mathrm{d}Q_n$, so that by Green's formula,

$$\begin{aligned} D_K(v, P_n) &= \int_K \mathrm{d}v \wedge \mathrm{d}Q_n = \int_{\partial K} v\,\mathrm{d}Q_n \\ &= nR^n \int_0^{2\pi} v(Re^{i\theta})\,\cos(n\theta)\,\mathrm{d}\theta \\ &= nR^n \int_0^{2\pi} u(Re^{i\theta})\,\cos(n\theta)\,\mathrm{d}\theta \\ &= \pi n R^{2n} a_n & \text{, and similarly,} \\ D_K(v, Q_n) &= \pi n R^{2n} b_n \end{aligned}$$

By setting $u = v = P_n$ or Q_n, we have the orthogonality relations

$$\begin{cases} D_K(P_m, Q_n) = 0 & \text{without exception} \\ D_K(P_m, P_n) = D_K(Q_m, Q_n) = 0 & \text{if } m \neq n \\ D_K(P_n) = D_K(Q_n) = \pi n R^{2n} & \text{without exception} \end{cases}$$

Also, by integrating under the polar coordinate, we have

$$\begin{cases} J_{z,K}(P_m, Q_n) = 0 & \text{without exception} \\ J_{z,K}(P_m, P_n) = J_{z,K}(Q_m, Q_n) = 0 & \text{if } m \neq n \\ J_{z,K}(P_n) = J_{z,K}(Q_n) = \frac{\pi}{2n+2} R^{2n+2} & \text{if } n > 0 \\ J_{z,K}(P_0) = \pi R^2 \end{cases}$$

Since $u(z) = a_0 + \sum_{n=1}^{\infty} (a_n P_n + b_n Q_n)$ converges uniformly, the orthogonality relation of D_K provides that

$$D_K(v) \geq D_K(u) = \sum_{n=1}^{\infty} \pi n R^{2n} (a_n^2 + b_n^2)$$

Similarly,

$$J_{z,K}(u) = \pi R^2 a_0^2 + \sum_{n=1}^{\infty} \frac{\pi}{2n+2} R^{2n+2} (a_n^2 + b_n^2)$$

Lemma 6 (p.103)**.** For all $v \in \mathfrak{M}(K)$, $\exists a \in \mathbb{R}$ such that $J_{z,K}(v-a) \leq const. D_K(v)$

Proof. On one hand, take $a = a_0$ with respect to $u = \overline{v}$, then

$$\begin{aligned} J_{z,K}(u - a_0) &= \sum_{n=1}^{\infty} \frac{\pi}{2n+2} R^{2n+2} (a_n^2 + b_n^2) \leq \frac{R^2}{4} \sum_{n=1}^{\infty} \pi n R^{2n} (a_n^2 + b_n^2) \\ &= \frac{R^2}{4} D_K(u) \end{aligned}$$

On the other hand, for $w = v - u$, which vanishes on ∂K,

$$w(\rho e^{i\theta}) = \int_R^\rho \frac{\partial w(z)}{\partial r} \, \mathrm{d}r$$

By Schwartz's inequality,

$$\begin{aligned} w(\rho e^{i\theta})^2 &= \left\{ \int_R^\rho \left[\frac{\partial w(z)}{\partial r} \sqrt{r} \right] \left[\frac{1}{\sqrt{r}} \right] \mathrm{d}r \right\}^2 \leq \int_R^\rho \left[\frac{\partial w(z)}{\partial r} \right]^2 r \, \mathrm{d}r \int_R^\rho \frac{1}{r} \, \mathrm{d}r \\ &= \int_\rho^R \left[\frac{\partial w}{\partial x} \cos\theta + \frac{\partial w}{\partial x} \sin\theta \right]^2 r \, \mathrm{d}r (\log R - \log \rho) \\ &= \int_\rho^R 2 \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] r \, \mathrm{d}r (\log R - \log \rho) \end{aligned}$$

Next, integrate the previous equation in order to yield that

$$\begin{aligned} J_{z,K}(w) &\leq \int_0^R \int_0^{2\pi} \int_\rho^R 2 \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial x} \right)^2 \right] r (\log R - \log \rho) \rho \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\rho \\ &= \int_0^R 2 \left\{ \int_{\rho \leq |z| \leq R} [\mathrm{d}w, \mathrm{d}w] \right\} (\log R - \log \rho) \rho \, \mathrm{d}\rho \\ &\leq 2 D_K(w) \int_0^R (\log R - \log \rho) \rho \, \mathrm{d}\rho = \frac{R^2}{4} D_K(w) \end{aligned}$$

Finally,

$$\begin{aligned}J_{z,K}(v-a_0) &= J_{z,K}((u-a_0)+w) \le 2(J_{z,K}(u-a_0)+J_{z,K}(w)) \\ &\le \frac{R^2}{2}[D_K(u)+D_K(w)] = \frac{R^2}{2}D_K(v)\end{aligned}$$

□

Proposition 7 (p.112)**.** *For all $K = B_z(0;R)$, there is a constant C so that for every $w \in \mathfrak{M}(S)$ that satisfies*

$$\int_{\partial K_0} w\,\mathrm{d}s = R_0\int_0^{2\pi} w\left(z_0^{-1}\left(R_0e^{i\theta}\right)\right)\,\mathrm{d}\theta = 0$$

we have $J_{z,K}(w) \le CD_S(w)$.

Proof. Recall that K_0 is the hole. Let each $1 \le j \le n$ be corresponded with K_j, which is a z_j-disk with radius R_j, such that $K_n = K$, $z_n = z$, and that $\forall 1 \le j \le n$, $K_{j-1} \cap K_j \ne \varnothing$. Set the constants c_j so that $\int_{\partial K_j}(w-c_j) = 0\,\mathrm{d}s$, where $c_0 = 0$.

We prove by induction. If $n = 0$, i.e., $K = K_0$, we take $C = \frac{R_0^2}{2}$ by Lemma 6.

It suffices to prove that if our claim holds on K_{n-1}, then it holds on K_n. Let $k \subset\subset K_{n-1} \cap K_n$ be a z_n-disk with radius tR_n, where $0 < t < 1$. Let m be an upper bound for $\left|\dfrac{\mathrm{d}z_n}{\mathrm{d}z_{n-1}}\right|$ on k. By the inductive hypothesis, there is a constant C' which only depends on K_{n-1} such that

$$J_{z_n,k}(w) \le m^2 J_{z_{n-1},k}(w) \le m^2 C' D_S(w)$$

In addition, by Lemma 6, we have

$$J_{z_n,k}(w-c_n) \le J_{z_n,K_n}(w-c_n) \le \frac{1}{2}R_n^2 D_k(w) \le \frac{1}{2}R_n^2 D_S(w)$$

It follows that

$$\begin{aligned}\pi c_n^2 t^2 R_n^2 = J_{z_n,k}(c_n) &\le 2(J_{z_n,k}(w)+J_{z_n,k}(w-c_n)) \\ &\le (2m^2C'+R_n^2)D_S(w)\end{aligned}$$

Finally, we have

$$\begin{aligned}J_{z,K}(w) &\le 2(J_{z_n,K_n}(w-c_n)+J_{z_n,K_n}(c_n)) \\ &\le 2\left(\frac{1}{2}R_n^2 D_K(w)+\pi c_n^2 R_n^2\right) \\ &\le 2\left(\frac{1}{2}R_n^2 D_S(w)+\frac{2m^2C'+R_n^2}{t^2}D_S(w)\right) \\ &= \left(R_n^2+\frac{4m^2C'+2R_n^2}{t^2}\right)D_S(w)\end{aligned}$$

□

3 Die Mittelwertfunktion

Recall. Let $z = x + iy$ be a local coordinate map and $K = B_z(0; R)$ be a open disk with "center" $p = z^{-1}(0)$. If v is harmonic, then

$$v(p) = \frac{1}{\pi R^2} \iint_K v(x + iy) \, \mathrm{d}x \, \mathrm{d}y$$

Notation (p.113)**.** From now on, let a point $p \in S$, a coordinate map z at p be fixed. In addition, let $K = B_z(0; R)$ be contained in the punched surface or the lid. Define a map $\mathbf{M}_{z,K} : \mathfrak{M}(K) \to \mathbb{R}$, which is abbreviated to $\mathbf{M}$, as following:

$$\mathbf{M}_{z,K}(w) = \frac{1}{\pi R^2} \iint_K v(x + iy) \, \mathrm{d}x \, \mathrm{d}y = \frac{1}{\pi R^2} \int_0^R \int_0^{2\pi} v(re^{i\theta}) \, r \, \mathrm{d}\theta \, \mathrm{d}r$$

If K is contained in the punched surface, one yields from Schwarz's inequality, and the Propositions 5 and 7 that for all $v_1, v_2 \in \mathscr{F}'$,

$$\begin{aligned}
(\mathbf{M}(v_1) - \mathbf{M}(v_2))^2 &= \left(\frac{1}{\pi R^2} \iint_K (v_1 - v_2) \, \mathrm{d}x \, \mathrm{d}y \right)^2 \\
&\leq \frac{1}{\pi R^2} \iint_K (v_1 - v_2)^2 \, \mathrm{d}x \, \mathrm{d}y = \frac{1}{\pi R^2} J_{z,K}(v_1 - v_2) \\
&\leq \frac{C}{\pi R^2} \left(\sqrt{D(v_1) - d} + \sqrt{D(v_2) - d} \right)^2
\end{aligned}$$

That is,

$$|\mathbf{M}(v_1) - \mathbf{M}(v_2)| \leq \frac{1}{R} \sqrt{\frac{C}{\pi}} \left(\sqrt{D(v_1) - d} + \sqrt{D(v_2) - d} \right) \tag{1}$$

Therefore $\lim\limits_v \mathbf{M}(v)$ exists. We denote the limit by $u(p)$. Then by the previous estimation,

$$|\mathbf{M}(v) - u(p)| \leq \frac{1}{R} \sqrt{\frac{C}{\pi}} \sqrt{D(v) - d} \tag{2}$$

For all $q \in K$, let $\mathbf{M}_q$ denote $\mathbf{M}_{z,k_q}$, where the disk $k_q := B_z(z(q); R - |z(q)|)$ is contained in K. Since we have an estimation which is similar to (1), the limit $u(q) := \lim\limits_v \mathbf{M}_q(v)$ exists. Moreover, in place of (2),

$$|\mathbf{M}_q(v) - u(q)| \leq \frac{1}{R - |z(q)|} \sqrt{\frac{C}{\pi}} \sqrt{D(v) - d}$$

It follows that $\mathbf{M}_q(v)$ converges uniformly to $u(q)$ on $q \in k$, where $k \subset\subset K$ is a disk (concentric with K).

Remark (p.114)**.** If K is contained in the lid, we can compute $u^*(p) := \lim\limits_v \mathbf{M}(v^*)$, which existence and estimations are given in a similar way. In particular, if K is contained in the lock-ring, $u = u^* + \Phi$ because Φ is harmonic.

Proposition 8 (p.114)**.** *$u : K \to \mathbb{R}$ or $u^* : K \to \mathbb{R}$ is harmonic (whenever any one of which is defined).*

Proof. For simplicity, we suppose that $K \subseteq S\backslash\overline{K_0}$ and consider $v \in \mathscr{F}'$. A similar argument holds for $K \subset K_0'$ and v^*.

Recall that $\overline{v} \in \mathfrak{M}(K)$ is harmonic. We define $\widetilde{v} \in \mathscr{F}$ by applying a smoothing process so that $\widetilde{v}$ coincides with v outside of K, but with $\overline{v}$ in $k = B_z(0; r)$, where $0 < r < R$. Let the smoothing be well chosen so that $D_K(\widetilde{v}) \to D_K(\overline{v})$ as $r \to R^-$.

By Lemma 1, $D_K(\overline{v}) \leq D_K(v)$, and it takes "=" if and only if v is harmonic, namely $v = \overline{v} = \widetilde{v}$. Therefore for sufficiently large r, we have $D_K(\widetilde{v}) \leq D_K(v)$. Notice that $\overline{v} = \overline{\widetilde{v}}$, so that $D_K(\overline{v}) \leq D_K(\widetilde{v})$. Hence $D_K(\overline{v}) \leq D_K(\widetilde{v}) \leq D_K(v)$. By Fact 3.4, $D(\widetilde{v}) \leq D(v)$.

We replace v_2 with $\widetilde{v_2}$ in Levi's inequality to yield that

$$\begin{aligned}\sqrt{D_K(v_1 - \widetilde{v_2})} &\leq \sqrt{D_S(v_1 - \widetilde{v_2})}\\ &\leq \sqrt{D(v_1) - d} + \sqrt{D(\widetilde{v_2}) - d}\\ &\leq \sqrt{D(v_1) - d} + \sqrt{D(v_2) - d}\end{aligned}$$

Take $r \to R^-$. Thus

$$\sqrt{D_K(v_1 - \overline{v_2})} \leq \sqrt{D(v_1) - d} + \sqrt{D(v_2) - d} \tag{3}$$

Similarly,

$$\sqrt{D_K(\overline{v_1} - \overline{v_2})} \leq \sqrt{D(v_1) - d} + \sqrt{D(v_2) - d}$$

Repeat the argument for (1). So $\lim\limits_{v} \mathbf{M}_q(\overline{v}) = u(q)$. Note that $\mathbf{M}_q(\overline{v}) = \overline{v}(q)$ because $\overline{v}$ is harmonic. Hence in place of (2),

$$|\overline{v}(q) - u(q)| \leq \frac{1}{R - |z(q)|}\sqrt{\frac{C}{\pi}}\sqrt{D(v) - d}$$

As a result, $\lim\limits_{v} \overline{v}(q) = u(q)$ uniformly on $q \in k$ for any $k \subset\subset K$. Therefore u is also harmonic. □

Lemma 9 (p.115)**.** For all $v \in \mathscr{F}'$, we have

- $D_K(v - \overline{v}) \leq 4(D(v) - d)$
- $J_{z,K}(v - \overline{v}) \leq R^2(D(v) - d)$

Proof. First, take $v_1 = v_2 = v$ in (3) to get the first estimation. Next, since $(v - \overline{v})$ vanishes on ∂K, $J_{z,K}(v - \overline{v}) \leq \frac{R^2}{4} D_K(v - \overline{v}) \leq R^2(D(v) - d)$ by the inequality for w in Lemma 6. □

In order to make u an ansatz, we need one more step:

Claim (p.114)**.** $u(p) := \lim_v \mathbf{M}_{z,K}(v)$ (or u^*, resp.) does not depend on z nor K.

Proof. Let $z' = x' + iy'$ be another coordinate, and $K' = B_{z'}(0; R')$ be a z'-disk with center p'. Observe that it suffices to prove for $K' \subset\subset K$ and $p = p'$.

Note that $\left|\frac{\mathrm{d}z}{\mathrm{d}z'}\right|$ has an lower bound $\frac{1}{m} > 0$ on K'. Therefore

$$\begin{aligned}
(\mathbf{M}_{z',K'}(v) - \mathbf{M}_{z',K'}(\overline{v}))^2 &\leq \frac{1}{\pi R'^2} \iint_{K'} (v - \overline{v})^2 \,\mathrm{d}x' \,\mathrm{d}y' \\
&\leq \frac{m^2}{\pi R'^2} \iint_{K} (v - \overline{v})^2 \,\mathrm{d}x \,\mathrm{d}y \\
&= \frac{m^2}{\pi R'^2} J_{z,K}(v - \overline{v}) \\
&\leq \frac{m^2 R^2}{\pi R'^2} (D(v) - d)
\end{aligned}$$

Because $\overline{v}$ is harmonic on K', we have $\mathbf{M}_{z',K'}(\overline{v}) = \overline{v}(p)$. Hence

$$u'(p) := \lim_v \mathbf{M}_{z',K'}(v) = \lim_v \overline{v}(p) = u(p)$$

$\square$

Proof of Theorem 2. We claim that (u, u^*) minimizes $D(\cdot)$.

First, observe that for B, a smaller z-disk concentric with K (the radius of B is smaller than the radius of K), $\lim_v D_B(v - \overline{v}) = 0$ follows from Lemma 9, and $\lim_v D_B(\overline{v} - u) = 0$ follows from the fact that the derivatives of $\overline{v}$ converge uniformly to those of u on B. Therefore $\lim_v D_B(v - u) = 0$ follows from the triangle inequality.

Next, associate each point p with a local coordinate z, a z-disk $K = K(p)$, and a smaller z-disk $B = B(p)$ such that $p \in B(p) \subset\subset K(p)$. Since $\{B(p)\}_{p\in S}$ covers S, there is a countable subcover $\{B(p_i)\}_{i=1}^{\infty}$ (by Lindelöf's covering theorem).

Next, we construct Diudonné factors μ_i by $\{K(p_i)\}$ and $\{B(p_i)\}$ such that $\sum_i \mu_i \equiv 1$ with each $\mu_i \in \mathcal{C}^1(S, [0,1])$, and vanishes outside $K(p_i)$. (See p.74)

The conclusions above lead to

$$\begin{aligned}
&\lim_v \int_S \mu_i[\mathrm{d}(v-u), \mathrm{d}(v-u)] \leq \lim_v \int_{K(p_i)} [\mathrm{d}(v-u), \mathrm{d}(v-u)] = 0 \\
\Rightarrow &\lim_v \int_S \mu_i[\mathrm{d}(v-u), \mathrm{d}(v-u)] = 0
\end{aligned} \tag{4}$$

In the statements above, $v - u \in \mathcal{C}^1(S)$. Naturally, for all $v_1, v_2 \in \mathscr{F}$, we define

$$D_i(v_1, v_2) = \int_{S\setminus\overline{K_0}} \mu_i[\mathrm{d}v_1, \mathrm{d}v_2] + \int_{K_0} \mu_i[\mathrm{d}v_1^*, \mathrm{d}v_2^*]$$

Observe that the triangle inequality of $\sqrt{D_i(\cdot)}$ holds. Hence

$$\left|\sqrt{D_i(v)} - \sqrt{D_i(u)}\right| \leq \sqrt{D_i(v-u)}$$

Combine this with (4). It follows that $\lim\limits_{v} \sum_{i=1}^{n} D_i(v) = \sum_{i=1}^{n} D_i(u)$. Observe that for all v, $\sum_{i=1}^{\infty} D_i(v)$ increases to $D(v)$. Therefore

$$D(u) = \lim_{n\to\infty} \sum_{i=1}^{n} D_i(u) = \lim_{n\to\infty} \lim_{v} \sum_{i=1}^{n} D_i(v) \leq \lim_{n\to\infty} \lim_{v} D(v) = d$$

By the definition of d, $D(u) \geq d$, so $D(u) = d$. As a result, for all $w \in \mathfrak{M}(S)$ and $\varepsilon \in \mathbb{R}$, $(u + \varepsilon w) \in \mathscr{F}$ implies $D(u + \varepsilon w) \geq D(u)$, so $D(u, w) = 0$.

Finally, we claim that the function U, given by u on the punched surface and $u^* + \Phi$ on the lid, minimizes $D_S(\cdot)$. It suffices to take any $w \in \mathfrak{M}(S)$ that vanishes in some neighborhood of every singularity of Φ, and check that $D_S(U, w) = 0$. We derive from the equation $D(u, w) = 0$ that

$$\begin{aligned} 0 = D(u,w) &= \int_{S\setminus\overline{K_0}} [\mathrm{d}u, \mathrm{d}w] + \int_{K_0} [\mathrm{d}u^*, \mathrm{d}w] \\ &= \int_{S\setminus\overline{K_0}} [\mathrm{d}U, \mathrm{d}w] + \int_{K_0} [\mathrm{d}(U - \Phi), \mathrm{d}w] \\ &= \int_{S} [\mathrm{d}U, \mathrm{d}w] - \int_{K_0} [\mathrm{d}\Phi, \mathrm{d}w] \\ &= D_S(U, w) - \int_{K_0} [\mathrm{d}\Phi, \mathrm{d}w] \\ &= D_S(U, w) - \int_{\partial K_0} w \frac{\partial \Phi}{\partial n}\, \mathrm{d}s \\ &= D_S(U, w) \end{aligned}$$

because $\frac{\partial\Phi}{\partial n} = 0$ along ∂K_0. $\square$

References

[1] Hermann Weyl, *The Concept of a Riemann Surface*, 3rd ed., Dover edition, translated by Gerald R. MacLane, Dover, Mineola, N.Y., 2009. pp.73–74, 93–118.